Some numerical homotopy invariants of certain classes of Flag manifolds.

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Abstract

We use knowlege of the cohomology ring of flag manifolds to compute values of a cup-length of certain classes of flag manifolds as well as a height of third Stiefel-Whitney class of canonical bundle of certain classes of flag manifolds. Using known theorem about Lusternik-Schnirelmann category, these results give lower bounds of a number of critical points of any smooth real function defined on these manifolds.

Introduction and motivation

In early 30's of the 20th century, L. Lusternik and L. Schnirelmann introduced new homotopy invariant of manifolds called category. The category of a space X, denoted by $\mathrm{cat}(X)$ is defined as the least n such that there is open cover $U_1, \dots, U_n$ of X with each U_i contractible in X (our definition is slightly different – shifted by one – from the one given in [2]). The motivation behind Lusternik-Schnirelmann category was to estimate the number of critical points of real function defined on smooth closed manifold. In particular, the Lusternik-Schnirelmann category gives lower bound for the number of critical points of a functions of aforementioned type,

$$\mathrm{cat}(M) \leq \mathrm{Crit}(M).$$

The Lusternik-Schnirelmann category was used e.g. in the solution of the problem of existence of closed geodesics on a surface of topological type of the sphere. The closed geodesics actually manifest as critical points of certain energy functional [2].

For a particular space it is usually not easy to find the value of Lusternik-Schnirelmann category and one is restricted to use some estimates which are usually easier to compute. One of fundamental bounds is given by a cup-length of a space.

For a space X and commutative ring R, the R-cup-length of the space X, denoted $\mathrm{cup}_R(X)$, is defined as a supremum of all integers m such that the cup product of m elements in reduced cohomology ring $\widetilde{H}^*(X;R)$ is nonzero, symbolically

$$\mathrm{cup}_R(X) = \sup\{m \in \mathbb{Z}; \prod_{i=1}^{m} x_i \neq 0,\ x_i \in \widetilde{H}^*(X;R)\},$$

where $\prod_{i=1}^{m} x_i$ denotes cup-product in cohomology. It is not very hard to show that [2]

$$\mathrm{cup}_R(X) + 1 \leq \mathrm{cat}(X).$$

Using this inequality and obvious open coverings one can easily show that e.g. $\mathrm{cat}(S^2) = 2$ and $\mathrm{cat}(T^2) = 3$, where S^2 is 2-sphere and $T^2 = S^1 \times S^1$ is 2-torus.

Closely related to cup-length is notion of height of cohomology class $x \in \widetilde{H}^*(X)$, which is defined as a number

$$\mathrm{ht}(x) = \sup\{n \in \mathbb{Z}; x^n \neq 0\}.$$

Hiller [4] computed the height of the first Stiefel-Whitney class of canonical bundle over Grassmannians with few exceptions. Later Stong [7] found all values of height the first Stiefel-Whitney class of canonical bundle over Grassmannians and used it to compute $\mathbb{Z}_2$-cup-length of some Grassmannians. Similarly Dutta and Khare [3] computed the height of second Stiefel-Whitney class of Grassmannians and following Stong, they used it to compute $\mathbb{Z}_2$-cup-length of more Grassmannians.

A natural generalization of Grassmannians are flag manifolds and the method introduced by Stong in [7] can be generalized and used to compute the cup-length of these manifolds.

Cohomology of flag manifolds

Given positive integers $n_1, \dots, n_q$, with $n_1 + \dots + n_q = n$ a flag of type $(n_1, \dots, n_q)$ is q-tuple $(S_1, \dots, S_q)$ of mutually orthogonal vector subspaces in $\mathbb{R}^n$ such that $\dim S_i = n_i$. Set of all flags of fixed type $(n_1, \dots, n_q)$ is denoted by $F(n_1, \dots, n_q)$ and can be identified with homogeneous space $O(n)/O(n_1) \times \dots \times O(n_q)$. This identification defines structure of closed manifold on the set $F(n_1, \dots, n_q)$. Special case of a flag manifold is Grassmann manifold $G_k(\mathbb{R}^{n+k}) = F(n, k)$ of linear k-subspaces of real $(n+k)$-vector space $\mathbb{R}^{n+k}$.

Let γ_j be the canonical vector bundle over $F(n_1, \dots, n_q)$ and denote by $w_i(\gamma_j) \in H^i(F(n_1, \dots, n_q); \mathbb{Z}_2)$ the Stiefel-Whitney class of γ_j. By Borel [1] the $\mathbb{Z}_2$-cohomology ring of flag manifold $H^*(F(n_1, \dots, n_q); \mathbb{Z}_2)$ is quotient polynomial ring

$$\mathbb{Z}_2[w_1(\gamma_1), \dots, w_{n_1}(\gamma_1), \dots, w_1(\gamma_q), \dots, w_{n_q}(\gamma_q)]/\mathcal{I}$$

where the ideal $\mathcal{I}$ is given by identity

$$\prod_{j=1}^{q}(1 + w_1(\gamma_j) + \dots + w_{n_j}(\gamma_j)) = 1.$$

The knowledge of the cohomology ring allows, in principle, to compute cup-length of flag manifolds and/or height of a (Stiefel-Whitney) class, however in general it is still not easy to determine, when given class is an element of the ideal. Considerable simplification of computations comes from the method used by Stong [7] for Grassmann manifolds which was generalized to flag manifolds by Lörinc and Korbaš [5].

Generalized Stong method

For the flag manifold $F(\underbrace{1, \dots, 1}_{m})$ denote $e_i = w_1(\gamma_i)$. Let

$$p : F(\underbrace{1, \dots, 1}_{n_1}, \dots, \underbrace{1, \dots, 1}_{n_q}) \to F(n_1, \dots, n_q)$$

be a map defined by

$$p(S_1, \dots, S_{n_1}, \dots, S_{\nu_{q-1}+1}, \dots, S_n) = (S_1 \oplus \dots \oplus S_{n_1}, \dots, S_{\nu_{q-1}+1} \oplus \dots \oplus S_n),$$

where $\nu_j = n_1 + \dots + n_j$.

Then we have that value of $u \in H^{top}(F(n_1, \dots, n_q), \mathbb{Z}_2)$ on the fundamental class of $F(n_1, \dots, n_q)$ is the same as the value of

$$p^*(u) \cdot e_1^{n_1-1} e_2^{n_1-2} \cdots e_{\nu_1-1} e_{\nu_1+1}^{n_2-1} e_{\nu_1+2}^{n_2-2} \cdots e_{\nu_2-1} \cdots e_{\nu_{q-1}+1}^{n_q-1} e_{\nu_{q-1}+2}^{n_q-2} \cdots e_{\nu_q-1}$$

on the fundamental class of $F(\underbrace{1, \dots, 1}_{n_1}, \dots, \underbrace{1, \dots, 1}_{n_q})$.

Moreover, the nonzero monomials in $H^{top}(F(\underbrace{1, \dots, 1}_{m}))$ are precisely those of the form $e_{\sigma(1)}^{m-1} \cdots e_{\sigma(i)}^{m-i} \cdots e_{\sigma(m)}^{0}$ for any permutation σ of the set $\{1, \dots, m\}$ (see [5]).

Selected results

Using generalized Stong's method we found cup-length of flag manifolds $F(2, 2, n_3)$ and $F(1, 3, 2^{s+1} - 3)$ and height of Stiefel-Whitney class $w_3(\gamma_4)$ of Grassmann manifold $F(4, n)$ for all $n_3 \geq 2$, $s \geq 2$ and n. Some of the obtained results are presented below.

Theorem 1. *For any integer $n_3 \geq 2$, let s be the unique integer such that $2^s < n_3 + 2 \leq 2^{s+1}$. Then*

$$\mathrm{cup}_{\mathbb{Z}_2}(F(2, 2, n_3)) = \begin{cases} 2^{s+1} + 2n_3, & \text{if } 2^s - 2 < n_3 \leq 2^{s+1} - 4, \\ 7 \cdot 2^s - 6, & \text{if } n_3 = 2^{s+1} - 3, \\ 2^{s+3} - 5, & \text{if } n_3 = 2^{s+1} - 2. \end{cases}$$

Theorem 2. *Let $s \geq 4$, $n + 4 = 2^{s+p-1} + 2^{s-2} + 2^{s-3} + 2 + t$, with*

$$\begin{aligned} &0 \leq t \leq 2^{s-3} - 1 && \text{if } p > 0, \\ &0 \leq t \leq 2^{s-3} && \text{if } p = 0. \end{aligned}$$

Then the height of w_3 in $H^*(F(4, n); \mathbb{Z}_2)$ is

$$ht(w_3(\gamma_4)) = 2^{s+p-1} + 2^{s-1} - 1.$$

Consequently, theorem 1 gives lower bound for Lusternik-Schnirelmann category of the flag manifold $F(2, 2, n_3)$.

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