Envelopes of planar curves in applications

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Abstract

One-parametric families of planar curves are a very fruitful topic in applications. We consider convex compact curves representing motion of a robot in a plane changing its local shape using a family of affine transformations. Another interpretations of the results are available as well. We discuss briefly theoretical issues such as singularities and the numerical aspects of the computation as well as approximation in a suitable spline space mostly required by applications.

Envelope of a family of hypersurfaces

- Let $f \colon \mathbb{R}^{m+1} \times \mathbb{R}^k \to \mathbb{R}$, $i = 1, \ldots, k$ be a smooth mapping. Let $Z(f(., t_1, \ldots, t_k))$ be the zero set of the mapping f for some fixed values of parameters $t_1, \ldots, t_k \in \mathbb{R}$. Assuming it is a variety of dimension m, we get a k-parametric family of such varieties defined implicitly.
- The conditions to be satisfied for the envelope (silhouette) points of the family is that the tangent vectors $\frac{\partial f}{\partial t_1} = 0, \ldots, \frac{\partial f}{\partial t_k} = 0$.
  A similar approach works for m-dimensional varieties in $\mathbb{R}^d$ with k-parameters. The condition leads to certain rank deficiency of the Jacobi matrix.
- We further consider the case $m = 1$ and $k = 1$ in this work – a one parameteric family of plane curves.

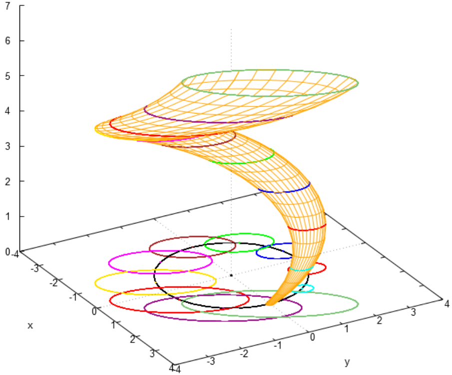
Particular cases in applications

There are many applications in low dimensional geometry which require envelopes of certain type

- offset curves/surfaces necessary in industry such as NC milling, cutting parts out of large material pieces (wide area developed by R. Farouki and others)
- tollerancing – curve with controlling its error in a certain region around the „skeletal“ curve (J. Wallner et al.)
- motion planning – non-colision control of the movement of a robot
- typeface design – construction of letters in fonts for typografical systems
- skinning – having samples (even just a few) of certain family of curves/surfaces, constructive approximate the envelope of the family

Family of curves and its envelope

It is very useful to consider the family of curves as a family of curves each in a separate plane parameterized by t. The envelope of one-parametric family of plane curves might also be defined using special projection of that surface $Z(f) \subset \mathbb{R}^2 \times \mathbb{R}$ into the plane $\mathbb{R}^2$.



How to compute the envelope

The envelope of a family of plane curves can be in general computed using the set of equations

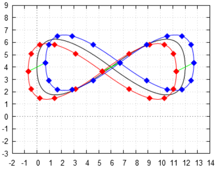
$$f(x, y, t) = 0, \quad \frac{\partial f}{\partial t}(x, y, t) = 0.$$

Geometrically, the equations can be interpreted in several useful ways.

- The common roots are the intersection of two implicitly defined surfaces in $\mathbb{R}^3$. Each point of intersection has tangent plane parallel to the axis t of the coordinate system x, y, t.
- Calculating in real plane only, it can be viewed as the computing of the curve parameterized by t consisting of points of the family such that tangent vector of the envelope generates the same space as the tangent vector of the corresponding curve in the family.

What else can be a part of the envelope

- If the curves in the family have endpoints, one has to consider them. They my produce part of the envelope which do not satisfy the above equations. In special cases, e.g. when the moving curve is segment, the envelope may contain parts traced by the endpoints only (on figure just trace is marked, the countour is missing in upper left and lower right).



- If the skeleton curve has endpoints (even in the case of a closed curve which is considered as a non-closed curve), some subarcs of the family curves can be a part of the envelope (see figure with ellipses above around the start/endpoint of the skeletal circle).

Setup for applications in parameteric form

In applications, a parameteric form of curves and surfaces is popular and dominant. We consider a plane curve $C(t)$, $t \in I$ some interval, called skeleton/path and a family of convex plane curves $D(s,t)$, $t \in I, s \in J$ for some interval J, representing e.g. a moving robot. The surface $P(s,t) = C(t) + D(s,t)$ is interpreted as the movement of the curve D along the skeletal curve C.
As stated above, we embed the plane with each curve of the family into $\mathbb{R}^3$ with standard basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Let $C\colon I \to \mathbb{R}^2$ be defined as $C(t) = (c_1(t), c_2(t), 0)$ and $D\colon \mathbb{R}^2 \to \mathbb{R}^3$ as $D(s,t) = (d_1(s,t), d_2(s,t), t)$. The surface obtained in this way provides the envelope as the contour of the surface in the projection to plane in the direction of the t-axis.

Clearly, it is given by the points at which the normal is perpendicular to the axis t. Computationally,

$$\langle P_s \times P_t, \vec{e}_3 \rangle = 0$$

or

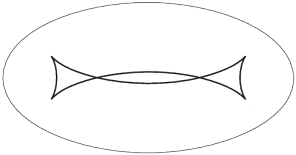
$$d_{1,s}(c_{2,t} + d_{2,t}) - d_{2,s}(c_{1,t} + d_{1,t}) = 0.$$

Clearly, this is a non-linear implicit equation of the set of parameters (s, t) in plane which identify the point of the surface $P(s, t)$ forming the envelope. Projecting them into the plane, we get the generic part of the envelope.

For the application purpose, it is enough that the family $D(s, t)$ is taken as the linear/affine map of some starting curve (e.g. a line, a circle, a convex curve even with corners – non C^1 continuous at finite number of points). In case of piecewise polynomial curves, it is advisable consider them piece by piece. The envelope might be computationally more difficult to obtain if the corners occur.

Regularity of the envelope

The properties of the envelope heavily depend on the properties of the curves in the family. Even for a smooth family, the envelope can have singular points. As an example, consider a family of circles with centers on an ellipse with constant radius greater than the radius of the smallest osculating curve.



There are many known properties helping the computation of singularities:

- if the projection line is the osculating line of the contour at the corresponding point, the cusp at the envelope appears;
- the simultaneous zero $\frac{\partial f}{\partial t} = 0, \ldots, \frac{\partial^r f}{\partial t^r} = 0$, $r > 2$ give another class of singularities of higher order at the envelope; clearly, the case $r = 2$ might give them all upto their internal structure.

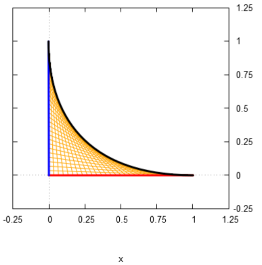
Detection of special cases

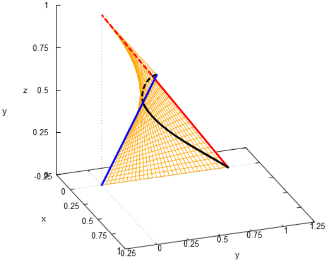
The special cases of the envelope are given either by local geometry of the family of the curves (see the ellipse above) or by the global properties such as self-intersection. The local properties can be computed as roots of special functions (e.g. difference in curvatures for skeleton curves and circle for offsets).

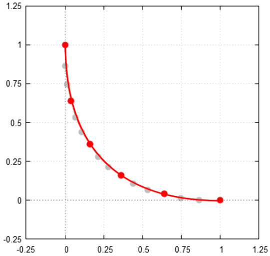
The global intersection of the envelope pose a more difficult problem. One has to compute singular points of the polynomial segments of the spline as well as the intersection of every pair of segments of the spline. The former task can be performed also as an intersection problem $(f = 0, \nabla f = \vec{\mathbf{0}})$. Since all equations are non-linear and singularities are isolated, the computation might be problematic using floating point arithmetic. However, envelope without the correct detection of singularities are usually not required.

Approximation of regular parts

The regular part of the envelope is approximated using a C^1 cubic spline (right figure), which is often required in the applications. Having sampled points and normals of the envelope, one might compute the tangent vectors at the points. Their length can be either used as a shaping parameter or it can be computed using various heuristic approaches based on asymptotic analysis. The spline is then computed using Hermite's formulas.







Conclusion

- We described a certain basic approach of computing envelope of a family of plane curves. The result is used in an application for the design area and some restrictions from the industrial partner were used.
- The complete solution of the task still requires considering singular situations which can be completely covered in case of bounded degree of the skeleton curve as well as curves of the family.
- The singularities in the envelope form many technical difficulties during the numerical computations. We plan to cover at least ordinary singularities.
- The higher order continuity splines could be useful, rational splines are challengeing in this case. A lot is known for PH hodograph curves which are essentially quintic.
- Identifying free parameters and approximation procedures minimizing certain functional such as L_p-norm might be used in order to minimize the number of segments.

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