D. KRUPKA / J. JANYŠKA

LECTURES ON DIFFERENTIAL INVARIANTS

UNIVERZITA J/ E/ PURKYNĚ V BRNĚ

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CONTENTS

Part 2

NATURAL GEOMETRIC OPERATIONS: EXAMPLES

LIST OF STANDARD SYMBOLS

$\emptyset$	empty set
$\times$	Cartesian product
$f\|_V$	restriction of a mapping f to a set V
id_X	identity mapping of a set X
$\circ$	composition of mappings, composition of jets
D	derivative
D^k	k-th derivative
D_i, $\partial/\partial x^i$	i-th partial derivative
d_k	k-th formal derivative
$\otimes$	tensor product
$\oplus$	direct sum, Whitney sum
$.$	group multiplication, action of a group
$[\ ,\ ]$	bracket, Lie bracket
$\mathrm{Mor}\,\mathscr{C}$	morphisms of a category $\mathscr{C}$
$\mathrm{Ob}\,\mathscr{C}$	objects of a category $\mathscr{C}$
proj	projection functor
dim	dimension
ker	kernel
im	image
rank	rank
det	determinant
R	real numbers
R^n	ordered n-tuples of real numbers
$GL_n(R)$	general linear group
$GL(E)$	group of linear transformations of a vector space E
T^r_sE	tensors of type (r, s) on E
E^*	dual vector space of a vector space E
T_xX	tangent space of a manifold X at a point x
T_xf	tangent mapping to f at a point x
T	tangent functor
e, e_G	identity of a group G
$L(G)$	Lie algebra of a group G
exp, $\exp_G$	exponential mapping of a Lie group G
$H\times_\varphi K$	exterior semi-direct product of groups H, K
$H\times_s K$	interior semi-direct product of subgroups $H, K \subset G$

G/H	quotient group
P/G	orbit space
$[q]$	equivalence class of a point q
$[q]_K$	K-orbit of a point q
$Y\times_G P$	fiber bundle with fiber P associated with principal fiber bundle Y
$J^r_x f$	r-jet of a mapping f at a point x
$J^r(X, Y)$	manifold of r-jets with source in X and target in Y
$J^r f$	r-jet prolongation of a mapping f
$J^r Y$	r-jet prolongation of a fibered manifold Y
$J^r\xi$	r-jet prolongation of a projectable vector field ξ
J^r	r-jet prolongation functor
L^r_n	r-th differential group of R^n
$F^r X$	bundle of r-frames over X
F^r	r-frame lifting
$F^r_Q X$	fiber bundle with fiber Q asociated with $F^r X$
F^r_Q	Q-lifting
∂_ξ	Lie derivative with respect to a vector field ξ
$T^r_n P$	manifold of r-jets with source at $0 \in R^n$ and target in P
G^r_n	(r, n)-prolongation of a Lie group G

PREFACE

Part 1 of the present work, written by the first author (D. K.), is based on the lectures prepared for the seminar "The Calculus of Variations and Its Applications" at Brno University in the years 1977-78, and published in 1979 as a preprint of the Department of Algebra and Geometry, Faculty of Science, J. E. Purkyně University, Brno (Czechoslovakia) under the title *Differential Invariants* (*Lecture Notes*). In this printing, small changes in the text have been made, and a few references have been added.

To make the text more self-contained we added an introduction to Lie groups (Chapter 1), some paragraphs in Chapter 2 and Chapter 3 on the jet structures, and the theory of invariant tensors (Chapter 4) which is very useful in practical calculations of differential invariants.

The primary purpose was to explain, in a relatively closed manner, the theory of those geometric structures which play a basic role in the theory of the so called generally invariant (or covariant) Lagrangean structures (see [10], [18]). Indeed, working on the subject we had been led to more general problems of the theory of invariant (geometric, natural) operations, requiring more complexity, and a more general approach. Our main sources for the fundamental concepts of the theory of jets and jet prolongations of geometric structures are Ehresmann, Libermann, and Kolář (see e.g. [5], [6], [8], [9], [19]). For the theory of fiber bundles and, in particular, of natural fiber bundles and their relationship to invariants we use Nijenhuis [20], Sulanke and Wintgen [23], and Terng [24]. The reader can consult the classical theory of invariants with Dieudonné and Carrell [4], Gurevich [7], Schouten [22], Thomas [25], and Weyl [26]. The exposition of the theory of differential invariants in Part 1 of this book follows the author's papers [10], [11], and [12] (see also [13] – [17]).

Main problem of the theory of differential invariants is to give a complete classification of them for concrete underlying geometric structures. To solve this problem, there are in fact at least four methods available: (1) the method of Lie equations, (2) the use of auxiliary formal connection, transforming the initial problem to a tensorial one, (3) the method of passing to a proper quotient group of the differential group, and to the quotient group action, and (4) the algebraic method. The aim of Part 2, written by the second author (J. J.), is to demonstrate the first and the second method on a few examples of natural differential operators and natural constructions with tensors, connections, and metric fields. The topics and results we discuss are due to Janyška [41], [42], Kolář [50], [51], Kolář and Michor [52], Kowalski and Sekizawa [54], and Krupka and Mikolášová [60]. A wide variety of further possible examples and illustrations of all of the methods (1)–(4) (e.g., invariants of a linear connection, invariant lagrangians and natural Lagrangean structures, liftings of tensor fields, invariants of a metric) can be found in the references.

Part 1 of this work has been completed during the first author's stay at Instituto di Matematica Applicata "Giovanni Sansone" in Florence (Italy). He is grateful to CNRS for creating excellent conditions for work, and to Professor M. Modugno for many valuable discussions on the subject.

Brno, December 1986

Demeter Krupka
Josef Janyška

PART 1

ELEMENTARY THEORY OF DIFERENTIAL INVARIANTS

1. LIE GROUPS

This introductory chapter is devoted to basic concepts of the theory of Lie groups and Lie group actions on smooth manifolds. We discuss some selected topics which are needed later on in the theory of differential invariants. Main notions of Sections 1.1 – 1.3 are the following: Lie group, Lie group homomorphism, Lie subgroup, Lie algebra of a Lie group, exponential mapping; Lie transformation group, orbit space, orbit manifold, equivariant mapping, fundamental vector field; semi-direct product of Lie groups.

1.1. Lie groups. A *Lie group* is a set G endowed with the structure of a smooth manifold and the structure of a group, with group operation denoted multiplicatively, such that the mapping $G \times G \ni (g, h) \to g \,.\, h^{-1} \in G$ is smooth. A *Lie group homomorphism* $f : G \to H$ is a smooth group homomorphism of a Lie group G into a Lie group H. A Lie group homomorphism which is a diffeomorphism is called a *Lie group isomorphism*.

The additive group of real numbers R together with its natural smooth structure is a Lie group. A Lie group homomorphism $\alpha : R \to G$ is called a *one-parameter subgroup* of G.

The identity of a Lie group G is usually denoted by e_G, or e, if there is no danger of confusion.

Recall that a mapping of manifolds $f : X \to Y$ is called an *immersion at a point* $x \in X$, if the tangent mapping $T_x f : T_x X \to T_{f(x)} Y$ is injective; f is called an *immersion* if it is an immersion at every point. A manifold X is called an *immersed submanifold* of a manifold Y, if (1) $X \subset Y$ (as a set), and (2) the canonical inclusion $\iota_X : X \to Y$ is an immersion. The smooth structure of X is uniquely determined by conditions (1) and (2). We note that the topology of an immersed submanifold $X \subset Y$ is not necessarily induced from Y.

A Lie group H is called a *Lie subgroup* of a Lie group G, if (1) $H \subset G$ (set theoretically) and H is a subgroup of G, and (2) H is an immersed submanifold of G.

Let G be a Lie group. For every $g \in G$ the mapping $G \ni h \to L_g(h) = g \, . \, h \in G$ (resp. $G \ni h \to R_g(h) = h \, . \, g \in G$) is called the *left* (resp. *right*) *translation* by g. Obviously both L_g and R_g are diffeomorphisms, $L_g \circ L_h = L_{g.h}$, $R_g \circ R_h = R_{h.g}$ so that $(L_g)^{-1} = L_{g^{-1}}$, $(R_g)^{-1} = R_{g^{-1}}$, and $L_e = R_e = \mathrm{id}_G$ (the identity mapping of G).

Recall that a (*real*) *Lie algebra* is a (real) vector space E together with a bilinear mapping $E \times E \ni (\xi, \zeta) \to [\xi, \zeta] \in E$ such that (1) $[\xi, \xi] = 0$ for all $\xi \in E$, and (2) $[\xi, [\zeta, \lambda]] + [\zeta, [\lambda, \xi]] + [\lambda, [\xi, \zeta]] = 0$ for all $\xi, \zeta, \lambda \in E$. The mapping $(\xi, \zeta) \to$ $\to [\xi, \zeta]$ is called the *bracket* of the Lie algebra E. The identity (2) is called the *Jacobi identity*.

Let E_1, E_2 be two vector subspaces of a Lie algebra E, and let $[E_1, E_2]$ denote the vector subspace of E generated by the vectors $[\xi, \zeta]$, where $\xi \in E_1$, $\zeta \in E_2$. A vector subspace $F \subset E$ is called a *Lie subalgebra* of E if $[F, F] \subset F$; F is called an *ideal* of E if $[F, E] \subset F$.

A linear mapping of Lie algebras $f : E \to F$ is called a *Lie algebra homomorphism* if $f([\xi, \zeta]) = [f(\xi), f(\zeta)]$ for all $\xi, \zeta \in E$. The kernel (resp. image) of f is then an ideal of E (resp. a Lie subalgebra of F).

Let E be a finite-dimensional Lie algebra, that is, E is finite-dimensional as a vector space, let $m = \dim E$. Let $(e_1, e_2, \ldots, e_m)$ be a basis of E. There exist uniquely determined real numbers γ^i_{jk} such that

$$(1.1.1) \qquad [e_j, e_k] = \gamma^i_{jk} e_i,$$

(summation on i). These numbers are called the *structure constants* of the Lie algebra E with respect to the basis $(e_1, e_2, \ldots, e_m)$. It follows from the definition of a Lie algebra that

$$(1.1.2) \qquad \begin{aligned} &\gamma^i_{jk} + \gamma^i_{kj} = 0, \\ &\gamma^i_{jp}\gamma^j_{kq} + \gamma^i_{jq}\gamma^j_{pk} + \gamma^i_{jk}\gamma^j_{qp} = 0. \end{aligned}$$

Conversely, if we have a vector space E of dimension m, a basis $(e_1, e_2, \ldots, e_m)$ of E, and a system of real numbers γ^i_{jk} satisfying these conditions, then there exists a unique Lie bracket on E for which γ^i_{jk} are the structure constants with respect to $(e_1, e_2, \ldots, e_m)$. This Lie bracket is obtained by defining the Lie bracket $[e_j, e_k]$ as in (1.1.1), and then extending it to $E \times E$ on the bilinearity condition.

A vector field ξ on a Lie group G is called *left invariant* if for every $g \in G$

$$(1.1.3) \qquad T_h L_g \, . \, \xi(h) = \xi(L_g(h))$$

for all $h \in G$. By definition ξ is left invariant if and only if the pair of vector fields

(ξ, ξ) is L_g-related for every $g \in G$. Hence the bracket $[\xi, \zeta]$ of any two left invariant vector fields ξ, ζ is also a left invariant vector field. Thus the set of left invariant vector fields on G has the structure of a subalgebra of the Lie algebra of vector fields on G.

A left invariant vector field ξ on G is always *complete*, i.e. every integral curve of ξ can be prolonged to an integral curve defined on R. Clearly, let $(-a, a)$ be an open interval in R, and let $\alpha : (-a, a) \to G$ be an integral curve of ξ; that is,

$$(1.1.4) \qquad \frac{d\alpha}{dt} = \xi(\alpha(t))$$

on $(-a, a)$. Let $g \in G$ be any point, and consider the curve $(-a, a) \ni t \to \beta(t) = g \cdot \alpha(t) = (L_g \circ \alpha)(t) \in G$. We have, since ξ is left invariant,

$$(1.1.5) \qquad \frac{d\beta}{dt} = T_{\alpha(t)}L_g \cdot \frac{d\alpha}{dt} = T_{\alpha(t)}L_g \cdot \xi(\alpha(t)) = \xi(L_g(\alpha(t))) = \xi(\beta(t)).$$

Thus β is also an integral curve of ξ. If α starts at $e \in G$, i.e. $\alpha(0) = e$, then $\beta(0) = g$, and β starts at g. Let $s \in (-a, a)$, $s > 0$. We take $g = \alpha(s)$ and put $\gamma(t) = \beta(t - s)$; γ is a curve in G defined on $(-a + s, a + s)$, and

$$(1.1.6) \qquad \frac{d\gamma}{dt} = \left\{\frac{d\beta}{dt'}\right\}_{t-s} = \xi(\beta(t - s)) = \xi(\gamma(t)).$$

Hence γ is an integral curve of ξ, and $\gamma(s) = \beta(0) = \alpha(s)$. By the uniqueness of integral curves, $\gamma = \alpha$ on $(-a, a) \cap (-a + s, a + s) = (-a + s, a)$, and α can be prolonged to $(-a, a + s)$. Thus α can be prolonged to R, and the vector field ξ is complete.

Let VG be the Lie algebra of vector fields on G, and let $V_L G$ be its subalgebra of left invariant vector fields. It is easily seen that $V_L G$ is finite-dimensional and its dimension is equal to the dimension of G. We shall show that $V_L G$ is linearly isomorphic to the vector space $T_e G$, the tangent space of G at the identity. For every tangent vector $\xi \in T_e G$ and every $h \in G$ we define a vector $\xi_L(h) \in T_h G$ by

$$(1.1.7) \qquad \xi_L(h) = T_e L_h \cdot \xi.$$

ξ_L is a smooth vector field on G. Let $g \in G$ be any point. We have $\xi_L(L_g(h)) = \xi_L(g \cdot h) = T_e L_{g \cdot h} \cdot \xi = T_e(L_g \circ L_h) \cdot \xi = T_h L_g \cdot (T_e L_h \cdot \xi) = T_h L_g \cdot \xi_L(h)$. Thus ξ_L is a left invariant vector field. We get a mapping $T_e G \ni \xi \to \xi_L \in V_L G$ which is obviously linear. On the other hand, the mapping $V_L G \in \xi_L \to \xi_L(e) \ni T_e G$ is also linear, and is the inverse to the mapping $\xi \to \xi_L$. Therefore, the vector spaces $V_L G$ and $T_e G$ are linearly isomorphic, and the dimension of $V_L G$ is equal to the dimension of G.

The vector field $\xi_L \in V_L G$ is said to be *associated* with the tangent vector $\xi \in T_e G$.

Using the linear isomorphism $\xi \to \xi_L$ we define a bracket on T_eG by the formula

(1.1.8) $$[\xi, \zeta] = [\xi_L, \zeta_L](e).$$

With this bracket, T_eG becomes a Lie algebra which is called the *Lie algebra* of the Lie group G, and is denoted by $L(G)$.

Let $\xi \in L(G)$ be any vector, ξ_L the left invariant vector field on G associated with ξ. Let $R \ni t \to \exp_G t\xi \in G$ denote the integral curve of ξ_L passing through the identity e at $t = 0$.

Theorem 1.1. *A mapping $\alpha : R \to G$ is a Lie group homomorphism if and only if there exists a vector $\xi \in L(G)$ such that $\alpha(t) = \exp_G t\xi$ for all $t \in R$.*

Proof. 1. Let $\xi \in L(G)$ be any vector. We want to show that

(1.1.9) $$\exp_G (s + t)\, \xi = \exp_G s\xi \,.\, \exp_G t\xi$$

for all $s, t \in R$. By the uniqueness of integral curves it is sufficient to show that the mappings $t \to \exp_G (s + t)\, \xi$, $t \to \exp_G s\xi \,.\, \exp_G t\xi = L_{\exp_G s\xi}(\exp t\xi)$ are both integral curves of ξ_L passing through the point $\exp s\xi$ at $t = 0$. Since

(1.1.10) $$\frac{\mathrm{d}}{\mathrm{d}t} \exp_G (s + t)\, \xi = \left\{\frac{\mathrm{d}}{\mathrm{d}t'} \exp_G t'\xi\right\}_{s+t} \cdot \frac{\mathrm{d}t'}{\mathrm{d}t} = \xi_L(\exp_G (s + t)\, \xi,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\exp_G s\xi}(\exp_G t\xi) = T_{\exp_G t\xi} L_{\exp_G s\xi} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \exp_G t\xi =$$
$$= T_{\exp_G t\xi} L_{\exp_G s\xi} \cdot \xi_L(\exp_G t\xi) =$$
$$= \xi_L(L_{\exp_G s\xi}(\exp_G t\xi)) = \xi_L(\exp_G (s + t)\, \xi),$$

we get that (1.1.9) holds.

Conversely, let $\alpha : R \to G$ be a one-parameter subgroup. Then $\alpha(0) = e$ and the tangent vector $\xi = \{d\alpha/\mathrm{d}t\}_0$ belongs to $L(G)$. Since for every $s, t \in R$, $\alpha(s + t) = \alpha(t) \,.\, \alpha(s) = L_{g(\cdot)}(\alpha(s))$, we have

(1.1.11) $$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \left\{\frac{\mathrm{d}}{\mathrm{d}s} \alpha(t + s)\right\}_0 = T_{\alpha(0)} L_{\alpha(t)} \cdot \left\{\frac{\mathrm{d}\alpha}{\mathrm{d}s}\right\}_0 = T_e L_{\alpha(t)} \cdot \xi = \xi_L(\alpha(t)).$$

Thus α is an integral curve of ξ_L passing through e at $t = 0$. By uniqueness of integral curves, $\alpha(t) = \exp_G t\xi$.

Theorem 1.2. *If $f : G \to H$ is a Lie group homomorphism, then $T_e f : L(G) \to L(H)$ is a Lie algebra homomorphism and for every $\xi \in L(G)$ and $t \in R$*

(1.1.12) $$f(\exp_G t\xi) = \exp_H t(T_e f \,.\, \xi).$$

Proof. Let $g, h \in G$ be any two points. Since $f(g \cdot h) = f(g) \cdot f(h)$, i.e., $f \circ L_g(h) = (L_{f(g)} \circ f)(h)$, we have

(1.1.13) $$T_{L_g(h)}f \circ T_h L_g = T_{f(h)} L_{f(g)} \circ T_h f.$$

Let $\xi \in L(G)$ be a vector, and let ξ_L be the left invariant vector field on G associated with ξ. Take $h = e_G$; then $\xi_L(g) = T_{e_G} L_g \cdot \xi$ and (1.1.13) yealds

(1.1.14) $$T_g f \cdot \xi_L(g) = T_{e_H} L_{f(g)} \circ T_{e_G} f \cdot \xi = T_{e_H} L_{f(e)} \cdot \zeta = \zeta_L(f(g)),$$

where $\zeta = T_{e_G} f \cdot \xi$. This shows that the vector fields ξ_L and ζ_L are f-related. Let $\xi_1, \xi_2 \in L(G)$ be any two vectors and denote $\zeta_1 = T_{e_G} f \cdot \xi_1, \zeta_2 = T_{e_G} f \cdot \xi_2$. Then $T_g f \cdot [\xi_{1L}, \xi_{2L}](g) = [\zeta_{1L}, \zeta_{2L}](f(g))$. Taking $g = e_G$ we obtain by definition of the Lie brackets in $L(G)$ and $L(H)$

(1.1.15) $$T_{e_G} f \cdot [\xi_1, \xi_2] = [\zeta_{1L}, \zeta_{2L}](e_H) = [T_{e_G} f \cdot \xi_1, T_{e_G} f \cdot \xi_2]$$

which proves that $T_{e_G} f$ is a Lie algebra homomorphism.

Let us prove the second assertion. Obviously, the mapping $t \to f(\exp_G t\xi)$ is a one-parameter subgroup of H: this mapping is smooth and for every $s, t \in R$, $f(\exp_G (s + t)\xi) = f(\exp_G s\xi \cdot \exp_G t\xi) = f(\exp_G s\xi) \cdot f(\exp_G t\xi)$ (Theorem 1.1). Since the tangent vector at $t = 0$ is

(1.1.16) $$\left\{\frac{\mathrm{d}}{\mathrm{d}t} f(\exp_G t\xi)\right\}_0 = T_e f \cdot \xi,$$

this one-parameter group must be precisely the one-parameter group $t \to \exp_H t(T_e f \cdot \xi)$. This proves the second assertion.

Let G be a Lie group. The mapping $L(G) \ni \xi \to \exp_G \xi \in G$, where $\exp_G \xi$ is the point on the curve $t \to \exp_G t\xi$ defined by $t = 1$, is called the *exponential mapping* of $L(G)$ into G.

Theorem 1.3. (1) *The exponential mapping* $L(G) \ni \xi \to \exp_G \xi \in G$ *is a local diffeomorphism at the point* $0 \in L(G)$.

(2) *If* $f: G \to H$ *is a Lie group homomorphism, then for each* $\xi \in L(G)$

(1.1.17) $$f(\exp_G \xi) = \exp_H (T_e f \cdot \xi).$$

Proof. 1. To show that $\exp_G$ is a smooth mapping define a vector field χ on $G \times L(G)$ by $\chi(g, \xi) = (\xi_L(g), 0)$. Since $\xi_L(g) = T_e L_g \cdot \xi$ is smooth on $G \times L(G)$, χ is also smooth, and the global flow $(t, g, \xi) \to \alpha(t, g, \xi)$ of χ is also smooth. But $\alpha(t, g, \xi) = (g \cdot \exp_G t\xi, \xi)$ since

(1.1.18) $$\left\{\frac{\mathrm{d}}{\mathrm{d}t}(g \cdot \exp_G t\xi, \xi)\right\}_0 = (T_e L_g \cdot \xi, 0) = (\xi_L(g), 0) = \chi(g, \xi).$$

Thus the mapping $R \times G \times L(G) \ni (t, g, \xi) \to (g \, . \exp_G t\xi, \xi) \in G \times L(G)$ is smooth, and so must be the mapping $L(G) \ni \xi \to e \, . \exp_G 1 \, . \, \xi = \exp_G \xi \in G$. Now let $\xi \in L(G)$ be any vector. We have

$$(1.1.19) \qquad \xi = \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \exp_G t\xi \right\}_0 = T_0 \exp_G . \, \xi,$$

and $T_0 \exp_G$ is the identity mapping of $L(G)$. In particular, the rank of $T_0 \exp_G$ is maximal, and $\exp_G$ is a local diffeomorphism at 0.

2. Relation (1.1.17) is a consequence of (1.1.12).

1.2. Semi-direct products of Lie groups. If G is a group we denote by Aut G the group of automorphisms of G.

Let H and K be groups, $\varphi : H \to \operatorname{Aut} K$ a homomorphism of groups. It is directly verified that the mapping

$$(1.2.1) \qquad \begin{aligned} &(H \times K) \times (H \times K) \ni ((h_1, k_1), (h_2, k_2)) \to (h_1, k_1) \, . \, (h_2, k_2) = \\ &= (h_1 \, . \, h_2, k_1 \, . \, \varphi(h_1)(k_2)) \in H \times K \end{aligned}$$

defines the structure of a group on the set $H \times K$. This group is called the *exterior semi-direct product* of the groups H and K, associated with the homomorphism φ, and is denoted by $H \times_\varphi K$.

If $(h_1, k_1), (h_2, k_2), (h_3, k_3) \in H \times_\varphi K$ are any elements, we have

$$(1.2.2) \qquad \begin{aligned} &(h_1, k_1) \, . \, (h_2, k_2) \, . \, (h_3, k_3) = \\ &= (h_1 h_2 h_3, k_1 \, . \, \varphi(h_1)(k_2) \, . \, \varphi(h_1)(\varphi(h_2)(k_3)). \end{aligned}$$

If e_H (resp. e_K) is the identity element of the group H (resp. K), then (e_H, e_K) is the identity element of $H \times_\varphi K$, and the inverse element of (h, k) is

$$(1.2.3) \qquad (h, k)^{-1} = (h^{-1}, \varphi(h^{-1})(k^{-1})).$$

The exterior semi-direct product of groups has the following elementary properties.

Theorem 1.4. *Let $H \times_\varphi K$ be the exterior semi-direct product of groups H and K, associated with a homomorphism $\varphi : H \to \operatorname{Aut} K$.*

(*a*) *The mapping $h \to (h, e_K)$ is an isomorphism of H onto the subgroup $H^* = H \times \{e_K\}$ of $H \times_\varphi K$.*

(*b*) *The mapping $k \to (e_H, k)$ (resp. $(h, k) \to h$) is an isomorphism (resp. homomorphism) of K (resp. $H \times_\varphi K$) onto the subgroup $K^* = \{e_H\} \times K$ of $H \times_\varphi K$ (resp. the group H), and K^* is a normal subgroup of $H \times_\varphi K$.*

(*c*) *$H \times_\varphi K$ is the product of H^* and K^*,*

$$(1.2.4) \qquad H \times_\varphi K = H^* \, . \, K^* = K^* \, . \, H^*.$$

(*d*) *The quotient group $(H \times_\varphi K)/K^*$ is isomorphic with H.*

Proof. All these assertions are immediate consequences of the definition.

Let G be a group, e its identity element. We say that G is the *interior semi-direct product* of its subgroups H and K if the following three conditions are satisfied: (1) K is a normal subgroup of G, (2) $H \cap K = \{e\}$, (3) $K . H = G$. If G is the semi-direct product we write $G = H \times_s K$.

Theorem 1.5. *Let $G = H \times_s K$ be the interior semi-direct product of its subgroups H and K. Then each element $g \in G$ has a unique decomposition*

$$g = k . h, \tag{1.2.5}$$

where $k \in K$, $h \in H$.

Proof. Let $g \in G$; the existence of $k \in K$ and $h \in H$ such that (1.2.5) holds, follows from the definition, condition (3). To prove the uniqueness, suppose that $g = k_1 . h_1 = k_2 . h_2$. Then $h_1 . h_2^{-1} = k_1^{-1} . k_2$; since this element belongs to $H \cap K$, condition (2) gives the uniqueness of the decomposition (1.2.5).

Notice that each element g has a unique decomposition $g = h . k$, where $h \in H$, $k \in K$; this follows from (1.2.5) by passing to the inverse on both sides.

According to Theorem 1.5, the decomposition (1.2.5) defines two mappings $\alpha : G \to K$, $\beta : G \to H$, where

$$g = \alpha(g) . \beta(g). \tag{1.2.6}$$

These mappings satisfy

$$\begin{aligned} \alpha(g_1 . g_2) &= \alpha(g_1) . \beta(g_1) . \alpha(g_2) . (\beta(g_1))^{-1}, \\ \beta(g_1 . g_2) &= \beta(g_1) . \beta(g_2) \end{aligned} \tag{1.2.7}$$

for all $g_1, g_2 \in G$.

Theorem 1.6. *Let $G = H \times_s K$ be the interior semi-direct product of its subgroups H and K. Let $\varphi : H \to$ Aut K be a homomorphism of groups defined by*

$$\varphi(h)(k) = h . k . h^{-1}. \tag{1.2.8}$$

Then the mapping $G \ni g \to (\beta \times \alpha)(g) = (\beta(g), \alpha(g)) \in H \times_\varphi K$ is an isomorphism of groups.

Proof. This assertion follows from the properties of the mappings α, β (1.2.7).

From now on, we study the semi-direct products of *Lie* groups.

Theorem 1.7. *Let $H \times_\varphi K$ be the exterior semi-direct product of groups H and K,* *associated with a homomorphism of groups* $\varphi : H \to \operatorname{Aut} K$. *Suppose that* H *and* K *are Lie groups, and the mapping*

$$(1.2.9) \qquad H \times K \ni (h, k) \to \varphi(h)(k) \in K$$

is analytic. Then the group $H \times_\varphi K$ *endowed with the product manifold structure, is a Lie group.*

Proof. It is sufficient to check that the mapping $((h_1, k_1), (h_2, k_2)) \to (h_1, k_1) \cdot (h_2, k_2)^{-1}$ is analytic. By (1.2.1) and (1.2.3),

$$(1.2.10) \qquad (h_1, k_1) \cdot (h_2, k_2)^{-1} = (h_1 \cdot h_2^{-1}, k_1 \cdot \varphi(h_1 h_2^{-1})(k_2^{-1}))$$

which shows that this mapping is composed of analytic mappings and hence is analytic.

If the condition of Theorem 1.7 is satisfied we say that the Lie group $H \times_\varphi K$ is the *exterior semi-direct product* of Lie groups, associated with the homomorphism φ.

We have the following analogue of Theorem 1.4.

Theorem 1.8. *Let* $H \times_\varphi K$ *be the exterior semi-direct product of Lie groups, associated with a homomorphism* $\varphi : H \to \operatorname{Aut} K$.

(*a*) *The mapping* $h \to (h, e_K)$ *is an isomorphism of* H *onto the Lie subgroup* $H^* = H \times \{e_K\}$ *of* $H \times_\varphi K$.

(*b*) *The mapping* $k \to (e_H, k)$ (*resp.* $(h, k) \to h$) *is an isomorphism* (*resp. a homomorphism*) *of* K (*resp.* $H \times_\varphi K$) *onto the Lie subgroup* $K^* = \{e_H\} \times K$ *of* $H \times_\varphi K$ (*resp. the Lie group* H), *and* K^* *is a normal Lie subgroup of* $H \times_\varphi K$.

(*c*) *The quotient Lie group* $(H \times_\varphi K)/K^*$ *is isomorphic with the Lie group* H.

Proof. All these assertions follow from the standard properties of Lie groups and their homomorphisms.

Let a Lie group G be the interior semi-direct product of its subgroups H and K, $G = H \times_s K$. Suppose that H and K are Lie subgroups of G. As a group, G is isomorphic with the exterior semi-direct product $H \times_\varphi K$, where $\varphi : H \to \operatorname{Aut} K$ is defined by (1.2.8). Since H and K are Lie subgroups of G, this mapping is analytic, and $H \times_\varphi K$ is the exterior semi-direct product of Lie groups, associated with φ. We say that G is the *exterior semi-direct product* of its Lie subgroups H and K if the isomorphism $g \to (\beta \times \alpha)(g)$ (Theorem 1.6) is an isomorphism of Lie groups.

Theorem 1.9. *Let a group* G *be the interior semi-direct product of its subgroups* H *and* K. *Suppose that* G *is a Lie group and* H *and* K *are its Lie subgroups. Then the following three conditions are equivalent:*

(1) *G is the interior semi-direct product of its Lie subgroups H and K.*
(2) *The mapping* $G \ni g \to \alpha(g) \in K$ *is analytic.*
(3) *The mapping* $G \ni g \to \beta(g) \in H$ *is analytic.*

Proof. If $g \to (\beta(g), \alpha(g))$ is an analytic mapping then both β and α are also analytic; hence (1) implies (2) and (3). If the mapping $g \to \alpha(g)$ is analytic then so is the mapping $g \to \beta(g) = (\alpha(g))^{-1} . g$; hence (2) implies (3); analogously (3) implies (2). Finally, if β is analytic, α is also analytic, and so is $\beta \times \alpha$; hence (3) implies (1).

Theorem 1.10. *Let* $p : G \to H$ *and* $s : H \to G$ *be homomorphisms of Lie groups such that*

(1.2.11) $\qquad p \circ s = \mathrm{id}_H .$

Then ker p *is a normal Lie subgroup of* G, $s(H)$ *is a Lie subgroup of* G, *and* G *is the interior semi-direct product of Lie subgroups* $s(H) \times_s \ker p$.

Proof. $\ker p$ is obviously a normal Lie subgroup of G, and $s(H)$ is a subgroup and a submanifold, since s is a homeomorphism of H onto $s(H)$; hence $s(H)$ is a Lie subgroup. Let us check the conditions (2) and (3) of the definition of the interior semi-direct product. If $g \in G$ is any element, we set $g_0 = g . (s(p(g)))^{-1}$; then $p(g_0) = e_H$, i.e. $g_0 \in \ker p$. Hence $G = (\ker p) . s(H)$ which proves (3). Suppose $g \in \ker p \cap s(H)$. Then $p(g) = e_H$, $g = s(p(g_0))$, where $g_0 \in G$, so that $p(g) = = p(s(p(g_0))) = p(g_0) = e_H$, $g = s(p(g_0)) = s(e_H) = e$, where e is the identity of G; this proves (2). It remains to show that the mapping $g \to (s \circ p)(g)$ is analytic (Theorem 1.9, (3)); since this is obviously true, we are done.

1.3. Lie group actions. Let G be a Lie group, P a manifold. A smooth mapping $\Phi : G \times P \to P$ is called a *left action* of G on P, if (1) $\Phi(e, p) = p$ for all $p \in P$, and (2) $\Phi(g, \Phi(h, p)) = \Phi(g . h, p)$ for every $g, h \in G$ and $p \in P$. A manifold P endowed with a left action of a Lie group G is called a *left G-manifold*; G is called the *Lie transformation group* of P.

Let P be a left G-manifold with a left action Φ of G. For every $g \in G$ we define a smooth mapping $\Phi_g : P \to P$ by $\Phi_g(p) = \Phi(g, p)$. Obviously,

(1.3.1) $\qquad \Phi_e = \mathrm{id}_P, \qquad \Phi_{g . h} = \Phi_g \circ \Phi_h,$

where $e \in G$ is the identity, id_P is the identity mapping of P, and $g, h \in G$ are any elements. In particular, the mappings Φ_g, $\Phi_{g^{-1}}$ satisfy $(\Phi_g)^{-1} = \Phi_{g^{-1}}$, and Φ_g is a diffeomorphism. We call Φ_g the *transformation* of P by g. For every $p \in P$ we define a smooth mapping $\Phi_p : G \to P$ by $\Phi_p(g) = \Phi(g, p)$. Φ_p is called the *orbit mapping* at the point p. For every p, the set $\Phi_p(G) = \{q \in P \mid q = \Phi_p(g), g \in G\}$ is called the *G-orbit*, or the *orbit* of the point p.

Lemma 1.1. *The orbit mapping $G \ni g \to \Phi_p(g) \in P$ has constant rank.*

Proof. Let $g \in G$ be any element. We have by the chain rule, $T_e(\Phi_p \circ L_g) = T_g\Phi_p \circ T_eL_g$. But $\Phi_p \circ L_g(h) = \Phi(g \,.\, h, p) = \Phi(g, \Phi(h, p)) = \Phi_g \circ \Phi_p(h)$, that is, $\Phi_p \circ L_g = \Phi_g \circ \Phi_p$. Thus $T_p\Phi_g \circ T_e\Phi_p = T_g\Phi_p \circ T_eL_g$. Since the mappings $T_p\Phi_g$ and T_eL_g are linear isomorphisms, we have rank $T_e\Phi_p =$ rank $T_g\Phi_p$ as required.

Now we shall introduce basic types of left actions of Lie groups on manifolds. Let $\Phi : G \times P \to P$ be a left action of G on P. We say that Φ is *transitive*, if for every $p \in P$, $\Phi_p(G) = P$, i.e. there is only one G-orbit in P. Φ is called *effective* if the condition $\Phi_g = \mathrm{id}_P$ implies $g = e$. Φ is called *free* if for each $g \in G$, $g \neq e$, the transformation Φ_g of P has no fixed points, i.e., the condition $\Phi_g(p) = p$, where $g \in G$, implies $g = e$. Let $\Phi' : G \times P \to P \times P$ be the mapping defined by $\Phi'(g, p) = (p, \Phi(g, p))$. We say that Φ is *proper*, if for every compact set $K \subset P \times P$, the set $(\Phi')^{-1}(K) \subset G \times P$ is compact.

As above, let G be a Lie group, P a manifold. A smooth mapping $\Psi : P \times G \to P$ is called a *right action* of G on P, if (1) $\psi(p, e) = p$ for all $p \in P$, and (2) $\Psi(\Psi(p, g), h) = \Psi(p, g \,.\, h)$ for every $g, h \in G$ and $p \in P$. A manifold endowed with a right action of a Lie group G is called a *right G-manifold*; G is then called the *Lie transformation group* of P.

If P is a right G-manifold with a right action Ψ and we set for every $(p, g) \in P \times G$

$$\Phi(g, p) = \Psi(p, g^{-1}), \tag{1.3.2}$$

P becomes a left G-manifold with the left action Φ. It is clear that all the notions defined above for left G-manifolds, transfer immediately with the help of this correspondence to right G-manifolds.

Let $\Phi : G \times P \to P$ be a left action of a Lie group G on a manifold P. The relation "$p \sim q$ if and only if p, q belong to the same G-orbit in P" is an equivalence relation on P. Let P/G denote the quotient space, i.e. the set of G-orbits, and $\pi : P \to P/G$ the canonical projection. We shall consider the set P/G with the final topology with respect to π, i.e. the quotient topology; a set $U \subset P/G$ is open in this topology if and only if $\pi^{-1}(U)$ is open in P. It is easily seen that with respect to this topology, π is an open mapping: If $W \subset P$ is an open set, then $\pi^{-1}(\pi(W)) = \cup \Phi_g(W)$ (union over $g \in G$ of the open sets $\Phi_g(W)$). Obviously, since P is second countable, P/G is also second countable. The topological space P/G is called the *orbit space* of the left G-manifold P.

If there is no danger of confusion the G-orbit $\Phi_p(G)$ of a point $p \in P$ is denoted by $[p]$. Thus $[p] = \pi(p)$ is the element of the orbit space P/G whose representative is p.

Lemma 1.2. *Let $\Phi : G \times P \to P$ be a left action of G on P, $\pi : P \to P/G$ the canonical projection onto the orbit space. The orbit space P/G is Hausdorff if and only if the set $\mathscr{R} = \{(p, q) \in P \times P \mid [p] = [q]\}$ is closed.*

Proof. 1. Suppose that $\mathscr{R}$ is a closed subset of $P \times P$. Let $p, q \in P$ be representatives of two different points $[p], [q] \in P/G$; clearly, $p \neq q$ and $(p, q) \notin \mathscr{R}$. Since $\mathscr{R}$ is closed there exist a neighborhood U of p and a neighborhood V of q such that $(U \times V) \cap \mathscr{R} = \emptyset$, and does not exist a G-orbit containing a point of U and a point of V. This implies that $\pi(U) \cap \pi(V) = \emptyset$. But the canonical projection π is an open mapping. This implies that $\pi(U)$ (resp. $\pi(V)$) is a neighborhood of $[p]$ (resp. $[q]$), and P/G must be Hausdorff.

2. Conversely, suppose that the orbit space P/G is Hausdorff. Let $\Delta_{P/G} = \{([p], [q]) \in P/G \times P/G \mid [p] = [q]\}$ be the diagonal in $P/G \times P/G$. Let $([p], [q]) \notin \Delta_{P/G}$. Then $[p] \neq [q]$ and there exist a neighborhood U of $[p]$ and a neighborhood V of $[q]$ such that $U \cap V = \emptyset$. We set $W = U \times V$; W is a neighborhood of $([p], [q])$ not containing the points of the diagonal $\Delta_{P/G}$. Thus the diagonal $\Delta_{P/G}$ is closed in $P/G \times P/G$. Since the mapping $\pi \times \pi : P \times P \to P/G \times P/G$ defined by $(\pi \times \pi)(p, q) = (\pi(p), \pi(q))$ is continuous, the set $(\pi \times \pi)^{-1}(\Delta_{P/G}) = \mathscr{R} \subset P \times P$ must also be closed.

Let X (resp. Y) be an n-dimensional (resp. m-dimensional) manifold, $f : X \to Y$ a differentiable mapping. Recall that f is called a *submersion at a point* $x \in X$, if the tangent mapping $T_x f : T_x X \to T_{f(x)} Y$ is surjective. f is a submersion at x if and only if $n \geqq m$ and there exist a chart (V, ψ) at x and a chart (U, φ) at $f(x)$ such that $U \supset f(V)$, $\psi(V) = \varphi(U) \times W$, where $W \subset R^{n-m}$ is an open set, and the chart expression of f, $\varphi f \psi^{-1} : \psi(V) \to \varphi(U)$ is the first canonical projection of the Cartesian product $\varphi(U) \times W$. f is called a *submersion* if it is a submersion at every point $x \in X$. A submersion is an open mapping.

Lemma 1.2 shows that in general, the orbit space P/G does not admit any smooth structure. Our aim now will be to give some necessary and sufficient conditions for P/G to have a smooth structure for which the canonical projection $\pi : P \to P/G$ is a submersion.

We need a lemma concerning submanifolds of the Cartesian product $X \times X$, where X is a manifold. Let $\Delta_X = \{(x, y) \in X \times X \mid x = y\}$ be the diagonal of $X \times X$.

Lemma 1.3. *Let X be an n-dimensional manifold, and let $\mathscr{R}$ be an r-dimensional submanifold of $X \times X$ such that $\Delta_x \subset \mathscr{R}$. Suppose that the restriction of the first canonical projection $\mathrm{pr}_1 : X \times X \to X$ to $\mathscr{R}$ is a submersion. Then to every point $x_0 \in X$ there exist a neighborhood U of x_0 and a chart $(U \times U, \Psi)$, $\Psi = (w^1, w^2, \ldots, \ldots, w^{2n})$, at (x_0, x_0) such that $\Psi(x_0, x_0) = 0$ and the following three conditions hold:*

(1) *A point* $(x, y) \in U \times U$ *belongs to* $\mathscr{R}$ *if and only if*

$$w^{r+1}(x, y) = 0, \ldots, w^{2n}(x, y) = 0. \tag{1.3.3}$$

(2) *There exists a chart* (U, ψ), $\psi = (u^1, u^2, \ldots, u^n)$, *at* x_0 *such that*

$$w^1 = u^1 \circ \mathrm{pr}_1, \ldots, w^n = u^n \circ \mathrm{pr}_1. \tag{1.3.4}$$

(3) *The functions* $w^{n+1}, w^{n+2}, \ldots, w^r$ *are independent of* x, *that is,*

$$w^{n+1}(x, y) = w^{n+1}(x_0, y), \ldots, w^r(x, y) = w^r(x_0, y) \tag{1.3.5}$$

for every $(x, y) \in U \times U$.

Proof. Let $x_0 \in X$ be a point. Since $(x_0, x_0) \in \mathscr{R}$ and $\mathscr{R} \subset X \times X$ is a submanifold, there exists a chart (W, Φ), $\Phi = (v^1, v^2, \ldots, v^{2n})$, on X adapted to $\mathscr{R}$ at (x_0, x_0), i.e. such that $\Phi(x_0, x_0) = 0$ and $(x, y) \in W \cap \mathscr{R}$ if and only if

$$v^{r+1}(x, y) = 0, \ldots, v^{2n}(x, y) = 0. \tag{1.3.6}$$

We may suppose without loss of generality that $\Phi(W) = W_1 \times W_2$, where $W_1 \subset R^r$, $W_2 \subset R^{2n-r}$ are open sets. Denote by $\pi : W_1 \times W_2 \to W_1$ the first canonical projection and write $W_{\mathscr{R}} = W \cap \mathscr{R}$, $\Phi_{\mathscr{R}} = \pi \circ \Phi|_{W \cap \mathscr{R}}$, i.e., $\Phi_{\mathscr{R}} = (v^1, v^2, \ldots, v^r)$, where $v^1 = v^1|_{W \cap \mathscr{R}}, \ldots, v^r = v^r|_{W \cap \mathscr{R}}$. Then $(W_{\mathscr{R}}, \Phi_{\mathscr{R}})$ is a chart on $\mathscr{R}$. Obviously, $\Phi_{\mathscr{R}}(W_{\mathscr{R}}) = W_1$ and the mapping $\pi_W : W \to W_{\mathscr{R}}$ defined by $\pi_W = \Phi_{\mathscr{R}}^{-1} \pi \Phi$ is a submersion.

By hypothesis, $\mathrm{pr}_1 : \mathscr{R} \to X$ is a submersion. Thus there exist a chart (Z, ζ), $\zeta = (z^1, z^2, \ldots, z^r)$, on $\mathscr{R}$ at (x_0, x_0), where $n \leqq r \leqq 2n$, and a chart (V, ψ), $\psi = (u^1, u^2, \ldots, u^n)$, on X at x_0 such that $\zeta(x_0, x_0) = 0$, $\mathrm{pr}(Z) \subset V$ and

$$z^1 = u^1 \circ \mathrm{pr}_1, \ldots, z^n = u^n \circ \mathrm{pr}_1. \tag{1.3.7}$$

Schrinking $W_{\mathscr{R}}$ and Z if necessary we may suppose that $W_{\mathscr{R}} = Z$.

Denoting $\chi = (z^1 \circ \pi_W, z^2 \circ \pi_W, \ldots, z^r \circ \pi_W, v^{r+1}, \ldots, v^{2n})$ we obtain another chart (W, χ) on $X \times X$ at (x_0, x_0). Since $\chi(x_0, y) = (u^1(x_0), \ldots, u^n(x_0), z^{n+1}(x_0, y), \ldots, z^r(x_0, y), v^{r+1}(x_0, y), \ldots, v^{2n}(x_0, y))$ and χ is a diffeomorphism, the mapping $y \to (z^{n+1}(x_0, y), \ldots, z^r(x_0, y), v^{r+1}(x_0, y), \ldots, v^{2n}(x_0, y))$ is also a diffeomorphism. In particular, this mapping is of maximal rank ($= n$) at x_0 which implies that the mapping $y \to (z^{n+1}(x_0, y), \ldots, z^r(x_0, y))$ must also be of maximal rank ($= r - n$) at x_0, i.e. of the same rank as the mapping $(x, y) \to (z^{n+1}(x, y), \ldots, z^r(x, y))$ at (x_0, x_0). Denote

$$\begin{aligned} w^1 = z^1 \circ \pi_W, \ldots, w^n = z^n \circ \pi_W, w^{n+1}(x, y) = z^{n+1}(x_0, y), \ldots, \\ \ldots, w^r(x, y) = z^r(x_0, y), w^{r+1} = v^{r+1}, \ldots, w^{2n} = v^{2n}. \end{aligned} \tag{1.3.8}$$

We have shown that the mapping $(x, y) \to (w^1(x, y), \ldots, w^{2n}(x, y))$ must be of maximal rank ($= 2n$) at (x_0, x_0). Therefore, there exists a neighborhood U of x_0 such that $U \times U \subset W$ and the restriction of this mapping to $U \times U$ is a diffeomorphism. We denote $\Psi = (w^1, w^2, \dots, w^{2n})$; then $(U \times U, \Psi)$ is a chart on $X \times X$ at (x_0, x_0). It is immediately verified that this chart satisfies conditions (1), (2), and (3) of Lemma 1.3.

Lemma 1.4. *Let $\pi : X \to Y$ be a submersion, $Z \subset Y$ a submanifold. Then $\pi^{-1}(Z) \subset X$ is a submanifold.*

Proof. Denote $k = \dim Z$, $n = \dim Y$, and $n + m = \dim X$. Let $x_0 \in \pi^{-1}(Z)$ be any point, $y_0 = \pi(x_0)$, and let (U, φ), $\varphi = (y^1, y^2, \dots, y^n, x^1, x^2, \dots, x^m)$, be a chart at x_0. We may suppose that there exists a chart (V, ψ), $\psi = (u^1, u^2, \dots, u^n)$, at y_0 such that $V = \pi(U)$ and $y^i = u^i \circ \pi$, $i = 1, 2, \dots, n$. Let (W, ζ), $\zeta = (z^1, z^2, \dots, z^n)$, be a chart at y_0 adapted to Z; that is, a point $y \in W$ belongs to $W \cap Z$ if and only if $z^{n+1}(y) = 0, \dots, z^n(y) = 0$. We may suppose without loss of generality that $W = V$. Let us express the mapping $\zeta\psi^{-1} : \psi(V) \to \zeta(V)$ by the equations $z^i = f^i(u^1, \dots, u^n)$, $i = 1, 2, \dots, n$, and set $\bar{U} = U \cap \pi^{-1}(V)$, $\bar{\varphi} = (\bar{y}^1, \bar{y}^2, \dots, \bar{y}^n, \bar{x}^1, \bar{x}^2, \dots, \bar{x}^m)$, where $\bar{y}^i = f^i(y^1, \dots, y^n)$, $\bar{x}^\sigma = x^\sigma$, $\sigma = 1, 2, \dots, m$. $(\bar{U}, \bar{\varphi})$ is a new chart at x_0, and $\bar{y}^i = f^i(y^1, \dots, y^n) = f^i(u^1 \circ \pi, \dots, u^n \circ \pi) = z^i \circ \pi$. Let $x \in \bar{U}$ be a point. If $x \in \bar{U} \cap \pi^{-1}(Z)$, then $\pi(x) \in V \cap Z$ and $\bar{y}^i(x) = z^i(\pi(x)) = 0$ for $i = k + 1, \dots, n$; conversely, if $\bar{y}^i(x) = 0$ for $i = k + 1, \dots, n$, where $x \in U$, then $z^i(\pi(x)) = 0$ for $i = k + 1, \dots, n$, which means that $\pi(x) \in V \cap Z$ and $x \in \pi^{-1}(Z)$ hence $x \in U \cap \pi^{-1}(Z)$. Thus the equations $\bar{y}^i(x) = 0$, $i = k + 1, \dots, n$, are equations of the set $U \cap \pi^{-1}(Z)$, and $\pi^{-1}(Z)$ is a submanifold of X.

Let Φ (resp. Ψ) be a left action of a Lie group G on a manifold P (resp. Q). A mapping $f : P \to Q$ is said to be *G-equivariant* if for all $g \in G$ and $p \in P$, $f(\Phi(g, p)) = \Psi(g, f(p))$, or, which is the same, for all $g \in G$,

$$f \circ \Phi_g = \Psi_g \circ f. \tag{1.3.9}$$

If f is G-equivariant then the G-orbit $\Phi_p(G)$ of a point $p \in P$ is transferred by f into the G-orbit $\Psi_{f(p)}(G)$ of the point $f(p) \in Q$.

The concept of a G-equivariant mapping is slightly generalized as follows. Let $U \subset P$ be an open set. We shall say that a mapping $f : U \to Q$ is *G-equivariant* if for each $p \in U$ and $g \in G$ such that $\Phi(g, p) \in U$

$$f \circ \Phi(g, p) = \Psi(g, f(p)). \tag{1.3.10}$$

If W is a subset of P we denote $\Phi_G(W) = \cup \Phi_g(W)$ (union over $g \in G$); $\Phi_G(W)$ is the union of the orbits of the points of the set W. If W is open, then $\Phi_G(W)$ is also open. W is said to be *G-invariant*, if $\Phi_G(W) = W$.

Let $f : U \to Q$ be a G-equivariant mapping. There exists a unique G-equivariant mapping $\tilde{f} : \Phi_G(U) \to Q$ such that $\tilde{f}|_U = f$. To see it we take any point $p \in \Phi_G(U)$, any point $g \in G$ such that $\Phi(g, p) \in U$, and set

$$(1.3.11) \qquad \tilde{f}(p) = \Psi(g^{-1}, f \circ \Phi(g, p)).$$

If $h \in G$ is another point such that $\Phi(h, p) \in U$, we have $\Phi(h, p) = \Phi(h \,.\, g^{-1} \,.\, g, p) = \Phi(h \,.\, g^{-1}, \Phi(g, p))$ and, since both $\Phi(g, p)$ and $\Phi(h, p)$ belong to U, $f \circ \Phi(h, p) = f \circ \Phi(h \,.\, g^{-1}, \Phi(g, p)) = \Psi(h \,.\, g^{-1}, f \circ \Phi(g, p))$; therefore, $\Psi(h^{-1}, f \circ \Phi(h, p)) = \Psi(h^{-1}, \Psi(h \,.\, g^{-1}, f \circ \Phi(g, p)) = \Psi(g^{-1}, f \circ \Phi(g, p))$, and the point $\tilde{f}(p)$ is well-defined, i.e. is independent of the choice of g. To show that the mapping $p \to \tilde{f}(p)$ is G-equivariant, choose any element $g_0 \in G$. We have for every $p \in \Phi_G(U)$, $\Psi(g_0, \tilde{f}(p)) = \Psi(g_0, \Psi(g^{-1}, f \circ \Phi(g, p)) = \Psi(g_0, \Psi(g^{-1}, f \circ \Phi(g \,.\, g_0^{-1} \,.\, g_0, p))) = \Psi(g_0 \,.\, g^{-1}, f \circ \Phi((g_0 \,.\, g^{-1})^{-1} \,.\, g_0, p)) = \Psi(g_0 \,.\, g^{-1}, f \circ \Phi((g_0 \,.\, g^{-1})^{-1}, \Phi(g_0, p))) = \tilde{f} \circ \Phi(g_0, p)$; hence $\tilde{f}$ is G-equivariant. If f is smooth, $\tilde{f}$ is also smooth. Clearly, let $p_0 \in \Phi_G(U)$ be any point, and let $g_0 \in G$ be such that $\Phi(g_0, p_0) \in U$. There exists a neighborhood V of p_0 in $\Phi_G(U)$ such that $\Phi_{g_0}(V) \subset U$. The restriction $\tilde{f}|_V$ can be expressed as the composition of two smooth mappings $V \ni p \to \Phi(g_0, p) \in U$, $U \ni q \to \Psi(g_0^{-1}, f(q)) \in Q$, and must be smooth. The uniqueness of $\tilde{f}$ is evident: if $F : \Phi_G(U) \to Q$ is any other G-equivariant mapping such that $F|_U = f$, and $p \in \Phi_G(U)$ is any point, we write $p = \Phi(g_0, p_0)$ for some $g_0 \in G$, $p_0 \in U$ and get $F(p) = f \circ \Phi(g_0, p_0) = \Psi(g_0, F(p_0)) = \Psi(g_0, f(p_0)) = \Psi(g_0, \tilde{f}(p_0)) = \tilde{f} \circ \Phi(g_0, p_0) = \tilde{f}(p)$.

A G-equivariant mapping $f : U \to R$, where R is the real line considered as the trivial left G-manifold defined by the left action $(g, t) \to t$ of G, is called a *G-invariant function* on U.

Lemma 1.5. *Let (V, ψ), $\psi = (y^1, y^2, \ldots, y^m)$, be a chart on a left G-manifold P. The following two conditions are equivalent:*

(1) *There exists an integer n, $1 \leq n \leq m$, such that the functions $y^1, y^2, \ldots, y^n : V \to R$ are G-invariant, and every (not necessarily smooth or continuous) G-invariant function $f : V \to R$ depends on $y^1, y^2, \ldots, y^n$ only.*

(2) *There exists an integer n, $1 \leq n \leq m$, such that for every $p, q \in V$, $y^1(p) = y^1(q), \ldots, y^n(p) = y^n(q)$ if and only if the G-orbits of p and q coincide, i.e. $[p] = [q]$.*

Proof. 1. Suppose that condition (1) holds. Then we can choose (V, ψ) in such a way that $\psi(V) = V_1 \times V_2$, where $V_1 \subset R^n$, $V_2 \subset R^{m-n}$ are open sets. Let $\pi : P \to P/G$ be the canonical projection, and denote $U = \pi(V)$; $U \subset P/G$ is an open set. Since the functions $y^1, y^2, \ldots, y^n$ are G-invariant, there exist unique functions $x^1, x^2, \ldots, x^n : U \to R$ such that $y^1 = x^1 \circ \pi, \ldots, y^n = x^n \circ \pi$ on V. Let $\mathrm{pr}_1 : V_1 \times$

$\times V_2 \to V_1$ be the first canonical projection. Denoting $\varphi = (x^1, x^2, \dots, x^n)$ we obtain a mapping $\varphi : U \to V_1$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\psi} & V_1 \times V_2 \\ \uparrow \pi & & \downarrow \mathrm{pr}_1 \\ U & \xrightarrow{\varphi} & V_1 \end{array} \tag{1.3.12}$$

commutes. We shall show that φ is injective. Let $[p], [q] \in U$, $[p] \neq [q]$. Then there exists a function $\chi : U \to R$ such that $\chi([p]) \neq \chi([q])$; for instance we may take $\chi([r]) = 0$ if $[r] \neq [q]$, $\chi([q]) = 1$. Then $\chi \circ \pi : V \to R$ is a G-invariant function. By hypothesis, $\chi \circ \pi$ depends on $y^1, \dots, y^n$ only which implies $(y^1(p), \dots, y^n(p)) \neq (y^1(q), \dots, y^n(q))$, otherwise $\chi([p]) = \chi([q])$. Thus $\varphi([p]) = (y^1(p), \dots, y^n(p)) \neq (y^1(q), \dots, y^n(q)) = \varphi([q])$,, and φ must be injective.

To show that the chart (V, ψ) satisfies condition (2), take $p, q \in V$. Suppose that $y^1(p) = y^1(q), \dots, y^n(p) = y^n(q)$. Then $\varphi([p]) = \varphi([q])$ and, since φ is injective, $[p] = [q]$. Conversely, if $[p] = [q]$, then $x^1([p]) = x^1([q]), \dots, x^n([p]) = x^n([q])$, i.e. $y^1(p) = y^1(q), \dots, y^n(p) = y^n(q)$. This means that condition (2) is satisfied.

2. To show that (2) implies (1), denote by Φ the left action of G on P and choose a point $p_0 \in V$. Obviously, if $g \in G$ is such that $\Phi(g, p_0) \in V$, then $y^1 \circ \Phi(g, p_0) = y^1(p_0), \dots, y^n \circ \Phi(g, p_0) = y^n(p_0)$. This implies that the functions $y^1, \dots, y^n$ are G-invariant. Let $(p^1, p^2, \dots, p^m)$, $(q^1, q^2, \dots, q^m) \in \psi(V)$ be two points such that $p^1 = q^1, \dots, p^n = q^n$, and denote $p = \psi^{-1}(p^1, p^2, \dots, p^m)$, $q = \psi^{-1}(q^1, q^2, \dots, q^m)$. Since $y^1(p) = y^1(q), \dots, y^n(p) = y^n(q)$, there exists, by hypothesis, an element $g \in G$ such that $q = \Phi(g, p)$. Let $f : V \to R$ be a G-invariant function. Then $f(q) = f(p)$, and we get $f\psi^{-1}(p^1, \dots, p^n, p^{n+1}, \dots, p^m) = f\psi^{-1}(p^1, \dots, p^n, q^{n+1}, \dots, q^m)$. This means that f is independent of $y^{n+1}, \dots, y^m$.

Let P be a left G-manifold, and let $m = \dim P$. A chart (V, ψ), $\psi = (y^1, y^2, \dots, y^m)$, on P satisfying one of the equivalent conditions (1), (2) of Lemma 1.5 is called a *G-flat* chart on P.

Simple examples show that in general, a G-flat chart need not exist at every point.

Let (V, ψ), $\psi = (y^1, y^2, \dots, y^m)$, be a G-flat chart on P. With the notation of the proof of Lemma 1.5, consider the pair (U, φ), where $U = \pi(V)$ and $\varphi = (x^1, x^2, \dots, x^n)$. We have shown that $\varphi : U \to V_1$ is injective; since φ is obviously surjective, it must be a bijection. Since P/G is endowed with the final topology with respect to π and the mapping $\mathrm{pr}_1 \circ \psi = \varphi \circ \pi$ is continuous, φ must be continuous. Analogously, since $\varphi^{-1} \circ \mathrm{pr}_1 = \pi \circ \psi^{-1}$ is a continuous mapping and pr_1 is open, the mapping φ^{-1} must also be continuous. Thus $\varphi : U \to V_1$ is a homeomorphism or, in other words, (U, φ) is a chart on the topological space P/G. This chart is said to be *associated* with the G-flat chart (V, ψ).

We note that by definition, none of the functions $y^{n+1}, y^{n+2}, \ldots, y^{m}$ is G-invariant. We are now in a position to prove the following result.

Theorem 1.11. *Let P be a left G-manifold, Φ the left action of G on P, $\pi : P \to P/G$ the canonical projection onto the orbit space, and $\mathscr{R} = \{(p, q) \in P \times P \mid \Phi_p(G) = \Phi_q(G)\}$. Suppose that P is connected. The following three conditions are equivalent:*

(1) The orbit space P/G has a smooth structure such that the canonical projection $\pi : P \to P/G$ is a submersion.

(2) $\mathscr{R}$ is a closed submanifold of $P \times P$.

(3) P can be covered by G-flat charts, and to every $p, q \in P$ such that $\Phi_p(G) \neq \Phi_q(G)$ there exist G-invariant open sets W_p, $W_q \subset P$ such that $p \in W_p$, $q \in W_q$, and $W_p \cap W_q = \emptyset$.

Proof. 1. We shall show that (1) implies (2). By Lemma 1.2, $\mathscr{R}$ is a closed subset of $P \times P$. It remains to verify that $\mathscr{R}$ is a submanifold. Let $\Delta_{P/G} = \{([p], [q]) \in P/G \times P/G \mid [p] = [q]\}$ be the diagonal. Since P/G is Hausdorff, $\Delta_{P/G}$ is a closed submanifold of $P/G \times P/G$, and the mapping $\pi \times \pi : P \times P \to P/G \times P/G$ defined by $(\pi \times \pi)(p, q) = (\pi(p), \pi(q))$ is a submersion. Hence $(\pi \times \pi)^{-1}(\Delta_{P/G}) = \{(p, q) \in P \times P \mid \pi(p) = \pi(q)\} = \mathscr{R}$ is a submanifold of $P \times P$ (Lemma 1.4).

2. Now we shall show that (2) implies (3). Suppose that $\mathscr{R}$ is a submanifold of $P \times P$ and denote $m = \dim P$, $r = \dim \mathscr{R}$. Since the diagonal Δ_P is an m-dimensional submanifold of $\mathscr{R}$, we have $m \leqq r \leqq 2m$.

Let $\mathrm{pr}_1 : \mathscr{R} \to P$ be the restriction of the first canonical projection of the Cartesian product $P \times P$. Clearly, pr_1 is surjective because $\Delta_P \subset \mathscr{R}$. Let $(p_0, q_0) \in \mathscr{R}$ be a point. By definition there exists an element $g_0 \in G$ such that $q_0 = \Phi(g_0, p_0)$. We define a mapping $\delta : P \to \mathscr{R}$ by $\delta(p) = (p, \Phi(g_0, p))$. Obviously, $\mathrm{pr}_1 \circ \delta = \mathrm{id}_P$. Since $\operatorname{rank} T_{(p_0, q_0)} \mathrm{pr}_1 = k \leqq m$ and $T_{p_0}(\mathrm{pr}_1 \circ \delta) = T_{(p_0, q_0)} \mathrm{pr}_1 \circ T_{p_0}\delta$, we get $\operatorname{rank} T_{p_0}(\mathrm{pr}_1 \circ \delta) = m \leqq \min \{\operatorname{rank} T_{(p_0, q_0)} \mathrm{pr}_1, \operatorname{rank} T_{p_0}\delta\} = \min\{k, m\} = k \leqq m$ so that $k = m$. Therefore, $\mathrm{pr}_1 : \mathscr{R} \to P$ a submersion at (p_0, q_0). Since the point $(p_0, q_0) \in \mathscr{R}$ is arbitrary, pr_1 is a submersion.

Hence by Lemma 1.3, to every point $p_0 \in P$ there exist a neighborhood U of p_0 and a chart $(U \times U, \Psi)$, $\Psi = (w^1, w^2, \ldots, w^{2m})$, at (p_0, p_0) such that $\Psi(p_0, p_0) = 0$ and the following conditions hold: (1) A point $(p, q) \in U \times U$ belongs to $(U \times U) \cap \mathscr{R}$ if and only if $w^{r+1}(p, q) = 0, \ldots, w^{2m}(p, q) = 0$, (2) there exists a chart (U, ψ), $\psi = (u^1, u^2, \ldots, u^m)$, at p_0 such that $w^1 = u^1 \circ \mathrm{pr}_1, \ldots, w^m = u^m \circ \mathrm{pr}_1$, and (3) the functions $w^{m+1}, \ldots, w^r$ satisfy $w^{m+1}(p, q) = w^{m+1}(p_0, q), \ldots, w^r(p, q) = w^r(p_0, q)$ for every $(p, q) \in U \times U$. Suppose that we have such a chart and denote $f = (w^{m+1}, w^{m+2}, \ldots, w^{2m})$; f is a mapping from $U \times U$ into R^m such that the partial mapping $U \ni q \to f(p_0, q) \in R^m$ is a diffeomorphism. Define the partial mappings $f_{1,q}, f_{2,p} : U \to R^m$ by $f_{1,q}(p) = f_{2,p}(q) = f(p, q)$. The mapping $T_{p_0} f_{2,p_0} : T_{p_0} P \to$

$\to R^m$ is a linear isomorphism and, by the implicit function theorem, there exist neighborhoods U_1, U_2 of p_0 in U with the following property: to every $p \in U_1$ there exists one and only one point $\chi(p) \in U_2$ such that

$$f(p, \chi(p)) = f(p_0, p_0) = 0; \tag{1.3.12}$$

the mapping $\chi : U_1 \to U_2$ satisfies $\chi(p_0) = p_0$ and is smooth. Differentiating (1.3.12) at p_0 we obtain

$$T_{p_0}\chi = -(T_{p_0}f_{2,p_0})^{-1} \circ T_{p_0}f_{1,p_0}. \tag{1.3.13}$$

Our aim now will be to show that on a neighborhood of p_0, χ is a submersion onto a $(2m - r)$-dimensional submanifold of U_2.

Let us consider the mapping $g : U \times U \to R^{2m-r}$ defined by $g = (w^{r+1}, w^{r+2}, \ldots, \ldots, w^{2m})$. By definition, $g(p_0, p_0) = 0$, $(U \times U) \cap \mathscr{R} = g^{-1}(0)$, and the partial mapping $q \to g(p_0, q)$ is of maximal rank ($= 2m - r$) at p_0, i.e. is a submersion at p_0. It is easily seen that the partial mapping $p \to g(p, p_0)$ is also of maximal rank ($= 2m - r$) at p_0, that is, a submersion at p_0. Let us define $g_{1,q}$ and $g_{2,p}$ by $g_{1,q}(p) = g_{2,p}(q) = g(p, q)$. To show that rank $T_{p_0}g_{1,p_0} =$ rank $T_{p_0}g_{2,p_0}$ it is enough to verify that the mappings $T_{p_0}g_{1,p_0}, T_{p_0}g_{2,p_0} : T_{p_0}P \to R^{2m-r}$ have the same kernels. Let ker F denote the kernel of a linear mapping F. Let $\xi \in \ker T_{p_0}g_{1,p_0}$, i.e., let $T_{p_0}g_{1,p_0} \cdot \xi = 0$. Then $T_{(p_0,p_0)}g \cdot (\xi, 0) = T_{p_0}g_{1,p_0} \cdot \xi = 0$, and $(\xi, 0) \in \ker T_{(p_0,p_0)}g$. Denote by j the diffeomorphism $(p, q) \to (q, p)$ of $P \times P$. Since the set $\mathscr{R} \subset P \times P$ is symmetric, we have $(U \times U) \cap \mathscr{R} = g^{-1}(0) = (g \circ j)^{-1}(0)$ and $\ker T_{(p_0,p_0)}g = T_{(p_0,p_0)}\mathscr{R} = \ker T_{(p_0,p_0)}(g \circ j)$. Thus $(\xi, 0) \in \ker T_{(p_0,p_0)}(g \circ j)$, $(0, \xi) \in \ker T_{(p_0,p_0)}g$ which implies that $\xi \in \ker T_{p_0}g_{2,p_0}$. Thus $\ker T_{p_0}g_{1,p_0} \subset \ker T_{p_0}g_{2,p_0}$. Using the same arguments we get $\ker T_{p_0}g_{1,p_0} \supset \ker T_{p_0}g_{2,p_0}$. This proves that $\ker T_{p_0}g_{1,p_0} = \ker T_{p_0}g_{2,p_0}$, and the mapping $p \to g(p, p_0) = g_{1,p_0}(p)$ is a submersion at p_0.

Since the rank of f_{2,p_0} at p_0 is maximal ($= m$), (1.3.13) implies that rank $T_{p_0}\chi =$ $=$ rank $T_{p_0}f_{1,p_0}$. But $f_{1,p_0}(p) = (w^{m+1}(p_0, p_0), \ldots, w^r(p_0, p_0), w^{r+1}(p, p_0), \ldots, \ldots, w^{2m}(p, p_0))$. Hence

$$\text{rank } T_{p_0}\chi = \text{rank } T_{p_0}g_{1,p_0} = 2m - r. \tag{1.3.14}$$

(1.3.12) implies that for every $p \in U_1$, $\chi(p)$ satisfies the equations $w^{m+1}(p_0, \chi(p)) = 0, \ldots, w^r(p_0, \chi(p)) = 0$. Consequently, χ takes its values in a $(2m - r)$-dimensional submanifold n of U_2 defined by the equations

$$w^{m+1}(p_0, q) = 0, \ldots, w^r(p_0, q) = 0, \tag{1.3.15}$$

and by (1.3.14), χ must be a submersion at p_0. Shrinking U_1 if necessary we may suppose that $\chi(U_1) \subset U_2 \cap \mathscr{N}$. is an open subset of $\mathscr{N}$. We set $\mathscr{M} = \chi(U_1) \cap U_1$,

$U_0 = \chi^{-1}(\mathscr{M}) \cap U_1$ .Then U_0 is a neighborhood of p_0, $\mathscr{M}$ is a $(2m - r)$-dimensional submanifold of U_0, and $\chi : U_0 \to \mathscr{M}$ is a submersion. Since for every $p \in U_0$

$$(1.3.16) \qquad w^{r+1}(p, \chi(p)) = 0, \ldots, w^{2m}(p, \chi(p)) = 0,$$

the pair $(p, \chi(p))$ belongs to $(U_0 \times U_0) \cap \mathscr{R}$, or, which is the same, the G-orbits $[p]$ and $[\chi(p)]$ coincide.

Summarizing our results we conclude that to each point $p_0 \in P$ there exist a neighborhood U_0 of p_0, a $(2m - r)$-dimensional submanifold $\mathscr{M} \subset U_0$ containing p_0, and a submersion $\chi : U_0 \to \mathscr{M}$ such that for every $p \in U_0$, $(p, \chi(p)) \in \mathscr{R}$, and if $(p, q) \in (U_0 \times \mathscr{M}) \cap \mathscr{R}$ then $q = \chi(p)$.

Denote $n = 2m - r$. Since χ is a submersion at p_0, there exist a chart (W, ψ), $\psi = (y^1, y^2, \ldots, y^m)$, on P at p_0 and a chart (V, φ), $\varphi = (x^1, x^2, \ldots, x^n)$, on $\mathscr{M}$ at p_0 such that $\chi(W) \subset V$ and $y^1 = x^1 \circ \chi, \ldots, y^n = x^n \circ \chi$ on W. Let $p, q \in W$ be two points such that $[p] = [q]$. Then $(p, q) \in \mathscr{R}$, $(p, \chi(p)) \in \mathscr{R}$, and $(q, \chi(q)) \in \mathscr{R}$ which implies, since $\mathscr{R}$ is a transitive relation, $(\chi(p), \chi(q)) \in \mathscr{R}$; but also $(\chi(p), \chi(p)) \in \mathscr{R}$, and by the uniqueness of χ, $\chi(p) = \chi(q)$. In particular, $y^i(q) = x^i \circ \chi(q) = x^i \circ \chi(p) = y^i(p)$ for every $i = 1, 2, \ldots, n$. Conversely, if for some $p, q \in W$, $y^i(p) = y^i(q)$, $i = 1, 2, \ldots, n$, then $\varphi \circ \chi(p) = \varphi \circ \chi(q)$ and, since (V, φ) is a chart, $\chi(p) = \chi(q)$. But $(p, \chi(p)), (q, \chi(q)) \in \mathscr{R}$ so that, by the transitivity of $\mathscr{R}$, $(p, q) \in \mathscr{R}$, i.e. $[p] = [q]$. Thus the chart (W, ψ) on P at p_0 is G-flat. Since p_0 is arbitrary, this shows that P can be covered by G-flat charts.

Suppose that $\mathscr{R}$ is a closed subset of $P \times P$. Then by Lemma 1.2, the orbit space P/G is Hausdorff, and to every $p, q \in P$ such that $[p] \neq [q]$, there exist a neighborhood U of $[p]$ and a neighborhood V of $[q]$ such that $U \cap V = 0$. Taking $W_p = \pi^{-1}(U)$ and $W_q = \pi^{-1}(V)$ we obtain G-invariant open sets in P such that $p \in W_p$, $q \in W_q$, and $W_p \cap W_q = 0$.

3. Now suppose that condition (3) holds. We want to show that P/G has a smooth structure such that the canonical projection $\pi : P \to P/G$ is a submersion.

Since by hypothesis every two points of P not belonging to the same G-orbit can be separated by G-invariant open sets, P/G must obviously be Hausdorff. It thus remains to show that the charts on P/G associated to the G-flat charts on P form an atlas; by definition of a G-flat chart, the mapping π will then automatically be a submersion.

Let $m = \dim P$. Let (V, ψ), $\psi = (y^1, y^2, \ldots, y^m)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^m)$, be two G-flat charts on P such that $V \cap \bar{V} \neq \emptyset$. Let n (resp. k) be the integer such that the functions $y^1, \ldots, y^n$ (resp. $\bar{y}^1, \ldots, \bar{y}^k$) are G-invariant and every G-invariant function defined on V (resp. $\bar{V}$) depends on $y^1, \ldots, y^n$)(resp. $\bar{y}^1, \ldots, \bar{y}^k$) only. We shall show that $n = k$. Let $p \in V \cap \bar{V}$ be a point. The Jacobi matrix $D\bar{\psi}\psi^{-1}(\psi(p))$ is of the form

$$(1.3.17)\quad \begin{bmatrix} \frac{\partial \bar{y}^1}{\partial y^1} & \cdots & \frac{\partial \bar{y}^1}{\partial y^n} & 0 & \cdots & 0 \\ \cdots & & & & & \\ \frac{\partial \bar{y}^k}{\partial y^1} & & \frac{\partial \bar{y}^k}{\partial y^n} & 0 & \cdots & 0 \\ \frac{\partial \bar{y}^{k+1}}{\partial y^1} & & \frac{\partial \bar{y}^{k+1}}{\partial y^n} & \frac{\partial \bar{y}^{k+1}}{\partial y^{n+1}} & \cdots & \frac{\partial \bar{y}^{k+1}}{\delta y^m} \\ \cdots & & & & & \\ \frac{\partial \bar{y}^m}{\partial y^1} & \cdots & \frac{\partial \bar{y}^m}{\partial y^n} & \frac{\partial \bar{y}^m}{\partial y^{n+1}} & \cdots & \frac{\partial \bar{y}^m}{\partial y^m} \end{bmatrix}.$$

since the functions $\bar{y}^1, \ldots, \bar{y}^k$ depend on $y^1, \ldots, y^n$ only, i.e.

$$(1.3.18)\qquad \frac{\partial \bar{y}^i}{\partial y^{n+1}} = 0, \ldots, \frac{\partial \bar{y}^i}{\partial y^m} = 0, \qquad i = 1, 2, \ldots, k.$$

Thus if $n < k$, or $n > k$, we get $\det D\bar{\varphi}\psi^{-1}(\psi(p)) = 0$ which is not possible. Consequently, $n = k$.

Let $p, q \in P$ be two different points. Let (V, ψ), $\psi = (y^1, y^2, \ldots, y^m)$ (resp. $(\bar{V}, \bar{\varphi})$, $\bar{\varphi} = (\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^m)$), be a G-flat chart at p (resp. q), and let n (resp. k) be the integer such that the functions $y^1, \ldots, y^n$ (resp. $\bar{y}^1, \ldots, \bar{y}^k$) are G-invariant and every G-invariant function defined on V (resp. $\bar{V}$) depends on $y^1, \ldots, y^n$ (resp. $\bar{y}^1, \ldots, \bar{y}^k$) only. It is easily seen that again $n = k$. Since P is connected, it is arcwise connected, and there exists a continuous mapping $\zeta : [0, 1] \to P$ such that $\zeta(0) = p$ and $\zeta(1) = q$. Clearly, there exists a finite numbei of G-flat charts (V_1, ψ_1), (V_2, ψ_2), $\ldots$, (V_N, ψ_N) such that $(V_1, \psi_1) = (V, \psi)$, $(V_N, \psi_N) = (\bar{V}, \bar{\varphi})$, $V_j \cap V_{j+1} \neq \emptyset$ for every $j = 1, 2, \ldots, N - 1$, and $\zeta([0, 1]) \subset \cup V_i$ (union over $i = 1, 2, \ldots, N$). This guarantees, however, that $n = k$.

Let (V, ψ) and $(\bar{V}, \bar{\varphi})$ be as above, and let (U, φ) (resp. $(\bar{U}, \bar{\varphi})$) be the chart on P/G associated with (V, ψ) (resp. $(\bar{V}, \bar{\varphi})$). To complete the proof it is sufficient to check that the mapping $\bar{\varphi}\varphi^{-1} : \varphi(U \cap \bar{U}) \to \bar{\varphi}(U \cap \bar{U})$ is smooth. Let the equations

$$(1.3.19)\qquad \bar{y}^i = f^i(y^1, y^2, \ldots, y^n), \qquad \bar{y}^\sigma = g^\sigma(y^1, y^2, \ldots, y^m),$$

where $1 \leqq i \leqq n$, $n + 1 \leqq \sigma \leqq m$, express the mapping $\bar{\varphi}\psi^{-1} : \psi(V \cap \bar{V}) \to \bar{\varphi}(V \cap \bar{V})$. Since $\bar{x}^i \circ \pi = \bar{y}^i$ and $x^i \circ \pi = y^i$, (1.3.19) implies that the mapping $\bar{\varphi}\varphi^{-1} : \varphi(U \cap \cap \bar{U}) \to \bar{\varphi}(U \cap \bar{U})$ is expressed by the equations

$$(1.3.20)\qquad \bar{x}^i = f^i(x^1, x^2, \ldots, x^n),$$

where $1 \leqq i \leqq n$, and must be smooth.

This completes the proof of Theorem 1.11.

Remark 1.1. We note that our assumption in Theorem 1.11 that P should be connected was only used to prove the implication (3) $\Rightarrow$ (1); we might suppose more generally that $P = \Phi_G(P_0)$, where P_0 is a connected component of P. This assumption guarantees that every G-flat chart on P has the same number of G-invariant coordinate functions which implies in turn that the manifold P/G has constant dimension (equal to the number of G-invariant functions in a G-flat chart). If P is not of the form $\Phi_G(P_0)$ and there are two connected components P_1, $P_2 \subset P$ such that $\Phi_G(P_1) \cap \Phi_G(P_2) = \emptyset$ then the dimensions of the manifolds $\Phi_G(P_1)/G$ and $\Phi_G(P_2)/G$ need not be the same.

Remark 1.2. Let $\pi : X \to Y$ be a surjective submersion, Q a manifold, $f : Y \to Q$ a mapping. If f is smooth then $f \circ \pi$ is smooth; we shall show that the converse is also true. Let $y \in Y$ be a point. There exist a neighborhood U of y and a section $\delta : U \to X$ i.e. a smooth mapping of U into X such that $\pi \circ \delta = \mathrm{id}_U$. Thus on U, $f = f \circ \pi \circ \delta$, and if $f \circ \pi$ is smooth, f must be smooth at y. Since y is arbitrary, f is smooth.

In particular, Y has a unique structure of a smooth manifold such that $\pi : X \to Y$ is a submersion. Let (U, φ), (U, ψ) be two charts on Y not necessarily belonging to the same smooth structure. We may suppose that there exists a chart (V, Φ) (resp. $(V, \overline{\Psi})$) belonging to the smooth structure of X such that $\pi(V) = U$ and $\varphi \circ \pi = \mathrm{pr}_1 \circ \Phi$ (resp. $\psi \circ \pi = \mathrm{pr}_1 \circ \Psi$), where $\mathrm{pr}_1 : R^n \times R^{m-n} \to R^n$ is the first canonical projection and $n = \dim Y$, $m = \dim X$. Then $\mathrm{pr}_1 \circ \Psi\Phi^{-1} = \psi \circ \pi \circ \Phi^{-1} = \psi\varphi^{-1} \circ \varphi\pi\Phi^{-1} = \psi\varphi^{-1} \circ \mathrm{pr}_1$ and since $\mathrm{pr}_1 \circ \Psi\Phi^{-1}$ is a smooth mapping $\psi\varphi^{-1} \circ \mathrm{pr}_1$ is also smooth. Since pr_1 is a surjective submersion the above argument shows that $\psi\varphi^{-1}$ is a smooth mapping. This proves the uniqueness of the smooth structure on Y.

Applying these remarks to a left G-manifold P we see that if the orbit space P/G has a smooth structure such that the canonical projection $\pi : P \to P/G$ is a submersion, then this smooth structure is unique. If in this case Q is a manifold and $f : P/G \to Q$ is a mapping, then f is smooth if and only if $f \circ \pi$ is smooth.

If the assumptions of Theorem 1.11 are satisfied we call the orbit space P/G with its smooth structure the *orbit manifold* of the left G-manifold P.

Let P be a left G-manifold, let Φ denote the left action of G on P. As above, we define a mapping $\Phi' : G \times P \to P \times P$ by

$$\Phi'(g, p) = (p, \Phi(g, p)). \tag{1.3.20}$$

Clearly, the image of Φ', im Φ', is precisely the set $\mathscr{R}$, i.e.

$$\mathrm{im}\, \Phi' = \mathscr{R}. \tag{1.3.21}$$

Thus the orbit space P/G has the orbit manifold structure if and only if im Φ' is a closed submanifold of $P \times P$. Some simple sufficient conditions imposed on the mapping Φ', ensuring the existence of the orbit manifold structure on P/G, are given by the following corollary.

Corollary 1. *Suppose that the left action Φ of G on P is proper and free. Then the orbit space has the orbit manifold structure.*

Proof. 1. We shall show that under our assumptions, Φ' is an injective immersion. Let $m = \dim P$, $n = \dim G$. Let $(g, p) \in G \times P$ be a point, $(\xi, \zeta) \in T_g G \times T_p P$ a tangent vector to $G \times P$ at this point. We have

$$(1.3.21) \qquad T_{(g,p)}\Phi' \,.\, (\xi, \zeta) = (\zeta, T_1\Phi(g, p) \,.\, \xi + T_2\Phi(g, p) \,.\, \zeta).$$

Thus rank $T_{(g,p)}\Phi' = m +$ rank $T_1\Phi(g, p)$. Since the orbit mapping $g \to \Phi(g, p) = \Phi_p(g)$ is of constant rank (Lemma 1.1), we have rank $T_{(g,p)}\Phi = m +$ rank $T_e\Phi_p$. But Φ is a free action and the mapping $g \to \Phi_p(g)$ is injective so it is of rank n and we have

$$(1.3.22) \qquad \text{rank } T_{(g,p)}\Phi' = p + n.$$

Thus Φ' is an immersion. It is directly seen that Φ' is injective: If $(p, \Phi(g, p)) = (q, \Phi(h, q))$ for some (g, p), $(h, q) \in G \times P$, then $p = q$ and $\Phi(g, p) = \Phi(h, q)$, i.e. $p = q$, $p = \Phi(g^{-1} \,.\, h, p)$. Since Φ is free, we have $g = h$ as required. Thus Φ is an injective immersion.

2. We have proved that Φ' is an injective immersion. On the other hand, Φ' is proper, and the Bolzano—Weierstrass theorem for Euclidean spaces implies that Φ' is a closed mapping. Therefore, Φ' is a homeomorphism of $G \times P$ onto a closed subspace of $P \times P$. Thus im $\Phi' = \mathscr{R}$ is a closed submanifold of $P \times P$ which proves our assertion.

Let $W \subset P$ be an open set, and denote by $\text{Inv}_G W$ the set of G-invariant functions defined on W. For every $f, f_1, f_2 \in \text{Inv}_G W$ and $a \in R$, and every $p \in W$ we set

$$(1.3.23) \qquad \begin{aligned} &(f_1 + f_2)(p) = f_1(p) + f_2(p), \qquad (c \,.\, f)(p) = c \,.\, f(p), \\ &(f_1 \cdot f_2)(p) = f_1(p) \cdot f_2(p). \end{aligned}$$

The mappings $(f_1, f_2) \to f_1 + f_2$, $(a, f) \to a \,.\, f$, and $(f_1, f_2) \to f_1 \,.\, f_2$ define the structure of a real, associative, commutative algebra with identity on the set $\text{Inv}_G W$: we call this algebra the *algebra of G-invariant functions* on W, or the *algebra of G-invariants* on W.

Let (V, ψ), $\psi = (y^1, y^2, \ldots, y^m)$, be a G-flat chart on P, and let n be the integer for which the functions $y^1, \ldots, y^n$ are G-invariant, and every G-invariant function on V depends on $y^1, \dots, y^n$ only. Then we call the functions $y^1, \dots, y^n$ the *G-invariant functions of the G-flat chart* (V, ψ).

Let $W \subset P$ be an open set. A system $(f^1, f^2, \dots, f^n)$ of G-invariant functions defined on W is called a *basis* of the algebra of G-invariants $\mathrm{Inv}_G W$, or a *basis of G-invariant functions* on W, if to every point $p \in W$ there exists a G-flat chart (V, ψ), $\psi = (y^1, y^2, \dots, y^m)$, at p with G-invariant functions $y^1, \dots, y^n$, such that $y^1 = f^1|_v, \dots, y^n = f^n|_v$. The number n is called the *dimension* of the algebra of G-invariants $\mathrm{Inv}_G W$.

Under the hypothesis of Theorem 1.11, the dimension of the algebra $\mathrm{Inv}_G W$ is independent of the choice of W; in this case the G-invariant functions are precisely functions on open subsets of the orbit manifold P/G.

Let us now consider an (algebraic) group G and suppose G is the semi-direct product of its subgroup H and its normal subgroup K, i.e., $G = H \times_s K$. Let Q be a set endowed with a left action of the group G, denoted multiplicatively. That is, for each $g_1, g_2 \in G$ and $q \in Q$, $(g_1 . g_2) . q = g_1 . (g_2 . q)$, $e . q = q$. The set Q is also endowed with an action of the subgroup K of G. Let $[q]_K$ denote the K-orbit of a point $q \in Q$, i.e. $[q]_K = \{q' \in Q \mid q' = g . q,\ g \in K\}$, let Q/K be the set of K-orbits, and let $\pi : Q \to Q/K$ denote the canonical projection $q \to [q]_K$ onto the quotient space. We set for each $h \in H$

(1.3.24) $\qquad h . [q]_K = [h . q]_K.$

If $q_1, q_2 \in [q]_K$, then there exists an element $g \in K$ such that $q_2 = g . q_1$ so that $h . q_2 = h . g . q_1 = h . g . h^{-1} . h . q_1$; since K is a normal subgroup of G, $h . g . h^{-1} \in K$ and $[h . q_2]_K = [h . q_1]_K$ which means that the element $h . [q]_K \in Q/K$ defined by (1.3.24) is correctly defined. The mapping of $H \times Q/K$ into Q/K defined by (1.3.24) is a left action of H on Q/K.

Let $\beta : H \times_s K \to H$ be the canonical homomorphism of groups defined by (1.2.36)

Lemma 1.6. *Suppose that we have a left action $(h, p) \to h . p$ of the group H on a set P, and a mapping $F : Q \to P$ such that for each $g \in G$ and $q \in Q$*

(1.3.25) $\qquad F(g . q) = \beta(g) . F(q).$

Then F has the form

(1.3.26) $\qquad F = F_P \circ \pi,$

where $F_P : Q/K \to P$ is a uniquely determined H-equivariant mapping.

Proof. (1.3.26) implies that if F_P exists it is unique. To prove the existence of F_P, choose any point $q \in Q$ and consider the point $F(q) \in P$. By (1.3.25), $F(q)$ does not depend on the choice of q in the equivalence class $[q]_K$. We put

$$(1.3.27) \qquad F_P([q]_K) = F(q).$$

This defines a mapping $F_P : Q/K \to P$ satisfying (1.3.26). Let $g \in G$ be any point. Since g is uniquely expressible in the form $g = g' . \beta(g)$, where $g' \in K$ (see Sec. 1.2), we have

$$(1.3.27) \qquad \begin{aligned} \pi(g . q) &= [g . q]_K = [\beta(g) . (\beta(g))^{-1} . g' . \beta(g) . q]_K = \\ &= \beta(g) . [(\beta(g))^{-1} . g' . \beta(g) . q]_K = \beta(g) . \pi(q). \end{aligned}$$

Thus for every $h \in H$ and $q \in Q$

$$(1.3.28) \qquad \begin{aligned} F_P(h . [q]_K) &= F_P(h . \pi(q)) = F_P(\pi(h . q)) = F(h . q) = \\ &= h . F(q) = h . F_P([q]_K), \end{aligned}$$

and F_P is H-equivariant.

Now let G be a Lie group. Suppose that G is the interior semi-direct product of its Lie subgroup H and its normal Lie subgroup K, in notations $G = H \times_s K$. Then the canonical homomorphism $\beta : H \times_s K \to H$ is a homomorphism of Lei groups. Let Q be a left G-manifold, and denote by $\pi : Q \to Q/K$ the canonical projection of Q onto the orbit space.

Theorem 1.12. *Suppose that Q/K has the structure of the orbit manifold. Then the formula* (1.3.24) *defines the structure of a left H-manifold on Q/K.*

Proof. We want to show that the mapping $H \times Q/K \ni (h, y) \to h . y \in Q/K$ is smooth. Let γ be a smooth local section of the projection $\pi : Q \to Q/K$ defined on an open set $U \subset Q/K$. We have for every $h \in H$ and $y \in U$

$$(1.3.29) \qquad h . y = h . \pi(\gamma(y)) = \pi(h . \gamma(y)).$$

Thus the mapping $(h, y) \to h . y$ is expressible as the composition of smooth mappings, locally, so it must be smooth on $H \times U$. Since to every point $y \in Q/K$ there exist a neighborhood U of y and a smooth section of π defined on U, this mapping is smooth.

Consider a left G-manifold Q with the left action Φ of G, and the Lie algebra $L(G)$ of the Lie group G. Let $\exp_G : L(G) \to G$ be the exponential mapping of the Lie group G, let $\xi \in L(G)$ be a vector. We set for each $q \in Q$

$$(1.3.30) \qquad \Phi'(\xi)(q) = \left\{\frac{d}{ds} \Phi_q(\exp s\xi)\right\}_0.$$

$\Phi'(\xi)$ is a vector field on Q, called the *fundamental vector field* on the left G-manifold Q, associated with the element $\xi \in L(G)$. Obviously

$$\Phi'(\xi)(q) = T_e\Phi_q \cdot \left\{\frac{\mathrm{d}}{\mathrm{d}s}(\exp s\xi)\right\}_0 = T_e\Phi_q \cdot \xi. \tag{1.3.31}$$

Let $g_0 \in G$ be any element. We define a mapping $\mathrm{Int}\, g_0 : G \to G$ by the formula

$$(\mathrm{Int}\, g_0)(g) = g_0 g g^{-1} \tag{1.3.32}$$

and put

$$\mathrm{Ad}\, g_0 = T_e \,\mathrm{Int}\, g_0 . \tag{1.3.33}$$

By definition, $\mathrm{Ad}\, g_0$ is a linear automorphism of the vector space T_eG. Since for any $g_1, g_2 \in G$, $\mathrm{Int}\,(g_1 g_2) = (\mathrm{Int}\, g_1) \circ (\mathrm{Int}\, g_2)$, we have

$$\mathrm{Ad}\,(g_1 g_2) = (\mathrm{Ad}\, g_1) \circ (\mathrm{Ad}\, g_2). \tag{1.3.34}$$

This means that the mapping $g \to \mathrm{Ad}\, g$ is a homomorphism of the group G into the group of linear transformations of $L(G)$, $GL(L(G))$. This homomorphism is called the *adjoint representation* of the Lie group G.

Theorem 1.13 *Let Q be a left G-manifold, Φ the left action of G on Q. Let $\xi, \xi_1, \xi_2 \in L(G)$ be any vectors, $g \in G$, $q \in Q$ any points. Then*

$$\Phi'(\xi) \circ \Phi_g = T\Phi_g \,.\, \Phi'(\mathrm{Ad}\, g^{-1} \,.\, \xi), \tag{1.3.35}$$

$$\Phi'([\xi_1, \xi_2]) = -[\Phi'(\xi_1), \Phi'(\xi_2)]. \tag{1.3.36}$$

Proof. To derive (1.3.35) write $(\Phi'(\xi) \circ \Phi_g)(q) = T_e\Phi_{\Phi(g,q)} \,.\, \xi$. Since for any $h \in G$

$$\begin{aligned}\Phi_{\Phi(g,q)}(h) &= \Phi(h, \Phi(g, q)) = \Phi(gg^{-1}hg, q) = \\ &= \Phi(g, \Phi(g^{-1}hg, q)) = (\Phi_g \circ \Phi_q \circ \mathrm{Int}\, g^{-1})(h),\end{aligned} \tag{1.3.37}$$

we have

$$\begin{aligned}T_e\Phi_{\Phi(g,q)} \,.\, \xi &= T_q\Phi_g \circ T_e\Phi_q \circ \mathrm{Ad}\, g^{-1} \,.\, \xi = \\ &= T_q\Phi_g \,.\, \Phi'(\mathrm{Ad}\, g^{-1} \,.\, \xi)(q),\end{aligned} \tag{1.3.38}$$

which gives (1.3.35).

To prove (1.3.36) denote by J the diffeomorphism $g \to g^{-1}$ of G onto G. We have for any $g, h \in G$ and $q \in Q$

$$(\Phi_{\Phi(h^{-1},q)} \circ J)(g) = \Phi((hg)^{-1}, q) = (\Phi_q \circ J \circ L_h)(q), \tag{1.3.39}$$

where L_h is the left translation on G by h. Thus

$$\Phi_q \circ J = \Phi_{\Phi(h^{-1},q)} \circ J \circ L_{h^{-1}}. \tag{1.3.40}$$

Let $\xi \in L(G)$ be any element. Applying $T_h(\Phi_q \circ J)$ to $\xi(h)$ we obtain

$$T_h(\Phi_q \circ J) \,.\, \xi(h) = T_e\Phi_{\Phi(h^{-1},q)} \circ T_eJ \circ T_hL_{h^{-1}} \circ T_eL_h \,.\, \xi(e) = \\ = T_e\Phi_{\Phi(h^{-1},q)} \circ T_eJ \,.\, \xi(e). \tag{1.3.41}$$

But

$$T_eJ \,.\, \xi(e) = T_eJ \,.\left\{\frac{\mathrm{d}}{\mathrm{d}t}(\exp t\xi)\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t}(\exp(-t\xi))_0 = -\xi(e), \tag{1.3.42}$$

so that by definition

$$T_h(\Phi_q \circ J) \,.\, \xi(h) = -T_e\Phi_{\Phi(h^{-1},q)} \,.\, \xi(e) = \\ = -\Phi'(\xi)\,(\Phi(h^{-1}, q)) = -\Phi'(\xi)\,((\Phi_q \circ J)\,(h)). \tag{1.3.43}$$

This means that the vector fields ξ and $-\Phi'(\xi)$ are $(\Phi_q \circ J)$–related,

$$T(\Phi_q \circ J) \,.\, \xi = -\Phi'(\xi) \circ (\Phi_q \circ J). \tag{1.3.44}$$

Using this equality we can easily compute $\Phi([\xi_1, \xi_2])$ for any $\xi_1, \xi_2 \in L(G)$. Since

$$T(\Phi_q \circ J)\,([\xi_1, \xi_2]) = -\Phi'([\xi_1, \xi_2]) \circ \Phi_q \circ J = \\ = [-\Phi'(\xi_1), -\Phi'(\xi_2)] \circ \Phi_q \circ J \tag{1.3.45}$$

for all $q \in Q$, we obtain (1.3.36) as required.

Corollary 1. *The mapping $\xi \to -\Phi'(\xi)$ of $L(G)$ into the Lie algebra of vector fields on Q is a homomorphism of Lie algebras. In particular, the fundamental vector fields on Q form a subalgebra of the Lie algebra of vector fields on Q.*

Proof. This assertion follows from (1.3.36).

Corollary 2. *Let Q be a left G-manifold, Φ the left action of G on Q.*

(a) If Φ is effective then the homomorphism of Lie algebras $\xi \to -\Phi'(\xi)$ is injective.

(b) If Φ is free then the Lie algebra of fundamental vector fields on Q consists of nowhere zero vector fields.

Proof. (a) Suppose that Φ is effective. Let $\Phi'(\xi) = 0$ for some $\xi \in L(G)$. Then for all $q \in Q$, $\Phi(\exp t\xi, q) = q$ or, which is the same, $\Phi_{\exp t\xi} = \mathrm{id}_Q$. Since Φ is effective, this gives $\exp t\xi = e$ and $\xi = 0$.

(b) Let $q \in Q$ be any point. By definition, $\Phi'(\xi)\,(q)$ is the tangent vector to the curve $t \to \Phi(\exp t\xi, q)$. If $\Phi'(\xi)\,(q) = 0$ then this curve is constant, contradicting the assumption that Φ is free. Thus $\Phi'(\xi)\,(q) \neq 0$ and we are done.

2. DIFFERENTIAL INVARIANTS

The theory explained in this chapter, reflects and extends the classical general theory of differential invariants. The fundamental notions for this theory are the following: r-jet, differential group, principal fiber bundle, associated fiber bundle, higher order frame bundle, r-frame lifting, P-lifting, differential invariant, realization of a differential invariant.

2.1. Manifolds of jets. Let X and Y be manifolds, $r \geqq 0$ an integer, $x \in X$ and $y \in Y$ arbitrary points. Let $C^\infty(x, y)$ be the set of smooth mappings f defined at x, with values in Y, such that $f(x) = y$. We say that two mappings $f_1, f_2 \in C^\infty(x, y)$ are *r-equivalent* if there exist a chart (U, φ) on X and a chart (V, ψ) on Y such that $x \in U$, $y \in V$, and for each k, $0 \leqq k \leqq r$,

$$D^k(\psi f_1 \varphi^{-1})(\varphi(x)) = D^k(\psi f_2 \varphi^{-1})(\varphi(x)). \tag{2.1.1}$$

The relation "f_1, f_2 are r-equivalent" is an equivalence relation on $C^\infty(x, y)$. An equivalence class with respect to this equivalence relation is called an *r-jet* with *source* x and *target* y. An r-jet P with source x and target y, whose representative is a mapping $f \in C^\infty(x, y)$, is denoted by $J^r_x f$. The set of r-jets with source $x \in X$ and target $y \in Y$ is denoted by $J^r_{(x,y)}(X, Y)$.

We denote

$$J^r(X, Y) = \bigcup_{x,y} J^r_{(x,y)}(X, Y), \tag{2.1.2}$$

and set for each $P \in J^r(X, Y)$, $P = J^r_x f$, and for each s, $0 \leqq s \leqq r$,

$$\pi^{r,s}(P) = J^s_x f. \tag{2.1.3}$$

$\pi^{r,s}$ is a well-defined surjection of $J^r(X, Y)$ onto $J^s(X, Y)$. For $s = 0$, $J^0(X, Y)$ is canonically identified with $X \times Y$, and we define

$$\pi^r = \pi^{r,0}, \qquad \pi^r_X = \mathrm{pr}_1 \circ \pi^r, \qquad \pi^r_Y = \mathrm{pr}_2 \circ \pi^r, \tag{2.1.4}$$

where pr_1 (resp. pr_2) is the first (resp. the second) projection of the Cartesian product $X \times Y$. The mappings $\pi^{r,s}$, π^r, π^r_X, and π^r_Y are called *canonical jet projections*.

Let X, Y, Z be three manifolds. We say that r-jets $P \in J^r_x(X, Y)$, $Q \in J^r_y(Y, Z)$ are *composable* if the target of P is equal to the source of Q. If $P = J^r_x f$ and $Q = J^r_y g$ then P and Q are composable if and only if $f(x) = y$. In this case we set

$$Q \circ P = J^r_x(g \circ f) \tag{2.1.5}$$

and call the r-jet $Q \circ P \in J^r(X, Z)$ the *composite* of P and Q. The mapping $(P, Q) \to Q \circ P$ of $J^r_{(x,y)}(X, Y) \times J^r_{(y,z)}(Y, Z)$ into $J^r_{(x,z)}(X, Z)$ is called the *composition of jets*.

Let $\dim X = \dim Y$. An r-jet $P \in J^r_{(x,y)}(X, Y)$ is called *invertible* if there exists an r-jet $Q \in J^r_{(y,x)}(Y, X)$ such that

$$(2.1.6) \qquad Q \circ P = J^r_x \operatorname{id}_X,$$

where id_X is the identity mapping of X. It follows from the inverse function theorem that an r-jet P is invertible if and only if it has a representative which is a diffeomorphism of a neighborhood of the point x onto a neighborhood of the point y. If P is invertible then the r-jet Q such that (2.1.6) holds is unique, and is called the *inverse* of P. We have

$$(2.1.7) \qquad P \circ Q = J^r_y \operatorname{id}_Y.$$

By definition of an r-jet, the set $J^r_{(0,0)}(R^n, R^m)$ is canonically identified with the product

$$(2.1.8) \qquad L^r_{n,m} = L(R^n, R^m) \times L^2_s(R^n, R^m) \times \ldots \times L^r_s(R^n, R^m),$$

where $L^k_s(R^n, R^m)$ denotes the vector space of k-linear, symmetric mappings of $R^n \times R^n \times \ldots \times R^n$ (k factors) into R^m. The *canonical* (global) *chart* on $L^r_{n,m}$ is defined by means of the *canonical* (global) *coordinates* $a^\sigma_{j_1 j_2 \ldots j_k}$, introduced as follows. If $A \in L^r_{n,m}$, $A = J^r_0\alpha$, where $\alpha = (\alpha^\sigma)$, then for each $\sigma, j_1, j_2, \ldots, j_k$ such that $1 \leqq \sigma \leqq m$, $1 \leqq j_1 \leqq j_2 \leqq \ldots \leqq j_k \leqq n$, $1 \leqq k \leqq r$,

$$(2.1.9) \qquad a^\sigma_{j_1 j_2 \ldots j_k}(A) = D_{j_1} D_{j_2} \ldots D_{j_k} \alpha^\sigma(0).$$

We note that (2.1.9) defines the functions $a^\sigma_{j_1 j_2 \ldots j_k}$ for all sequences $(j_1, j_2, \ldots, j_k)$, not only for non-decreasing ones; we choose, however, for the canonical coordinates those of them which are independent, and satisfy $j_1 \leqq j_2 \leqq \ldots \leqq j_k$.

Let U (resp. V) be an open set in R^n (resp. R^m), and let t_x denote the translation $y \to y - x$ of R^n. Assigning to an r-jet $P \in J^r(U, V)$, $P = J^r_x f$, the triple $(x, f(x), J^r_0(t_{f(x)} f t_{-x}))$, we obtain the canonical identification of $J^r(U, V)$ with $U \times V \times L^r_{n,m}$; $J^r(U, V)$ thus becomes an open set in the topological space $R^n \times R^m \times L^r_{n,m}$. Obviously,

$$(2.1.10) \qquad \begin{aligned} \dim J^r(U, V) &= n + m\left(1 + n + \binom{n+1}{2} + \ldots + \right. \\ &\left. + \binom{n+r-1}{r}\right) = n + m\binom{n+r}{n}. \end{aligned}$$

Let X and Y be two manifolds, $\dim X = n$, $\dim Y = m$. Using the smooth structures of the manifolds X, Y, we can introduce a smooth structure on the set $J^r(X, Y)$ of r-jets with source in X and target in Y. Let (U, φ), $\varphi = (x^i)$, $1 \leqq i \leqq n$

(resp. (V, ψ), $\psi = (y^\sigma)$, $1 \leqq \sigma \leqq m$) be a chart on X (resp. Y). Denote $W = U \times V$, $W^r = (\pi^r)^{-1}(W)$, and put for each $P \in W^r$, $P = J^r_x f$,

$$(2.1.11) \qquad \chi^r(P) = (\varphi(x), \psi(f(x)), J^r_0(t_{\psi(f(x))}\psi f \varphi^{-1} t_{-\varphi(x)})).$$

It is easily seen that χ^r is a bijection of W^r onto the set $J^r(\varphi(U), \psi(V)) \subset R^n \times \times R^m \times L^r_{n,m}$. This bijection can be described by means of components. Writing for $j_1 \leqq j_2 \leqq \dots \leqq j_k$

$$(2.1.12) \qquad y^\sigma_{j_1 j_2 \dots j_k} = a^\sigma_{j_1 j_2 \dots j_k} \circ \chi^r,$$

we obtain, without explicit mentioning that the sequences $(j_1, j_2, \dots, j_r)$ should be non-decreasing, $\chi^r = (x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1 j_2 \dots j_r})$. Notice that similarly as in (2.1.9), the functions (2.1.12) are defined for all sequences $(j_1, j_2, \dots, j_k)$. The set $J^r(X, Y)$ can be endowed with a unique smooth structure such that for any charts (U, φ), (V, ψ), the pair (W^r, χ^r) is a chart on $J^r(X, Y)$; with this smooth structure, $J^r(X, Y)$ is called the *manifold of r-jets* with *source* in X and *target* in Y. The chart (W^r, χ^r) is called *associated* with the charts (U, φ), (V, ψ).

It is easily verified that with respect to this smooth structure on $J^r(X, Y)$ the canonical jet projections (2.1.4) become smooth mappings; each of these mappings is, moreover, a surjective submersion or, in another terminology, a fibered manifold. The composition of jets (2.1.5), considered as a mapping of $J^r(X, Y) \oplus J^r(Y, Z)$ into $J^r(X, Z)$, where $\oplus$ denotes the fiber product of the canonical projections $J^r(X, Y) \to X$ and $J^r(Y, Z) \to Y$, is also smooth.

The construction of the manifold of r-jets with source in X and target in Y immediately applies to the case when Y is replaced by $J^s(X, Y)$, where s is a non-negative integer. In this way we obtain the manifold of r-jets $J^r(X, J^s(X, Y))$ whose elements are usually called *non-holonomic jets* (with precise specification of the integers r, s if necessary). If (U, φ), (V, ψ), and (W^s, χ^s) are the above charts then we get the *associated chart* on $J^r(X, J^s(X, Y))$ of the form $((W^s)^r, (\chi^s)^r)$.

Let $U \subset X$ be an open set, $f : U \to Y$ a mapping. We define a mapping $U \ni x \to \to J^r f(x) \in J^r(X, Y)$ by

$$(2.1.13) \qquad J^r f(x) = J^r_x f.$$

Clearly, $J^r f$ is smooth. We call it the *r-jet prolongation* of f.

Let $Z \in J^{r+s}(X, Y)$ be a point. Choose a representative f of Z so that $Z = J^{r+s}_x f$. Then the r-jet $J^r_x(J^s f)$ is an element of $J^r(X, J^s(X, Y))$. Expressing $J^s_x(J^s f)$ in the chart $((W^s)^r, (\chi^s)^r)$ we can easily see that this r-jet depends on Z only. Thus putting

$$(2.1.14) \qquad \iota(Z) = J^r_x(J^s f)$$

we obtain a mapping $\iota : J^{r+s}(X, Y) \to J^r(X, J^s(X, Y))$ which is an embedding; we call it the *canonical embedding*.

2.2. Higher order frames. Let $n, r \geqq 1$ be any integers, and let us consider the manifold $L^r_{n,n}$ (2.1.8). Let $a^i_{j_1 j_2 \dots j_k}$ be the canonical coordinates (2.1.9) on $L^r_{n,n}$, and denote by $L^r_n \subset L^r_{n,n}$ the subset of *invertible* r-jets. Obviously,

$$L^r_n = L^r_{n,n} \backslash \{A \in L^r_{n,n} \mid \det(a^i_k(A)) = 0\}. \tag{2.2.1}$$

It is directly verified that L^r_n is a Lie group. By (2.2.1), L^r_n is an open subset in $L^r_{n,n}$. Any two elements $A, B \in L^r_n$ are composable r-jets, and the composition of jets (2.1.5) defines a group structure on L^r_n. Since the smooth structure of L^r_n is defined by the (global) coordinates $a^i_{j_1 j_2 \dots j_k}$ and the coordinates $a^i_{j_1 j_2 \dots j_k}(A \circ B)$ depend polynomially on $a^p_{q_1 q_2 \dots q_k}(A)$ and $a^p_{q_1 q_2 \dots q_k}(B)$, the group multiplication is analytic, and L^r_n is a Lie group. We call L^r_n the *r-th differential group* of R^n, or just the *differential group*. The functions $a^i_{j_1 j_2 \dots j_k}$, where $1 \leqq i \leqq n$, $1 \leqq j_1 \leqq j_2 \leqq \dots \leqq j_k \leqq n$, $1 \leqq k \leqq r$ (the restrictions of (2.1.9) to L^r_n) are called the *canonical* (global) *coordinates* on L^r_n.

Example 2.1. To illustrate the group multiplication in differential groups we describe the multiplication of L^3_n in the canonical coordinates. Using the chain rule we obtain for any two points $A, B \in L^3_n$

$$\begin{aligned} &a^i_j(A \circ B) = a^i_k(A)\, a^k_j(B),\\ &a^i_{j_1 j_2}(A \circ B) = a^i_{k_1 k_2}(A)\, a^{k_1}_{j_1}(B)\, a^{k_2}_{j_2}(B) + a^i_k(A)\, a^k_{j_1 j_2}(B),\\ &a^i_{j_1 j_2 j_3}(A \circ B) = a^i_{k_1 k_2 k_3}(A)\, a^{k_1}_{j_1}(B)\, a^{k_2}_{j_2}(B)\, a^{k_3}_{j_3}(B) +\\ &+ a^i_{k_1 k_2}(A)\,(a^{k_1}_{j_1 j_2}(B)\, a^{k_2}_{j_3}(B) + a^{k_1}_{j_3 j_1}(B)\, a^{k_2}_{j_2}(B) +\\ &+ a^{k_1}_{j_2 j_3}(B)\, a^{k_2}_{j_1}(B)) + a^i_k(A)\, a^k_{j_1 j_2 j_3}(B). \end{aligned} \tag{2.2.2}$$

In deriving these formulas, we have used some representatives of the r-jets A, B, and (2.1.5).

There are some useful global coordinates on the group L^r_n, differing from the canonical ones. Let $A \in L^r_n$, $A = J^r_0\alpha$. Then $A^{-1} = J_0\alpha^{-1}$, and we put for all i, $j_1, j_2, \dots, j_k$

$$b^i_{j_1 j_2 \dots j_k}(A) = a^i_{j_1 j_2 \dots j_k}(A^{-1}) \tag{2.2.3}$$

or, which is the same, $b^i_{j_1 j_2 \dots j_k} = a^i_{j_1 j_2 \dots j_k} \circ J$, where J is the diffeomorphism $A \to A^{-1}$. The functions $b^i_{j_1 j_2 \dots j_k}$, where $1 \leqq i \leqq n$, $1 \leqq j_1 \leqq j_2 \leqq \dots \leqq j_k \leqq n$, $1 \leqq k \leqq r$, define a global chart on L^r_n. The transformation formulas to the canonical coordinates and vice versa may be easily obtained from definitions.

Let for example $r = 3$. Putting $B = A^{-1}$ in (2.2.2) and using

$$\begin{aligned} &a^i_j(J^3_0\,\mathrm{id}) = \delta^i_j, \qquad a^i_{j_1 j_2}(J^3_0\,\mathrm{id}) = 0,\\ &a^i_{j_1 j_2 j_3}(J^3_0\,\mathrm{id}) = 0, \end{aligned} \tag{2.2.4}$$

where id $= \mathrm{id}_{R^n}$, we obtain

$$a^i_k b^k_j = \delta^i_j,$$
$$(2.2.5)\qquad a^i_{k_1k_2} b^{k_1}_{j_1} b^{k_2}_{j_2} + a^i_k b^k_{j_1j_2} = 0,$$
$$a^i_{k_1k_2k_3} b^{k_1}_{j_1} b^{k_2}_{j_2} b^{k_3}_{j_3} + a^i_{k_1k_2}(b^{k_1}_{j_1j_2} b^{k_2}_{j_3} + b^{k_1}_{j_3j_1} b^{k_2}_{j_2} + b^{k_1}_{j_2j_3} b^{k_2}_{j_1}) + a^i_k b^k_{j_1j_2j_3} = 0.$$

The first of these equations determines b^k_j as the elements of the inverse matrix of the matrix (a^i_k) and, multiplying both sides of the second and the third equations by b^p_i we obtain the remaining transformation formulas.

Let X be an n-dimensional manifold. An invertible r-jet $\zeta \in J^r_{(0,x)}(R^n, X)$ is called an *r-frame* at the point $x \in X$. Let F^rX denote the set of all r-frames ζ, where the target of ζ runs over X, and let π^r_X denote the canonical jet projection of F^rX onto X. Consider a chart (U, φ), $\varphi = (x^i)$, on X. Denote $W^r = (\pi^r_X)^{-1}(U)$ and put for each $\zeta \in W^r$, $\zeta = J^r_0\mu$,

$$(2.2.6)\qquad \varphi^r(\zeta) = (\varphi(\mu(0)), J^r_0(t_{\varphi(\mu(0))}\varphi\mu)).$$

φ^r is a bijection of W^r onto $\varphi(U)\times L^r_n \subset R^n \times L^r_{n,n}$. We set for all $i, j_1, j_2, \ldots, j_k$ and for all $\zeta \in W^r$

$$(2.2.7)\qquad \zeta^i_{j_1j_2\ldots j_k}(\zeta) = a^i_{j_1j_2\ldots j_k}(J^r_0(t_{\varphi(\mu(0))}\varphi\mu)) = D_{j_1}D_{j_2}\ldots D_{j_k}(x^i\mu)(0),$$

where $a^i_{j_1j_2\ldots j_k}$ are the canonical coordinates on L^r_n. The set F^rX can be endowed with a unique smooth structure such that for each chart (U, φ) on X, (W^r, φ^r) is a chart on F^rX. The set F^rX with this smooth structure is called the *manifold of r-frames* over X. The chart (W^r, φ^r) is called *associated* with the chart (U, φ); we write x^i instead of $x^i \circ \pi^r_X$ and, without explicit mentioning that the sequences $(j_1, j_2, \ldots, j_k)$ should be non-decreasing, $\varphi^r = (x^i, \zeta^i_{j_1}, \ldots, \zeta^i_{j_1j_2\ldots j_r})$.

The manifold of r-frames F^rX is endowed with a natural structure of a principal L^r_n-bundle. To check it, notice that the mapping π^r_X is smooth, and the differential group L^r_n acts on F^rX to the right by the composition of jets,

$$(2.2.8)\qquad \zeta . A = \zeta \circ A.$$

This action is free, i.e., no element except the identity of L^r_n has a fixed point, and its orbits coincide with the sets $(\pi^r_X)^{-1}(x)$, where x runs over X. Moreover, slightly modifying the mapping φ^r (2.2.6) and putting for each $\zeta \in (\pi^r_X)^{-1}(U)$, $\zeta = J^r_0\mu$,

$$(2.2.9)\qquad \bar{\varphi}^r(\zeta) = (\mu(0), J^r_0(t_{\varphi(\mu(0))}\varphi\mu)),$$

we obtain a diffeomorphism $\bar{\varphi}^r : (\pi^r_X)^{-1}(U) \to U\times L^r_n$ such that for each $\zeta \in (\pi^r_X)^{-1}(U)$, $\zeta = J^r_0\mu$, and $A \in L^r_n$

(2.2.10) $\bar{\varphi}^r(\zeta \,.\, A) = (\mu(0), J_0^r(t_{\varphi(\mu(0))}\varphi\mu) \circ A),$

(multiplication in the group L_n^r on the right). These observations show that the manifold of r-frames F^rX has the structure of a right principal L_n^r bundle over the manifold X; we call it the *bundle of r-frames* over X. The bundles of r-frames are also called the *higher order frame bundles.*

Example 2.2. We shall write down explicitly the right action (2.2.8) of L_n^3 on F^3X.. Let (U, φ) be a chart on X. For any $\zeta \in (\pi_X^3)^{-1}(U)$ and $A \in L_n^3$

$$\begin{aligned}
&\zeta_j^l(\zeta \,.\, A) = \zeta_k^l(\zeta)\, a_j^k(A),\\
&\zeta_{j_1j_2}^l(\zeta \,.\, A) = \zeta_{k_1k_2}^l(\zeta)\, a_{j_1}^{k_1}(A)\, a_{j_2}^{k_2}(A) + \zeta_k^l(\zeta)\, a_{j_1j_2}^k(A),\\
&\zeta_{j_1j_2j_3}^l(\zeta \,.\, A) = \zeta_{k_1k_2k_3}^l(\zeta)\, a_{j_1}^{k_1}(A)\, a_{j_2}^{k_2}(A)\, a_{j_3}^{k_3}(A) +\\
&+ \zeta_{k_1k_2}^l(\zeta)\,(a_{j_1j_2}^{k_1}(A)\, a_{j_3}^{k_2}(A) + a_{j_3j_1}^{k_1}(A)\, a_{j_2}^{k_2}(A) +\\
&+ a_{j_2j_3}^{k_1}(A)\, a_{j_1}^{k_2}(A)) + \zeta_k^l(\zeta)\, a_{j_1j_2j_3}^k(A).
\end{aligned} \tag{2.2.11}$$

These formulas define the right action (2.2.8) in terms of the chart associated with (U, φ).

2.3. Fundamental categories. The purpose of this section is to introduce the fundamental categories used in this book. We begin by recalling some definitions.

Let us consider a system $\mathscr{C} = (\mathrm{Ob}\,\mathscr{C}, (\mathrm{Mor}\,(X, Y),\ X, Y \in \mathrm{Ob}\,\mathscr{C}))$, where $\mathrm{Ob}\,\mathscr{C}$ is a class of elements $X, Y, \ldots$, and $\mathrm{Mor}\,(X, Y)$ is a set. The system $\mathscr{C}$ is called a *category*, if to any three elements $X, Y, Z \in \mathrm{Ob}\,\mathscr{C}$ there is assigned a mapping $(f, g) \to g \circ f$ of the set $\mathrm{Mor}\,(X, Y) \times \mathrm{Mor}\,(Y, Z)$ into $\mathrm{Mor}\,(X, Z)$ such that the following conditions hold:

(1) $\mathrm{Mor}\,(X, Y) \cap \mathrm{Mor}\,(X', Y') \neq \emptyset$ if and only if $X = X'$, $Y = Y'$;

(2) each set $\mathrm{Mor}\,(X, X)$ contains an element id_X such that for any $f \in \mathrm{Mor}\,(X, X)$, $\mathrm{id}_X \circ f = f \circ \mathrm{id}_X = f$;

(3) the mapping $(f, g) \to g \circ f$ satisfies the associative law.

The elements of the class $\mathrm{Ob}\,\mathscr{C}$ are called the *objects* of the category $\mathscr{C}$, and the elements of the sets $\mathrm{Mor}\,(X, Y)$ are called the *morphisms* of $\mathscr{C}$. The mapping $(f, g) \to g \circ f$ is called the *composition* of the category $\mathscr{C}$. Two morphisms $f \in \mathrm{Mor}\,(X, Y)$, $g \in \mathrm{Mor}\,(Y', Z)$ are called *composable* if $Y = Y'$. A morphism $f \in \mathrm{Mor}\,(X, Y)$ is called an *isomorphism* (in the category $\mathscr{C}$) if there exists a morphism $g \in \mathrm{Mor}\,(Y, X)$ such that $f \circ g = \mathrm{id}_Y$, $g \circ f = \mathrm{id}_X$.

A *covariant functor* $\tau : \mathscr{A} \to \mathscr{B}$ from a category $\mathscr{A}$ into a category $\mathscr{B}$ is a mapping assigning to each element $X \in \mathrm{Ob}\,\mathscr{A}$ an element $\tau(X) \in \mathrm{Ob}\,\mathscr{B}$, and to each morphism $f \in \mathrm{Mor}\,(X, Y)$ of the category $\mathscr{A}$ a morphism $\tau(f) \in \mathrm{Mor}\,(\tau(X), \tau(Y))$ in such a way that the following two conditions are satisfied:

(1) for any two composable morphisms f, g of the category $\mathscr{A}$,

(2.3.1) $\tau(g \circ f) = \tau(g) \circ \tau(f)$,

(2) for each object $X \in \mathrm{Ob}\, \mathscr{A}$,

(2.3.2) $\tau(\mathrm{id}_X) = \mathrm{id}_{\tau(X)}$.

Replacing in this definition condition (2.3.1) by the condition $\tau(g \circ f) = \tau(f) \circ \tau(g)$ we obtain a *contravariant functor* $\tau : \mathscr{A} \to \mathscr{B}$.

Let μ and ν be two covariant functors from a category $\mathscr{A}$ into a category $\mathscr{B}$. A *natural transformation* of the functor μ into the functor ν is a system $t = (t_X)$ of morphisms $t_X : \mu(X) \to \nu(X)$ of the category $\mathscr{B}$, where X runs over Ob $\mathscr{A}$, such that for any morphism $f \in \mathrm{Mor}\,(X, Y)$ of the category $\mathscr{A}$ the diagram

(2.3.3)
$$\begin{array}{ccc} \mu(X) & \xrightarrow{t_X} & \nu(X) \\ \downarrow \mu(f) & & \downarrow \nu(f) \\ \mu(Y) & \xrightarrow{t_Y} & \nu(Y) \end{array}$$

commutes. An analogous definition can be given for contravariant functors.

We now introduce the categories used in the theory of differential invariants. The category of real, n-dimensional, smooth Hausdorff manifolds satisfying the second axiom of countability, and their embeddings, is denoted by $\mathscr{D}_n$. $\mathscr{PB}_n$ detnotes the category whose objects are smooth right principal fiber bundles over the objects of the category $\mathscr{D}_n$, and whose morphisms are homomorphisms of these principal fiber bundles over the morphisms of the category $\mathscr{D}_n$. If G is a Lie group, then $\mathscr{PB}_n(G)$ denotes the category formed by smooth right principal G-bundles over the objects of $\mathscr{D}_n$, and their G-homomorphisms over the morphisms of $\mathscr{D}_n$.

By a (*left*) *G-manifold* we mean a manifold endowed with a left action of a Lie group G. A mapping $f : P \to Q$ of G-manifolds is called *G-equivariant* if $f(g \,.\, p) = g \,.\, f(p)$ for all $g \in G$ and $p \in P$.

To introduce a category of fiber bundles we should fix some notation. Let Y be a principal G-bundle with projection $\pi : Y \to X$, P a left G-manifold, and $Y \times_G P$ the bundle with fiber P, associated with the principal G-bundle Y; the projection of the bundle $Y \times_G P$ is denoted by π_P. A point of $Y \times_G P$ is by definition an equivalence class $z = [y, p]$ of a pair $(y, p) \in Y \times P$ relative to the right action $((y, p), g) \to (y \cdot g, g^{-1} \,.\, p)$ of G on $Y \times P$.

Let Y_1 (resp. Y_2) be a principal G-bundle with projection $\pi : Y_1 \to X_1$ (resp. $\varrho : Y_2 \to X_2$), P (resp. Q) a left G-manifold, and consider the fiber bundles $Y_1 \times_G P$, $Y_2 \times_G Q$. We say that a mapping $\Phi : Y_1 \times_G P \to Y_2 \times_G Q$ is a *homomorphism* of $Y_1 \times_G P$ into $Y_2 \times_G Q$ if each point $x \in X_1$ has a neighborhood U such that there exist a G-homomorphism of principal G-bundles $\sigma_U : \pi^{-1}(U) \to Y_2$ and a smooth mapping of manifolds $F_U : P \to Q$ such that

$$\Phi([y, p]) = [\sigma_U(y), F_U(p)] \tag{2.3.4}$$

for all $(y, p) \in \pi^{-1}(U) \times P$.

This condition is equivalent to saying that there exist U, σ_U, and F_U such that the diagram

$$\begin{array}{ccc} U \times_G P & \xrightarrow{\Phi} & Y_2 \times_G Q \\ \uparrow & & \uparrow \\ U \times P & \xrightarrow{\sigma_U \times F_U} & Y_2 \times Q \end{array} \tag{2.3.5}$$

where $U \times P \to U \times_G P$ and $Y_2 \times Q \to Y_2 \times_G Q$ are the canonical projections onto the quotient, commutes.

Any pair (σ_U, F_U) satisfying (2.3.4) is called a *representative* of the homomorphism Φ over $U \subset X_1$. We denote by $\Phi_U = [\sigma_U, F_U]$ the restriction of the homomorphism Φ to $\pi_P^{-1}(U)$. If $U = X_1$ and there exists a representative (σ, F) of Φ over X_1 we write $\Phi = [\sigma, F]$.

We have the following simple consequences of the definition of a homomorphism of fiber bundles.

Theorem 2.1. *Let $\Phi : Y_1 \times_G P \to Y_2 \times_G Q$ be a homomorphism of fiber bundles.*

(a) Φ is a homomorphism of fibered manifolds, i.e., there exists a unique smooth mapping of manifolds $\Phi_0 : X_1 \to X_2$ such that $\varrho_Q \circ \Phi = \Phi_0 \circ \pi_P$.

(b) If (σ_U, F_U) is a representative of Φ over an open set $U \subset X_1$, then $F_U : P \to Q$ is a G-equivariant mapping.

(c) Let $U \subset X_1$ be an open set and $\Phi_U = [\sigma_U, F_U] = [\bar{\sigma}_U, \bar{F}_U]$. Then there exists a unique mapping $h_U : \pi^{-1}(U) \to G$ such that

$$\bar{\sigma}_U(y) = \sigma_U(y) \, . \, h_U(y), \qquad F_U(p) = h_U(y) \, . \, \bar{F}_U(p) \tag{2.3.6}$$

for all $y \in Y_1$, $g \in G$, and $p \in P$. This mapping satisfies

$$h_U(y \, . \, g) = g^{-1} \, . \, h_U(y) \, . \, g. \tag{2.3.7}$$

Proof. (a) Let $\Phi_U = [\sigma_U, F_U]$, $\Phi_V = [\sigma_V, F_V]$ for some open sets $U, V \subset X_1$; it is enough to show that there exists a unique smooth mapping of manifolds $\Phi_0 : U \cup V \to X_2$ such that $\varrho_Q \circ \Phi = \Phi_0 \circ \pi_P$ on $\pi_P^{-1}(U \cup V)$. Denote by proj σ_U (resp. proj σ_V) the projection of σ_U (resp. σ_V). Let $x \in U \cap V$, $z \in \pi_P^{-1}(x)$, $z = [y, p]$. We have $(\text{proj}\, \sigma_U)(x) = \text{proj}\, \sigma_U \circ \pi(y) = \varrho \circ \sigma_U(y) = \varrho \circ \Phi_U(z) = \varrho_Q \circ \Phi(z) = \varrho_Q \circ \Phi_V(z) = \text{proj}\, \sigma_V(x)$. Putting $\varrho_0(x) = (\text{proj}\, \sigma_U)(x)$ for $x \in U$,

$\varrho_0(x) = (\text{proj}\, \sigma_V)(x)$ for $x \in V$ we obtain a well-defined mapping $\Phi_0 : U \cup V \to X_2$. Obviously, $\Phi_0 \circ \pi_P = \varrho_Q \circ \Phi$ on $\pi_P^{-1}(U \cup V)$ as required.

(b) Let $z = [y, p] \in \pi_P^{-1}(U)$. For any point $g \in G$, $z = [y \cdot g, g^{-1} \cdot p]$, and we have $\Phi(z) = [\sigma_U(y), F_U(p)] = [\sigma_U(y) \cdot g, g^{-1} \cdot F_U(p)]$, $\Phi(z) = [\sigma_U(y \cdot g), F_U(g^{-1} \cdot p)]$; hence (b) follows from the property $\sigma_U(y) \cdot g = \sigma_U(y \cdot g)$ of σ_U.

(c) The proof of this assertion is straightforward.

Theorem 2.1 describes the freedom in the choice of the pair (σ_U, F_U) representing Φ. If in particular, $\sigma_U = \bar{\sigma}_U$, then necessarily $F_U = \bar{F}_U$; if $F_U = \bar{F}_U$, then $h_U(y)$ belongs to the isotropy group of the point $F_U(p) \in Q$ for each $p \in P$. Theorem 2.1 may also be used to define global homomorphisms of fiber bundles by means of their local representatives.

Fiber bundles (with various fibers) associated with principal G-bundles from the category $\mathscr{P}\mathscr{B}_n(G)$, and homomorphisms of fiber bundles over morphisms of the category $\mathscr{D}_n$, form a category which will be denoted by $\mathscr{F}\mathscr{B}_n(G)$.

If $X \in \text{Ob}\,\mathscr{D}_n$, then the bundle of r-frames over X, F^rX, is a right principal L^r_n-bundle, i.e. an element of $\text{Ob}\,\mathscr{P}\mathscr{B}_n(L^r_n)$. Let $\alpha \in \text{Mor}\,\mathscr{D}_n$, $\alpha : X_1 \to X_2$, and let $\pi^r_{X_1}$ (resp. $\pi^r_{X_2}$) be the projection of F^rX_1 (resp. F^rX_2). α defines an L^r_n-homomorphism $F^r\alpha : F^rX_1 \to F^rX_2$ of principal L^r_n-bundles by the formula

$$F^r\alpha(\zeta) = J^r_0\alpha \circ \zeta, \tag{2.3.8}$$

(composition of jets on the right). The projection of $F^r\alpha$ is equal to α, that is,

$$\pi^r_{X_2} \circ F^r\alpha = \alpha \circ \pi^r_{X_1}. \tag{2.3.9}$$

The correspondence $X \to F^rX$, $\alpha \to F^r\alpha$ is a covariant functor from the category $\mathscr{D}_n$ to the category $\mathscr{P}\mathscr{B}_n(L^r_n)$, called the *r-frame lifting*, and denoted by F^r. The morphism $F^r\alpha$ is called the *r-frame lift*, or simply the *lift*, of α.

Let F^r be the r-frame lifting, Q a left L^r_n-manifold. For a manifold $X \in \text{Ob}\,\mathscr{D}_n$, denote by F^r_QX the fiber bundle with fiber Q, associated with the principal L^r_n-bundle F^rX. Consider a morphism $\alpha \in \text{Mor}\,\mathscr{D}_n$, $\alpha : X_1 - X_2$, and put for each $z \in F^r_QX_1$, $z = [\zeta, q]$,

$$F^r_Q\alpha(z) = [F^r\alpha(\zeta), q]. \tag{2.3.10}$$

It is easily seen that this defines a morphism $F^r_Q\alpha \in \text{Mor}\,\mathscr{F}\mathscr{B}_n(L^r_n)$, in our previous notation, $F^r_Q\alpha = [F^r\alpha, \text{id}_Q]$. The correspondence $X \to F^r_QX$, $\alpha \to F^r_Q\alpha$ is a covariant functor from the category $\mathscr{D}_n$ to the category $\mathscr{F}\mathscr{B}_n(L^r_n)$. We call this functor the *Q-lifting* associated with the r-frame lifting F^r, and denote it by F^r_Q. The morphism $F^r_Q\alpha$ is called the *Q-lift*, or simply the *lift*, of α.

2.4. Differential invariants and their realizations. We state the following definition. A *differential invariant* is an L^r_n-equivariant mapping $f : P \to Q$ of an L^r_n-manifold P into an L^r_n-manifold Q.

Let P and Q be two L_n^r-manifolds, $f : P \to Q$ a differential invariant. For each $X \in \mathrm{Ob}\, \mathscr{D}_n$ and each $z \in \mathrm{Ob}\, \mathscr{F}\mathscr{B}_n(L_n^r)$, $z = [\zeta, p]$, the formula

(2.4.1) $$f_X(z) = [\zeta, f(p)]$$

establishes a morphism $f_X \in \mathrm{Mor}\, \mathscr{F}\mathscr{B}_n(L_n^r)$, $f_X : F_P^r X \to F_Q^r X$, whose projection is id_X. This morphism is called the *realization* of the differential invariant f on the manifold X.

Theorem 2.2. *Let $X \in \mathrm{Ob}\, \mathscr{D}_n$, $\Phi \in \mathrm{Mor}\, \mathscr{F}\mathscr{B}_n(L_n^r)$, $\Phi : F_P^r X \to F_Q^r X$. The following two conditions are equivalent:*

(1) *For each $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : U \to X$, where U is an open subset of X,*

(2.4.2) $$F_Q^r\alpha \circ \Phi = \Phi \circ F_L^r\alpha.$$

(2) *There exists a unique differential invariant $f : P \to Q$ whose realization on X is Φ, i.e. such that $f_X = \Phi$.*

Proof. Suppose that condition (1) holds. Denote by π_X^r (resp. $\pi_{X,P}^r$, resp. $\pi_{X,Q}^r$) the projection of the bundle F^rX (resp. F_P^rX, resp. F_Q^rX). We shall show that the projection of the morphism Φ, denoted by Φ_0, is id_X. We have, using (2.4.2) and the properties of morphisms of fiber bundles, $\pi_{X,Q}^r \circ F_Q^r\alpha \circ \Phi = \pi_{X,Q}^r \circ \Phi \circ F_P^r\alpha$, $\alpha \circ \pi_{X,Q}^r \circ \Phi = \Phi_0 \circ \pi_{X,P}^r \circ F_P^r\alpha$, $\alpha \circ \Phi_0 \circ \pi_{X,P}^r = \Phi_0 \circ \alpha \circ \pi_{X,P}^r$, that is, $\alpha \circ \Phi_0 = \Phi_0 \circ \alpha$; since this holds for each α, we must have $\Phi_0 = \mathrm{id}_X$. Let $x_0 \in X$ be any point, $\zeta \in (\pi_X^r)^{-1}(x_0)$ an r-frame. ζ defines a mapping $\Phi_\zeta : P \to Q$ by the relation

(2.4.3) $$\Phi(z) = [\zeta, \Phi_\zeta(p)],$$

where $z = [\zeta, p]$. We shall show that the mapping Φ_ζ is independent of ζ (over x_0). Fix $\zeta_0 \in (\pi_X^r)^{-1}(x_0)$, $\zeta_0 = J_0^r\mu$, choose $A \in L_n^r$, $A = J_0^r\sigma$, and put $\alpha = \mu\sigma\mu^{-1}$. By the definition of the r-frame lifting, $\zeta_0 \circ A = J_0^r(\mu\sigma) = J_{x_0}^r(\mu\sigma\mu^{-1}) \circ J_0^r\mu = F^r\alpha(\zeta_0)$. Using our assumption, we obtain for each $p \in P$

(2.4.4) $$\begin{aligned} F_Q^r\alpha \circ \Phi([\zeta_0, p]) &= [F^r\alpha(\zeta_0), \Phi_{\zeta_0}(p)] = \Phi \circ F_P^r\alpha([\zeta_0, p]) = \\ &= [F^r\alpha(\zeta_0), \Phi_{F^r\alpha(\zeta_0)}(p)] = [F^r\alpha(\zeta_0), \Phi_{\zeta_0 \circ A}(p)], \end{aligned}$$

which implies $\Phi_{\zeta_0} = \Phi_{\zeta_0 \circ A}$. By the transitivity of the action of L_n^r on F^rX, Φ_ζ must be independent of the choice of ζ in the fiber over x_0 and depends only on x_0. We shall show that in fact it is independent of x_0.

For $x \in X$ we define Φ_x as Φ_ζ, where ζ is any r-frame over the point x. Let x_1, $x_2 \in X$ be any points. Choose an element $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$ sending x_1 to x_2. For each $z \in (\pi_{X,P}^r)^{-1}(x_1)$, $z = [\zeta, p]$

(2.4.5) $$F_Q^r\alpha \circ \Phi(z) = [F^r\alpha(\zeta), \Phi_{x_1}(p)] = \Phi \circ F_P^r\alpha(z) = [F^r\alpha(\zeta), \Phi_{x_2}(p)],$$

which implies $\Phi_{x_1}(p) = \Phi_{x_2}(p)$.

Choose any point $x \in X$ and put $f = \Phi_x$. Let $z \in (\pi^r_{X,P})^{-1}(x)$, $z = [\zeta, p]$. For any element $A \in L^r_n$, z may be represented in the form $z = [\zeta . A, A^{-1} . p]$. We have, using this representation, $\Phi(z) = [\zeta, \Phi_x(p)] = [\zeta, f(p)] = [\zeta . A, f(A^{-1} . p)] = [\zeta, A . f(A^{-1} . p)]$, which implies $f(A^{-1} . p) = A^{-1} . f(p)$. Thus (2.4.3) is rewritten in the form $\Phi(z) = [\zeta, f(p)]$, where f is a differential invariant; by (2.4.1), Φ is the realization of f on X. The uniqueness of f follows from Theorem 2.1, (c). This completes the proof of Theorem 2.2.

2.5. Natural transformations of liftings, associated with the r-frame lifting. By definition, each differential invariant f gives rise to a correspondence $X \to f_X$, where X runs over $\mathrm{Ob}\, \mathscr{D}_n$, and $f_X \in \mathrm{Mor}\, \mathscr{F}\mathscr{B}_n(L^r_n)$. We shall now study this correspondence in more detail.

Theorem 2.3. *Let P and Q be two L^r_n-manifolds, $f : P \to Q$ a differential invariant. Then for each $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to X_2$, the realizations f_{X_1} and f_{X_2} of f satisfy*

$$F^r_Q\alpha \circ f_{X_1} = f_{X_2} \circ F^r_P\alpha. \tag{2.5.1}$$

In other words, the correspondence $T_f : X \to f_X$, where $X \in \mathrm{Ob}\, \mathscr{D}_n$, $f_X \in \mathrm{Mor}\, FB_n(L^r_n)$, is a natural transformation of the P-lifting F^r_P into the Q-lifting F^r_Q.

Proof. Let $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to X_2$, and choose an element $z \in F^r_P X_1$, $z = [\zeta, p]$. Then by definitions,

$$\begin{aligned} F^r_Q\alpha \circ f_{X_1}(z) &= F^r_Q\alpha([\zeta, f(p)]) = [F^r\alpha(\zeta), f(p)], \\ f_{X_2} \circ F^r_P\alpha(z) &= f_{X_2}([F^r\alpha(\zeta), p]) = [F^r\alpha(\zeta), f(p)] \end{aligned} \tag{2.5.2}$$

proving our assertion.

With the notation of Theorem 2.3, we have the following result.

Theorem 2.4. *The correspondence $f \to T_f$ is a bijection between the set of differential invariants $f : P \to Q$ and the set of natural transformations of the P-lifting F^r_P into the Q-lifting F^r_Q.*

Proof. Firstly, we shall show that the correspondence $f \to T_f$ is injective. Assuming that for some differential invariants f_1, f_2, the relation $T_{f_1} = T_{f_2}$ holds we obtain for each $X \in \mathrm{Ob}\, \mathscr{D}_n$, $(f_1)_X = (f_2)_X$, and the apply Theorem 2.2.

Secondly, we shall show that the correspondence $f \to T_f$ is surjective. Let T be a natural transformation of F^r_P into F^r_Q, le $X \in \mathrm{Ob}\, \mathscr{D}_n$. By Theorem 2.2, there must exist a differential invariant $f : P \to Q$ such that $f_X = T_X$, and this differential invariant is, for fixed X, unique. In order to show its independence of X, suppose that for some $X_1, X_2 \in \mathrm{Ob}\, \mathscr{D}_n$, $T_{X_1} = f_{X_1}$, $T_{X_2} = f_{X_2}$. Take $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to$

$\to X_2$, and apply Theorem 2.4. We see that $F^r_Q\alpha \circ T_{X_1} = T_{X_2} \circ F^r_P\alpha$. Substituting f_{X_1} and $\bar{f}_{X_2}$ into this equality we obtain $F^r_Q\alpha \circ f_{X_1} = \bar{f}_{X_2} \circ F^r_P\alpha$ which directly leads to the equality $f = \bar{f}$, by the same calculation as in (2.5.2). This ends the proof.

3. DIFFERENTIAL INVARIANTS AND LIE DERIVATIVES

We describe one of the effective methods for direct computations of differential invariants, the "infinitesimal" method, based on the concept of the Lie derivative. Main result are first order partial differential equations for differential invariants.

3.1. Jets of sections of a submersion. Let X (resp. Y) be an n-dimensional (resp. p-dimensional) manifold. We remind that a (smooth) mapping $\pi : Y \to X$ is called a *submersion* if to each point $y_0 \in Y$ there exist a chart (V, ψ), $\psi = (u^1, \ldots, u^n, y^1, \ldots, y^{p-n})$ with $y_0 \in V$ and a chart (U, φ), $\varphi = (x^1, \ldots, x^n)$ on X with $\pi(y_0) \in U$ such that $\pi(V) = U$ and

$$(3.1.1) \qquad u^1 = x^1 \circ \pi, u^2 = x^2 \circ \pi, \ldots, u^n = x^n \circ \pi.$$

Condition (3.1.1) may be equivalently expressed by

$$(3.1.2) \qquad \varphi \circ \pi = \mathrm{pr}_1 \circ \psi,$$

where $\mathrm{pr}_1 : R^n \times R^{p-n} \to R^n$ is the first projection. This shows that $\pi(V)$ is an open set in X; for (3.1.2) gives $\varphi\pi(V) = \mathrm{pr}_1\, \psi(V)$ which means that $\psi(V) = \varphi\pi(V) \times W$ and by definition both sets $\varphi\pi(V) \subset R^n$ and $W \subset R^{p-n}$ must be open. In particular, the charts (V, ψ) and $(\pi(V), \varphi)$, where φ is considered restricted to $\pi(V)$, also satisfy (3.1.2). We say that a chart (V, ψ) on Y is a ***fiber chart***, or is *adapted* to the submersion $\pi : Y \to X$ if there exists a chart (U, φ) on X such that (3.1.2) holds and $U = \pi(V)$. The chart (U, φ) is then obviously unique. In view of (3.1.1) the coordinates of a fiber chart (V, ψ) are usually denoted by $\psi = (x^i, y^\sigma)$, where $1 \leqq i \leqq n$, $1 \leqq \sigma \leqq m$, $m = p - n$; the coordinates of the corresponding chart (U, φ) on X are then denoted, with the obvious convention, by $\varphi = (x^i)$, where $1 \leqq i \leqq n$.

From the existence of fiber charts it immediately follows that a submersion is an open mapping.

A surjective submersion is also called a fibered manifold. More precisely, a *fibered manifold* is a triple (Y, π, X) in which Y and X are manifolds and π is a surjective submersion of Y onto X. X (resp. π) is called the *base* (resp. the *projection*) of this fibered manifold, the set $\pi^{-1}(x) \subset Y$, where $x \in X$ is a point, is called the *fiber* over x. A mapping $\gamma : U \to Y$, where $U \subset X$ is an open set, is called a *section* of the fibered manifold (Y, π, X), if $\pi \circ \gamma = \mathrm{id}_U$.

Many examples of fibered manifolds which we need later on were discussed in Sections 2.1, 2.2, and 2.4.

Let $\pi : Y \to X$ be a fibered manifold, and let $J^r_\pi(X, Y)$, or simply J^rY, denote the set of r-jets in $J^r(X, Y)$ which may be represented by local sections of π. It is easily seen that J^rY is a closed submanifold of $J^r(X, Y)$. Let $n = \dim X$, $n + m = \dim Y$. The manifold Y can be covered by fiber charts. Let (V, ψ) be a fiber chart on Y, (U, φ) the chart on X associated with (V, ψ); write for convenience $\psi = (z^\nu)$, $1 \leqq \nu \leqq n + m$, $\varphi = (x^i)$, $1 \leqq i \leqq n$. Put $W = U \times V$ and denote by (W^r, χ^r), $\chi^r = (x^i, z^\nu, z^\nu_{j_1}, \ldots, z^\nu_{j_1 j_2 \ldots j_r})$ the chart on $J^r(X, Y)$ associated with the charts (U, φ), (V, ψ) (Sec. 2.1, (2.1.11)). Let $P \in J^r(X, Y) \cap W^r$, $P = J^r_x\gamma$ for some section γ of π defined on a neighborhood of $x \in U$. γ has equations of the form

$$
\begin{aligned}
&z^i \circ \gamma = x^i, \qquad 1 \leqq i \leqq n, \\
&z^{n+\sigma} \circ \gamma = f^\sigma(x^1, x^2, \ldots, x^n), \qquad 1 \leqq \sigma \leqq m,
\end{aligned}
\tag{3.1.3}
$$

which implies that $z^i_j(P) = \delta^i_j$, $z^i_{j_1 j_2}(P) = 0, \ldots, z^i_{j_1 j_2 \ldots j_r}(P) = 0$. Thus the set $J^r(X, Y) \cap W^r$ has the equations

$$
z^i = x^i,\ z^i_j = \delta^i_j,\ z^i_{j_1 j_2} = 0. \ldots, z^i_{j_1 j_2 \ldots j_r} = 0. \tag{3.1.4}
$$

Since the charts (W^r, χ^r) cover $J^r(X, Y)$, J^rY is a closed submanifold of $J^r(X, Y)$. The manifold J^rY is called the *r-jet prolongation* of the fibered manifold $\pi : Y \to X$.

The chart representations of the canonical jet projections (2.1.3) show that their restrictions to the submanifold J^rY of $J^r(X, Y)$ are surjective submersions. We denote these restrictions by $\pi^{r,s} : J^rY \to J^sY$, $0 \leqq s \leqq r$, and $\pi^r : J^rY \to X$; we also call them the *canonical jet projections*.

If (V, ψ), $\psi = (x^i, y^\sigma)$, is a fiber chart on Y, we have $J^r(X, Y) \cap W^r = (\pi^{r,0})^{-1}(V)$, and denote this set by V^r. (3.1.4) shows that the pair (V^r, ψ^r), where $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, $1 \leqq i \leqq n$, $1 \leqq \sigma \leqq m$, $1 \leqq j_1 \leqq j_2 \leqq \ldots \ldots \leqq j_r \leqq n$, is a chart on J^rY. This chart is a fiber chart with respect to each of the canonical jet projections $\pi^{r,s}$, π^r. We shall say that this fiber chart is *associated* with the fiber chart (V, ψ).

Let $\pi_1 : Y_1 \to X_1$ and $\pi_2 : Y_2 \to X_2$ be two fibered manifolds. A mapping $\alpha : Y_1 \to Y_2$ is called a *homomorphism* of fibered manifolds if there exists a mapping $\alpha_0 : X_1 \to X_2$ such that

$$
\pi_2 \circ \alpha = \alpha_0 \circ \pi_1 . \tag{3.1.5}
$$

If such a mapping α_0 exists, it is unique, and is called the *projection* of α. If there is no danger of confusion we denote $\alpha_0 = \operatorname{proj} \alpha$.

It is clear that the composition of two homomorphisms of fibered manifolds is a homomorphism of fibered manifolds.

Let γ be a section of the fibered manifold $\pi : Y \to X$, defined on an open subset U of X. We set for each $x \in U$

$$J^r\gamma(x) = J^r_x\gamma. \tag{3.1.6}$$

We get a mapping $U \ni x \to J^r\gamma(x) \in J^rY$ which is a section of the fibered manifold $\pi^r : J^rY \to X$. $J^r\gamma$ is called the *r-jet prolongation* of γ.

The construction of the r-jet prolongation of a fibered manifold immediately applies to the s-jet prolongation of this fibered manifold. For a fibered manifold $\pi : Y \to X$ we obtain in this way a fibered manifold $(\pi^s)^r : J^r(J^sY) \to X$. Prolongations of this kind are usually referred to as the *non-holonomic prolongations* of fibered manifolds.

Let r and s be non-negative integers, $Z \in J^{r+s}Y$ a point. Choose a representative γ of the r-jet Z, so that $Z = J^{r+s}_x\gamma$, and consider the mapping $x \to J^s\gamma(x)$. The r-jet of $J^s\gamma$, $J^r_x(J^s\gamma)$, is an element of $J^r(J^sY)$. Expressing this r-jet with respect to the fiber chart $((V^s)^r, (\psi^s)^r)$, where (V, ψ) is a fiber chart on Y, we easily see that $J^r_x(J^s\gamma)$ depends only on Z. Therefore, putting

$$\imath(Z) = J^r_x(J^s\gamma), \tag{3.1.7}$$

we obtain a well defined mapping $\imath : J^{r+s}Y \to J^r(J^sY))$. The chart expression of $\imath$ is the mapping $(\psi^s)^r \circ \imath \circ (\psi^{r+s})^{-1}$; finding the explicit expression for this mapping we obtain at once that $\imath$ is an embedding. We call it the *canonical embedding* of $J^{r+s}Y$ into $J^r(J^sY)$. Clearly, $\imath$ is also a homomorphism of fibered manifolds over the identity mapping of J^sY.

Let $\pi_1 : Y_1 \to X_1$ and $\pi_2 : Y_2 \to X_2$ be two fibered manifolds over n-dimensional bases and let $\alpha : Y_1 \to Y_2$ be a homomorphism of fibered manifolds. Suppose that the projection α_0 of α is a diffeomorphism of X_1 onto an open subset $\alpha_0(X_1)$ of X_2. If γ is a section of π_1, then $\alpha\gamma\alpha_0^{-1}$ is a section of π_2, by (3.1.5). Thus the formula

$$J^r\alpha(Z) = J^r_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}), \tag{3.1.8}$$

where $Z = J^r_x\gamma$, defines a mapping $J^r\alpha : J^rY_1 \to J^rY_2$. Since the composition of jets is smooth, this mapping is also smooth. The following identities immediately follow from definitions:

$$\pi_2^r \circ J^r\alpha = \alpha_0 \circ \pi_1^r, \qquad \pi_2^{r,s} \circ J^r\alpha = J^s\alpha \circ \pi_1^{r,s}. \tag{3.1.9}$$

Thus $J^r\alpha$ is a homomorphism of fibered manifolds π_1^r and π_2^r and a homomorphism of fibered manifolds $\pi_1^{r,s}$ and $\pi_2^{r,s}$. Moreover, for any two composable homomorphisms α and β such that the projections α_0 and β_0 are diffeomorphisms,

$$J^r(\alpha \circ \beta) = J^r\alpha \circ J^r\beta. \tag{3.1.10}$$

3.2. Lie algebras of differential groups. Let us consider the differential group L_n^r, its Lie algebra $L(L_n^r)$ and the exponential mapping $\exp: L(L_n^r) \to L_n^r$.

Lemma 3.1. *Let A_t be a one-parameter subgroup of L_n^r. There exist a neighborhood U of the origin $0 \in R^n$ and a vector field ξ on U such that $\xi(0) = 0$ and $J_0^r\alpha_t = A_t$, where α_t is the local one-parameter group of ξ.*

Proof. Let $\bar{\xi} \in L(L_n^r)$ be the generator of A_t, that is, $\exp t\bar{\xi} = A_t$. Let us consider $L(L_n^r)$ as the tangent space $T_e L_n^r$ and denote by $a^i_{j_1 \dots j_k}$ the canonical coordinates on L_r. $\bar{\xi}$ has a unique expression

$$(3.2.1) \qquad \bar{\xi} = \bar{\xi}^i_j \left\{\frac{\partial}{\partial a^i_j}\right\}_e + \sum \bar{\xi}^i_{j_1 j_2} \left\{\frac{\partial}{\partial a^i_{j_1 j_2}}\right\}_e + \dots + \sum \bar{\xi}^i_{j_1 \dots j_r} \left\{\frac{\partial}{\partial a^i_{j_1 \dots j_r}}\right\}_e.$$

We denote by x^i the canonical coordinates on R^n and put

$$(3.2.2) \qquad \xi = \xi^i \frac{\partial}{\partial x^i},$$

where

$$(3.2.3) \qquad \xi^i = \bar{\xi}^i_j x^j + \sum \bar{\xi}^i_{j_1 j_2} x^{j_1} x^{j_2} + \dots + \sum \bar{\xi}^i_{j_1 \dots j_r} x^{j_1} \dots x^{j_r}.$$

In (3.2.1) and (3.2.3) summation over sequences $j_1 \leqq \dots \leqq j_k$ is assumed. ξ is a vector field on R^n. Obviously, $\xi(0) = 0$. Let α_t be the local one-parameter group of ξ. Choose $\delta > 0$ and a neighborhood U of the point $0 \in R^n$ in such a way that for each t, $|t| < \delta$, and $x \in U$, $\alpha_t(x)$ be defined. Then $\alpha_t(0) = 0$, and $J_0^r\alpha_t \in L_n^r$ for each t. Let us compute the tangent vector to the curve $t \to \alpha_t(x)$ at $t = 0$. Since $\xi^i = \left\{\frac{\mathrm{d}}{\mathrm{d}t} x^i\alpha_t\right\}_0$ we obtain in the canonical coordinates, using commutativity of partial derivatives,

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} a^i_j(J_0^r\alpha_t)\right\}_0 = \left\{\frac{\partial \xi^i}{\partial x^j}\right\}_0 = \bar{\xi}^i_j,$$

$$(3.2.4) \qquad \left\{\frac{\mathrm{d}}{\mathrm{d}t} a^i_{j_1 j_2}(J_0^r\alpha_t)\right\}_0 = \left\{\frac{\partial^2 \xi^i}{\partial x^{j_1} \partial x^{j_2}}\right\}_0 = \bar{\xi}^i_{j_1 j_2},$$

$$\dots$$

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} a^i_{j_1 \dots j_r}(J_0^r\alpha_t)\right\}_0 = \left\{\frac{\partial^2 \xi^i}{\partial x^{j_1} \dots \partial x^{j_r}}\right\}_0 = \bar{\xi}^i_{j_1 \dots j_r}.$$

This means that

$$(3.2.5) \qquad \left\{\frac{\mathrm{d}}{\mathrm{d}t} J_0^r\alpha_t\right\}_0 = \bar{\xi}.$$

Taking into account that $J_0^r\alpha_{t+s} = J_0^r\alpha_t \circ J_0^r\alpha_s$ for all s and t such that $|s|$, $|t|$, $|s+t| \leqq \delta$ and using the uniqueness of the one-parameter subgroup of L_n^r generated by $\bar{\xi}$ we obtain at once that $A_t = J_0^r\alpha_t$ for all t such that $|t| < \delta$. This proves Lemma 3.1.

Let TR^n be the tangent bundle of R^n. Since the projection of TR^n onto R^n is a surjective submersion, there is defined, for any integer $r > 0$, the r-jet prolongation J^rTR^n (Sec. 3.1). We shall now consider a subset of J^rTR^n, denoted by $\Gamma^r_{(0,0)}TR^n$, consisted of r-jets of vector fields $\xi : U \to TR^n$, where U is a neighborhood of the origin $0 \in R^n$, such that $\xi(0) = 0$. The operations

$$J_0^r\xi + J_0^r\zeta = J_0^r(\xi + \zeta), \qquad c \, . \, J_0^r\xi = J_0^r(c \, . \, \xi), \tag{3.2.6}$$

where $c \in R$, define a vector space structure on $\Gamma^r_{(0,0)}TR^n$.

We shall construct a natural linear isomorphism between $\Gamma^r_{(0,0)}TR^n$ and the tangent space $T_eL_n^r$. Let $J_0^r\xi \in \Gamma^r_{(0,0)}TR^n$ be any element. The r-jet $J_0^r\xi$ is expressible in the form

$$J_0^r\xi = J_0^r\left\{\frac{\mathrm{d}}{\mathrm{d}t}\chi_t\right\}_0, \tag{3.2.7}$$

where χ_t is a one-parameter family of diffeomorphisms of a neighborhood U of $0 \in R^n$ onto $\chi_t(U) \subset R^n$ such that $\chi_t(0) = 0$ for all t, and $J_0^r\chi_0 = J_0^r\,\mathrm{id}_{R^n}$. One may take for χ_t the local one-parameter group of a vector field representing the r-jet $J_0^r\xi$. Let x^i be the canonical coordinates on R^n, $(x^i_{j_1}, x^i_{j_1j_2}, \ldots, x^i_{j_1\ldots j_r})$, $1 \leqq i \leqq n$, $1 \leqq j_1 \leqq \ldots \leqq j_r \leqq n$, the corresponding canonical coordinates on $\Gamma^r_{(0,0)}TR^n$. These coordinates are defined by the relations

$$\begin{aligned} x^i_{j_1}(J_0^r\xi) = \left\{\frac{\partial \xi^i}{\partial x^j}\right\}_0, \; x^i_{j_1j_2}(J_0^r\xi) = \left\{\frac{\partial^2\xi^i}{\partial x^{j_1}\,\partial x^{j_2}}\right\}_0, \ldots, x^i_{j_1\ldots j_r}(J_0^r\xi) = \\ = \left\{\frac{\partial^r\xi^i}{\partial x^{j_1}\ldots\partial x^{j_r}}\right\}_0, \end{aligned} \tag{3.2.8}$$

where ξ^i are the components of the vector field ξ and the right side expressions are considered at the point $0 \in R^n$. Denoting the mapping $(t, x) \to \chi_t(x)$ by χ and its components by $\chi^i = x^i\chi$, condition (3.2.7) reads

$$\begin{aligned} \left\{\frac{\partial \xi^i}{\partial x^{j_1}}\right\}_0 = \left\{\frac{\partial^2\chi^i}{\partial t\,\partial x^{j_1}}\right\}_{(0,0)}, \ldots, \left\{\frac{\partial^r\xi^i}{\partial x^{j_1}\ldots\partial x^{j_r}}\right\}_0 = \\ = \left\{\frac{\partial^{r+1}\chi^i}{\partial t\,\partial x^{j_1}\ldots\partial x^{j_r}}\right\}_{(0,0)}, \end{aligned} \tag{3.2.9}$$

where $(0, 0)$ is the origin of $R \times R^n$. Let χ_t, ζ_t be two one-parameter families satysfying (3.2.7). Then, since partial derivatives in (3.2.9) commute,

$$(3.2.10) \qquad \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^r_0\chi_t\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^r_0\zeta_t\right\}_0.$$

Thence the relation

$$(3.2.11) \qquad \nu(J^r_0\xi) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^r_0\chi_t\right\}_0,$$

where χ_t is any one-parameter family satisfying (3.2.7), determines a well-defined tangent vector $\nu(J^r_0\xi) \in T_e L^r_n$.

Lemma 3.2. *The mapping* $\nu : \Gamma^r_{(0,0)}TR^n \to T_e L^r_n$ *is a linear isomorphism.*

Proof. Let $J^r_0\xi, J^r_0\zeta \in \Gamma^r_{(0,0)}TR^n$ be any elements, let α^ξ_t (resp. α^ζ_t) be the local one-parameter group of the vector field ξ (resp. ζ) representing $J^r_0\xi$ (resp. $J^r_0\zeta$), and let α^ξ (resp. α^ζ) denote the global flow of ξ (resp. ζ). We have

$$(3.2.12) \qquad J^r_0\xi + J^r_0\zeta = J^r_0\left\{\frac{\mathrm{d}}{\mathrm{d}t} \alpha^\xi_t \circ \alpha^\zeta_t\right\}_0.$$

To prove this equality notice that for any point x of a neighborhood of $0 \in R^n$

$$(3.2.13) \qquad \begin{aligned} &\left\{\frac{\mathrm{d}}{\mathrm{d}t} \alpha^\xi_t \circ \alpha^\zeta_t(x)\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t} \alpha^\xi(t, \alpha^\zeta(t, x))\right\}_0 = \\ &= T_1\alpha^\xi(0, x) + T_2\alpha^\xi(0, x) \,.\, \left\{\frac{\mathrm{d}}{\mathrm{d}t} \alpha^\zeta(t, x)\right\}_0 = \\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}t} \alpha^\xi_t(x)\right\}_0 + \left\{\frac{\mathrm{d}}{\mathrm{d}t} \alpha^\zeta_t(x)\right\}_0 = \xi(x) + \zeta(x), \end{aligned}$$

where $T_1\alpha^\xi(0, x)$ is the tangent mapping of the mapping $t \to \alpha^\xi_t(t, x) = \alpha^\xi_t(x)$ at $t = 0$, and $T_2\alpha^\xi(0, x)$ is the tangent mapping of the mapping $x \to \alpha^\xi(0, x) = x$, i.e. the identity mapping of T_xR^n. Consequently

$$(3.2.14) \qquad \begin{aligned} &\nu(J^r_0\xi + J^r_0\zeta) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^r_0(\alpha^\xi_t \circ \alpha^\zeta_t)\right\} = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^r_0\alpha^\xi_t \circ J^r_0\alpha^\zeta_t\right\}_0 = \\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}t} \Phi(J^r_0\alpha^\xi_t, J^r_0\alpha^\zeta_t)\right\}_0 = T_1\Phi(e, e) \,.\, \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^r_0\alpha^\xi_t\right\}_0 + \\ &+ T_2\Phi(e, e) \,.\, \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^r_0\alpha^\zeta_t\right\}_0 = \nu(J^r_0\xi) + \nu(J^r_0\zeta), \end{aligned}$$

where Φ denotes the group operation in L^r_n. Let now $c \in R$ be any number, and consider the tangent vector $\nu(c \,.\, J^r_0\xi)$. Let $\alpha^{c.\xi}_t$ be the local one-parameter group of the vector field $c \,.\, \xi$. We have for any point x of a neighborhood of $0 \in R^n$

$$(3.2.15)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\alpha_t^{c\,.\,\xi}(x) = c\,.\,\xi(\alpha_t^\xi(x)) = c\,.\,\frac{\mathrm{d}}{\mathrm{d}t}\alpha_t^\xi(x) = \frac{\mathrm{d}}{\mathrm{d}t}\alpha_{ct}^\xi(x),$$

so that

$$(3.2.16)\qquad \nu(c\,.\,J_0^r\xi) = \nu(J_0^r(c\,.\,\xi)) = \left\{\frac{\mathrm{d}}{\mathrm{d}t}J_0^r\alpha_{ct}^\xi\right\} = c\,.\,\nu(J_0^r\xi).$$

Equalities (3.2.14) and (3.2.16) show that the mapping ν is linear. It remains to verify that it is bijective. If $\nu(J_0^r\xi) = 0$ then using (3.2.9) and (3.2.11) we see at once that $J_0^r\xi = 0$ which means that ν is injective; that it is also surjective it follows directly from Lemma 3.1. This proves Lemma 3.2.

There is one and only one Lie algebra structure on the vector space $\Gamma^r_{(0,\,0)}TR^n$ such that ν is an isomorphism of Lie algebras; this Lie algebra structure is defined by the bracket

$$(3.2.17)\qquad \{J_0^r\xi, J_0^r\zeta\} = \nu^{-1}([\nu(J_0^r\xi), \nu(J_0^r\zeta)]),$$

where $[\nu(J_0^r\xi), \nu(J_0^r\zeta)]$ is the Lie algebra bracket in $T_eL_n^r$.

We shall interprete the bracket operation (3.2.17) in terms of the vector fields ξ, ζ representing the r-jets $J_0^r\xi$, $J_0^r\zeta$. To do this, consider any vector field ξ defined on a nieghborhood U of $0 \in R^n$ such that $\xi(0) = 0$. As before, denote by α_t^ξ the local one-parameter group of ξ. Since $J_0^r\alpha_t^\xi \in L_n^r$ for all t from a neighborhood of $0 \in R$, α_t^ξ induces a local one-parameter group of transformations of L_n^r, $(t, A) \to J_0^r\alpha^\xi \circ A$ (multiplication in L_n^r). We put for each $A \in L_n^r$

$$(3.2.18)\qquad \xi^r(A) = \left\{\frac{\mathrm{d}}{\mathrm{d}t}J_0^r\alpha_t^\xi \circ A\right\}_0.$$

ξ^r is a vector field on the Lie group L_n^r. We assert that

$$(3.2.19)\qquad [\xi^r, \zeta^r] = [\xi, \zeta]^r$$

for any two vector fields ξ, ζ. Obviously, we have for any element $B \in L_n^r$, $B =$ $= J_0^r\beta$,

$$(3.2.20)\qquad [\xi, \zeta]^r(B) = \left\{\frac{\mathrm{d}}{\mathrm{d}t}J_0^r\alpha_t^{[\xi,\zeta]} \circ B\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t}J_0^r(\alpha_t^{[\xi,\zeta]} \circ \beta)\right\}_0,$$

and

$$(3.2.21)\qquad \begin{aligned}[\xi^r, \zeta^r](B) &= \left\{\frac{\mathrm{d}}{\mathrm{d}t}(J_0^r\alpha^\xi_{-\sqrt{t}} \circ J_0^r\alpha^\zeta_{-\sqrt{t}} \circ J_0^r\alpha^\xi_{\sqrt{t}} \circ J_0^r\alpha^\zeta_{\sqrt{t}} \circ B)\right\}_0 = \\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}t}J_0^r(\alpha^\xi_{-\sqrt{t}}\alpha^\zeta_{-\sqrt{t}}\alpha^\xi_{\sqrt{t}}\alpha^\zeta_{\sqrt{t}} \circ \beta)\right\}_0,\end{aligned}$$

(the limit of vectors in $L(L_n^r)$ with respect to t). Using the commutativity of partia derivatives and the identity

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}\alpha_t^{[\xi,\zeta]}\circ\beta(x)\right\}_0=\left\{\frac{\mathrm{d}}{\mathrm{d}t}\alpha^{\xi}_{-\sqrt{t}}\alpha^{\zeta}_{-\sqrt{t}}\alpha^{\xi}_{\sqrt{t}}\alpha^{\zeta}_{\sqrt{t}}\circ\beta(x)\right\}_0= \tag{3.2.22}$$
$$=[\xi,\zeta]\circ\beta(x)$$

we obtain from (3.2.20) and (3.2.21) that (3.2.19) holds as required.

Notice that the vector fields ξ^r do not belong to the Lie algebra $L(L_n)$ for they are not left invariant. Denote by $J : L_n^r \to L_n^r$ the diffeomorphism $A \to A^{-1}$, and by L_B the left translation on L_n^r by an element B. We have

$$\xi^r(B)=\left\{\frac{\mathrm{d}}{\mathrm{d}t}J\circ L_{B^{-1}}\circ J^{-1}(J_0^r\alpha_t)\right\}_0= \tag{3.2.23}$$
$$=T_e(J\circ L_{B^{-1}}\circ J^{-1})\,.\left\{\frac{\mathrm{d}}{\mathrm{d}t}J_0^r\alpha_t\right\}=T_e(J\circ L_{B^{-1}}\circ J^{-1})\,.\,\xi^r(e).$$

Since $J\circ L_{B^{-1}}\circ J^{-1}$ is the right translation $A\to A\circ B$, ξ^r is a *right invariant* vector field. A left invariant vector field is obtained when we set

$$\bar{\xi}^r(B)=\left\{\frac{\mathrm{d}}{\mathrm{d}t}B\circ J_0^r\alpha_t^{-1}\right\}_0=\left\{\frac{\mathrm{d}}{\mathrm{d}t}J\circ L_{J_0^r\alpha_t}\circ J^{-1}(B)\right\}_0. \tag{3.2.24}$$

Obviously, then

$$T_eL_B\,.\,\bar{\xi}^r(e)=\left\{\frac{\mathrm{d}}{\mathrm{d}t}B\circ J_0^r\alpha_t^{-1}\right\}_0=\bar{\xi}\,(B). \tag{3.2.25}$$

Since moreover

$$\bar{\xi}^r(e)=-\xi^r(e)=-\nu(J_0^r\xi) \tag{3.2.26}$$

and ν is a linear isomorphism, all left invariant vector fields on L_n^r are obtained in this way.

We are now in a position to prove the following result.

Theorem 3.1. *For any* $J_0^r\xi, J_0^r\zeta\in\Gamma^r_{(0,0)}TR^n$

$$\{J_0^r\xi, J_0^r\zeta\}=-J_0^r[\xi,\zeta]. \tag{3.2.7}$$

Proof. It is enough to verify that $[\nu(J_0^r\xi),\nu(J_0^r\zeta)]=-\nu(J_0^r[\xi,\zeta])$. Notice that for each $B\in L_n^r$

$$\bar{\xi}^r(B)=T_{B^{-1}}J\,.\,\xi^r(J^{-1}(B)). \tag{3.2.28}$$

This relation means that the vector fields $\bar{\xi}^r$ and ξ^r are J-related. Consequently, their Lie brackets are also J-related, (3.2.29) $[\bar{\xi}^r, \bar{\zeta}^r] \circ J = TJ \, . \, [\xi^r, \zeta^r]$.

Now using (3.2.25), the definition of bracket in $T_e L_n^r$, and (3.2.19) we obtain

$$[\nu(J_0^r\xi), \nu(J_0^r\zeta)] = [\bar{\xi}^r(e), \bar{\zeta}^r(e)] = [\bar{\xi}^r, \bar{\zeta}^r](e) =$$

$$(3.2.30) \qquad = T_eJ \, . \, [\xi^r, \zeta^r](e) = T_eJ \, . \, [\xi, \zeta]^r(e) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J_0^r \alpha_{-t}^{[\xi,\zeta]}\right\}_0 =$$

$$= -[\xi, \zeta]^r(e) = -\nu(J_0^r[\xi, \zeta]).$$

This proves Theorem 3.1.

Corollary 1. *The Lie algebra* $L(L_n^r)$ *can be identified with the vector space* $\Gamma_{(0,0)}^r TR^n$ *endowed with the bracket* (3.2.27).

Remark 3.1. Some other equivalent descriptions of the Lie algebra $L(L_n^r)$ may be given by means of effective actions of L_n^r on differential manifolds (see Corollary 2 to Theorem 1.13). The simplest case of an action of this type, the prolongation of the tensor action of the general linear group $GL_n(R)$ on R^n, will be discussed later.

Remark 3.2. The Lie bracket $[\xi, \zeta]$ of two vector fields is a vector field whose value at a point x depends on 1-jets $J_x^1\xi$ and $J_x^1\zeta$, not only on $\xi(x)$ and $\zeta(x)$. It should be pointed out that the definition of the bracket $\{,\}$ on $T_{(0,0)}^r TR^n$ is correct, i.e., the right side of (3.2.27) depends on r-jets $J_0^r\xi$, $J_0^r\zeta$ only, not on the $(r+1)$-jets $J_0^{r+1}\xi$, $J_0^{r+1}\zeta$. Obviously, this is guaranteed by the conditions $\xi(0) = 0$, $\zeta(0) = 0$.

To give an explicit illustration we shall derive the general form of a left invariant vector field on L_n^3.

Example 3.1. Let $\bar{\xi} \in T_e L_n^3$ be any vector. We shall express in the canonical coordinates the left invariant vector field ξ on L_n^3 defined by $\bar{\xi}$. Let $\bar{\xi}$ be expressed by

$$(3.2.31) \qquad \bar{\xi} = \bar{\xi}_q^p \left\{\frac{\partial}{\partial a_q^p}\right\} + \Sigma\, \bar{\xi}_{q_1q_2}^p \left\{\frac{\partial}{\partial a_{q_1q_2}^p}\right\}_e + \Sigma\, \bar{\xi}_{q_1q_2q_3}^p \left\{\frac{\partial}{\partial a_{q_1q_2q_3}^p}\right\}_e,$$

(summation over non-decreasing sequences). By definition, $\xi(A) = T_eL_A \, . \, \bar{\xi}$. ξ is of the form

$$(3.2.32) \qquad \xi = \xi_j^i \frac{\partial}{\partial a_j^i} + \sum \xi_{j_1j_2}^i \frac{\partial}{\partial a_{j_1j_2}^i} + \sum \xi_{j_1j_2j_3}^i \frac{\partial}{\partial a_{j_1j_2j_3}^i},$$

where

$$\xi^i_j = \left\{\frac{\partial a^i_j L_A}{\partial a^p_q}\right\}_e . \bar{\xi}^p_q ,$$

$$\xi^i_{j_1 j_2} = \left\{\frac{\partial a^i_{j_1 j_2} L_A}{\partial a^p_q}\right\}_e . \bar{\xi}^p_q + \sum \left\{\frac{\partial a^i_{j_1 j_2} L_A}{\partial a^p_{q_1 q_2}}\right\}_e . \bar{\xi}^p_{q_1 q_2} ,$$

(3.2.33)

$$\xi^i_{j_1 j_2 j_3} = \left\{\frac{\partial a^i_{j_1 j_2 j_3} L_A}{\partial a^p_q}\right\}_e . \bar{\xi}^p_q + \sum \left\{\frac{\partial a^i_{j_1 j_2 j_3} L_A}{\partial a^p_{q_1 q_2}}\right\}_e . \bar{\xi}^p_{q_1 q_2} + $$

$$+ \sum \left\{\frac{\partial a^i_{j_1 j_2 j_3} L_A}{\partial a^p_{q_1 q_2 q_3}}\right\}_e . \bar{\xi}^p_{q_1 q_2 q_3} .$$

Taking the equations of the mapping L_A from (2.2.2) and remembering that in (3.2.33) one should take $j_1 \leqq j_2 \leqq j_3$, we obtain after some calculation

$$\xi^i_j = a^i_p \bar{\xi}^p_j ,$$

$$\xi^i_{j_1 j_2} = (a^i_{p j_2} \delta^q_{j_1} + a^i_{j_1 p} \delta^q_{j_2}) \bar{\xi}^p_q + a^i_p \bar{\xi}^p_{q_1 q_2} ,$$

(3.2.34)

$$\xi^i_{j_1 j_2 j_3} = (a^i_{p j_2 j_3} \delta^q_{j_1} + a^i_{j_1 p j_3} \delta^q_{j_2} + a^i_{j_1 j_2 p} \delta^q_{j_3}) \bar{\xi}^p_q +$$

$$+ (a^i_{p j_3} \delta^{q_1}_{j_1} \delta^{q_2}_{j_2} + a^i_{p j_2} \delta^{q_1}_{j_1} \delta^{q_2}_{j_3} + a^i_{p j_1} \delta^{q_1}_{j_2} \delta^{q_2}_{j_3}) \bar{\xi}^p_{q_1 q_2} + a^i_p \bar{\xi}^p_{j_1 j_2 j_3} .$$

The left invariant vector field ξ such that $\xi(e) = \bar{\xi}$ is now obtained by substituting these expressions in (3.2.31). In particular, a basis of left invariant vector fields ϑ^q_p, $\vartheta^{q_1 q_2}_p$, $\vartheta^{q_1 q_2 q_3}_p$ is obtained when we express ξ in the form

(3.2.35) $$\xi = \bar{\xi}_q . \vartheta^q_p + \sum \bar{\xi}^p_{q_1 q_2} . \vartheta^{q_1 q_2}_p + \sum \bar{\xi}^p_{q_1 q_2 q_3} . \vartheta^{q_1 q_2 q_3}_p .$$

3.3. Lifting and fundamental vector fields. In this section we suppose that we have an L^r_n-manifold P and the P-lifting F^r_P associated with the r-frame lifting F^r (see Sec. 3.3).

Let $X \in \mathrm{Ob}\, \mathscr{D}_n$. Recall that each morphism $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : U \to X$, where $U \subset X$ is an open subset, induces a morphism $F^r_P \alpha \in \mathrm{Mor}\, \mathscr{F}\mathscr{B}_n(L^r_n)$ – the F^r_P-lift of α. This construction is directly transferred to vector fields. Let ξ be a vector field on X and α_t its local one-parameter group. We put for each $z \in F^r_P X$

(3.3.1) $$F^r_P \xi(z) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} F^r_P \alpha_t(z)\right\}_0 .$$

This formula defines a vector field $F^r_P \xi$ on $F_P X$ which we call the F^r_P-*lift* of the vector field ξ.

Let π^r_X (resp. $\pi^r_{X,P}$) denote the projection of the principal L^r_n-bundle $F^r_P X$ (resp. associated fiber bundle $F^r_P X$). Each r-frame $\zeta \in F^r X$ determines an isomorphism $\varkappa_\zeta$ of the fiber in $F^r_P X$ over the point $x = \pi^r_X(\zeta)$ onto the fiber P of $F^r_P X$. If $z \in F^r_P X$, $z = [\zeta, p]$, then $\varkappa_\zeta(z)$ is defined by

(3.3.2) $\varkappa_\zeta(z) = p.$

Obviously, for each $\zeta \in F^rX$ and $A \in L^r_n$,

(3.3.3) $\varkappa_{\zeta \cdot A} = A^{-1} \cdot \varkappa_\zeta.$

The isomorphism of manifolds $\varkappa_\zeta$ is called the *framing* of the fiber $(\pi^r_{X,P})^{-1}(x)$ defined by the r-frame ζ.

Let $x \in X$ be a point, $\zeta \in F^rX$ an r-frame at this point, and let ξ be a vector field, defined on a neighborhood of x such that $\xi(x) = 0$. Since the local one-parameter group α_t of ξ preserves the point x, we have $F^r\alpha_t(\zeta) = \zeta \circ J^r_x\alpha_t \in \in (\pi^r_X)^{-1}(x)$ for all t, and $F^r_P\alpha_t(z) = [F^r\alpha_t(\zeta), p] \in (\pi^r_{X,P})^{-1}(x)$ for all t, where $z = [\zeta, p]$. In particular, the F^r_P-lift of ξ, restricted to the fiber $(\pi^n_X s_P)^{-1}(x)$, defines a vector field along this fiber. Since $\varkappa_\zeta$ is an isomorphism, there exists a unique vector field ξ_ζ on P such that

(3.3.4) $T\varkappa_\zeta \cdot F^r_P\xi = \xi_\zeta \circ \varkappa_\zeta$

along $(\pi^r_{X,P})^{-1}(x)$.

We shall show that for each $p \in P$ and $A \in L^r_n$

(3.3.5) $\xi_{\zeta \cdot A}(A^{-1} \cdot p) = T_p\Phi_{A^{-1}} - \xi_\zeta(p),$

where ϕ denotes the action $(A, p) \to A \cdot p$ of the group L^r_n on P. Let $z = [\zeta, p] = = [\zeta \cdot A, A^{-1} \cdot p]$. We have, using (3.3.2) and (3.3.4)

(3.3.6) $\xi_{\zeta \cdot A}(A^{-1} \cdot p) = \xi_{\zeta \cdot A}(\varkappa_{\zeta \cdot A}(z)) = T_z\varkappa_{\zeta \cdot A} \cdot F^r_P\xi(z).$

Differentiating the mapping $z \to \varkappa_{\zeta \cdot A}(z) = A^{-1} \cdot \varkappa_\zeta(z) = \Phi(A^{-1} \cdot \varkappa_\zeta(z)) = \Phi_{A^{-1}} \circ \circ \varkappa_\zeta(z)$ and substituting its tangent mapping into (3.3.6) we obtain

(3.3.7) $\xi_{\zeta \cdot A}(A^{-1} \cdot p) = T_{\varkappa_\zeta(z)}\, \Phi_{A^{-1}} \circ T_z\varkappa_\zeta \cdot F^r_P\xi(z).$

Since $\varkappa_\zeta(z) = p$, this equality coincides with (3.3.5).

The following assertion, together with the definition (3.3.4), establishes a relation between the fundamental vector fields on P and the F^r_P-lifts of vector fields on a manifold $X \in \mathrm{Ob}\, \mathscr{D}_n$.

Theorem 3.2. *Let X be an n-dimensional manifold, $x \in X$ a point, $\zeta \in F^rX$ an r-frame at x, and let P be an L^r_n-manifold.*

(a) Let ξ be a vector field defined on a neighborhood of x such that $\xi(x) = 0$. Then the vector field ξ_ζ is a fundamental vector field on P.

(b) Let Ξ be a fundamental vector field on P. There exists a vector field ξ defined on a neighborhood of x such that $\xi(x) = 0$ and $\Xi = \xi_\zeta$.

Proof. (a) With the notation of Theorem 3.2, let α_t be the local one-parameter group of ξ. Then $\alpha_t(x) = x$ for all t, and there exists a unique curve A_t in L_n^r such that $F^r\alpha_t(\zeta) = A_t$. In fact, A_t is a local one-parameter subgroup of L_n^r. From any z from the fiber in $F_P^r X$ over x, $z = [\zeta, p]$, we obtain

$$\begin{aligned}\varkappa_\zeta \circ F_P^r\alpha_t(z) &= \varkappa_\zeta([F^r\alpha_t(\zeta), p]) = \varkappa_\zeta([\zeta, A_t \, . \, p]) = \\ &= A_t \, . \, p = \Phi(A_t, \varkappa_\zeta(z)),\end{aligned} \tag{3.3.8}$$

where ϕ is the action of L_n^r on P. Differentiating both sides of (3.3.8) with respect to t at $t = 0$ we obtain

$$T_z\varkappa_\zeta \, . \left\{\frac{\mathrm{d}}{\mathrm{d}t} F^r\alpha_t(z)\right\} = T_e\Phi_{\varkappa_\zeta}(z) \, . \left\{\frac{\mathrm{d}}{\mathrm{d}t} A_t\right\}_0 . \tag{3.3.9}$$

Since the expression on the left is equal to $\xi_\zeta(p)$ and the point z is arbitrary, ξ_ζ is a fundamental vector field.

(b) Consider a fundamental vector field Ξ on P. By definition, there exists an element $\bar{\xi} \in L(L_n^r)$ such that $\Xi(p) = T_e\Phi_p \, . \, \bar{\xi}$. According to Lemma 3.1 there exists a vector field λ on a neighborhood of $0 \in R^n$ such that $\lambda(0) = 0$, whose local one-parameter group α_t satisfies $J_0^r\alpha_t = \exp t\bar{\xi}$. Let (U, φ) be a chart on X such that $x \in U$, $\varphi(x) = 0$, and $\zeta = J_0^r\varphi^{-1}$. Put $\beta_t = \varphi^{-1}\alpha_t\varphi$. β_t is a one-parameter family of diffeomorphisms of a neighborhood V of x onto $\beta_t(V) \subset U$. In fact, it is a local one-parameter group, satisfying $\beta_t(x) = x$ for all t. Denote by ξ the vector field generated by β_t, i.e., $\xi = \left\{\frac{\mathrm{d}}{\mathrm{d}t}\beta_t\right\}_0$, and compute the vector field ξ_ζ (3.3.4). We get for any $z = [\zeta, p]$

$$\begin{aligned}\xi_\zeta(p) &= T_z\varkappa_\zeta \, . \, F_P^r\xi(z) = \left\{\frac{\mathrm{d}}{\mathrm{d}t}\varkappa_\zeta \circ F_P^r\beta_t(z)\right\}_0 = \\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}t}\varkappa_\zeta([J_0^r\varphi^{-1} \circ J_0^r\alpha_t \circ J_x^r\varphi \circ J_0^r\varphi^{-1}, p])\right\}_0 = \\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}t}\varkappa_\zeta([\zeta, \Phi(J_0^r\alpha_t, p)])\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\exp t\bar{\xi} \, . \, p)\right\}_0 = T_e\Phi_p \, . \, \bar{\xi} = \Xi(p).\end{aligned} \tag{3.3.10}$$

Hence $\Xi = \xi_\zeta$.

3.4. Differential invariants and Lie derivatives. In this section we discuss "infinitesimal versions" of Theorem 2.2. The resulting criterion gives us a possibility of computing differential invariants by solving some systems of partial differential equations. To this purpose we first define the notion of the Lie derivative of a morphism of fiber bundles.

Let P and Q be two L_n^r-manifolds, $X \in \mathrm{Ob}\,\mathscr{D}_n$, $\Phi \in \mathrm{Mor}\,\mathscr{F}\mathscr{B}_n(L_n^r)$, $\Phi : F_P^r X \to F_Q^r X$.

Let ξ be a vector field on X, α_t its local one-parameter group. For each $z \in F_P^r X$, $t \to (F_Q^r \alpha_t \circ \Phi \circ F_P^r \alpha_{-t})(z)$ is a curve in $F_Q^r X$ passing through the point $\Phi(z)$. Put

$$(3.4.1) \qquad \partial_\xi \Phi(z) = \left\{ \frac{\mathrm{d}}{\mathrm{d}t} (F_Q^r \alpha_t \circ \Phi \circ F_P^r \alpha_{-t})(z) \right\}_0.$$

The correspondence $z \to \partial_\xi \Phi(z)$ is a *vector field along the morphism* Φ. This vector field is called the *Lie derivative* of the morphism Φ with respect to the vector field ξ.

Before going on to the connection between the Lie derivatives and differential invariants we need, for the proof of the next theorem, a lemma on *extension of vector fields* defined on closed submanifolds. In order to formulate this lemma, we first recall some definitions.

Let Y be an m-dimensional manifold, $X \subset Y$ a non-empty set. We say that X is as *n-dimensional submanifold* of Y if to each point $x \in X$ there exists a chart (V, ψ), $\psi = (y^\sigma)$, on Y such that $x \in V$ and the set $V \cap X$ is defined by the equations

$$(3.4.2) \qquad y^1 = 0, \ldots, y^{m-n} = 0.$$

In this case the chart (V, ψ) is called *adapted* to the submanifold X at the point x. The induced topology together with the adapted charts define on X the structure of an n-dimensional manifold.

Let $X \subset Y$ be an n-dimensional submanifold, ξ a vector field on X, $\bar{\xi}$ a vector field defined on a neighborhood of X in Y. We say that $\bar{\xi}$ (resp. ξ) is an *extension* of ξ (resp. the *restriction* of $\bar{\xi}$) if for each $x \in X$, $\bar{\xi}(x) = \xi(x)$.

Lemma 3.3. *To each vector field ξ defined on X there exists an extension $\bar{\xi}$ of ξ. If the submanifold X of Y is closed then there exists an extension $\bar{\xi}$ of ξ defined on Y.*

Proof. Let X be an n-dimensional submanifold of an m-dimensional manifold Y. By definition to each $x \in X$ there exists a chart (V, ψ), $\psi = (y^\sigma)$, where V is a neighborhood of x in Y, adapted to X at x; we may suppose without loss of generality that $\psi(V) \subset R^m$ is an open rectangle. Denote by $\pi : R^m \to R^n$ the second projection of the Cartesian product $R^{m-n} \times R^n$. With this chart there is associated a mapping $V \ni y \to \psi^{-1}\pi\psi(y) \in V \cap X$.

Let ξ be a vector field on X. Denote by t_z the translation of R^m sending a point $z \in R^m$ into the origin; that is, for each $z' \in R^m$, $t_z(z') = z' - z$. We set for each $y \in V$

$$(3.4.3) \qquad \xi_V(y) = T_{\psi^{-1}\pi\psi(y)}(\psi^{-1} t_{-\psi(y)+\pi\psi(y)} \psi) \cdot \xi(\psi^{-1}\pi\psi(y)).$$

$\bar{\xi}_V$ is a smooth vector field on V.

Let (V_i, ψ_i), $i = 1, 2, \ldots$, be adapted charts such that $\psi_i(V_i) \subset R^m$ is an open rectangle for each i and $\cup V_i \supset X$. Let (χ_i), $i = 1, 2, \ldots$, be a partition of unity on $\cup V_i$ subordinate to its covering (V_i). We put for each $y \in \cup V_i$

$$\bar{\xi}(y) = \sum \chi_i(y) \cdot \bar{\xi}_{V_i}(y). \tag{3.4.4}$$

$\bar{\xi}$ is a smooth vector field on $\cup V_i$ provided ξ is smooth.

Let $y = x \in X$ be any point and let $i_1, \ldots, i_k$ be the indices for which $\chi_{i_1}(x), \ldots, \ldots, \chi_{i_k}(x) \neq 0$, $\chi_{i_1}(x) + \ldots + \chi_{i_k}(x) = 1$. We may suppose that $i_1 = 1, \ldots, i_k = k$. For every $j = 1, 2, \ldots, k$ we have $\pi\psi_j(x) = \psi_j(x)$ so that

$$t_{-\psi_j(x)+\pi\psi_j(x)} = t_0 = \mathrm{id}_{R^m}, \tag{3.4.5}$$

$$T_z(\psi^{-1} t_{-\psi_j(x)+\pi\psi_j(x)}\psi_j) = \mathrm{id}_{T_x Y}, \tag{3.4.6}$$

where $z = \psi_j^{-1}\pi\psi_j(x)$. Therefore

$$\bar{\xi}_{V_1}(x) = \xi(x). \tag{3.4.7}$$

Thus at x

$$\bar{\xi}(x) = \chi_1(x)\,\bar{\xi}_{V_1}(x) + \ldots + \chi_k(x)\,\bar{\xi}_{V_k}(x) = \xi(x) \tag{3.4.8}$$

which proves the first part of the lemma.

If X is a closed submanifold then the complement $Y \setminus X$ is an open subset, and we may consider the open covering (V_i) of X, where $i = 0, 1, 2, \ldots$, and $V_0 = Y \setminus X$. Setting $\bar{\xi}_{V_0} = 0$ (on V_0) and defining $\bar{\xi}$ by (3.2.4) again, where (χ_i) is a partition of unity subordinate to the covering (V_i), $i = 0, 1, 2, \ldots$, we obtain an extension of ξ defined on Y.

Notice that the differential group L_n^r consists of two components. The first one, denoted by $L_n^{r(+)}$, is the maximal connected subgroup of L_n^r, and its formed by the r-jets $A \in L_n^r$ admitting a representation $A = J_0^r\alpha$, where $\det D\alpha(0) > 0$. The second component, denoted by $L_n^{r(-)}$, is the complement of $L_n^{r(+)}$ in L_n^r, and is formed by the r-jets $A \in L_n^r$, admitting a representation $A = J_0^r\alpha$, where $\det D\alpha(0) < 0$.

The following theorem gives us a condition equivalent to any of the conditions (1) and (2) of Theorem 2.2, provided the base manifold X is connected.

Theorem 3.3. *Let $X \in \mathrm{Ob}\,\mathscr{D}_n$ be a connected manifold, $\Phi \in \mathrm{Mor}\,\mathscr{F}\mathscr{B}_n(L_n^r)$, $\Phi : F_P^r X \to F_Q^r X$, a morphism. The following two conditions are equivalent:*

(1) *For each vector field ξ defined on an open subset of X,*

$$\partial_\xi \Phi = 0, \tag{3.4.9}$$

and there exist a point $x_0 \in X$ and a morphism $\alpha_0 \in \mathrm{Mor}\,\mathscr{D}_n$, $\alpha_0 : U \to X$, where U is a neighborhood of x_0, such that $\alpha_0(x_0) = x_0$, $\det \mathscr{D}\alpha_0(x_0) < 0$, and

(3.4.10) $\quad F_Q^r\alpha_0 \circ \Phi = \Phi \circ F_P^r\alpha_0.$

(2) *There exists a unique differential invariant* $f : P \to Q$ *whose realization on* X *is* Φ, *i.e. such that* $f_X = \Phi$.

Proof. If condition (2) holds then obviously, by Theorem 2.2, condition (1) must also hold. We have therefore to show that (1) implies (2).

Let ξ be any vector field on X. If (1) holds then to each point $x \in X$ there exists a neighborhood U of x and $\delta > 0$ such that

(3.4.11) $\quad F_Q^r\alpha_t \circ \Phi = \Phi \circ F_P^r\alpha_t$

on $(\pi_X^r)^{-1}(U)$ for all t such that $|t| < \delta$. We shall show that this implies

(3.4.12) $\quad \operatorname{proj} \Phi = \mathrm{id}_X,$

where proj Φ is the projection of the morphism Φ. Suppose that for some $x \in X$, proj $\Phi(x) = x' \neq x$, and choose a vector field ξ on X such that $\xi(x) = 0$, $\xi(x') \neq 0$. The local one-parameter group α_t of ξ satisfies $\alpha_t(x) = x$ and $\alpha_t(x') \neq x'$. Applying the projection $\pi_{X,Q}^r$ of the fiber bundle F_Q^rX on both sides of (3.4.4) we get $\pi_{X,Q}^r \circ F_Q^r\alpha_t \circ \Phi = \alpha_t \circ \pi_{X,Q}^r \circ \Phi = \alpha_t \circ \operatorname{proj} \Phi \circ \pi_{X,P}^r = \pi_{X,Q}^r \circ \Phi \circ F_P^r\alpha_t = \operatorname{proj} \Phi \circ \pi_{X,P}^r \circ F_P^r\alpha_t = \operatorname{proj} \Phi \circ \alpha_t \circ \pi_{X,P}^r$, i.e.

(3.4.13) $\quad \alpha_t \circ \operatorname{proj} \Phi = \operatorname{proj} \phi \circ \alpha_t.$

This means that $\alpha_t \circ \operatorname{proj} \Phi(x) = \alpha_t(x')$, $\operatorname{proj} \Phi \circ \alpha_t(x) = \operatorname{proj} \Phi(x) = x'$ which is a contradiction, and (3.4.12) must hold.

We now relate to each r-frame $\zeta \in F^rX$ a mapping Φ_ζ of P into Q by (2.4.3) and, using (3.4.4), we deduce that this mapping does not depend on ζ.

Let $x_0 \in X$ be a point, ξ a vector field on X such that $\xi(x_0) = 0$, α_t its local one-parameter group. Then the one-parameter family of transformations $F_P^r\alpha_t$ preserves the fiber $(\pi_{X,P}^r)^{-1}(x_0) \subset F_P^rX$. Denote $\chi_t = F^r\alpha_t$. From (3.2.4) we obtain for any r-frame $\zeta \in (\pi_X^r)^{-1}(x_0) \subset F^rX$ and each $z \in (\pi_{X,P}^r)^{-1}(x_0)$ represented in the form $z = [\zeta, p]$,

$$\Phi \circ F_P^r\alpha_t(z) = [\chi_t(\zeta), \Phi_{\chi_t(\zeta)}(p)] = F_Q^r\alpha_t \circ \Phi(z) = = [F^r\alpha_t(\zeta), \Phi_\zeta(p)], \tag{3.4.14}$$

which implies, since $p \in P$ is quite arbitrary,

(3.4.15) $\quad \Phi_{\chi_t(\zeta)} = \Phi_\zeta,$

where by (3.4.4), this equality holds for $|t| < \delta$, and the domain of α_t is U.

Now fix $\zeta_0 \in (\pi_X^r)^{-1}(x_0)$. Restricted U if necessary we may suppose without loss of generality that there exists a chart on X of the form (U, φ), where $\varphi(x_0) = 0$ and $\zeta_0 = J_0^r\varphi^{-1}$. Put for each t, $|t| < \delta$

(3.4.16) $\quad \beta_t = \varphi\alpha_t\varphi^{-1}.$

$\beta_t : \varphi(U) \to R^n$ is a one-parameter family of transformations such that $\beta_t(0) = 0$. That is, $J_0^r\beta_t \in L_n^r$ for all t for which α_t is defined. Denote

(3.4.17) $$\bar{\xi} = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J_0^r\beta_t\right\}_0.$$

$\bar{\xi}$ belongs to the Lie algebra $L(L_n^r) \approx T_eL_n^r$ and by the uniqueness of the integral curves of vector fields,

(3.4.18) $\quad \exp t\bar{\xi} = J_0^r\beta_t$

for all t such that $|t| < \delta$. Thus we get

(3.4.19) $$\begin{aligned}&\zeta_0 \cdot \exp t\bar{\xi} = \zeta_0 \cdot J_0^r\beta_t = J_0^r(\varphi^{-1}\beta_t\varphi) \circ J_0^r\varphi^{-1} = \\ &= J_0^r\alpha_t \circ \zeta_0 = F^r\alpha_t(\zeta_0).\end{aligned}$$

Let us consider the group element $A(s) = \exp s\bar{\xi} \in L_n^r$ for arbitrary $s \in R$. There exists a positive integer K and $t_0 \in R$, such that $|t_0| < \delta$, $s = K \cdot t_0$. For this group element

(3.4.20) $$\begin{aligned}&\zeta_0 \cdot A(s) = \zeta_0 \cdot \exp(Kt_0\bar{\xi}) = \zeta_0 \cdot (\exp t_0 \cdot \bar{\xi})^K = \\ &= \zeta_0 \cdot A(t_0)^K = \zeta_0 \cdot A(t_0)^{K-1} \cdot A(t_0) = F^r\alpha_t(\zeta_0 \cdot A(t_0))^{K-1},\end{aligned}$$

(K factors $\exp t_0\bar{\xi}$), where we used (3.4.19). Applying (3.4.15) we obtain after K steps

(3.4.21) $\quad \Phi_{\zeta_0 \cdot A(s)} = \Phi_{\zeta_0}.$

Thus, Φ_ζ is constant along the orbits of one-parameter subgroups of L_n^r in the fiber $(\pi_X^r)^{-1}(x_0)$.

The fiber $(\pi_X^r)^{-1}(x_0)$ of the principal L_n^r-bundle F^rX is, however, diffeomorphic with the Lie group L_n^r. Since the one-parameter subgroups of a Lie group fill a neighborhood of its identity, $\Phi_\zeta = \Phi_{\zeta_0}$ for all ζ from a neighborhood of ζ_0. Thus the mapping $\zeta \to \Phi_\zeta(p)$ is constant for each p on the connected component of the fiber $(\pi_X^r)^{-1}(x_0)$. In other words this says that

(3.4.22) $\quad \Phi_{\zeta_0 \cdot A} = \Phi_{\zeta_0}$

for all $A \in L_n^{r(+)}$.

Let us consider condition (3.4.10). Let ζ_0 be an r-frame from the domain of definition of $F^r\alpha_0$ and denote $\chi = F^r\alpha_0$. Then (3.4.22) gives, for each $p \in P$,

$$(3.4.23) \qquad [\chi(\zeta_0), \Phi_{\zeta_0}(p)] = [\chi(\zeta_0), \Phi_{\chi(\zeta_0)}(p)],$$

which implies $\Phi_{\zeta_0} = \Phi_{\chi(\zeta_0)}$. Writing as before $\zeta_0 = J_0^r\varphi^{-1}$ and

$$(3.4.24) \qquad \chi(\zeta_0) = \zeta_0 \circ J_0^r(\varphi\alpha_0\varphi^{-1}) = \zeta_0 \circ A_0,$$

we obtain an element $A_0 \in L_n^{r(-)}$. Thus $\Phi_{\zeta_0 \circ A_0} = \Phi_{\zeta_0}$, which implies that (3.4.22) holds for all $A \in L_n^r$.

Φ_ζ may therefore depend on the projection of ζ only. We define

$$(3.4.25) \qquad \Phi_x = \Phi_\zeta,$$

where ζ is any r-frame from the fiber $(\pi_X^r)^{-1}(x)$.

We shall now show that Φ_x is independent of $x \in X$ provided X is a connected manifold. Let x_1, x_2 be two different points of X. Since X is connected there exists a curve $I \ni t \to \gamma(t) \in X$, where $I \subset R$ is an open interval, such that (a) $\gamma(a) = x_1$, $\gamma(b) = x_2$ for some points $a, b \in I$, $a < b$, and (b) the tangent vector $\dot{\gamma}(t) = \mathrm{d}\gamma/\mathrm{d}t \in T_{\gamma(t)}X$ does not vanish along γ. That is, γ is an immersion, and the interval $[a, b]$ can be covered by open intervals $I_1, \ldots, I_k$, where $I_i = (a_i, b_i)$, such that the restriction of γ to I_i, denoted by γ_i, is an embedding for each i. Consider the subset $\gamma(I_i) \subset X$. This subset is a submanifold of X, and by Lemma 3.3 the vector field $\dot{\gamma}$, defined on $\gamma(I_i)$, can be extended to a vector field ξ_i, defined on a neighborhood of $\gamma(I_i)$. If α_t^i is the local one-parameter group of ξ_i, we have for any $z \in F_P^rX$, $z = [\zeta, p]$, over the point $\gamma(a_i) = x_i$

$$(3.4.26) \qquad F_P^r\alpha_t^i(z) = [F^r\alpha_t^i(\zeta), \Phi_{x_i}(p)] = [F^r\alpha_t^i(\zeta), \Phi_{y_{i,t}}(p)],$$

where $y_{i,t} = \alpha_t^i(x_i)$, which implies that $\Phi_{x_i} = \Phi_{y_{i,t}}$, or $\Phi_{x_i} = \Phi_x$ for all $x \in \gamma(I_i)$. Thus Φ_x does not depend on x on each $\gamma(I_i)$. In particular, $\Phi_{x_1} = \Phi_{x_2}$ as required.

We now take any $x \in X$ and set $f = \Phi_x$. To verify that f is a differential invariant whose realization on X is Φ, we proceed in the same way as in the proof of Theorem 2.2.

Let G be a Lie group, P and Q two left G-manifolds, and let $\Phi: G \times P \to P$ (resp. $\Psi : G \times Q \to Q$) denote the left action of G on P (resp. Q). Consider a mapping $f : P \to Q$ and a point $p \in P$. Let ζ be an element of the Lie algebra $L(G)$ of the Lie group G. ζ defines a curve $R \ni t \to f_t(p) \in Q$, where

$$(3.4.27) \qquad \begin{aligned} f_t(p) &= \Psi(\exp t\zeta, f(\Phi(\exp(-t\zeta), p))) = \\ &= (\Psi_{\exp t\zeta} \circ f \circ \Phi_{\exp(-t\zeta)})(p). \end{aligned}$$

Denote by $(\partial_\zeta f)(p)$ the tangent vector to this curve at $t = 0$,

$$(3.4.28) \qquad (\partial_\zeta f)(p) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} f_t(p)\right\}_0.$$

Lemma 3.4. *For each $p \in P$ and $t \in R$*

$$(3.4.29)\qquad \begin{aligned} &(\partial_\zeta f)(p) = \Psi'(\zeta)(f(p)) - T_p f \cdot \Phi'(\zeta)(p), \\ &\frac{\mathrm{d}}{\mathrm{d}t} f_t(p) = T_{f(\Phi_{\exp(-t\zeta)}(p))} \Psi_{\exp t\zeta} \cdot \partial_\zeta f(\Phi_{\exp(-t\zeta)}(p)), \end{aligned}$$

where $\Phi'(\zeta)$ (resp. $\Psi'(\zeta)$) is the fundamental vector field on P (resp. Q) associated with ζ.

Proof. By definition,

$$(3.4.30)\qquad (\partial_\zeta f)(p) = T_e \Psi_{f(p)} \cdot \zeta + T_{f(p)} \Psi_e \cdot T_p f \cdot T_e \Phi_p \cdot (-\zeta),$$

which gives the first formula. Further,

$$(3.4.31)\qquad \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f_t(p) &= \left\{ \frac{\mathrm{d}}{\mathrm{d}s} \Psi_{\exp(s+t)\zeta} f \Phi_{\exp(-s-t)\zeta}(p) \right\}_0 = \\ &= \partial_\zeta f_t(p) = \left\{ \frac{\mathrm{d}}{\mathrm{d}s} \Psi_{\exp t\zeta} \circ f_s \circ \Phi_{\exp(-t\zeta)}(p) \right\}_0 = \\ &= T_{f(\Phi_{\exp(-t\zeta)}(p))} \Psi_{\exp t\zeta} \cdot \left\{ \frac{\mathrm{d}}{\mathrm{d}s} f_s(\Phi_{\exp(-t\zeta)}(p)) \right\}_0, \end{aligned}$$

which gives the second formula.

The vector field $p \to \partial_\zeta f(p)$ along the mapping f is called the *Lie derivative* of the mapping f, relative to the vector $\zeta \in L(G)$.

Lemma 3.5. *Let G be a Lie group, G_0 the connected component of G, $f : P \to Q$ a mapping of left G-manifolds. Then f is G_0-equivariant if and only if for each $p \in P$ and $\zeta \in L(G)$*

$$(3.4.32)\qquad \partial_\zeta f(p) = 0.$$

Proof. Suppose that for some $\zeta \in L(G)$ and $p \in P$ (3.4.32) holds. Then by the second formula (3.4.29), $f_t(p) = f_0(p) = f(p)$ for all t. Using (3.4.27) we obtain that the mapping f must be ($\exp t\zeta$)-equivariant. Since ζ is arbitrary this ensures that f is G_0-equivariant.

Let us now consider the differential group L^r_n and its Lie algebra $L(L^r_n)$. We have the following infinitesimal criterion for a mapping between two L^r_n-manifolds to be a differential invariant.

Theorem 3.4. *Let P and Q be two L^r_n-manifolds, $f : P \to Q$ a mapping. The following conditions are equivalent:*

(1) *f is a differential invariant.*
(2) *For each element $\zeta \in L(L_n^r)$*

(3.4.33) $\qquad \partial_\zeta f = 0$

and there exists an element $A_0 \in L_n^{r(-)}$ such that for all $p \in P$

(3.4.34) $\qquad f(A_0 \, . \, p) = A_0 \, . \, f(p).$

Proof. Obviously, only the implication (2) $\Rightarrow$ (1) needs proof. By Lemma 3.5, (3.4.33) ensures that f is $L_n^{r(+)}$-equivariant. Moreover, if $A \in L_n^{r(-)}$ is any element, we have for each $p \in P$, $f(A \, . \, p) = f(A_0 \, . \, A_0^{-1} \, . \, A \, . \, p) = A_0 \, . \, f(A_0^{-1} \, . \, A \, . \, p)$. Since $A_0^{-1} \, . \, A \in L_n^{r(+)}$, $f(A_0^{-1} \, . \, A \, . \, p) = A_0^{-1} \, . \, A \, . \, f(p)$ and we have $f(A \, . \, p) = A \, . \, f(p)$ as required.

Remark 3.3. The following example shows that condition (3.4.34) of Theorem 3.4 cannot be omitted. Consider the tensor action of the general linear group $GL_n(R)$ on R^n, $(A, \xi) \to A \, . \, \xi$, where, in the canonical coordinates on $GL_n(R)$ and on R^n, $A \, . \, \xi$ is defined by the equations $\bar{\xi}^i = A_j^i \xi^j$. Let a mapping f from $R^n \times \ldots \times R^n$ into R be given by

(3.4.35) $\qquad f(\xi_1, \ldots, \xi_n) = |\det(\xi_1, \ldots, \xi_n)|,$

where ξ_n in the matrix $(\xi_1, \ldots, \xi_n)$ stands in the k-th column. It can be directly verified that f is a $GL_n^{(+)}(R)$-equivariant mapping which is not $GL_n(R)$-equivariant, if we consider R as a left $GL_n(R)$-manifold defined by the action $A \, . \, t = (\det A) \, . \, t$.

Condition (3.4.33) may be regarded as a system of first order partial differential equations for differential invariants from P to Q. Notice that these equations contain the fundamental vector fields on the left L_n^r-manifolds P and Q. The following remark shows that the structure of the Lie algebra $L(L_n^r)$ plays an important role in solving these equations.

Remark 3.3. *Let $f: P \to Q$ be as above, and suppose that for some vectors ξ, $\zeta \in L(L_n^r)$,*

(3.4.36) $\qquad \partial_\xi f = 0, \qquad \partial_\zeta f = 0.$

Then also

(3.4.37) $\qquad \partial_{[\xi, \zeta]} f = 0,$

where $[\xi, \zeta]$ is the bracket of the vectors ξ and ζ in $L(L_n^r)$. Clearly, the vector fields $\Phi'(\xi)$ and $\Psi'(\xi)$ are f-related, i.e. $[\Psi'(\xi), \Psi'(\zeta)] = Tf \, . \, [\Phi'(\xi), \Phi'(\zeta)]$. But (1.1.7) implies that for every $p \in P$, $[\Psi'(\xi), \Psi'(\zeta)](p) - T_e \Psi_p \, . \, [\xi, \zeta] = 0$, $[\Psi'(\xi),$

$\Psi'(\zeta)](p) = T_e\Phi_p \cdot [\xi, \zeta]$ which gives $\omega'([\xi, \zeta]) - Tf \cdot \Phi'([\xi, \zeta]) = 0$ as desired. Therefore, f is a solution of the system (3.4.33) for every $\xi \in L(L_n)$ if and only if f is a solution of this system for every ξ belonging to a vector subspace of $L(L_n^r)$ generating the Lie algebra $L(L_n^r)$.

4. INVARIANT TENSORS

In this chapter we explain the classical theory of invariant tensors on a finite dimensional vector space. We completely describe the structure of these tensors, and apply them to the problem of finding multilinear invariants of the general linear group $GL_n(R)$. Main notions are the following: Tensor representation of $GL_n(R)$, invariant tensor, weight.

4.1. Absolute invariants tensors. A finite-dimensional vector space endowed with a linear representation of a group G will be called a *G-module*. Each finite-dimensional vector space E can be regarded as a $GL(E)$-module, where $GL(E)$ is the group of linear transformations o E.

Let E be an n-dimensional vector space, E^* its dual vector space, and denote by T_s^rE the vector space of tensors of type (r, s) on E, $T_s^rE = E \otimes \ldots \otimes E \otimes E^* \otimes \ldots \ldots \otimes E^*$ (r factors E, s factors E^*). We shall consider T_s^rE with its canonical structure of a $GL(E)$-module, defined by the tensor representation of the group $GL(E)$.

Let (e_i), $1 \leqq i \leqq n$, be a basis of E, (e^i) the dual basis of E^*. If $A \in GL(E)$ we define a matrix (A_j^i) by

$$A \cdot e_j = A_j^i e_i. \tag{4.1.1}$$

(A_j^i) is the matrix of the linear isomorphism A with respect to the basis (e_i). The action of $GL\ E)$ on E^* is defined by the condition $(A.e^i)(A \cdot e_j) = \delta_j^i$ (the *Kronecker symbol*) or, which is the same, by the formula

$$A \cdot e^i = B_j^i e^j, \tag{4.1.2}$$

where (B_j^i) is the inverse matrix of the matrix (A_j^i). The action of $GL(E)$ on T_s^rE is defined by the formula

$$\begin{aligned} &A \cdot (e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{j_1} \otimes \ldots \otimes e^{j_s}) = \\ &= (A \cdot e_{i_1}) \otimes \ldots \otimes (A \cdot e_{i_r}) \otimes (A \cdot e^{j_1}) \otimes \ldots \otimes (A \cdot e^{j_s}) \end{aligned} \tag{4.1.3}$$

and by the linearity requirements. If $t \in T_s^rE$ is any tensor,

$$t = t^{i_1 \dots i_r}_{j_1 \dots j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s}, \tag{4.1.4}$$

then

$$A \,.\, t = \bar{t}^{p_1 \dots p_r}_{q_1 \dots q_s} e_{p_1} \otimes \dots \otimes e_{p_r} \otimes e^{q_1} \otimes \dots \otimes e^{q_s}, \tag{4.1.5}$$

where

$$\bar{t}^{p_1 \dots p_r}_{q_1 \dots q_s} = A^{p_1}_{k_1} \dots A^{p_r}_{k_r} B^{j_1}_{q_1} \dots B^{j_s}_{q_s} t^{k_1 \dots k_r}_{j_1 \dots j_s}. \tag{4.1.6}$$

A tensor $t \in T^r_s E$ is called an *absolute invariant tensor*, or simply an *invariant tensor*, if for each element $A \in GL(E)$, $A \,.\, t = t$. Expressing t as in (4.1.4) we obtain that t is absolute invariant if and only if for all $A \in GL(E)$

$$t^{p_1 \dots p_r}_{q_1 \dots q_s} = A^{p_1}_{k_1} \dots A^{p_r}_{k_r} B^{j_1}_{q_1} \dots B^{j_s}_{q_s} t^{k_1 \dots k_r}_{j_1 \dots j_s}. \tag{4.1.7}$$

Let $t \in T^r_s E$ be an invariant tensor, (e_i) and $(\bar{e}_i)$ two bases of E. Expressing t with respect to (e_i) and $(\bar{e}_i)$ we obtain (4.1.4) and

$$t = \bar{t}^{i_1 \dots i_r}_{j_1 \dots j_s} \bar{e}_{i_1} \otimes \dots \otimes \bar{e}_{i_r} \otimes \bar{e}^{j_1} \otimes \dots \otimes \bar{e}^{j_s}. \tag{4.1.8}$$

There exists a unique element $A \in GL(E)$ such that $\bar{e}_j = A^i_j e_i$. Since t is invariant, $A \,.\, t = t$, and $\bar{t}^{i_1 \dots i_r}_{j_1 \dots j_s} = t^{i_1 \dots i_r}_{j_1 \dots j_s}$. Thus the components of an invariant tensor do not depend on the basis. We write for simplicity

$$t = (t^{i_1 \dots i_r}_{j_1 \dots j_s}) \tag{4.1.9}$$

not specifying the basis. If F is another n-dimensional vector space, we may define an invariant tensor $t \in T^r_s F$ by means of the same formula, (4.1.9). It is therefore sufficient to study invariant tensors on one particular n-dimensional vector space, for example R^n.

Example 4.1. The Kronecker tensor

$$\delta = (\delta^i_j) \tag{4.1.10}$$

is an absolute invariant tensor of type (1,1); this is directly verified by means of (4.1.6).

Let S_r be the *r-th order symmetric group*, i.e. the group of permutations (= bijections) $\sigma : \{1, 2, \dots, r\} \to \{1, 2, \dots, r\}$ of the set $\{1, 2, \dots, r\}$, and let $GL_n(R)$ be the general linear group. The connected component of identity in $GL_n(R)$ is denoted by $GL^{(+)}_n(R)$, and its complement in $GL_n(R)$ is denoted by $GL^{(-)}_n(R)$. Notice that the group $GL_n(R)$ is canonically identified with $GL(R^n)$, the group of linear transformations of the vector space R^n; this identification is defined by means of the canonical basis of R^n.

Theorem 4.1. *Let $t \in T^r_s R^n$ be a tensor.*

(a) If $r \neq s$, then t is invariant if and only if $t = 0$.

(b) If $r = s$, then the following four conditions are equivalent:

(1) t is an invariant tensor.

(2) $A \,.\, t = t$ for all $A \in GL_n^{(+)}(R)$.

(3) For any integers $i, j, p_1, \ldots, p_r, q_1, \ldots, q_r = 1, 2, \ldots, n$.

$$\begin{aligned} \delta^{p_1}_i t^{jp_2\ldots p_r}_{q_1\ldots q_r} + \delta^{p_2}_i t^{p_1 j p_3\ldots p_r}_{q_1\ldots q_r} + \ldots + \delta^{p_r}_i t^{p_1\ldots p_{r-1}j}_{q_1\ldots q_r} - \\ - \delta^j_{q_1} t^{p_1\ldots p_r}_{iq_2\ldots q_r} - \delta^j_{q_2} t^{p_1\ldots p_r}_{q_1 i q_3\ldots q_r} - \ldots - \delta^j_{q_r} t^{p_1\ldots p_r}_{q_1\ldots q_{r-1}i} = 0. \end{aligned} \tag{4.1.11}$$

(4) $t = (t^{p_1\ldots p_r}_{q_1\ldots q_r})$, where

$$t^{p_1\ldots p_r}_{q_1\ldots q_r} = \sum_{\sigma\in S_r} c_\sigma \,.\, \delta^{p_1}_{q_{\sigma(1)}} \ldots \delta^{p_r}_{q_{\sigma(r)}} \tag{4.1.12}$$

for some $c_\sigma \in R$.

Proof. (a) Suppose that $r \neq s$, and choose an element $A \in GL_n(R)$, $A = (A^i_j)$, of the form

$$A^i_j = \lambda^i \delta^i_j. \tag{4.1.13}$$

Obviously, $\lambda^i \neq 0$, and $A^{-1} = (1/\lambda^j)\, \delta^i_j$. Then for any $t \in T^r_s R^n$,

$$A^{i_1}_{p_1} \ldots A^{i_r}_{p_r} B^{q_1}_{j_1} \ldots B^{q_s}_{j_s} t^{p_1\ldots p_r}_{q_1\ldots q_s} = \lambda^{i_1} \ldots \lambda^{i_r} \frac{1}{\lambda^{j_1}} \ldots \frac{1}{\lambda^{j_s}} t^{i_1\ldots i_r}_{j_1\ldots j_s} \tag{4.1.14}$$

(no summation on the right side), where $t^{p_1\ldots p_r}_{q_1\ldots q_s}$ are the components of t with respect to the canonical basis of R^n. Hence if t is invariant,

$$\lambda^{i_1} \ldots \lambda^{i_r} \frac{1}{\lambda^{j_1}} \ldots \frac{1}{\lambda^{j_s}} t^{i_1\ldots i_r}_{j_1\ldots j_s} = t^{i_1\ldots i_r}_{j_1\ldots j_s} \tag{4.1.15}$$

for all $\lambda^i \neq 0$; this obviously implies $t^{i_1\ldots i_r}_{j_1\ldots j_s} = 0$.

(b) Suppose that $r = s$. We shall prove the second part of Theorem 4.1 in four steps according to the following schema: (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

1. Condition (1) obviously implies (2).

2. Let $t \in T^r_r R^n$ be a tensor satisfying (2), let ξ^k_i, $1 \leq i, k \leq n$, be the fundamental vector field on $T^r_r R^n$, associated with the elements of the canonical basis of the Lie algebra $gl_n(R)$ of the Lie group $GL_n(R)$. By definition, each of the vector fields ξ^k_i vanishes at t. Since the left action of $GL_n(R)$ on $T^r_r R^n$ is defined by (4.1.6), we have

$$\xi^k_i = \left\{ \frac{\partial}{\partial A^i_k} (A^{p_1}_{j_1} \ldots A^{p_r}_{j_r} B^{m_1}_{q_1} \ldots B^{m_r}_{q_r}) \right\}_e . \, t^{j_1\ldots j_r}_{m_1\ldots m_r} \frac{\partial}{\partial t^{p_1\ldots p_r}_{q_1\ldots q_r}}, \tag{4.1.16}$$

where e is the identity of $GL_n(R)$. Using the equality

$$(4.1.17) \qquad \frac{\partial B^p_q}{\partial A^i_k} = -B^p_i B^k_q$$

and the conditions $\xi^k_i(t) = 0$ we obtain (3).

3. If $n = 1$, (3) is satisfied by every tensor $t \in T^r_r R$, and so is (4), hence (3) implies (4). We shall now suppose that $n \geqq 2$.

Firstly, we shall show that (3) implies (4) for $r = 1$. In this case (4.1.11) takes the form $\delta^p_i t^j_q - \delta^j_q t^p_i = 0$. Contracting the left hand side in i and p we get $nt^j_q - \delta^j_q t^p_p = 0$ that is, $t^j_q = (1/n) \,.\, t^p_p \delta^j_q$ which coincides with (4.1.12).

Secondly, we shall show that if a tensor $t = (t^{p_1 \ldots p_r}_{q_1 \ldots q_r})$ satisfies condition (3) then its contraction in any of its superscript and subscript also satisfies (3). Consider for instance the tensor $s = (s^{p_1 \ldots p_{r-1}}_{q_1 \ldots q_{r-1}})$, where $s^{p_1 \ldots p_{r-1}}_{q_1 \ldots q_{r-1}} = t^{p_1 \ldots p_{r-1} l}_{q_1 \ldots q_{r-1} l}$. Contracting (4.1.11) in p_r and q_r we get

$$(4.1.18) \qquad \begin{aligned} &\delta^{p_1}_i t^{j p_2 \ldots p_{r-1} k}_{q_1 \ldots q_{r-1} k} + \delta^{p_2}_i t^{p_1 j p_3 \ldots p_{r-1} k}_{q_1 \ldots q_{r-1} k} + \ldots + \\ &+ \delta^k_i t^{p_1 \ldots p_{r-1} j}_{q_1 \ldots q_{r-1} k} - \delta^j_{q_1} t^{p_1 \ldots p_{r-1} k}_{i q_2 \ldots q_{r-1} k} - \\ &- \delta^j_{q_2} t^{p_1 \ldots p_{r-1} k}_{q_1 i q_3 \ldots q_{r-1} k} - \ldots - \delta^j_k t^{p_1 \ldots p_{r-1} k}_{q_1 \ldots q_{r-1} i} = 0, \end{aligned}$$

that is,

$$(4.1.19) \qquad \begin{aligned} &\delta^{p_1}_i s^{j p_2 \ldots p_{r-1}}_{q_1 \ldots q_{r-1}} + \delta^{p_2}_i s^{p_1 j p_3 \ldots p_{r-1}}_{q_1 \ldots q_{r-1}} + \ldots + \\ &+ \delta^{p_{r-1}}_{i} {}^{p_1 \ldots p_{r-2} j}_{q_1 \ldots q_{r-1}} - \delta^j_{q_1} s^{p_1 \ldots p_{r-1}}_{i q_2 \ldots q_{r-1}} - \\ &- \delta^j_{q_2} s^{p_1 \ldots p_{r-1}}_{q_1 i q_3 \ldots q_{r-1}} - \ldots - \delta^j_{q_{r-1}} s^{p_1 \ldots p_{r-1}}_{q_1 \ldots q_{r-2} i} = 0. \end{aligned}$$

Finally, supposing that a tensor $t = (t^{p_1 \ldots p_r}_{q_1 \ldots q_r})$ satisfies condition (3) we deduce that it must be of the form (4.1.12), i.e., it must be a linear combination of tensors $\delta^{p_1}_{q_{\sigma(1)}} \delta^{p_2}_{q_{\sigma(2)}} \ldots \delta^{p_r}_{q_{\sigma(r)}}$ where σ runs over $r!$ elements of the symmetric group S_r. Fix two r-tuples $(p_1, \ldots, p_r)$, $(q_1, \ldots, q_r)$. Contracting (4.1.11) we obtain for every permutation $(s_1, \ldots, s_r)$ of $(p_1, \ldots, p_r)$,

$$(4.1.20) \qquad \begin{aligned} &n t^{s_1 \ldots s_r}_{q_1 \ldots q_r} + t^{s_2 s_1 s_3 \ldots s_r}_{q_1 q_2 q_3 \ldots q_r} + \ldots + t^{s_r s_2 \ldots s_{r-1} s_1}_{q_1 q_2 \ldots q_r} = \\ &= \delta^{s_1}_{q_1} t^{i s_2 \ldots s_r}_{i q_2 \ldots q_r} + \delta^{s_1}_{q_2} t^{i s_2 s_3 \ldots s_r}_{q_1 i q_3 \ldots q_r} + \ldots + \delta^{s_1}_{q_r} t^{i s_2 \ldots s_{r-1} s_r}_{q_1 q_2 \ldots q_{r-1} i} \end{aligned}$$

(summation over i on the right side). (4.1.20) can be considered as a system of n^r linear, non-homogeneous equations for $t^{s_1 \ldots s_r}_{q_1 \ldots q_r}$, with given right-hand side. It is easily seen that because $n \geqq 2$, this system can be solved by direct eliminating all n^r unknowns $t^{s_1 \ldots s_r}_{q_1 \ldots q_r}$, where $(q_1, \ldots, q_r)$ is fixed and $(s_1, \ldots, s_r)$ runs over all permutations of $(p_1, \ldots, p_r)$. Hence writing this system in the form

$$(4.1.21) \qquad Q \,.\, t = t',$$

where t' is the right side of (4.1.20) we get a regular matrix Q, and the column t must be of the form

(4.1.22) $\quad t = Q^{-1} \,.\, t'.$

Therefore, to complete the proof it is enough to show that t' has the form (4.1.12).

We can proceed by induction and suppose that every solution of the system (4.1.11) with r replaced by $r - 1$, has the form (4.1.12). Since the contractions of t satisfy (4.1.11), we get for the right side of (4.1.20) the expression

$$\sum_{\nu} (c_\nu^1 \delta_{q_1}^{s_1} \delta_{q_2}^{s_{\nu(2)}} \dots \delta_{q_r}^{s_{\nu(r)}} + c_\nu^2 \delta_{q_1}^{s_{\nu(2)}} \delta_{q_2}^{s_1} \delta_{q_3}^{s_{\nu(3)}} \dots \delta_{q_r}^{s_{\nu(r)}} + \\ + \dots + c_\nu^r \delta_{q_1}^{s_{\nu(2)}} \dots \delta_{q_{r-1}}^{s_{\nu(r)}} \delta_{q_r}^{s_1}) \tag{4.1.23}$$

(summation over all elements $\nu \in S_{r-1}$). Clearly, (4.1.23) can be regarded, with a proper choice of constants $c_\sigma \in R_r$, as a sum over $\sigma \in S_r$, and we are done.

4. Let a tensor $t \in T_r^r R^n$ be given by (4.1.12), let $A \in GL_n(R)$ be any element, $A = (A_j^i)$. Computing the components of the tensor $\bar{t} = A \,.\, t$ we get, with the summation over S_r on the right,

$$\begin{aligned} \bar{t}_{j_1 \dots j_r}^{i_1 \dots i_r} &= \sum_\sigma c_\sigma A_{p_1}^{i_1} \dots A_{p_r}^{i_r} B_{j_1}^{q_1} \dots B_{j_r}^{q_r} \delta_{q_{\sigma(1)}}^{p_1} \dots \delta_{q_{\sigma(r)}}^{p_r} = \\ &= \sum_\sigma c_\sigma A_{q_{\sigma(1)}}^{i_1} \dots A_{q_{\sigma(r)}}^{i_r} B_{j_1}^{q_1} \dots B_{j_r}^{q_r} = \\ &= \sum_\sigma c_\sigma A_{q_{\sigma(1)}}^{i_1} \dots A_{q_{\sigma(r)}}^{i_r} B_{j_{\sigma(1)}}^{q_{\sigma(1)}} \dots B_{j_{\sigma(r)}}^{q_{\sigma(r)}} = \\ &= \sum_\sigma c_\sigma A_{k_1}^{i_1} \dots A_{k_r}^{i_r} B_{j_{\sigma(1)}}^{k_1} \dots B_{j_{\sigma(r)}}^{k_r} = \\ &= \sum_\sigma c_\sigma \delta_{j_{\sigma(1)}}^{i_1} \dots \delta_{j_{\sigma(r)}}^{i_r} = t_{j_1 \dots j_r}^{i_1 \dots i_r}, \end{aligned} \tag{4.1.24}$$

so that $A \,.\, t = t$. Hence (4) implies (1), and the proof is complete.

Remark 4.1. (4.1.12) implies that if $(q_1, \dots, q_r)$ is not a permutation of $(p_1, \dots, p_r)$, then

(4.1.25) $\quad t_{q_1 \dots q_r}^{p_1 \dots p_r} = 0.$

This can be shown directly by solving the system (4.1.11). Clearly, in this case there exists an integer k which is contained in the r-tuple $(p_1, \dots, p_r)$ more times than in $(q_1, \dots, q_r)$. For simplicity of notation, suppose that $p_1 = p_2 = \dots = p_m = k$, $q_1 = q_2 = \dots = q_s = k$, where $m > s$, and $p_{m+1}, \dots, p_r, q_{s+1}, \dots, q_r \neq k$, and take $i = j = k$ in (4.1.11). Then we get $(m - s)\, t_{k \dots k q_{s+1} \dots q_r}^{k \dots k p_{m+1} \dots p_r} = 0$ and because $m - s \neq 0$ we get (4.1.25).

Corollary 1. *Each invariant tensor $t = (t_j^i)$ of type* (1,1) *is a scalar multiple of the Kronecker tensor δ,*

(4.1.26) $\quad t_j^i = c\delta_j^i.$

Corollary 2. *Each invariant tensor* $t = (t^{ij}_{km})$ *of type* (2,2) *is of the form*

$$t^{ij}_{km} = c_1\delta^i_k\delta^j_m + c_2\delta^i_m\delta^j_k = c'_1\frac{1}{2}(\delta^i_k\delta^j_m + \delta^i_m\delta^j_k) + \tag{4.1.27}$$
$$+ c'_2\frac{1}{2}(\delta^i_k\delta^j_m - \delta^i_m\delta^j_k),$$

where $c_1, c_2 \in R$ *and* $c_1 = (1/2)(c'_1 + c'_2)$, $c_2 = (1/2)(c'_1 - c'_2)$.

Proof. This follows directly from (4.1.11) or (4.1.12).

Corollary 3. *Let* $(e_1, e_2, \ldots, e_n)$ *be a basis of the vector space* R^n, $(e^1, e^2, \ldots, e^n)$ *the dual basis of* R^{n*}. *A tensor* $t \in T^r_r R^n$ *is invariant if and only if it is a linear combination of tensors*

$$t_\sigma = \Sigma e_{p_{\sigma(1)}} \otimes \ldots \otimes e_{p_{\sigma(p)}} \otimes e^{p_1} \otimes \ldots \otimes e^{p_r} \tag{4.1.28}$$

(*summation over* $p_1, \ldots, p_r$), *where* σ *runs over* S_r.

Corollary 4. *If* $t = (t^{p_1\ldots p_r}_{q_1\ldots q_r})$ *is an invariant tensor, then for any permutation* $\nu \in S$

$$t^{p_{\nu(1)}\ldots p_{\nu(r)}}_{q_{\nu(1)}\ldots q_{\nu(r)}} = t^{p_1\ldots p_r}_{q_1\ldots q_r}. \tag{4.1.29}$$

Proof. By (4.1.12) we get, using commutativity of multiplication in R,

$$t^{p_{\nu(1)}\ldots p_{\nu(r)}}_{q_{\nu(1)}\ldots q_{\nu(r)}} = \sum_\sigma c_\sigma \delta^{p_{\nu(1)}}_{q_{\sigma\nu(1)}} \ldots \delta^{p_{\nu(r)}}_{q_{\sigma\nu(r)}} = \tag{4.1.30}$$
$$= \sum_\sigma c_\sigma \delta^{p_1}_{q_{\sigma(1)}} \ldots \delta^{p_r}_{q_{\sigma(r)}} = t^{p_1\ldots p_r}_{q_1\ldots q_r}$$

(summation over $\sigma \in S_r$), as desired.

4.2. Characters of the general linear group. By a (real) *character* of a group G we mean a homomorphism of G into the multiplicative group R^* of real numbers. Notice that R^* is canonically isomorphic with the group $GL(R)$. Define a function sign: $R\setminus\{0\} \to \{+1, -1\}$ by putting sign $t = 1$ if $t > 0$ and sign $t = -1$ if $t < 0$. The following theorem describes all continuous characters of the general linear group $GL_n(R)$.

Theorem 4.2. *Each continuous character* χ *of the group* $GL_n(R)$ *has one of the following two forms:*

$$\chi(A) = |\det A|^c, \tag{4.2.1}$$

$$\chi(A) = \text{sign} \det A \,.\, |\det A|^c, \tag{4.2.2}$$

where $c \in R$.

Proof. If $\chi : GL_n(R) \to R^*$ is a continuous character then χ is smooth and $T_e\chi : gl_n(R) \to R$, where $gl_n(R)$ (resp. R) is the Lie algebra of $GL_n(R)$ (resp. R^*), is a homomorphism of Lie algebras. That is, $T_e\chi$ is linear and $T_e\chi \cdot [\xi, \zeta] = [T_e\chi \cdot \xi, T_e\chi \cdot \zeta] = 0$ for all $\xi, \zeta \in gl_n(R)$.

Let $f : gl_n(R) \to R$ be any homomorphism of Lie algebras. Since the Lie algebra R is commutative, for any $\xi, \zeta \in gl_n(R)$, $f([\xi, \zeta]) = f(\xi . \zeta - \zeta . \xi) = 0$ (matrix multiplication). In the canonical coordinates on $gl_n(R)$

(4.2.3) $$f(\xi) = A^i_j \xi^j_i.$$

Hence

(4.2.4) $$f([\xi, \zeta]) = A^i_j \xi^j_k \zeta^k_i - A^i_j \zeta^j_k \xi^k_i = (A^i_j \delta^m_k - A^m_k \delta^i_j)\, \xi^j_m \zeta^k_i = 0$$

and we have $A^i_j \delta^m_k - A^m_k \delta^i_j = 0$. Contracting the expression on the left in m, k we obtain $A^i_j = c . \delta^i_j$ for some $c \in R$. Thus f must have the form

(4.2.5) $$f(\xi) = c . \xi^j_j = c . \operatorname{tr} \xi.$$

Let c be fixed, and denote by $R^*_{(+)}$ the connected component of the identity of R (the multiplicative group of positive real numbers). There is a unique homomorphism of Lie groups $\chi_c : GL^{(+)}_n(R) \to R^*_{(+)}$ such that $T_e\chi_c = f$. It is easily seen that the homomorphism $A \to (\det A)^c$ satisfies this condition. Let i, j, $1 \leqq i$, $j \leqq n$, be any integers. By the Laplace's theorem

(4.2.6) $$\det A = A^i_1 P^1_i + A^i_2 P^2_i + \ldots + A^i_n P^n_i,$$

(no summation over i), where P^k_j is the algebraic complement of the element A^j_k in the matrix (A^i_j). Since $P^k_j = B^k_j . \det A$, where (B^k_m) is the inverse matrix of the matrix (A^i_j), we have

(4.2.7) $$\frac{\partial}{\partial A^i_j} \det A = B^j_i . \det A.$$

Hence for any $\xi \in gl_n(R)$, $\xi = (\xi^i_j)$,

(4.2.8) $$T_e\chi_c \cdot \xi = \left\{ \frac{\partial}{\partial A^i_j} (\det A)^c \right\}_e . \xi^i_j = c . \delta^j_i \xi^i_j = f(\xi).$$

This formula proves that each continuous character of the group $GL^{(+)}(R)$ is of the form

(4.2.9) $$\chi_c(A) = (\det A)^c,$$

where $c \in R$.

Let now $\chi : GL_n(R) \to R^*$ be a continuous character. The restriction of χ to the subgroup $GL^{(+)}(R)$ of $GL_n(R)$ must be of the form $A \to (\det A)^c$ for some $c \in R$. Let $A, B \in GL^{(-)}(R)$ be any two elements. Since $A . B \in GL^{(+)}(R)$, we have

$$\chi(A \, . \, B) = \chi(A) \, . \, \chi(B) = (\det (A \, . \, B))^c = |\det (A \, . \, B)|^c = = |\det A|^c \, . \, |\det B|^c \tag{4.2.10}$$

and the number $\chi(A_0/|\det A|^c = |\det B|^c/\chi(B)) = \lambda$ should be independent of the choice of A and B. Hence for any $A \in GL_n^{(-)}(R)$, $\chi(A) = \lambda \, . \, |\det A|^c$. Condition (4.2.10) now gives $\lambda^2 = 1$, $\lambda = +1, -1$, and $\chi(A)$ must be of the form (4.2.1) or (4.2.2).

Corollary. *For any integer* k,

$$\chi(A) = (\det A)^k \tag{4.2.11}$$

is a character of $GL_n(R)$, *and all algebraic characters of* $GL_n(R)$ *are of this form.*

Proof. If c is even (4.2.1) reduces to (4.2.11). Since sign $\det A \, . \, |\det A|^c = \det A \, . \, |\det A|^{c-1}$, if c is odd, (4.2.2) reduces to (4.2.11).

We note that for $c = 0$ (4.2.1) defines the trivial character, and (4.2.2) gives

$$\chi(A) = \text{sign} \det A. \tag{4.2.12}$$

4.3. Relative invariant tensors. Let us consider an n-dimensional vector space S and a tensor $t \in T_s^r E$. We say that t is a *relative invariant tensor*, if there exists a function $\chi : GL(E) \to R^*$ such that for each element $A \in GL(E)$

$$A \, . \, t = \chi(A) \, . \, t. \tag{4.3.1}$$

If $t \neq 0$ and such a function χ exists, it is unique, and is a (real) character of the group $GL(E)$. We call χ the *weight* of the relative invariant tensor t.

An absolute invariant tensor is a relative invariant tensor of weight $\chi = 1$.

Let $t \in T_s^r E$ be a tensor, (e_i) a basis of E. Expressing t as in (4.1.4) we can see at once that t is a relative invariant tensor of weight χ if and only if for each element $A \in GL(E)$, $A = (A_j^i)$,

$$A_{i_1}^{p_1} \dots A_{i_r}^{p_r} B_{q_1}^{j_1} \dots B_{q_s}^{j_s} t_{j_1 \dots j_s}^{i_1 \dots i_r} = \chi(A) \, . \, t_{q_1 \dots q_s}^{p_1 \dots p_r}. \tag{4.3.2}$$

Example 4.2. Let us consider the vector space R^n. Let (e_i) be the canonical basis of R^n, and denote by $(\varepsilon^{i_1 \dots i_n})$, where $1 \leqq i_1, \dots, i_n \leqq n$, the system of real numbers defined by the conditions $\varepsilon^{i_1 \dots i_n} = 1$ (resp. $\varepsilon^{i_1 \dots i_n} = -1$, resp. $\varepsilon^{i_1 \dots i_n} = 0$) if $(i_1, \dots, i_n)$ is an even (resp. odd, resp. is not) permutation of the set $(1, 2, \dots, n)$. Define an element $\varepsilon \in T_0^n R^n$ by

$$\varepsilon = \varepsilon^{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{i_n}. \tag{4.3.3}$$

If $A \in GL(R^n)$ is any element, $A = (A_j^i)$ with respect to the canonical basis, we have

$$
\begin{aligned}
A \,.\, \varepsilon &= A^{i_1}_{j_1} \dots A^{i_n}_{j_n} \varepsilon^{j_1 \dots j_n} e_{i_1} \otimes \dots \otimes e_{i_n} = \\
&= \varepsilon^{i_1 \dots i_n} A^{1}_{j_1} \dots A^{n}_{j_n} \varepsilon^{j_1 \dots j_n} e_{i_1} \otimes \dots \otimes e_{i_n} = \\
&= \det (A^k_j) \,.\, \varepsilon^{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{i_n} = \det A \,.\, \varepsilon.
\end{aligned} \tag{4.3.4}
$$

Thus ε is a relative invariant tensor of weight $\chi(A) = \det A$.

Analogously, define $\varepsilon_{i_1 \dots i_n} = \varepsilon^{i_1 \dots i_n}$ and set

$$\eta = \varepsilon_{i_1 \dots i_n} e^{i_1} \otimes \dots \otimes e^{i_n}, \tag{4.3.5}$$

where (e^i) is the dual basis. $\eta \in T^0_n R^n$, and η is a relative invariant tensor of weight $\chi(A) = (\det A)^{-1}$.

Each of the tensors ε and η is usually referred to as the *Levi–Civita tensor*.

Let $t \in T^r_s E$ be a relative invariant tensor of weight χ. Let (e_i) (resp. $(\bar{e}_i)$) be a basis of E, (e^j) (resp. $(\bar{e}^j)$) the dual basis of E^*. Expressing t as in (4.1.4) and (4.1.8) and using definition (4.3.1) we obtain

$$\bar{t}^{i_1 \dots i_r}_{j_1 \dots j_s} = \chi(A) \,.\, t^{i_1 \dots i_r}_{j_1 \dots j_s}, \tag{4.3.6}$$

for any sequences $(i_1, \dots, i_r)$, $(j_1, \dots, j_s)$, where A is an element of the group $GL(E)$ defined by the conditions $\bar{e}_i = A \,.\, e_i$. In particular, the components of a relative invariant tensor on E depend, in general, on the basis of E.

Let $\Phi : E \to F$ be a linear isomorphism of n-dimensional vector spaces, (e_i) a basis of E. Φ induces a linear isomorphism $\Phi' : T^r_s E \to T^r_s F$ as follows. We take any element $t \in T^r_s E$, express t by (4.1.4) and set $\bar{e}_i = \Phi(e_i)$, and

$$\Phi'(t) = t^{i_1 \dots i_r}_{j_1 \dots j_s} \bar{e}_{i_1} \otimes \dots \otimes \bar{e}_{i_r} \otimes \bar{e}^{j_1} \otimes \dots \otimes \bar{e}^{j_s}. \tag{4.3.7}$$

Relative invariant tensors of weight χ form a vector subspace of $T^r_s E$, and Φ', restricted to this subspace, is an isomorphism onto the vector subspace of relative invariant tensors of weight χ on F. For if $A \in GL(E)$ and $t \in T^r_s E$ then one easily gets

$$A \,.\, \Phi'(t) = \Phi'(A \,.\, t) = \chi(A). \tag{4.3.8}$$

Our main problem in this section is to determine all relative invariant tensors on an n-dimensional vector space E; the above remark says that this is equivalent to determining the relative invariant tensors on the vector space R^n.

Theorem 4.3. *Let $t \in T^r_s R^n$ be a tensor.*

(a) Suppose that $r - s \neq k \,.\, n$ for all integers k. Then t is a relative invariant tensor if and only if $t = 0$.

(b) Suppose that there exists an integer k such that $r - s = k \,.\, n$. Then the following three conditions are equivalent:

(1) *t is a relative invariant tensor.*

(2) *For all $A \in GL^{(+)}_n(R)$*

$$A \,.\, t = (\det A)^k \,.\, t. \tag{4.3.9}$$

(3) *For any integers* $i, j, p_1, \dots, p_r, q_1, \dots, q_s = 1, 2, \dots, n,$

$$\begin{aligned} &\delta_i^{p_1} t^{j p_2 \dots p_r}_{q_1 \dots q_s} + \delta_i^{p_2} t^{p_1 j p_3 \dots p_r}_{q_1 \dots q_s} + \dots + \delta_i^{p_r} t^{p_1 \dots p_{r-1} j}_{q_1 \dots q_s} - \\ &- \delta^j_{q_1} t^{p_1 \dots p_r}_{i q_2 \dots q_s} - \delta^j_{q_2} t^{p_1 \dots p_r}_{q_1 i q_3 \dots q_s} + \dots - \\ &- \delta^j_{q_s} t^{p_1 \dots p_r}_{q_1 \dots q_{s-1} i} = k \,.\, \delta^j_i t^{p_1 \dots p_r}_{q_1 \dots q_s}, \end{aligned} \tag{4.3.10}$$

where $t^{i_1 \dots i_r}_{j_1 \dots j_s}$ *are the components of* t *with respect to the canonical basis of* R^n.

Proof. (a) Let $t \in T^r_s R^n$ be a relative invariant tensor. We shall show that if $t \neq 0$ then $r - s = k \,.\, n$ for some integer k. According to Theorem 4.2 we distinguish two cases. Suppose first that t is of weight χ, where $\chi(A) = (\det A)^c$. If $c = 0$ then t is absolute invariant and the condition $t \neq 0$ implies $r - s = 0$ (Theorem 4.1) so we can take $k = 0$. Suppose that $c \neq 0$, and consider condition (4.3.2). Let k, m be any integers and choose A of the form $A^i_j = \delta^i_j, j \neq k, m$, $A^i_j = \delta^{k-m+i}_k$, $j = k$, $A^i_j = \delta^{m-k+i}_m$, $j = m$. Then for any $i_1, \dots, i_r, j_1, \dots, j_s$,

$$\begin{aligned} &\lambda^{i_1} \dots \lambda^{i_r} \frac{1}{\lambda^{j_1}} \dots \frac{1}{\lambda^{j_s}} t^{i_1 \dots i_r} = (\lambda^1)^{k_1} \dots (\lambda^n)^{k_n} t^{i_1 \dots i_r}_{j_1 \dots j_s} = \\ &= |\lambda^1|^c \dots |\lambda^n|^c t^{i_1 \dots i_r}_{j_1 \dots j_s}, \end{aligned} \tag{4.3.11}$$

where $k_1, \dots, k_n$ are some integers. By hypothesis there exist $i_1, \dots, i_r, j_1, \dots, j_s$ such that $t^{i_1 \dots i_r}_{j_1 \dots j_s} \neq 0$. This implies that

$$(\lambda^1)^{k_1} \dots (\lambda^n)^{k_n} = |\lambda^1|^c \dots |\lambda^n|^c. \tag{4.3.12}$$

Since this relation holds for all non-zero $\lambda^1, \dots, \lambda^n \in R$ we get for each i, $1 \leqq i \leqq n$, $(\lambda^i)^{k_i} = |\lambda^i|^c$. This is possible only if k_i is even and $k_i = c$. That is, c must be an even integer, and

$$k_1 = k_2 = \dots = k_n = c. \tag{4.3.13}$$

Since k_i is the difference between the number of indices $i_1, \dots, i_r$, equal to i, and the number of indices $j_1, \dots, j_s$, equal to i, we have $k_1 + k_2 + \dots + k_n = r - s = n \,.\, c$.

Now suppose that t is of weight χ, where $\chi(A) = \operatorname{sign} \det A \,.\, |\det A|^c$ for some $c \in R$. Using (4.3.2) again and substituting $A \in GL_n(R)$ of the form $A^i_j = \delta^i_j$, $j \neq k, m$, $A^i_j = \delta^{k-m+i}_k$, $j = k$, $A^i_j = \delta^{m-k+1}_m$, $j = m$, we get in place of (4.3.11)

$$\begin{aligned} &\lambda^{i_1} \dots \lambda^{i_r} \frac{\lambda}{\lambda^{j_1}} \dots \frac{\lambda}{\lambda^{j_s}} t^{i_1 \dots i_r}_{j_1 \dots j_s} = (\lambda^1)^{k_1} \dots (\lambda^n)^{k_n} t^{i_1 \dots i_r}_{j_1 \dots j_s} = \\ &= \operatorname{sign} \det A \,.\, |\lambda^1|^c \dots |\lambda^n|^c \,.\, t^{i_1 \dots i_r}_{j_1 \dots j_s}, \end{aligned} \tag{4.3.14}$$

where $k_1, \ldots, k_n$ are some integers. By hypothesis there exist $i_1, \ldots, i_r, j_1, \ldots, j_r$ such that $t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \neq 0$. Then, since $\operatorname{sign} \det A = (\det A)/|\det A| = (\lambda^1/|\lambda^1|) \ldots \ldots (\lambda^n/|\lambda^n|)$,

$$(4.3.15) \qquad (\lambda^1)^{k_1} \ldots (\lambda^n)^{k_n} = \lambda^1 |\lambda^1|^{c-1} \ldots \lambda^n |\lambda^n|^{c-1}.$$

Since this equality holds for all non-zero $\lambda^1, \ldots, \lambda^n \in R$, we get for each i, $1 \leqq i \leqq n$, $(\lambda^i)^{k_i - 1} = |\lambda^i|^{c-1}$. Thus k_i must be odd, and $k_i = c$. Similarly as above we get $r - s = n \,.\, c$.

Summarizing both cases we see that if $(r - s)/n$ is not an integer, then $t = 0$ as desired.

(b) 1. Condition (1) obviously implies (2).

2. Suppose that (2) holds. To derive (3), we compute the fundamental vector fields (4.1.16) and find at once that their components coincide with the left sides of (4.3.10). On the other hand, differentiating the right side expression (4.3.9) with respect to A^i_j at the identity element $e \in GL_n(R)$ we obtain, using (4.2.7),

$$(4.3.16) \qquad \left\{ \frac{\partial}{\partial A^i_j} (\det A)^k \,.\, t^{p_1 \ldots p_r}_{q_1 \ldots q_s} \right\}_e = k \,.\, \delta^j_i t^{p_1 \ldots p_r}_{q_1 \ldots q_s}.$$

Hence (3) holds.

3. To complete the proof, we shall show that (3) implies (1). Let $\Phi : GL_n(R) \times \times T^r_s R^n \to T^r_s R^n$ denote the tensor action of $GL_n(R)$, let ζ^j_i be the canonical basis of the Lie algebra $gl_n(R)$ and let $\xi^j_i = \Phi'(\zeta^j_i)$ be the fundamental vector fields of this action, associated with the vectors ζ^j_i. Consider also another action $\Psi : GL_n(R) \times T^r_s R^n \to T^r_s R^n$ defined by

$$(4.3.17) \qquad \Psi(A, s) = (\det A)^k \,.\, s,$$

where k is defined by the condition $r - s = k \,.\, n$. Using (4.2.7) we easily obtain the fundamental vector fields of this action, associated with the vectors ζ^j_i. With our standard notation,

$$(4.3.18) \qquad \begin{aligned} \Psi'(\zeta^j_i)(t) &= T_e \Psi_t \,.\, \zeta^j_i = k \,.\, (\det A)^{k-1} \frac{\partial}{\partial A^p_q} (\det A) (\zeta^j_i)^p_q \,.\, t = \\ &= k \,.\, (\det A)^{k-1} \frac{\partial}{\partial A^j_i} (\det A) \,.\, t = k \,.\, (\det A)^k \,.\, B^j_i \,.\, t = k \,.\, \delta^j_i \,.\, t, \end{aligned}$$

where we have used the fact that ζ^j_i is the matrix whose unique non-zero element stands in the i-th column and j-th row, that is, $(\zeta^j_i)^p_q = \delta^{jp} \delta_{iq}$, and have set $A = e$ on the right.

With this notation, (4.3.10) can be written in the form

$$(4.3.19) \qquad \Phi'(\zeta^j_i)(t) - \Psi'(\zeta^j_i)(t) = 0.$$

Notice that if a tensor t satisfies this equation, then so does any scalar multiple of t; in particular, for any $A \in GL_n(R)$

(4.3.20) $\quad \Phi'(\zeta_i^j)(\Psi(A, t)) - \Psi'(\zeta_i^j)(\Psi(A, t)) = 0.$

Now take in Lemma 3.4, $f = \mathrm{id}_Q$, $P = Q = T_s^r R^n$, and consider the curve $s \to \gamma_s(t)$ in $T_s^r R^n$ defined by

(4.3.21) $\quad \gamma_s(t) = \Phi(\exp s\zeta, \Psi(\exp(-s\zeta), t)),$

where $\zeta \in gl_n(R)$ is any element. By (3.4.29)

(4.3.22)
$$\left\{\frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(t)\right\}_0 = \Phi'(\zeta)(t) - \Psi'(\zeta)(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(t) = T_{\Psi(\exp -s\zeta), t)}\Phi_{\exp s\zeta} \cdot (\Phi(\zeta)(\Psi(\exp(-s\zeta), t)) - - \Psi'(\zeta)(\Psi(\exp(-s\zeta), t)).$$

Applying (4.3.20) we get $\gamma_s(t) = t$ for all $s \in R$. Since the element $\zeta \in gl_n(R)$ is arbitrary we have for each A from a neighborhood of the identity $\Phi(A, \Psi(A^{-1}, t)) = t$ or, which is the same,

(4.3.23) $\quad A \,.\, t - (\det A)^k \,.\, t = 0.$

This relation holds, however, for any A, because the expression $A \,.\, t - (\det A)^k \,.\, t$ (resp. $(1/(\det A)^k)\, A \,.\, t - t$) is a polynomial in the components of A, if $k \geqq 0$ (resp. $k < 0$), and (4.3.23) means that all the coefficients in this polynomial must vanish.

It remains to show that (4.3.23) holds for all $A \in GL_n^{(-)}(R)$. Notice that if for some $A_0 \in GL_n^{(-)}(R)$, $A_0 \,.\, t = (\det A_0)^k \,.\, t$, then for any $A \in GL_n^{(-)}(R)$, $A \,.\, t = (\det A)^k \,.\, t$. Clearly, in this case

(4.3.24)
$$\begin{aligned}(A \,.\, A_0) \,.\, t &= \det(A \,.\, A_0) \,.\, t = A_0 \,.\, (A \,.\, t) = \\ &= (\det A)^k (\det A_0)^k \,.\, t = (\det A)^k \,.\, A_0 \,.\, t = \\ &= A_0 \,.\, (\det A)^k\, t,\end{aligned}$$

because $A \,.\, A_0 \in GL_n^{(+)}(R)$; clearly this implies $A \,.\, t = (\det A)^k \,.\, t$. We take A_0 of the form $A_0 = (A_j^i)$, where

(4.3.25) $\quad A_j^i = \varkappa(i)\,\delta_j^i,$

where $\varkappa(i) = -1$ if $i = 1$ and $\varkappa(i) = 1$ if $2 \leqq i \leqq n$. Then the inverse matrix $A_0^{-1} = (B_j^i)$ has the same elements, and the components of the tensor $A \,.\, t$ with respect to the canonical basis of R^n satisfy

(4.3.26)
$$\begin{aligned}A_{p_1}^{i_1} \dots A_{p_r}^{i_r} B_{j_1}^{q_1} \dots B_{j_s}^{q_s} t_{q_1 \dots q_s}^{p_1 \dots p_r} &= \varkappa(i_1) \dots \varkappa(i_r)\, \varkappa(j_1) \,..\, \varkappa(j_s)\, t_{j_1 \dots j_s}^{i_1 \dots i_r} = \\ &= (-1)^m\, t_{j_1 \dots j_s}^{i_1 \dots i_r},\end{aligned}$$

where m is the number of indices equal to 1 in $(i_1, \ldots, i_r, j_1, \ldots, j_s)$. If 1 enters $(p_1, \ldots, p_r)$ m_1 times and $(q_1, \ldots, q_s)$ $m_2 = m_1 + k$ times (see part (a) of this proof), then $m = m_1 + m_2 = 2m_1 + k$, and we have

$$(4.3.27) \qquad A_0 \, . \, t = (-1)^k \, . \, t.$$

On the other hand, $(\det A_0)^k \, . \, t = (-1)^k \, . \, t$ proving that $A_0 \, . \, t = (\det A_0)^k \, . \, t$. This completes the proof.

Corollary 1. *The weight of a relative invariant tensor is always of the form* $\chi(A) = (\det A)^k$, *where k is an integer. Any other character of the group* $GL_n(R)$ *cannot be the weight of a relative invariant tensor.*

Proof. This has been shown in the proof of Theorem 4.3.

Corollary 2. *If* $r - s = k \, . \, n$ *for some integer* $k \leqq 0$, *and t is a relative invariant tensor, then for any component* $t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \neq 0$, $(j_1, \ldots, j_s)$ *is a permutation of* $(i_1, \ldots, i_r, 1, 2, \ldots, n, \ldots, 1, 2, \ldots, n)$, *where the n-tuple* $(1, 2, \ldots, n)$ *enters* $|k|$ *times. An analogous expression holds for* $k > 0$.

Proof. Suppose that $t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \neq 0$. By (4.3.13), the difference between the number of indices m in $(i_1, \ldots, i_r)$ and in $(j_1, \ldots, j_s)$ is equal to k. Since each of the integers $1, 2, \ldots, n$ should enter $(j_1, \ldots, j_s)$, otherwise $t^{i_1 \ldots i_r}_{j_1 \ldots j_s} = 0$, by (4.3.11), our assertion follows.

Notice that if $t \in T^r_s R^n$ is a relative invariant tensor of weight $(\det A)^k$ with $k < 0$ then the tensor product $t \otimes \varepsilon \otimes \ldots \otimes \varepsilon \in T^r_s R^n$ of t with **k** factors the Levi–Civita tensor ε is an absolute invariant tensor; if $k > 0$ then the tensor product $t \otimes \eta \otimes \ldots \otimes \eta \in T^r_s R^n$ with k factors the Levi–Civita tensor η is an absolute invariant tensor. Notice also the relation

$$(4.3.28) \qquad \varepsilon_{i_1 \ldots i_n} \varepsilon^{i_1 \ldots i_n} = n!.$$

Theorem 4.4. (*a*) *Each relative invariant tensor* $t \in T^0_n R^n$ *is a scalar multiple of the Levi–Civita tensor* η, $t = c \, . \, \eta$. *Each relative invariant tensor* $t \in T^n_0 R^n$ *is a scalar multiple of the Levi–Civita tensor* ε, $t = c \, . \, \varepsilon$.

(*b*) *Let* $t \in T^r_s R^n$ *be a relative invariant tensor, let* $r - s = k \, . \, n$, *where* $k < 0$. *Then t is expressible as a complete contraction of an absolute invariant tensor* $u \in T^r_s R^n$ *and the tensor* $\eta \otimes \ldots \otimes \eta$ ($-k$ *factors* η). *If* $k > 0$ *then t is expressible as a complete contraction of an absolute invariant tensor* $v \in T^r_s R^n$ *and the tensor* $\varepsilon \otimes \ldots \otimes \varepsilon$ (*k factors* ε).

Proof. (a) Let $t \in T_n^0 R^n$ be a relative invariant tensor, let χ be its weight. By Corollary 1 to Theorem 4.3, $\chi(A) = (\det A)^{-1}$, and for any element $A \in GL_n(R)$, $A = (A_j^i)$,

$$B_{j_1}^{q_1} \dots B_{j_n}^{q_n} t_{q_1 \dots q_n} = (\det A)^{-1} \cdot t_{j_1 \dots j_n}, \tag{4.3.29}$$

where $A^{-1} = (B_j^i)$. Choose $A_j^i = \lambda^i \delta_j^i$ (no summation). Then $B_j^i = (1/\lambda_j)\, \delta_j^i$, and (4.3.29) gives

$$\frac{1}{\lambda^{j_1}} \cdots \frac{1}{\lambda^{j_n}} t_{j_1 \dots j_n} = (\lambda^1 \dots \lambda^n)^{-1} t_{j_1 \dots j_n}. \tag{4.3.30}$$

Since for each i, $\lambda^i \neq 0$ and λ^i is arbitrary, if $(j_1, \dots, j_n)$ is not a permutation of $(1, 2, \dots, n)$, $t_{j_1 \dots j_n}$ must be equal to 0. Now take A of the form

$$A_j^i = \begin{cases} \delta_j^i, & j \neq k, m, \\ \delta_k^{k-m+i}, & j = k, \\ \delta_m^{m-k+i}, & j = m. \end{cases} \tag{4.3.31}$$

The matrix (A_j^i) has the form

$$\begin{array}{c} \\ i \\ \downarrow \\ \\ i = k \\ \\ \\ \\ \\ i = m \\ \\ \\ \end{array} \begin{array}{c} j \rightarrow \qquad j = k \qquad j = m \\ \left[\begin{array}{ccccccccc} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & \dots & 1 & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & 1 & \dots & 0 & & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{array}\right] \end{array}, \tag{4.3.32}$$

where all the missing non-diagonal elements are zero. From (4.3.32) it is evident that the inverse matrix $A^{-1} = (B_j^i)$ has the elements

$$B_j^i = \begin{cases} \delta_j^i, & i \neq k, m, \\ \delta_{k-m+j}^k, & i = k, \\ \delta_{m-k+j}^m, & i = m. \end{cases} \tag{4.3.33}$$

Because A differs from the identity matrix by one transposition of rows, $\det A = -1$. Suppose that in (4.3.29), $(j_1, \dots, j_n)$ is a permutation of $(1, 2, \dots, n)$. Substituting (4.3.31) in (4.3.29) we obtain that the interchange of the integers k and m in $(j_1, \dots, j_n)$ changes the sign of the component $t_{1_1 \dots j_n}$, that is, $t_{\dots m \dots k \dots} =$

$= -t_{\dots k \dots m \dots}$. Since k and m are arbitrary, $t_{j_1 \dots j_n}$ must be completely antisymmetric. This means that $t_{1_1 \dots j_n} = c\delta_{j_1 \dots j_n}$ for some $c \in R$.

In the case $t \in T_0^r R^n$ our assertion can be obtained in the same way.

(b) To prove the second statement, consider a relative invariant tensor $t \in T_s^r R^n$ such that $r - s = kn$, where $k < 0$. Denote

$$u = t \otimes \eta \otimes \dots \otimes \eta \tag{4.3.34}$$

($-k$ factors η). Contracting u and the tensor $\varepsilon \otimes \dots \otimes \varepsilon$ ($-k$ factors ε) and using (4.3.28) we obtain $(n!)^{-k} . t$ as desired. The rest is proved in the same way.

Corollary 1. *A nontrivial relative invariant tensor $t \in T_s^r R^n$ exists if and only if $r - s = k . n$ for some integer k. The weight of t is $\chi(A) = (\det A)^k$.*

Proof. The existence (resp. non-existence) of t follows from Theorem 4.4 (resp. Theorem 4.3, (a)). The second assertion has been verified within the last part of the proof of Theorem 4.3.

We note that Corollary 1 includes absolute invariant tensors for which $k = 0$.

Relative invariant tensors form a subalgebra of the tensor algebra of the vector space R^n. This subalgebra can be completely characterized as follows.

Corollary 2. *The algebra of relative invariant tensors is generated by the Kronecker tensor δ and by the Levi–Civita tensor ε.*

Proof. This is a consequence of Theorem 4.1 and Theorem 4.4.

Theorem 4.4 describes the structure of a relative invariant tensor. We shall illustrate it by an example.

Example 4.3. Let $t \in T_s^r R^n$ be a relative invariant tensor, and let $r - s = 2n$. Then, in the canonical basis of R^n

$$\begin{aligned} &t_{j_1 \dots j_s}^{i_1 \dots i_s p_1 \dots p_n q_1 \dots q_n} = \\ &= u_{j_1 \dots j_s k_1 \dots k_n m_1 \dots m_n}^{i_1 \dots i_s p_1 \dots p_n q_1 \dots q_n} \varepsilon^{k_1 \dots k_n} \varepsilon^{m_1 \dots m_n}, \end{aligned} \tag{4.3.35}$$

where $u_{j_1 \dots j_s k_1 \dots k_n m_1 \dots m_n}^{i_1 \dots i_s p_1 \dots p_n q_1 \dots q_n}$ are components of an absolute invariant tensor of type (r, r); the general form of this tensor is determined by Theorem 4.1.

4.4. Multilinear invariants of the general linear group. We now apply the theory of invariant tensors to linear $GL_n(R)$-equivariant mappings of tensor spaces, and to multiplinear relative invariants of $GL_n(R)$.

Theorem 4.5. *Let $\Phi : T^k_m R^n \to T^r_s R^n$ be a linear mapping,*

(4.4.1) $$t^{p_1 \dots p_r}_{q_1 \dots q_s} = \Phi^{p_1 \dots p_r i_1 \dots i_m}_{q_1 \dots q_s j_1 \dots j_k} s^{j_1 \dots j_k}_{i_1 \dots i_m},$$

its representation relative to a basis of R^n. Then Φ is $GL(R)$-equivariant if and only if the coefficients $\Phi^{p_1 \dots p_r i_1 \dots i_m}_{q_1 \dots q_s j_1 \dots j_k}$ are components of an absolute invariant tensor.

Proof. One can find by a direct computation that the coefficients $\Phi^{p_1 \dots p_r i_1 \dots i_m}_{q_1 \dots q_s j_1 \dots j_k}$ satisfy (4.1.7) if and only if Φ is a $GL_n(R)$-equivariant mapping.

Theorem 4.5. establishes a one-to-one correspondence between linear $GL_n(R)$-equivariant mappings of tensor spaces and absolute invariant tensors.

Corollary 1. *A non-trivial linear $GL_n(R)$-equivariant mapping $\Phi : T^k_m R^n \to T^r_s R^n$ exists if and only if and only if $r + m = k + s$.*

Proof. This follows from Theorem 4.5 and Theorem 4.1.

Corollary 2. *Let R be endowed with the trivial action $A \to 1$ of $Gl_n(R)$. Then each linear $GL_n(T)$-equivariant mapping $\Phi : T^1_1 R^n \to R$ is of the form $(t^i_j) \to c \,.\, t^i_i$ (summation over i), where $c \in R$ is some constant. Each linear $GL_n(R)$-equivariant mapping $\Phi : R \to T^1_1 R^n$ is of the forms $\to (c\delta^i_j \,.\, s)$, where $c \in R$ is some constant.*

Proof. Both parts of this assertion follow immediately from the expression of Φ in the form (4.4.1).

Remark 4.2. We may paraphraze Theorem 4.5 in such a way that any linear $GL_n(R)$-equivariant mapping $\Phi : T^k_m R^n \to T^r_s R^n$ can be obtained by a sequence of the following constructions:

(a) contraction in one superscript and one subscript;
(b) tensor multiplication by the Kronecker tensor;
(c) vector operations: addition and scalar multiplication.

Remark 4.3. Theorem 4.5 can be used to find $GL_n(R)$- equivariant decompositions of tensor spaces into the direct sum. Let for example $t \in T^r_s R^n$, $t = (t^{i_1 \dots i_r}_{j_1 \dots j_r})$, be an absolute invariant tensor, and consider the mapping $\Phi : T^0_r R^n \to T^0_r R^n$ defined by the equations

(4.4.2) $$\bar{s}_{j_1 \dots j_r} = t^{i_1 \dots i_r}_{j_1 \dots j_r} s_{i_1 \dots i_r}.$$

By Theorem (4.1, (4), t is of the form

(4.4.3) $$t^{i_1 \dots i_r}_{j_1 \dots j_r} = \sum_{\sigma \in S_r} c_\sigma \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_r}_{j_{\sigma(r)}}$$

for some $c_\sigma \in R$. It is easily seen that Φ is a projection, i.e. $\Phi \circ \Phi = \Phi$, if and only if for each $\nu \in S_r$

(4.4.4) $$c_\nu - \sum_\sigma c_\sigma c_{\sigma^{-1}\nu} = 0.$$

Obviously, if ϕ is expressed by (4.4.2) then for any $s \in T_r^0 R^n$, $s = (s_{i_1 \ldots i_r})$, $\phi(\phi(s))$ is given by

(4.4.5) $$\bar{\bar{s}}_{j_1 \ldots j_r} = \sum_\sigma c_\sigma \bar{s}_{j_{\sigma(1)} \ldots j_{\sigma(r)}}.$$

Using Corollary 4 to Theorem 4.1 we get

(4.4.6) $$\begin{aligned} \bar{s}_{j_{\sigma(1)} \ldots j_{\sigma(r)}} &= \sum_\nu c_\nu s_{j_{\sigma\nu(1)} \ldots j_{\sigma\nu(r)}}, \\ \bar{\bar{s}}_{j_1 \ldots j_r} &= \sum_\sigma c_\sigma \sum_\nu c_\nu s_{j_{\sigma\nu(1)} \ldots j_{\sigma(\nu)r}}. \end{aligned}$$

On the other hand, $\Phi(s)$ is expressed by

(4.4.7) $$\begin{aligned} \bar{s}_{j_1 \ldots j_r} &= \sum_\varrho c_\varrho \delta^{i_1}_{j_{\varrho(1)}} \ldots \delta^{i_r}_{j_{\varrho(r)}} s_{i_1 \ldots i_r} = \\ &= \sum_\varrho c_\varrho s_{j_{\varrho(1)} \ldots j_{\varrho(r)}}. \end{aligned}$$

Putting $\sigma\nu = \varrho$ we obtain $\nu = \sigma^{-1}\varrho$ and

(4.4.8) $$\bar{\bar{s}}_{j_1 \ldots j_r} = \sum_\sigma c_\sigma \sum_\varrho c_{\sigma^{-1}\varrho} s_{j_{\varrho(1)} \ldots j_{\varrho(r)}}.$$

The condition $\Phi \circ \Phi = \Phi$ is now equivalent to $\bar{\bar{s}}_{j_1 \ldots j_r} = \bar{s}_{j_1 \ldots j_r}$ which is in turn equivalent to (4.4.4).

We shall now turn to multilinear relative invariants of the group $GL_n(R)$.

Let G be a group, Q a set endowed with a left action of G. A function $f : Q \to R$ is called a *relative invariant* of the group G if there exists a function $\chi : G \to R^*$ such that for each $q \in Q$ and $g \in G$

(4.4.9) $$f(g \cdot q) = \chi(g) \cdot f(q).$$

If such a function χ exists and for some $q \in Q$, $f(q) \neq 0$, then it is unique, and is a homomorphism of G into the multiplicative group of real numbers. In this case χ is called the *weight* of the relative invariant f.

Our main concern here will be multilinear relative invariants of vectors in R^n and vectors in R^{n*}. We have the following simple observation, analogous to Theorem 4.5.

Theorem 4.6. *Let $f : (R^n)^s \times (R^{n*})^r \to R$ be a multilinear mapping,*

(4.4.10) $$f(\xi_1, \ldots, \xi_s, \omega^1, \ldots, \omega^r) = t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \xi_1^{j_1} \ldots \xi_s^{j_s} \omega^1_{i_1} \ldots \omega^r_{i_r},$$

its expression for a basis of R^n. Then f is a relative invariant if and only if $t^{i_1\dots i_r}_{j_1\dots j_s}$ are components of a relative invariant tensor of type (r, s).

Proof. The proof is analogous to the proof of Theorem 4.5 and is straightforward.

Corollary 1. *If $(r - s)/n$ is not an integer, there is no relative invariant of s vectors in R^n and r vectors in R^{n*}.*

Proof. This follows from Theorem 4.6 and Theorem 4.3, (a).

Corollary 2. *If $r - s = k \cdot n$ for some integer k, then the weight of each relative invariant of s vectors in R^n and r vectors in R^{n*} is $\chi(A) = (\det A)^k$.*

Proof. This follows from Theorem 4.6 and Corollary 1 to Theorem 4.3.

We shall now formulate a consequence of Theorem 4.1, Theorem 4.4, and Theorem 4.6 which is known as the *main theorem on relative invariants* of the group $GL_n(R)$.

Corollary 3. *Each relative multilinear invariant of vectors in R^n and R^{n*} is a linear combination of products of relative multilinear invariants of the following three types:*

$$\det(\xi_1, \xi_2, \dots, \xi_n) = \varepsilon_{i_1\dots i_n}\xi_1^{i_1}\xi_2^{i_2}\dots\xi_n^{i_n}, \tag{4.4.11}$$

$$\det(\omega^1, \omega^2, \dots, \omega^n) = \varepsilon^{i_1\dots i_n}\omega^1_{i_1}\omega^2_{i_2}\dots\omega^n_{i_n}, \tag{4.4.12}$$

$$\xi^i\omega_i, \tag{4.4.13}$$

where $\xi, \xi_i \in R^n$, $\omega, \omega^i \in R^{n*}$. More precisely, each relative multilinear invariant of s vectors $\xi_j \in R^n$ and r vectors $\omega^i \in R^{n*}$ of weight $(\det A)^k$, where $k < 0$, is expressible as a linear combination of products of invariants of type (1) and (3); if $k > 0$, it is expressible as a linear combination of products of invariants of type (2) and (3); if $k = 0$, it is expressible as a linear combination of products of invariants of type (3).

Proof. Let f be a relative multilinear invariant of s vectors $\xi_j \in R^n$ and r vectors $\omega^i \in R^{n*}$, let $r - s = k \cdot n$, where $k > 0$ is an integer. Let $t^{i_1\dots i_r}_{j_1\dots j_s}$ be the components of a relative invariant tensor of weight k, associated to f by (4.4.10). By Theorem 4.4, this tensor if of the form

$$\begin{aligned} t^{i_1\dots i_s(p_1)(p_2)\dots(p_k)}_{j_1\dots j_s} &= \\ &= u^{i_1\dots i_s(p_1)(p_2)\dots(p_k)}_{j_1\dots j_s(q_1)(q_2)\dots(q_k)} \cdot \varepsilon^{(q_1)}\varepsilon^{(q_2)}\dots\varepsilon^{(q_k)}, \end{aligned} \tag{4.4.14}$$

where $(p_m) = (p_{m1}, p_{m2}, \ldots, p_{mn})$ denotes n indices with values between 1 and n (compare with Example 4.3). Substituting (4.4.14) in (4.4.10) and using Corollary 4 to Theorem 4.1 we obtain that the tensor (4.4.14) is antisymmetric in $p_{m1}, p_{m2}, \ldots, \ldots, p_{mn}$ for each m, $1 \leqq m \leqq k$. Hence (4.4.10) contains factors of the form

(4.4.15) $$\det(\omega^1, \ldots, \omega^n) = \varepsilon^{p_1 \ldots p_n} \omega^1_{p_1} \ldots \omega^n_{p_n},$$

and factors of the form $\xi^i \omega_i$, that is, factors of type (2) and (3).

If $k < 0$ we proceed in the same way; if $k = 0$ we apply Theorem 4.1.

5. PROLONGATIONS OF LIFTINGS

This chapter is devoted to the formal theory of prolongations of those geometric structures which appear in the theory of differential invariants: Lie groups, smooth manifolds endowed with Lie group actions, principal and associated fiber bundles. These geometric prolongation constructions lead naturally to the prolongations of the r-frame lifting, and the prolongations of liftings, associated with the prolonged r-frame lifting. To the end of this chapter basic definitions concerning natual differential operators are given.

5.1. Prolongations of Lie groups. Let G be a Lie group, and denote by $T^r_n G$ the set of r-jets with source at the origin $0 \in R^n$ and target in G. $T^r_n G$ is a subset of the manifold $J^r(R^n, G)$ of r-jets with source in R^n and target in G (Sec. 2.1). More precisely, $T^r_n G$ is the fiber over the point $0 \in R^n$ with respect to the canonical jet projection of $J^r(R^n, G)$ onto R^n. In particular, $T^r_n G$ is a closed submanifold of $J^r(R^n, G)$.

Let $S, T \in T^r_n G$, $S = J^r_0 f$, $T = J^r_0 g$, be any elements. We define a multiplication in $T^r_n G$ by

(5.1.1) $$S \,.\, T = J^r_0(f \,.\, g),$$

where $(f \,.\, g)(x) = f(x) \,.\, g(x)$ (the group multiplication in G). It is obvious that the mapping $(S, T) \to S \,.\, T$ satisfies the associative law; moreover, the element $J^r_0 e_G$, where e_G denotes the identity element of G and the constant mapping of R^n onto e_G, is the identity element of this multiplication, and the element $J^r_0 f^{-1}$, where $f^{-1}(x) = (f(x))^{-1}$ (the inversion is taken in the group G), is the inverse element to $J^r_0 f$. Thus $T^r_n G$ is a group. Denoting for a moment the group multiplication in G by Ψ, we obtain (5.1.1) in the form

(5.1.2) $$S \,.\, T = J^r_{(f(0), g(0))} \Psi \circ J^r_0(f \times g) = J^r_{(f(0), g(0))} \Psi \circ (S, T).$$

Since the composition of jets is smooth, the product $S\,.\,T$ depends smoothly on S and T, and we see that the group structure in $T_n^r G$ is compatible with its smooth structure. Thus $T_n^r G$ is a Lie group.

Let us consider the r-th differential group of R^n, L_n^r. Each element $A \in L_n^r$ defines a mapping $\varphi(A) : T_n^r G \to T_n^r G$ by the formula

$$\text{(5.1.3)} \qquad \varphi(A)\,(S) = S \circ A^{-1}.$$

Since for any $S, T \in T_n^r G$ and $A, B \in L_n^r$,

$$\text{(5.1.4)} \qquad \varphi(A)\,(S\,.\,T) = (S\,.\,T) \circ A^{-1} = (S \circ A^{-1})\,.\,(T \circ A^{-1}),$$

$$\text{(5.1.5)} \qquad \begin{aligned} &\varphi(A\,.\,B)\,(S) = S \circ (A\,.\,B)^{-1} = (S \circ B^{-1}) \circ A^{-1} = \\ &= \varphi(A)\,(\varphi(B)\,(S)) = \varphi(A) \circ \varphi(B)\,(S), \end{aligned}$$

$\varphi(A)$ is an automorphism of the Lie group $T_n^r G$, and the mapping $A \to \varphi(A)$ is a homomorphism of the Lie group L_n^r into the group Aut $T_n^r G$ of automorphisms of $T_n^r G$. The mapping $(A, S) \to \varphi(A)\,(S) = S \circ A^{-1}$ is obviously analytic. Consider the exterior semi-direct product $L_n^r \times_\varphi T_n^r G$, associated with the homomorphism φ. By Theorem 4.4, $L_n^r \times_\varphi T_n^r G$ is a Lie group. We call this Lie group the (r, n)-*prolongation*, or simply the *prolongation* of the Lie group G, and denote it by G_n^r. By (1.2.1), the group multiplication in G_n^r is defined by the formula

$$\text{(5.1.6)} \qquad (A, S)\,.\,(B, T) = (A\,.\,B, S\,.\,(T \circ A^{-1})).$$

Remark 5.1. We shall give a less formal interpretation of the group G_n^r. Let e_G denote the identity element of G. Let α be an isomorphism of the trivial principal G-bundle $R^n \times G$, α_0 the projection of α, $\alpha_0 = \operatorname{proj} \alpha$. Writing

$$\text{(5.1.7)} \qquad \begin{aligned} &\alpha(x, g) = \alpha(x, e_G)\,.\,g = \gamma(\alpha_0(x))\,.\,g = \\ &= (\alpha_0(x), \alpha'(\alpha_0(x)))\,.\,g \end{aligned}$$

we obtain a section $x \to \gamma(x) = (x, \alpha'(x))$ of $R^n \times G$. Thus α gives rise to the pair (α_0, α'). Conversely, each pair (α_0, α'), where $\alpha_0 : R^n \to R^n$ is an isomorphism and $\alpha' : R^n \to G$ is a mapping, defines an isomorphism of the principal G-bundle $R^n \times G$ by the same formula (5.1.7). Thus we have a bijection between the set of isomorphisms α and the set of pairs (α_0, α'). Consider the composition $\alpha \circ \beta$ f two isomorophisms α and β. We get

$$\text{(5.1.8)} \qquad \begin{aligned} &\alpha\beta(x, e) = \alpha(\beta(x, e)) = \alpha(\beta_0(x), \beta'(\beta_0(x))) = \\ &= \alpha(\beta_0(x), e_G)\,.\,\beta'(\beta_0(x)) = \\ &= (\alpha_0 \circ \beta_0(x), \alpha'(\alpha_0 \circ \beta_0(x))\,.\,\beta'(\beta_0(x)) = \\ &= (\alpha_0 \circ \beta_0(x), \alpha'(\alpha_0 \circ \beta_0(x))\,.\,\beta'(\alpha_0^{-1} \circ \alpha_0 \circ \beta_0(x)), \end{aligned}$$

so that the isomorphism $\alpha \circ \beta$ is identified with the pair $(\alpha_0 \circ \beta_0, \alpha' . (\beta' \circ \alpha_0^{-1}))$. Restrict ourselves to isomorphisms α whose projection α_0 satisfies $\alpha_0(0) = 0$, and transfer this bijection to jets. Identifying the r-jet $J^r_{(0,e)}\alpha$ (resp. $J^r_{(0,e)}\beta$) with the element $(J^r_0\alpha_0, J^r_0\alpha') \in L^r_n \times T^r_n G$ (resp. $(J^r_0\beta_0, J^r_0\beta') \in L^r_n \times T^r_n G$), the r-jet $J^r_{(0,e)}(\alpha \circ \beta)$ is identified with the element $(J^r_0(\alpha_0 \circ \beta_0), J^r_0(\alpha' . (\beta' \circ \alpha_0^{-1}))) \in L^r_n \times T^r_n G$. The arising multiplication law $((J^r_0\alpha_0, J^r_0\alpha'), (J^r_0\beta_0, J^r_0\beta')) \to (J^r_0(\alpha_0 \circ \beta_0), J^r_0(\alpha' . . (\beta' \circ \alpha_0^{-1})))$ coincides with (5.1.6).

5.2. Prolongations of left G-manifolds. Let G be a Lie group, P a left G-manifold. Consider the set $T^r_n P$ of r-jets with source $0 \in R^n$ and target in P. $T^r_n P$ is a subset of the manifold of r-jets $J^r(R^n, P)$, and is a closed submanifold of this manifold. Let $(A, S) \in G^r_n$ be any element, $A = J^r_0\alpha$, $S = J^r_0 f$. For each $p \in T^r_n P$, $p = J^r_0\tau$ define

(5.2.1) $$(A, S) . p = S . (p \circ A^{-1}) = J^r_0(f . (\tau \circ \alpha^{-1})),$$

where $(f . (\tau \circ \alpha^{-1}))(x) = f(x) . \tau(\alpha^{-1}(x))$ (the action of G on P). We shall show that the mapping $((A, S), p) \to (A, S) . p$ is a left action of the group G^r_n on $T^r_n P$. We have, by (5.1.6),

(5.2.2) $$\begin{aligned} &((A, S) . (B, T)) . p = (A . B, S . (T \circ A^{-1})) . p = \\ &= S . (T \circ A^{-1}) . (p \circ (A . B)^{-1}, \end{aligned}$$

where (A, S), $(B, T) \in G^r_n$ are any elements. On the other hand,

(5.2.3) $$\begin{aligned} &(A, S) . (B, T) . p = (A, S) . (T . (p \circ B^{-1})) = \\ &= S . (T \circ A^{-1}) . (p \circ B^{-1} \circ A^{-1}). \end{aligned}$$

Since the expressions (5.2.2) and (5.2.3) coincide, the mapping $((A, S), p) \to (A, S) . p$ defines the structure of a left G^r_n-manifold on $T^r_n P$. With this structure, $T^r_n P$ is called the (r, n)-*prolongation*, or simply the *prolongation* of the left G-manifold P.

5.3. Prolongations of a principal G-bundle. Let Y be a principal G-bundle over a manifold X, π its projection. Let $F^r X$ (resp. $J^r Y$) be the bundle of r-frames over X (resp. the r-jet prolongation of the fibered manifold π (Sec. 3.1)). Consider the fiber product $W^r Y = F^r X \oplus J^r Y$, i.e. the submanifold in $F^r X \times J^r Y$ of pairs (ζ, Z) such that ζ and Z belong to the fiber over the same point of X. We put for each $(\zeta, Z) \in W^r Y$, $\zeta = J^r_0\mu$, $Z = J^r_x\gamma$, where $x = \mu(0)$, and each $(A, S) \in G^r_n$, $A = J^r_0\alpha$, $S = J^r_0 f$

(5.3.1) $$(\zeta, Z) . (A, S) = (\zeta . A, Z . (S \circ \zeta^{-1})) = (J^r_0\mu\alpha, J^r_x(\gamma . (f \circ \mu^{-1})),$$

where $(\gamma . (f \circ \mu^{-1}))(x) = \gamma(x) . f(\mu^{-1}(x))$ (the action of G on Y). We shall show that the mapping $((\zeta, Z), (A, S)) \to (\zeta, Z) . (A, S)$ defines a right action of the group G^r_n on W^rY. Using (5.1.6) we get for any (A, S), $(B, T) \in G^r_n$ and $(\zeta, Z) \in W^rY$

$$(5.3.2)\qquad \begin{aligned}&(\zeta, Z) . ((A, S) . (B, T)) = (\zeta, Z) . (A . B, S . (T \circ A^{-1})) = \\ &= (\zeta . (A . B), Z . (S \circ \zeta^{-1}) . (T \circ A^{-1} \circ \zeta^{-1})).\end{aligned}$$

On the other hand,

$$(5.3.3)\qquad \begin{aligned}&((\zeta, Z) . (A, S)) . (B, T) = (\zeta . A, Z . (S \circ \zeta^{-1})) . (B, T) = \\ &= (\zeta . A . B, Z . (S \circ \zeta^{-1}) . (T \circ (\zeta . A)^{-1})).\end{aligned}$$

Since these two expressions coincide, (5.3.1) defines a right action of the group G^r_n on W^rY, and W^rY becomes a right G^r_n-manifold.

Theorem 5.1. *The group action (5.3.1) defines the structure of a principal G^r_n-bundle on W^rY. The base of this bundle is X, and its projection is the projection of the fiber product.*

Proof. We shall show that the action (5.3.1) of G^r_n on W^rY free. Suppose that for some (ζ, Z) and (A, S), $(\zeta, Z) . (A, S) = (\zeta, Z)$. Then $\zeta = \zeta . A$ and $Z = Z . (S \circ \zeta^{-1})$. Since L^r_n acts freely on F^rX, A must be the identity of L^r_n, $A = J^r_0 \,\mathrm{id}_{R^n}$. Let us consider the second condition. If $Z = J^r_x\gamma$, $S = J^r_0 f$, and $\zeta = J^r_0\mu$, this condition reads

$$(5.3.4)\qquad J^r_x\gamma = J^r_x(\gamma . (f \circ \mu^{-1})).$$

Let U be a neighborhood of the point $x \in X$ and $\varphi : \pi^{-1}(U) \to U \times G$ a diffeomorphism such that $\varphi(y . g) = \varphi(y) . g$ for all $y \in \pi^{-1}(U)$ and $g \in G$, and $\mathrm{pr}_1 \circ \varphi = \pi$; since Y is a principal G-bundle, such a diffeomorphism φ exists. Denote

$$(5.3.5)\qquad \varphi(\gamma(x)) = (x, \varphi_0(\gamma(x))).$$

Then

$$(5.3.6)\qquad \begin{aligned}&\varphi(\gamma(x) . f(\mu^{-1}(x))) = \varphi(\gamma(x)) . f(\mu^{-1}(x)) = \\ &= (x, \varphi_0(\gamma(x))) . f(\mu^{-1}(x)),\end{aligned}$$

and (5.3.4) holds if and only if $J^r_x(\varphi_0\gamma) = J^r_x(\varphi_0\gamma . f\mu^{-1})$. Applying $J^r_0\mu$ on both sides we get

$$(5.3.7)\qquad J^r_0(\varphi_0\gamma\mu) = J^r_0(\varphi_0\gamma\mu) . J^r_0 f$$

(multiplication in the group T^r_nG). Thus $S = J^r_0 f = J^r_0 e_G$, where e_G is the identity element of G, and S is the identity of T^r_nG. This proves that the action (5.3.1) is free.

It remains to show that two elements (ζ_1, Z_1), $(\zeta_2, Z_2) \in W^rY$ belong to the same fiber over X if and only if there exists an element $(A, S) \in G^r_n$ such that $(\zeta_2, Z_2) = (\zeta_1, Z_1) \cdot (A, S)$, and to verify the local triviality of W^rY over X. The proof of these assertions is straightforward, and will be omitted.

Remark 5.2. A less formal way of obtaining the left action (5.3.1) is the following. Let β be an isomorphism of the principal G-bundle $U \times G$, where $U \subset R^n$ is a neighborhood of the origin, onto an open subbundle of the principal G-bundle Y. β defines the projection of β, $\beta_0 = \text{proj } \beta$, and a mapping $\beta' : \beta_0(U) \to Y$ by the condition $\beta'(x') = \beta(\beta_0^{-1})(x'), e_G)$. Since $\pi\beta(x, g) = \beta_0(x)$, we have $\pi\beta'(x') = x'$, and β' is a section of π over $\beta_0(U)$. β' may equivalently be defined by

$$(5.3.8) \qquad \beta(x, g) = \beta(x, e) \cdot g = \beta'\beta_0(x) \cdot g.$$

Conversely, a pair (β_0, β'), where $\beta_0 : U \to X$ is a diffeomorphism and $\beta' : \beta_0(U) \to \to Y$ is a section, defines an isomorphism $\beta : U \times G \to Y$ by the same formula. Let $\alpha : U \times G \to U \times G$ be an isomorphism of principal G-bundles, $\alpha_0 = \text{proj } \alpha$, and suppose that $0 \in U$ and $\alpha_0(0) = 0$. Let β be as above. Then β and α are composable, and $\beta \circ \alpha$ is an isomorphism of $U \times G$ onto an open subbundle of the principal G-bundle Y. We have, using the notation of (5.1.6), and (5.3.8),

$$(5.3.9) \qquad \begin{aligned} \beta\alpha(x, g) &= (\beta\alpha)' \circ \beta_0\alpha_0(x) \cdot g = \beta(\alpha_0(x), \alpha'\alpha_0(x) \cdot g) = \\ &= \beta'\beta_0\alpha_0(x) \cdot \alpha'\alpha_0(x) \cdot g, \end{aligned}$$

which implies

$$(5.3.10) \qquad (\beta\alpha)' = \beta' \cdot \alpha'\beta_0^{-1}.$$

Thus, the isomorphism $\beta \circ \alpha$ is identified with the pair $(\beta_0 \circ \alpha_0, \beta' \cdot \alpha'\beta_0^{-1})$. Consider the r-jet $J^r_{(0,e)}(\beta \circ \alpha)$. (5.3.10) shows that under our identification, this r-jet is identified with the element $(J^r_0(\beta_0 \circ \alpha_0), J^r_{\beta_0(0)}(\beta' \cdot \alpha'\beta_0^{-1})) \in W^rY$. Expressing this element explicitly in terms of the pairs (β_0, β'), (α_0, α'), we get

$$(5.3.11) \qquad \begin{aligned} &(J^r_0(\beta_0 \circ \alpha_0), J^r_{\beta_0(0)}(\beta' \cdot \alpha'\beta_0^{-1})) = \\ &= (J^r_0\beta_0 \circ J^r_0\alpha_0, J^r_{\beta_0(0)}\beta' \cdot (J^r_0\alpha' \circ J^r_{\beta_0(0)}\beta_0^{-1})), \end{aligned}$$

and we come to (5.3.1).

The principal G^r_n-bundle W^rY is called the *r-prolongation*, or simply the *prolongation* of the principal G-bundle Y.

We note that the r-jet prolongation of the fibered manifold $\pi : Y \to X$, J^rY, has no natural structure of a principal fiber bundle, even though Y is a principal G-bundle.

Let $\alpha \in \operatorname{Mor} \mathscr{PB}_n(G)$, and define

$$W^r\alpha = (F^r\alpha_0, J^r\alpha), \tag{5.3.12}$$

where F^r is the r-frame lifting, $\alpha_0 = \operatorname{proj} \alpha$, and $J^r\alpha$ is the r-jet prolongation of the homomorphism of fibered manifolds (Sec. 3.1).

Theorem 5.2. *The correspondence $Y \to W^rY$, $\alpha \to W^r\alpha$ is a covariant functor from the category $\mathscr{PB}_n(G)$ to the category $\mathscr{PB}_n(G_n^r)$.*

Proof. Let $\alpha \in \operatorname{Mor} \mathscr{PB}_n(G)$, $\alpha : Y_1 \to Y_2$. We shall show by a direct calculation that for each $(\zeta, Z) \in W^rY_1$ and $(A, S) \in G_n^r$,

$$W^r\alpha((\zeta, Z) . (A, S)) = W^r\alpha(\zeta, Z) . (A, S). \tag{5.3.13}$$

Choose $(\zeta, Z) = (J_0^r\mu, J_x^r\gamma)$, where $x = \mu(0)$, $(A, S) = (J_0^r\beta, J_0^rf)$, and denote $\beta_0 = \operatorname{proj} \beta$. By definition, $W^r\alpha((\zeta, Z) . (A, S)) = (F^r\alpha_0(\zeta \circ A),\ J^r\alpha(Z . (S \circ \zeta^{-1}))$. Since $F^r\alpha_0 \in \operatorname{Mor} \mathscr{PB}_n(L_n^r)$, we have for the first component in this pair, $F^r\alpha_0(\zeta \circ A) = F^r\alpha_0(\zeta) \circ A$. For the second component we obtain from (3.1.8)

$$\begin{aligned} J^r\alpha(J_x^r(\gamma . (f\mu^{-1}))) &= J_{\alpha_0(x)}^r(\alpha \circ (\gamma . f\mu^{-1}) \circ \alpha_0^{-1}) = \\ &= J_{\alpha_0(x)}^r(\alpha \circ (\gamma\alpha_0^{-1} . f\mu^{-1}\alpha_0^{-1})) = J_{\alpha_0(x)}^r(\alpha\gamma\alpha_0^{-1} . f\mu^{-1}\alpha_0^{-1}), \end{aligned} \tag{5.3.14}$$

where the last equality follows from the fact that α is a homomorphism of principal G-bundles. This expression is equal to

$$\begin{aligned} J^r\alpha(J_x^r\gamma) . (J_0^rf \circ J_{\alpha_0(x)}^r(\mu^{-1}\alpha_0^{-1})) &= \\ = J^r\alpha(Z) . (S \circ (F^r\alpha_0(\zeta))^{-1}). \end{aligned} \tag{5.3.15}$$

Comparing (5.3.15) to (5.3.1) we obtain (5.3.13).

The rest of the assertion is obvious.

5.4. Prolongations of a fiber bundle. Let G be a Lie group, Y a principal G-bundle, and P a left G-manifold. Let $Y \times_G P$ be the fiber bundle with fiber P, associated with the principal G-bundle Y (Sec. 2.3); we write for short $Y \times_G P = Y_P$. The projection of Y_P is denoted by π_P, where π is the projection of Y. Recall that the equivalence class of a pair $(y, p) \in Y \times P$ with respect to the equivalence relation defined by the right action of G, is an element of Y_P denoted by $[y, p]$.

Let $\alpha \in \operatorname{Mor} \mathscr{PB}_n(G)$, $\alpha : Y_1 \to Y_2$. We put for each $z \in (Y_1)_P$, $z = [y, p]$,

$$\alpha_P(z) = [\alpha(y), p]. \tag{5.4.1}$$

We obtain a morphism $\alpha_P \in \operatorname{Mor} \mathscr{PB}_n(G)$, $\alpha_P : (Y_1)_P \to (Y_2)_P$ which is called *associated* with the morphism α of principal G-bundles. It is easily seen that the correspondence $Y \to Y_P$, $\alpha \to \alpha_P$ is a covariant functor from the category $\mathscr{PB}_n(G)$ to the category $\mathscr{PB}_n(G)$.

Theorem 5.3. *Let $Y \in \mathrm{Ob}\, \mathscr{PB}_n(G)$. The r-jet prolongation $J^r Y_P$ has the structure of a fiber bundle with fiber $T^r_n P$, associated with the principal G^r_n-bundle $W^r Y$.*

Proof. Let us consider the fiber bundle $(W^r Y)_Q$, where $Q = T^r_n P$; $(W^r Y)_Q$ is a fiber bundle with fiber Q, associated with the principal G^r_n-bundle $W^r Y$. To prove Theorem 5.3 it is enough to show that there exists an isomorphism of manifolds $\Psi : (W^r Y)_Q \to J^r Y_P$, commuting with the projections onto the base X of Y.

With the notation of Remark 5.2 consider an isomorphism β of the trivial principal G-bundle $U \times G$, where U is a neighborhood of $0 \in R^n$, onto an open subbundle of Y, and its projection β_0. Putting

$$\beta_P(x, \tau) = [\beta' \beta_0(x), \tau], \tag{5.4.2}$$

we obtain a mapping $\beta_P : U \times P \to Y_P$. Let now τ be a mapping of U into P. Then the mapping $x \to \beta_P(\beta_0^{-1}(x), \tau\beta_0^{-1}(x))$ is a local section of Y_P, defined on $\beta_0(U) \subset X$. Now consider the r-jet $J^r_{\beta_0(0)}(\beta_P \circ (\mathrm{id}_U \times \tau) \circ \beta_0^{-1})$ of this local section. To show that this r-jet does not depend on the pair $((J^r_0\beta_0, J^r_{\beta_0(0)}\beta'), J^r_0\tau) \in W^r Y \times \times T^r_n P$ provided this pair runs over an equivalence class in $(W^r Y)_Q$, we take an element $(A, S) \in G^r_n$, $A = J^r_0\alpha$, $S = J^r_0 f$, and construct the pair $((J^r_0\beta_0, J^r_{\beta_0(0)}\beta') \,.\, (A, S), (A, S)^{-1} \,.\, J^r_0\tau)$. By (5.3.1),

$$\begin{aligned} &(J^r_0\beta_0, J^r_{\beta_0(0)}\beta') \,.\, (J^r_0\alpha, J^r_0 f) = \\ &= (J^r_0(\beta_0\alpha), J^r_{\beta_0(0)}(\beta' \,.\, (f\beta_0^{-1}))). \end{aligned} \tag{5.4.3}$$

Using (5.1.6) we get $(A, S)^{-1} = (A^{-1}, S^{-1} \circ A)$, so that, by (5.2.1)

$$(J^r_0\alpha^{-1}, J^r_0 f^{-1} \circ J^r_0\alpha) \,.\, J^r_0\tau = J^r_0((f^{-1} \circ \alpha) \,.\, (\tau \circ \alpha)). \tag{5.4.4}$$

Denote $\sigma_0 = \beta_0\alpha$, $\sigma' = \beta' \,.\, (f\beta_0^{-1})$, and $\varrho = (f^{-1}\alpha) \,.\, (\tau\alpha)$. Let σ be an isomorphism of $U \times G$ onto an open subbundle of Y, defined by the pair (σ_0, σ'), and define σ_P by (5.4.2). Consider the r-jet $J^r_{\sigma_0(0)}(\sigma_P \circ (\mathrm{id}_U \times \varrho) \circ \sigma_0^{-1})$ and its representative $x \to \sigma_P(\sigma_0^{-1}(x), \varrho\sigma_0^{-1}(x))$. By definition,

$$\begin{aligned} &\sigma_P(\sigma_0^{-1}(x), \varrho\sigma_0^{-1}(x)) = [\sigma'(x), \varrho\sigma_0^{-1}(x)] = \\ &= [\beta'(x) \,.\, f\beta_0^{-1}(x), (f^{-1}\alpha \,.\, \tau\alpha)(\alpha^{-1}\beta_0^{-1}(x))] = \\ &= [\beta'(x) \,.\, f\beta_0^{-1}(x), f^{-1}(\beta_0^{-1}(x)) \,.\, \tau\beta_0^{-1}(x)] = \\ &= \beta_L(\beta_0^{-1}(x), \tau\beta_0^{-1}(x)). \end{aligned} \tag{5.4.5}$$

This proves the independence of the r-jet $J^r_{\beta_0(0)}(\beta_P \circ (\mathrm{id}_U \times \tau) \circ \beta_0^{-1})$ of the choice of $((J^r_0\beta_0, J^r_{\beta_0(0)}\beta'), J^r_0\tau)$, and we have a well defined mapping

$$(5.4.6)\qquad \begin{aligned}&(W^rY)_Q \ni [(\zeta, Z), p] \to \Psi([(\zeta, Z), p]) = \\ &= J^r_{\beta_0(0)}(\beta_P \circ (\mathrm{id}_U \times \tau) \circ \beta_0^{-1}) \in J^rY_L,\end{aligned}$$

where $\zeta = J^r_0\beta_0$, $Z = J^r_x\beta'$, $x = \beta_0(0)$, and $p = J^r_0\tau$. This mapping is a bijection. Clearly, its inverse is defined by the formula

$$(5.4.7)\qquad \Psi^{-1}(J^r_x\gamma) = [(J^r_0\beta_0, J^r_{\beta_0(0)}\beta'), J^r_0(\mathrm{pr}_2 \circ \beta_P^{-1}\gamma\beta_0)],$$

where $(J^r_0\beta_0, J^r_{\beta_0(0)}\beta')$ is any element of W^rY and $\mathrm{pr}_2 : R^n \times G \to G$ is the second projection. The differentiability of both Ψ and Ψ^{-1} follows from the differentiability of β_P and the composition of jets and is obvious. The commutativity of the mapping Ψ with the projection onto X is also obvious. This completes the proof.

Theorem 5.4. *The correspondence $Y_P \to J^rY_P$, $\alpha \to J^r\alpha_P$ is a covariant functor from the category $\mathscr{PB}_n(G)$ to the category $\mathscr{FB}_n(G^r_n)$.*

Proof. By (3.1.10) it is enough to verify that if $\alpha \in \mathrm{Mor}\,\mathscr{PB}_n(G)$, $\alpha : Y_1 \to Y_2$, then $J^r\alpha_P$ is a morphism of the category $\mathscr{FB}_n(G^r_n)$. We shall show that $J^r\alpha_P$ is expressible in the form

$$(5.4.7a)\qquad J^r\alpha_P = [F^r\alpha_0 \times J^r\alpha, \mathrm{id}_Q],$$

where $\alpha_0 = \mathrm{proj}\,\alpha$, $Q = T^r_nP$, and $\alpha_P = [\alpha, \mathrm{id}_P]$. Consider a point $J^r_x\gamma \in J^r(Y_1)_P$. With the notation of the proof of Theorem 5.3 we shall determine the element $\Psi^{-1}(J^r\alpha_P(J^r_x\gamma)) \in (W^rY_2)_Q$, where Ψ is defined by (4.5.6). Using (4.5.7) and (3.1.6) we obtain

$$(5.4.8)\qquad \begin{aligned}&\Psi^{-1}(J^r_{\alpha_0(x)}\alpha_P\gamma\alpha_0^{-1}) = \\ &= [(J^r\beta_0, J^r_{\beta_0(0)}\beta'), J^r_0(\mathrm{pr}_2 \circ \beta_P^{-1}\alpha_P\gamma\alpha_0^{-1}\beta_0)],\end{aligned}$$

where β is any isomorphism of the trivial principal G-bundle $U \times G$ into Y_2, such that $\beta_0(0) = \alpha_0(x)$, and β_0, β' are determined as in Remark 5.2. Choose an isomorphism χ of $R^n \times G$ into Y_1 such that $\chi_0(0) = x$ and take $\beta = \alpha\chi$. Then

$$(5.4.9)\qquad \beta_0 = \alpha_0\chi_0.$$

For each (x, g) from the domain of definition of β, $\beta(x, g) = \beta'\beta_0(x) . g = = \alpha\chi(x, g) = \alpha(\chi'\chi_0(x)) . g$ which implies $\alpha\chi'\chi_0(x) = \beta'\beta_0(x) = \beta'\alpha_0\chi_0(x)$, i.e.

$$(5.4.10)\qquad \beta' = \alpha\chi'\alpha_0^{-1}.$$

Then by (5.4.2), $\beta_P(x', \tau) = [\beta'\beta_0(x'), \tau] = [\alpha\chi'\alpha_0^{-1}\alpha_0\chi_0(x'), \tau] = [\alpha\chi'\chi_0(x')\ \tau]$. Using definition (5.4.1) we obtain

$$(5.4.11)\qquad \beta_P = \alpha_P\chi_P.$$

Substituting (5.4.9) – (5.4.11) into (3.1.8) we obtain

$$\Psi^{-1}(J^r_{\alpha_0(x)}\alpha_P\gamma\alpha_0^{-1}) =$$

(5.4.12) $$= [(J^r_0(\alpha_0\chi_0), J^r_{\alpha_0\eta_0(0)}\alpha\chi'\alpha_0^{-1}), J^r_0(\mathrm{pr}_2 \circ \chi_P^{-1}\alpha_P^{-1}\alpha_P\gamma\alpha_0^{-1}\alpha_0\chi_0)] =$$

$$= [(F^r\alpha_0(J^r_0\chi_0), J^r\alpha(J^r{}_{\alpha(0)}\chi')), J^r_0(\mathrm{pr}_2 \circ \chi_P^{-1}\gamma\chi_0)].$$

Since

(5.4.13) $$\Psi^{-1}(J^r_x\gamma) = [(J^r_0\chi_0, J^r_{\alpha_0(0)}\chi'), J^r_0(\mathrm{pr}_2 \circ \chi_P^{-1}\gamma\chi_0)],$$

formula (5.4.7a) follows on comparing (5.4.12) and (3.1.8).

5.5. Prolongations of the r-frame lifting and of the associated liftings. We shall now apply the theory of prolongations explained in the previous sections to the fundles of r-frames.

Let r and s be some positive integers, and consider the Lie groups L^{r+s}_n, and $(L^s_n)^r_n = L^r_n \times_\varphi T^r_n L^s_n$ (the semi-direct product of L^r_n and $T^r_n L^r_n$, defined in Sec. 5.1). To each $A \in L^{r+s}_n$, $A = J^{r+s}_0\alpha$, we assign an element $\nu(A)$ of the prolongation $(L^s_n)^r_n$ of L^s_n as follows. We identify the principal L^s_n-bundle $R^n \times L^s_n$ with $F^r R^n$ by putting $J^r_0\mu = (\mu(0), J^r_0(t_{\mu(0)}\mu))$, there t_x denotes the translation of R^n sending a point $x \in R^n$ to the origin. As in Remark 5.1, we identify the automorphism $F^s\alpha$ of $U \times L^s_n$, where U is the domain of α, with a pair $(\alpha, F^s\alpha)$, where $F^r\alpha$ is a mapping compossable with α, with values in L^s_n. We put

(5.5.1) $$\nu(A) = (J^r_0\alpha, J^r_0 F^s\alpha).$$

Obviously, $\nu : L^{r+s}_n \to (L^s_n)^r_n$ is a well-defined mapping.

Lemma 5.1. *ν is a homomorphism of groups and an embedding.*

Proof. Firstly, we shall show that ν is a homomorphism of groups. Clearly, for any $A, B \in L^{r+s}_n$, $A = J^{r+s}_0\alpha$, $B = J^{r+s}_0\beta$

(5.5.2) $$\nu(J^{r+s}_0\alpha \circ J^{r+s}_0\beta) = (J^r_0(\alpha\beta), J^r_0 F^s(\alpha\beta)).$$

Similarly as in Remark 5.1 we obtain $J^r_0 F^s(\alpha\beta) = J^r_0(F^s\alpha \,.\, (F^s\beta \circ \alpha^{-1}))$ which gives

(5.5.3) $$\nu(J^{r+s}_0\alpha \circ J^{r+s}_0\beta) = (J^r_0\alpha \circ J^r_0\beta, J^r_0 F^s\alpha \,.\, (J^r_0 F^s\beta \circ J^r_0\alpha^{-1})).$$

On comparing this relation to (5.1.6) we obtain

(5.5.4) $$\nu(A \,.\, B) = \nu(A) \,.\, \nu(B).$$

Secondly, we shall show that ν is an injective homomorphism of Lie groups; this property of ν will imply that ν is an embedding. Since ν is obviously smooth, it is enough to verify that it is injective. Let us choose an element $A \in L^{r+s}_n$, $A =$

$= J_0^{r+s}\alpha$, and an s-frame $\zeta \in F^sX$, $\zeta = J_0^s\mu$, from the domain of definition of $F^s\alpha$. We obtain, using our standard identification,

$$\begin{aligned} F^s\alpha(\zeta) &= J_0^s(\alpha\mu) = (\alpha\mu(0), J_0^s(t_{\alpha\mu(0)}\alpha\mu)) = \\ (5.5.5) \qquad &= (\alpha\mu(0), J_0^s(t_{\alpha\mu(0)}\alpha t_{-\mu(0)} t_{\mu(0)}\mu)) = \\ &= (\alpha\mu(0), J_0^s(t_{\alpha\mu(0)}\alpha t_{-\mu(0)})) \,.\, J_0^s(t_{\mu(0)}\mu). \end{aligned}$$

Writing $\mu(0) = x$ we obtain $F^s\alpha(x) = J_0^s(t_{\alpha(x)}\alpha t_{-x})$. Now suppose that for some $A \in L_n^{r+s}$, $A = J_0^{r+s}\alpha$,

$$(5.5.6) \qquad \nu(A) = (J_0^r \,\mathrm{id}_{R^n}, J_0^r(J_0^s \,\mathrm{id}_{R^n})),$$

where $J_0^s \,\mathrm{id}_{R^n}$ stands for the constant mapping of R^n into the identity element of the group L_n^s. It follows from this assumption that all partial derivatives of the mapping $x \to F^s\alpha(x) = J_0^s(t_{\alpha(x)}\alpha t_{-x})$ up to r-th order, vanish at the point $0 \in R^n$. To see it, notice that for each j and k,

$$(5.5.7) \qquad D_k(t_{\alpha(x),j}\alpha t_{-x})(0) = D_k\alpha_j(x),$$

where $t_{x,j}$ (resp. α_j) is the j-the component of the mapping t_x (resp. α). This means that the mapping $F^s\alpha$ is precisely the mapping $x \to (D_k\alpha_j(x), \ldots, D_{k_1}, \ldots, D_{ks}\alpha_j(x))$. Now applying our assumption (5.5.6) we obtain for all $j, k_1, \ldots, k_{r+s}$

$$(5.5.8) \qquad D_{k_1}\alpha_j(0) = 0, \ldots, D_{k_1}D_{k_2} \ldots D_{k_{r+s}}\alpha_j(0) = 0,$$

which implies that $A = J_0^{r+s} \,\mathrm{id}_{R^n}$, and the homomorphism ν is injective. This proves Lemma 5.1.

Let $X \in \mathrm{Ob}\, \mathscr{D}_n$, and let W^rF^sX be the r-jet prolongation of the bundle of s-frames F^sX. To each $\zeta \in F^{r+s}X$, $\zeta = J_0^{r+s}\mu$, we assign an element $\nu_X(\zeta) \in W^rF^sX$ as follows. As in Remark 5.2, we identify the isomorphism of principal L_n^s-bundles $F^s\mu : U \times L_n^s \to F^sX$, where U is the domain of definition of μ, with the pair $(\mu, F^s\mu)$, where $F^s\mu$ is a section of the bundle of s-frames F^sX composable with μ. We set

$$(5.5.9) \qquad \nu_X(\zeta) = (J_0^r\mu, J_{\mu(0)}^r S^s\mu)).$$

Lemma 5.2. *The pair (ν_X, ν) is a reduction of the principal $(L_n^s)_n^r$-bundleW^rF^sX to the principal L_n^{r+s}-bundle $F^{r+s}X$.*

Proof. By Lemma 5.1, we may restrict ourselves to examining the mapping ν_X. Obviously, proj $\nu_X = \mathrm{id}_X$. To show that ν_X is a ν-morphism of principal fiber bundles choose an element $A \in L_n^{r+s}$, $A = J_0^{r+s}\alpha$, an r-frame $\zeta \in F^{r+s}X$, $\zeta = J_0^{r+s}\mu$, and consider the point $\nu_X(\zeta \,.\, A)$. Using the definition of the action of L_n^{r+s} on $F^{r+s}X$ we obtain

(5.5.10) $\quad \nu_X(\zeta \,.\, A) = (J^r_0(\mu\alpha), J^r_{\mu(0)}S^s(\mu\alpha)).$

Then, since F^s is a covariant functor, $F^s(\mu\alpha) = F^s\mu \,.\, (\bar{F}^s\alpha \circ \mu^{-1})$, where $\bar{F}^s\alpha$ is a mapping with values in L^s_n, composable with α, determined by the condition $F^s\alpha = (\alpha, \bar{F}^s\alpha)$. This expression implies

(5.5.11) $\quad \nu_X(\zeta \,.\, A) = (J^r_0\mu \circ J^r_0\alpha, J^r_{\mu(0)}\bar{F}^s\mu \,.\, (J^r_0\bar{F}^s\alpha \circ J^r_{\mu(0)}\mu^{-1})) = \nu_X(\zeta) \,.\, \nu(A).$

Hence ν_X is a homomorphism of principal fiber bundles over id_X. It remains to, show that ν_X is injective. The proof of this fact consists in verifying that over the domain U of each chart (U, φ) on X, $W^rF^s\varphi \circ \nu_X = \nu \circ F^{r+s}\varphi$, and is routine.

The correspondence $X \to W^rF^sX$, $\alpha \to W^rF^s\alpha$, where $X \in \mathrm{Ob}\,\mathscr{D}_n$, may be considered as the composition of two covariant functors, F^s and W^r, and is therefore a covariant functor from the category $\mathscr{D}_n$ to the category $\mathscr{PB}_n((L^s_n)^r_n)$. We call this functor the *r-jet prolongation* of the *r*-frame lifting.

The following theorem shows that there is a relation of the functor W^rF^s and the $(r+s)$-lifting F^{r+s}.

Theorem 5.5. *There exists a natural transformation $X \to T_X$ of the functor F^{r+s} to W^rF^s such that for each $X \in \mathrm{Ob}\,\mathscr{D}_n$, T_X is a reduction of the principal $(L^s_n)^r_n$-bundle W^rF^sX to the principal L^{r+s}_n-bundle $F^{r+s}X$.*

Proof. For each $X \in \mathrm{Ob}\,\mathscr{D}_n$, put $T_X = (\nu_X, \nu)$, where ν_X (resp. ν) is defined by (5.5.9) (resp. (5.1.1)). By Lemma 5.1 and Lemma 5.2 it is sufficient to verify that for each $\alpha \in \mathrm{Mor}\,\mathscr{D}_n$, $\alpha : X_1 \to X_2$,

(5.5.12) $\quad W^rF^s\alpha \circ \nu_{X_1} = \nu_{X_2} \circ F^{r+s}\alpha.$

Choose $\zeta \in F^{r+s}X_1$, $\zeta = J^{r+s}_0\mu$. We have, by (5.5.9) and Theorem 5.2,

(5.5.13)
$$\begin{aligned} W^rF^s\alpha \circ \nu_{X_1}(\zeta) &= W^rF^s\alpha(J^r_0\mu, J^r_{\mu(0)}\bar{F}^s\mu) = \\ &= (F^r\alpha(J^r_0\mu), J^rF^s\alpha(J^r_{\mu(0)}\bar{F}^s\mu)) = \\ &= (J^r_0(\alpha\mu), J^r_{\alpha\mu(0)}(F^s\alpha \circ \bar{F}^s\mu \circ \alpha^{-1})), \end{aligned}$$

and

(5.5.14)
$$\begin{aligned} \nu_{X_2} \circ F^{r+s}\alpha(\zeta) &= \nu_{X_2}(J^{r+s}_0(\alpha\mu)) = \\ &= (J^r_0(\alpha\mu), J^r_{\alpha\mu(0)}\bar{F}^s(\alpha\mu)). \end{aligned}$$

To prove (5.5.12) it therefore suffices to show that

(5.5.15) $\quad J^r_{\alpha\mu(0)}(F^s\alpha \circ \bar{F}^s\mu \circ \alpha^{-1}) = J^r_{\alpha\mu(0)}\bar{F}^s(\alpha\mu).$

Using the functoriality of F^s we obtain

(5.5.16)
$$\begin{aligned} F^s(\alpha\mu)\,(x, g) &= (\bar{F}^s(\alpha\mu) \circ \alpha\mu(x)) \,.\, g = F^s\alpha \circ F^s\mu(x, g) = \\ &= F^s\alpha(\bar{F}^s\mu \circ \mu(x)) \,.\, g \end{aligned}$$

for each x from the domain of definition of μ and for each $g \in G$. That is, $F^s(\alpha\mu) = F^s\alpha \circ F^s\mu \circ \alpha^{-1}$, which proves (5.5.12).

Let P be a left L^s_n-manifold. Theorem 5.4 implies that the correspondence $X \to J^rF^s_PX$, $\alpha \to J^rF^s_P\alpha$ is a covariant functor from the category $\mathscr{D}_n$ to the category $\mathscr{FB}_n((L^s_n)^r_n)$. A natural question arises, as to whether this correspondence is a lifting.

Theorem 5.6. *The correspondence $X \to J^rF^s_PX$, $\alpha \to J^rF^s_P\alpha$ is a T^r_nP-lifting, associated with the $(r+s)$-frame lifting F^{r+s}.*

Proof. By Theorem 5.5, each fiber bundle $J^rF^s_PX$, where $X \in \text{Ob}\,\mathscr{D}_n$, may be regarded as a fiber bundle with fiber T^r_nP, associated with the bundle of $(r+s)$-frames $F^{r+s}X$. This means that

$$(5.5.17) \qquad J^rF^s_P = F^{r+s}_Q,$$

where $J^rF^s_P$ is the covariant functor from Theorem 5.5, and $Q = T^r_nP$.

The T^r_nP-lifting $J^rF^s_P$ is called the *r-jet prolongation* of the P-lifting F^s_P.

5.6. Natural differential operators. Let $\pi_1 : Y_1 \to X$ and $\pi_2 : Y_2 \to X$ be two fibered manifolds over the same base, let $C^\infty(Y_i)$ denote the set of smooth (local) sections of π_i, $i = 1, 2$. We state the following definition. A mapping $D : C^\infty(Y_1) \to C^\infty(Y_2)$ is said to be a *differential operator*, if there exist an integer $r \geqq 0$ and a homomorphism of fibered manifolds $D^r : J^rY_1 \to Y_2$ over id_X such that for every section $\gamma \in C^\infty(Y_1)$

$$(5.6.1) \qquad D(\gamma) = D^r \circ J^r\gamma.$$

Clearly, if such an integer r exists, then for every $s \geqq r$ there exists a homomorphism $D^s : J^sY_1 \to Y_2$ over id_X such that $D(\gamma) = D^s \circ J^s\gamma$; it is sufficient to take $D^s = D^r \circ \pi^{s,r}_1$, where $\pi^{s,r}_1 : J^sY_1 \to J^rY_1$ is the canonical jet projection. The smallest integer r for which there exists a homomorphism D^r satisfying (5.6.1) for every section $\gamma \in C^\infty(Y_1)$, is called the *order* of the differential operator D; we also say that D is *of order r*.

Every homomorphism of fibered manifolds $\nu : J^rY_1 \to Y_2$ over id_X defines a differential operator $D_\nu : C^\infty(Y_1) \to C^\infty(Y_2)$ by the formula

$$(5.6.2) \qquad D_\nu(\gamma) = \nu \circ J^r\gamma.$$

Let $\pi : Y \to X$ be a fibered manifold, and let $\tau : Z \to Y$ be a fibered manifold with base Y. Denote by $C^\infty_\pi(Y)$ (resp. $C^\infty_{\pi\tau}Z)$) the set of smooth sections of π

(resp. $\pi \circ \tau$). A differential operator $D : C^\infty_\pi(Y) \to C^\infty_{\pi\tau}(Z)$ is said to be a *prolongation operator*, if for every $\gamma \in C^\infty(Y)$

$$(5.6.3) \qquad \tau \circ D(\gamma) = \gamma.$$

Let P and Q be two left L^r_n-manifolds, let $X \in \mathrm{Ob}\,\mathscr{D}_n$ be an n-dimensional manifold, and let $F^s_P X$ (resp. $F^s_Q X$) be the fiber bundle with fiber P (resp. Q) associated to the bundle of s-frames $F^s X$. A differential operator $D : C^\infty(F^s_P X) \to C^\infty(F^s_Q X)$ is said to be *natural*, if for every open set $U \subset X$ and every diffeomorphism $\alpha : U \to \alpha(U) \subset X$,

$$(5.6.4) \qquad D(F^s_P\alpha \circ \gamma \circ \alpha^{-1}) = F^s_Q\alpha \circ D(\gamma) \circ \alpha^{-1}.$$

The following theorem says that the natural differential operators can be identified with certain differential invariants.

Theorem 5.7. *A necessary and sufficient condition for a differential operator $D : C^\infty(F^s_P X) \to C^\infty(F^s_Q X)$ to be a natural differential operator is that there exist an integer r, a homomorphism $D^r : J^r F^s_P X \to F^s_Q X$ over id_X, and a differential invariant $\Delta : T_n P \to Q$ whose realization on X is D^r, i.e. such that*

$$(5.6.5) \qquad \Delta_X = D^r.$$

Proof. Let D be a natural differential operator. There exist an integer r and a homomorphism $D^r : J^r F^s_P X \to F^s_Q X$ over id_X such that for every section $\gamma \in F^s_P X$, condition (5.6.1) holds. Since D is natural, (5.6.4) is satisfied for every diffeomorphism $\alpha : U \to \alpha(U) \subset X$. Combining these two relations we obtain

$$(5.6.6) \qquad D^r \circ J^r(F^s_P\alpha \circ \gamma \circ \alpha^{-1}) = F^s_Q\alpha \circ D^r \circ J^r\gamma \circ \alpha^{-1}.$$

Since by (3.1.8),

$$(5.6.7) \qquad J^r(F^s_P\alpha \circ J^r\gamma \circ \alpha^{-1}) = J^r F^s_P\alpha \circ J^r\gamma \circ \alpha^{-1},$$

where $J^r F^s_P\alpha$ is the r-jet prolongation of $F_P\alpha$, we get

$$(5.6.8) \qquad D^r \circ J^r F^s_P\alpha \circ J^r\gamma = F^s_Q\alpha \circ D^r \circ J^r\gamma.$$

The section γ being arbitrary, the homomorphism D^r satisfies

$$(5.6.9) \qquad D^r \circ J^r F^s_P\alpha = F^s_Q\alpha \circ D^r.$$

Therefore, by Theorem 2.2, D^r must be of the form (5.6.5) for some differential invariant $\Delta : T_r P \to Q$.

To prove the converse, it is sufficient to apply Theorem 2.3.

6. FUNDAMENTAL VECTOR FIELDS ON PROLONGATIONS OF $GL_n(R)$-MODULES

Vector fields used in partial differential equations for differential invariants are constructed explicitly, and their basic properties are studied.

6.1. Projectable vector fields and their prolongations. Let $\pi : Y \to X$ be a fibered manifold, $n = \dim X$, $m = \dim Y - n$, $\pi^r : J^rY \to X$ its r-jet prolongation, $\pi^{r,s} : J^rY \to J^sY$, where $0 \leqq s \leqq r$, the canonical jet projections. Let ξ be a vector field on Y. We say that ξ is *π-projectable*, if there exists a vector field ξ_0 on X such that

$$(6.1.1) \qquad T\pi \,.\, \xi = \xi_0 \circ \pi.$$

If such a vector field ξ_0 exists, it is unique, and is called the *π-projection* of ξ. We say that ξ is *π-vertical*, if it is π-projectable and its π-projection is the zero vector field.

Let ξ be a vector field on Y, α_t its local one-parameter group. It is easily verified that ξ is π-projectable if and only if each point $y \in Y$ has a neighborhood V such that α_t is defined on V for any sufficiently small t, and is an isomorphism of the fibered manifold $\pi|_V$ onto $\pi|_{\alpha(V)}$.

Let ξ be a π-projectable vector field on Y, ξ_0 its π-projection, α_t (resp. β_t) the local one-parameter group of ξ (resp. ξ_0). Tor each t, denote by $J^r\alpha_t$ the r-jet prolongation of the isomorphism α_t of fibered manifolds (Sec. 3.1). The isomorphisms $J^r\alpha_t$ form a local one-parameter group of transformations of the manifold J^rY. We put for each point $Z \in J^rY$

$$(6.1.2) \qquad J^r\xi(Z) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^r\alpha_t(Z)\right\}.$$

This defines a vector field $J^r\xi$ on J^rY, called the *r-jet prolongation* of the π-projectable vector field ζ. $J^r\zeta$ is $\pi^{r,s}$-projectable for any s, $0 \leqq s \leqq r$ (resp. π^r-projectable), and its $\pi^{r,s}$-projection (resp. π^r-projection) is equal to $J^s\xi$ (resp. ξ_0).

To obtain the chart expression of the r-jet prolongation of a π-projectable vector field, we now introduce a notion related to fiber charts. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart on Y, (V^s, ψ^s), $\psi^s = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 \ldots j_s})$, the associated fiber chart on J^sY (Sec. 3.1). Let $f : V^r \to R$ be a function. We put

$$(6.1.3) \qquad d_kf = \frac{\partial f}{\partial x^k} + \sum_{m=0}^{r} \sum_{j_1 \leqq \ldots \leqq j_m} \frac{\partial f}{\partial y^\sigma_{j_1 \ldots j_m}} y^\sigma_{j_1 \ldots j_m k}.$$

d_kf is a function on V^{r+1}, called the *k-th formal derivative* of the function f. Notice that

$$(6.1.4) \qquad d_k y^\sigma_{j_1 \ldots j_m} = y^\sigma_{j_1 \ldots j_m k}.$$

Lemma 6.1. *Let ξ be a π-projectable vector field on Y, (V, ψ), $\psi = (x^i, y^\sigma)$, a fiber chart on Y,*

$$(6.1.5) \qquad \xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma},$$

the expression of Ξ in (V, ψ). Then the vector field $J^r\xi$ has with respect to the associated chart (V^r, ψ^r) an expression

$$(6.1.6) \qquad J^r\xi = \xi^i \frac{\partial}{\partial x^i} + \sum_{m=0}^{r} \sum_{j_1 \leq \ldots \leq j_m} \Xi^\sigma_{j_1 \ldots j_m} \frac{\partial}{\partial y^\sigma_{j_1 \ldots j_m}},$$

where $\Xi_{j_1 \ldots j_m}$ are functions on V^r determined by the recurrent formula

$$(6.1.7) \qquad \Xi^\sigma_{j_1 \ldots j_m} = \mathrm{d}_{j_m} \Xi^\sigma_{j_1 \ldots j_{m-1}} - y^\sigma_{j_1 \ldots j_{m-1} k} \frac{\partial \xi^k}{\partial x^{j_m}}.$$

Proof. Let (U, φ), $\varphi = (x^i)$, be the chart on X, associřted with the fiber chart (V, ψ). Since $J^r\xi$ is $\pi^{r,0}$-projectable, the coefficient at $\partial/\partial x^i$ (resp. $\partial/\partial y^\sigma$) in the chart expression of $J^r\xi$ is equal to ξ^i (resp. Ξ^σ). It therefore remains to verify the recurrent formula (6.1.7).

Let $Z \in V^r$, $Z = J^r_x\gamma$, be an r-jet. Using the definition of $J^r\xi$ we get

$$(6.1.8) \qquad \Xi^\sigma_{j_1 \ldots j_m}(Z) = \left\{ \frac{\mathrm{d}}{\mathrm{d}t} y^\sigma_{j_1 \ldots j_m} \circ J^r\alpha_t(Z) \right\}_0,$$

where α_t is the local one-parameter group of ξ. Denote by β_t the π-projection of α_t. By (3.1.8)

$$(6.1.9) \qquad \begin{aligned} & y^\sigma_{j_1 \ldots j_m} \circ J^r\alpha_t(Z) = y^\sigma_{j_1 \ldots j_m}(J^r_{\beta_t(x)} \alpha_t \gamma \beta_t^{-1}) = \\ & = \left\{ \frac{\partial^m}{\partial x^{j_1} \ldots \partial x^{j_m}} y^\sigma \alpha_t \gamma \beta_t^{-1} \varphi^{-1} \right\}_{\varphi\beta_t(x)}. \end{aligned}$$

Differentiating the identity

$$(6.1.10) \qquad \begin{aligned} & \left\{ \frac{\partial}{\partial x^{j_m}} \frac{\partial^{m-1}}{\partial x^{j_1} \ldots \partial x^{j_{m-1}}} y^\sigma \alpha_t \gamma \beta_t^{-1} \varphi^{-1} \right\}_{\varphi\beta_t(x)} = \\ & = \left\{ \frac{\partial}{\partial x^k} \left(\frac{\partial^{m-1}}{\partial x^{j_1} \ldots \partial x^{j_{m-1}}} y^\sigma \alpha_t \gamma \beta_t^{-1} \varphi^{-1} \right) \circ \varphi \beta_t \varphi^{-1} \right\}_{\varphi(x)} \cdot \\ & \cdot \left\{ \frac{\partial}{\partial x^{j_m}} x^k \beta_t^{-1} \varphi^{-1} \right\}_{\varphi\beta_t(x)}. \end{aligned}$$

with respect to t at the point $t = 0$ we obtain

$$\Xi^{\sigma}_{j_1 \dots j_m}(Z) =$$

(6.1.11) $$= \left\{ \frac{\partial}{\partial x^k} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial^{m-1}}{\partial x^{j_1} \dots \partial x^{j_{m-1}}} y^{\sigma} \alpha_t \gamma \beta_t^{-1} \varphi^{-1} \right) \circ \varphi \circ \beta_t \varphi^{-1} \right\}_{\varphi(x)} \cdot \delta^k_{j_m} -$$
$$- y^{\sigma}_{j_1 \dots j_{m-1} k} \frac{\partial \xi^k}{\partial x^j} .$$

In the parentheses, we have the derivative of the mapping

(6.1.12) $$(t, x^1, \dots, x^n) \rightarrow$$
$$\rightarrow \left(\left(\frac{\partial^{m-1}}{\partial x^{j_1} \dots \partial x^{j_{m-1}}} y^{\sigma} \alpha_t \gamma \beta_t^{-1} \varphi^{-1} \right) \circ \varphi \beta_t \varphi^{-1} \right) (x^1, \dots, x^n).$$

Since the derivative at $t = 0$ of this mapping is the component $\Xi^{\sigma}_{j_1 \dots j_{m-1}}$ of the vector field $J^r\xi$, (6.1.11) leads directly to (6.1.7).

The following lemma concerns the Lie bracket of projectable vector fields. We prove it on induction by means of "non-holonomic" jets (see Sec. 3.1), although some other proof, using appropriate curves tangent to the Lie bracket of vector fields, is also available.

Lemma 6.2 *Let ξ and ζ be two π-projectable vector fields on Y. Then the Lie bracket $[\xi, \zeta]$ is also a π-projectable vector field, and*

(6.1.13) $$J^r[\xi, \zeta] = [J^r\xi, J^r\zeta].$$

Proof. 1. We shall show that (6.1.13) holds for $r = 1$. Let (V, ψ), $\psi = (x^i, y^{\sigma})$, be a fiber chart on Y, and let ξ and ζ be with respect to (V, ψ) expressed by

(6.1.14) $$\xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^{\sigma} \frac{\partial}{\partial y^{\sigma}}, \qquad \zeta = \zeta^i \frac{\partial}{\partial x^i} + Z^{\sigma} \frac{\partial}{\partial y^{\sigma}} .$$

Then

(6.1.15) $$[\xi, \zeta] = \left(\xi^k \frac{\partial \zeta^i}{\partial x^k} - \zeta^k \frac{\partial \xi^i}{\partial x^k} \right) \frac{\partial}{\partial x^i} +$$
$$+ \left(\xi^k \frac{\partial Z^{\nu}}{\partial x^k} + \Xi^{\sigma} \frac{\partial Z^{\nu}}{\partial y^{\sigma}} - \zeta^k \frac{\partial \Xi^{\nu}}{\partial x^k} - Z^{\sigma} \frac{\partial \Xi^{\nu}}{\partial y^{\sigma}} \right) \frac{\partial}{\partial y} .$$

By Lemma 6.1, the component of $J^1[\xi, \zeta]$ at $\partial/\partial y^{\nu}_i$ is defined by

(6.1.16) $$d_i \left(\xi^k \frac{\partial Z^{\nu}}{\partial x^k} + \Xi^{\sigma} \frac{\partial Z^{\nu}}{\partial y^{\sigma}} - \zeta^k \frac{\partial \Xi^{\nu}}{\partial x^k} - Z^{\sigma} \frac{\partial \Xi^{\nu}}{\partial y^{\sigma}} \right) -$$
$$- y_k \frac{\partial}{\partial x^i} \left(\xi^j \frac{\partial \xi^k}{\partial x^j} - \zeta^j \frac{\partial \xi^k}{\partial x^j} \right),$$

and the component of $[J^1\xi, J^1\zeta]$ at $\partial/\partial y_i^\nu$ is defined by

$$\xi^k \frac{\partial Z_i^\nu}{\partial x^k} + \Xi^\sigma \frac{\partial Z_i^\nu}{\partial y^\sigma} + \Xi_j^\sigma \frac{\partial Z_i^\nu}{\partial y_j^\sigma} - \zeta^k \frac{\partial \Xi_i^\nu}{\partial x^k} - Z^\sigma \frac{\partial \Xi_i^\nu}{\partial y^\sigma} - Z_j^\sigma \frac{\partial \Xi_i^\nu}{\partial k^\sigma}, \tag{6.1.17}$$

where

$$Z_i^\nu = d_i Z^\nu - y_m^\nu \frac{\partial \zeta^m}{\partial x^i}, \qquad \Xi_i^\nu = d_i \Xi^\nu - y_m^\nu \frac{\partial \xi^m}{\partial x^i}. \tag{6.1.18}$$

Now substituting the expressions

$$\begin{aligned} &\frac{\partial}{\partial x^k} d_i Z^\nu = d_i \frac{\partial Z^\nu}{\partial x^k}, \qquad \frac{\partial}{\partial y^\sigma} d_i Z^\nu = d_i \frac{\partial Z^\nu}{\partial k^\sigma}, \\ &\frac{\partial}{\partial y_j^\sigma} d_i Z^\nu = \frac{\partial}{\partial y_j^\sigma}\left(\frac{\partial Z^\nu}{\partial y^\varrho} y_i^\varrho\right) = \frac{\partial Z^\nu}{\partial y^\sigma}\delta_i^j, \end{aligned} \tag{6.1.19}$$

in (6.1.17) we obtain after some calculation that (6.1.17) is equal to (6.1.16) as desired.

2. Now suppose that $J^{r-1}[\xi, \zeta] = [J^{r-1}\xi, J^{r-1}\zeta]$ and show that (6.1.13) holds. Let $\iota : J^rY \to J^1(J^{r-1}Y)$ be the canonical embedding $J_x^r\gamma \to J_x^1(J^{r-1}\gamma)$. Let $\alpha : V \to Y$ be an isomorphism of fibered manifolds, defined on an open set $V \subset Y$, α_0 its π-projection. For any point Z from the domain of definition of $J^r\alpha$, $Z = J_x^r\gamma$,

$$\iota \circ J^r\alpha(Z) = \iota(J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}) = J^1_{\alpha_0(x)}(J^{r-1}\alpha\gamma\alpha_0^{-1}). \tag{6.1.20}$$

Let $J^1(J^{r-1}\alpha)$ be the 1-jet prolongation of the isomorphism of fibered manifolds $J^{r-1}\alpha$. We have

$$\begin{aligned} &J^1(J^{r-1}\alpha)(\iota(Z)) = J^1(J^{r-1}\alpha)(J_x^1(J^{r-1}\gamma)) = \\ &= J^1_{\alpha_0(x)}(J^{r-1}\alpha \circ J^{r-1}\gamma \circ \alpha_0^{-1}). \end{aligned} \tag{6.1.21}$$

But $J^{r-1}\alpha \circ J^{r-1}\gamma \circ \alpha_0^{-1} = J^{r-1}(\alpha\gamma\alpha_0^{-1}) \circ \alpha_0 \circ \alpha_0^{-1} = J^{r-1}\alpha\gamma\alpha_0^{-1}$, so that

$$J^1(J^{r-1}\alpha)(\iota(Z)) = J^1_{\alpha_0(x)}(J^{r-1}\alpha\gamma\alpha^{-1}) = \iota \circ J^r\alpha(Z), \tag{6.1.22}$$

that is,

$$J^1(J^{r-1}\alpha) \circ \iota = \iota \circ J^r\alpha. \tag{6.1.23}$$

Denote by $J^1J^{r-1}\xi$ (resp. $J^1J^{r-1}\zeta$) the 1-jet prolongation of the π^{r-1}-projectable vector field $J^{r-1}\xi$ (resp. $J^{r-1}\zeta$). Then (6.1.23) gives

$$J^1J^{r-1}\xi \circ \iota = T\iota \,.\, J^r\xi, \qquad J^1J^{r-1}\zeta \circ \iota = T\iota \,.\, J^r\zeta. \tag{6.1.24}$$

Let us consider the vector field $J^r[\xi, \zeta]$ on J^rY. By (6.1.24)

$$\begin{aligned} &T\iota \,.\, J^r[\xi, \zeta] = J^1(J^{r-1}[\xi, \zeta]) \circ \iota = J^1([J^{r-1}\xi, J^{r-1}\zeta]) \circ \iota = \\ &= [J^1J^{r-1}\xi, J^1J^{r-1}\zeta] \circ \iota, \end{aligned} \tag{6.1.25}$$

where we applied the first part of this proof. But (6.1.24) implies

(6.1.26) $$[J^1 J^{r-1}\xi, J^1 J^{r-1}\zeta] \circ \iota = T\iota \,.\, [J^r\xi, J^r\zeta].$$

Comparing (6.1.25) and (6.1.26) we get

(6.1.27) $$T\iota \,.\, (J^r[\xi, \zeta] - [J^r\xi, J^r\zeta]) = 0,$$

and (6.1.13) follows from the fact that $T\iota$ is injective.

6.2. Fundamental vector fields on prolongations of $GL_n(R)$-modules. Let G be a group. A *G-module* is by definition a finite-dimensional vector space endowed with a linear representation of the group G. In this section we consider a fixed $GL_n(R)$-module E, the E-lifting F_E, associated with the frame lifting F (Sec. 2.3), and the r-jet prolongation J^rF_E of this lifting (Sec. 5.5). We denote $m = \dim E$. The r-th differential group of R^n is denoted by L^r_n, and we identify the general linear group $GL_n(R)$ with L^1_n. Let $(A, q) \to A \,.\, q$ be the left action of $GL_n(R)$ on E, defined by the representation of $GL_n(R)$. Let (q^σ), $1 \leqq \sigma \leqq m$, be some global coordinates on E, defined by a basis of the vector space E. The action of $GL_n(R)$ on E is expressed in these coordinates by a matrix function $GL_n(R) \ni A \to (\Theta^\lambda_\nu(A)) \in$ $\in GL_m(R)$, where

(6.2.1) $$q^\sigma(A \,.\, q) = \Theta^\lambda_\nu(A) \,.\, q^\nu(q)$$

for each $q \in E$. Since this mapping is a homomorphism of groups, we have for all $A, B \in GL_n(R)$

(6.2.2) $$\Theta^\lambda_\nu(A \,.\, B) = \Theta^\lambda_\sigma(A) \,.\, \Theta^\sigma_\nu(B).$$

We shall introduce some charts which will be needed in the discussion of fundamental vector fields.

Let $X \in \mathrm{Ob}\, \mathscr{D}_n$, and consider the fiber bundle $F_EX \in \mathrm{Ob}\, \mathscr{F}\mathscr{B}_n(L^1_n)$, associated with the bundle of frames FX. Let π_X (resp. $\pi_{X,E}$) denote the projection of FX (resp. F_EX). Choose a chart (U, φ), $\varphi = (x^i)$, on X. This chart defines a section of FX over U, $x \to \zeta(x)$, where

(6.2.3) $$\zeta(x) = J^1_0\beta_x,$$

(6.2.4) $$\beta_x = \varphi^{-1} \circ t_{-\varphi(x)},$$

and $t_{-\varphi(x)}(y) = y + \varphi(x)$. Let $Z \in \pi^{-1}_{X,E}(U)$ be any point, $x = \pi_{X,E}(Z)$, and let $\varkappa_{\zeta(x)} : \pi^{-1}_{X,E}(x) \to E$ denote the framing defined by the frame $\zeta(x) \in FX$ (Sec. 3.3). We put

(6.2.5) $$\bar{q}^\sigma(Z) = q^\sigma \circ \varkappa_{\zeta(x)}(Z).$$

The pair (V, φ_E), where $V = \pi_{X,E}^{-1}(U)$, $\varphi_E = (x^i, \bar{q}^\sigma)$, is easily seen to be a fiber chart on $F_E X$. This fiber chart is said to be *associated* with the chart (U, φ) on X (and with the coordinates (q^σ) on E; the coordinates $(x^i, \bar{q}^\sigma)$ are said to be *associated* with the coordinates x^i (and with q^σ).

The concept of an associated chart is directly transferred to higher jet spaces. Let π_X^r (resp. $\pi_{X,E}^r$) denote the projection of the bundle of r-frames $F^r X$ (resp. of the r-jet prolongation $J^r F_E X$ of $F_E X$). Let (U, φ) be as above, and put

$$\zeta^{r+1}(x) = J_0^{r+1} \beta_x. \tag{6.2.6}$$

$\zeta^{r+1}(x)$ is an $(r+1)$-frame at $x \in U$. The framing defined by this $(r+1)$-frame is a linear isomorphism $\varkappa_{\zeta^{r+1}(x)} : (\pi_{X,E}^r)^{-1}(x) \to T_n^r E$ (Sec. 4.5). Let $q^\sigma, q_{j_1}^\sigma, \ldots, \ldots, q_{j_1 j_2 \ldots j_r}^\sigma$ be the coordinates on $T_n^r E$, associated with the coordinates q^σ on E. We put for each $Z \in (\pi_{X,E}^r)^{-1}(U)$

$$\begin{aligned} \bar{q}^\sigma(Z) &= q^\sigma \circ \varkappa_{\zeta^{r+1}(x)}(Z), \\ q_{j_1}^\sigma(Z) &= q_{j_1}^\sigma \circ \varkappa_{\zeta^{r+1}(x)}(Z), \\ &\ldots \\ \bar{0}_{j_1 \ldots j_r}^\sigma(Z) &= q_{j_1 \ldots j_r}^\sigma \circ \varkappa_{\zeta^{r+1}(x)}(Z), \end{aligned} \tag{6.2.7}$$

where $x = \pi_{X,E}^r(Z)$. The pair $((\pi_{X,E}^r)^{-1}(U), \varphi_E^r)$, where $\varphi_E^r = (x^i, \bar{q}^\sigma, \bar{q}_{j_1}^\sigma, \ldots, \ldots, \bar{q}_{j_1 \ldots j_r}^\sigma)$, $1 \leqq j_1 \leqq \ldots \leqq j_r \leqq n$, is a fiber chart on $J^r F_E X$. Notice that in introducing this fiber chart, we used the identification of $J^r F_E X$ and the fiber bundle $F_Q^{r+1} X$, where $Q = T_n^r E$, associated with the bundle of $(r+1)$-frames $F_Q^{r+1} X$ (Sec. 5.5, (5.5.17)). This chart is said to be *associated* with the chart (U, φ) on X (and with the coordinates (q^σ) on E); the coordinates $(x^i, \bar{q}^\sigma, \bar{q}_{j_1}^\sigma, \ldots, \bar{q}_{j_1 \ldots j_r}^\sigma)$ are said to be *associated* with the coordinates x^i (and with q^σ).

Remark 6.1. According to the above constructions, we have two fiber charts on the fiber bundle $J^r F_E X$, the fiber chart $((\pi_{X,E}^r)^{-1}(U), \varphi_E^r)$, and the fiber chart associated with the fiber chart (V, φ_E) on $F_E X$ in the sense of Sec. 3.1. It can be shown, however, that these two fiber charts coincide.

Let a_k^i be the canonical coordinates on the group L_n^1, let $e = (\delta_k^i)$ be the identity element, and let the functions $\bar{\Theta}_\nu^\lambda(a_k^i)$ express the action (6.2.1) of the group L_n^1 on E.

Theorem 6.1. (*a*) *Let* ξ *be a vector field on a manifold* $X \in \operatorname{Ob} \mathscr{D}_n$, (U, φ), $\varphi = (x^i)$, *a chart on* X,

$$\xi = \xi^i \frac{\partial}{\partial x^i}, \tag{6.2.8}$$

the expression of ξ in this chart. Then in the associated coordinates $(x^i, \bar{q}^\sigma)$, the vector field $F_E\xi$ has an expression

$$(6.2.9) \qquad F_L\xi = \xi^i \frac{\partial}{\partial x^i} + C^{i\sigma}_{k\lambda} \frac{\partial \xi^k}{\partial x^i} \bar{q}^\lambda \frac{\partial}{\partial \bar{q}^\sigma},$$

where $C^{i\sigma}_{k\lambda}$ are real numbers determined by

$$(6.2.10) \qquad C^{i\sigma}_{k\lambda} = \left\{\frac{\partial \Theta^\sigma_\lambda}{\partial a^i_k}\right\}_e,$$

and

$$(6.2.11) \qquad C^{i\sigma}_{k\lambda} \,.\, C^{j\nu}_{m\sigma} = \delta^i_m \,.\, C^{j\nu}_{k\lambda}.$$

(*b*) *For any two vector fields ξ, η on X,*

$$(6.2.12) \qquad F_E[\xi, \eta] = [F_E\xi, F_E\eta].$$

Proof. 1. Let $Z \in \pi^{-1}_{X,E}(U)$, $x = \pi_{X,E}(Z)$. Z is expressible in the form

$$(6.2.13) \qquad Z = [\zeta(x), q],$$

where $\zeta(x) = J^1_0\beta_x$, $\beta_x = \varphi^{-1} \circ t_{-\varphi(x)}$. Let α_t be the local one-parameter group of ξ. For all sufficiently small t,

$$(6.2.14) \qquad \begin{aligned} F_E\alpha_t(Z) &= [J^1_0(\alpha_t \circ \beta_x), q] = \\ &= [J^1_0(\varphi^{-1} \circ t_{-\varphi\alpha_t(x)} \circ t_{\varphi\alpha_t(x)} \circ \varphi \circ \alpha_t \circ \beta_x), q] = \\ &= [J^1_0\beta_{\alpha_t}(x) \circ J^1_0(t_{\varphi\alpha_t}(x)\, \varphi\alpha_t\beta_x), q] = \\ &= [J^1_0\beta_{\alpha_t}(x), J^1_0(t_{\varphi\alpha_t}(x)\, \varphi\alpha_t\beta_x) \,.\, q]. \end{aligned}$$

Hence with the usual convention

$$(6.2.15) \qquad \begin{aligned} x^i \circ F_E\alpha_t(Z) &= x^i\alpha_t(x), \\ \bar{q}^\sigma \circ F_E\alpha_t(Z) &= \Theta^\sigma_\lambda(J^1_0(t_{\varphi\alpha_t}(x)\, \varphi\alpha_t\beta_x) \,.\, \bar{q}^\nu(Z). \end{aligned}$$

Differentiating both sides with respect to t we obtain

$$(6.2.16) \qquad \begin{aligned} &\left\{\frac{\mathrm{d}}{\mathrm{d}t} x^i \circ F_E\alpha_t(Z)\right\}_0 = \xi^i(x), \\ &\left\{\frac{\mathrm{d}}{\mathrm{d}t} q^\sigma \circ F_E\alpha_t(Z)\right\}_0 = \left\{\frac{\partial \Theta^\sigma_\lambda}{\partial a^k_i}\right\}_e . \left\{\frac{d}{\mathrm{d}t} a^k_i(J^1_0(t_{\varphi\alpha_t(x)}\varphi\alpha_t\beta_x))\right\}_0 \bar{q}^\lambda(Z) = \\ &= \left\{\frac{\partial \Theta^\sigma_\lambda}{\partial a^k_i}\right\}_e . \left\{\frac{\mathrm{d}}{\mathrm{d}t} D_i(x^k\alpha_t\varphi^{-1})(\varphi(x))\right\}_0 . \bar{q}^\lambda(Z) \end{aligned}$$

proving (6.2.9).

2. Next we shall prove relation (6.2.12). Consider two vector fields ξ, η on X

and their Lie bracket $[\xi, \eta]$. Let α_t (resp. β_t, resp. δ_t) be the local one-parameter group of ξ (resp. η, resp. $[\xi, \eta]$). We define a one-parameter family of transformations ε_t, where $t \geqq 0$, by

$$\varepsilon_t = \beta_{-\sqrt{t}} \circ \alpha_{-\sqrt{t}} \circ \beta_{\sqrt{t}} \circ \alpha_{\sqrt{t}}. \tag{6.2.17}$$

It is known that for each $x \in X$

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_t(x)\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t}\delta_t(x)\right\}_0 = [\xi, \eta](x), \tag{6.2.18}$$

where the derivative on the left is the limit of vectors at $t = 0$. Consider the local one-parameter group $F_E\alpha_t$ (resp. $F_E\beta_t$, resp. $F_E\delta_t$) of the lift $F_E\xi$ (resp. $F_E\eta$, resp. $F_E[\xi, \eta]$), and put

$$\bar{\varepsilon}_t = F_E\beta_{-\sqrt{t}} \circ F_E\alpha_{-\sqrt{t}} \circ F_E\beta_{\sqrt{t}} \circ F_E\alpha_{\sqrt{t}}. \tag{6.2.19}$$

Since F_E is a covariant functor, we have $\bar{\varepsilon}_t = F_E\varepsilon_t$, and for each $z \in F_EX$,

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}F_E\varepsilon_t(z)\right\}_0 = [F_E\xi, F_E\eta](z). \tag{6.2.20}$$

To prove that

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}F_E\varepsilon_t(z)\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t}F_E\delta_t(z)\right\}_0, \tag{6.2.21}$$

we shall use a chart (U, φ), $\varphi = (x^i)$, on X and the associated coordinates x^i, $\bar{q}^\sigma$ on F_EX defined above. Let $z \in \pi_{X,E}^{-1}(U)$, $x = \pi_{X,E}(z)$. The components of the vector on the left of (6.2.21) with respect to these coordinates are expressed by

$$\begin{aligned} &\left\{\frac{\mathrm{d}}{\mathrm{d}t}x^i \circ F_E\varepsilon_t(z)\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t}x^i \circ \varepsilon_t(x)\right\}_0, \\ &\left\{\frac{\mathrm{d}}{\mathrm{d}t}\bar{q}^\sigma \circ F_E\varepsilon_t(z)\right\}_0 = \left\{\frac{\partial \Theta^\sigma_\lambda}{\partial a^k_i}\right\}_e . D_i\left\{\frac{\mathrm{d}}{\mathrm{d}t}x^k\varepsilon_t\varphi^{-1}\right\}_0(\varphi(x)) . \bar{q}^\lambda(z), \end{aligned} \tag{6.2.22}$$

and analogous expressions are obtained for the components of the vector on the right. Equality (6.2.21) now follows from (6.2.18). Now since

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}F_E\delta_t(z)\right\}_0 = F_E[\xi, \zeta](z), \tag{6.2.23}$$

we obtain (6.2.12) on comparing (6.2.21) and (6.2.20).

3. It remains to prove relation (6.2.11) for the coefficients $C^{i\sigma}_{k\lambda}$; but this relation follows immediately from (6.2.12).

Let $X \in \mathrm{Ob}\,\mathscr{D}_n$, and consider the r-jet prolongation J^rF_EX of the fiber bundle F_EX. We shall consider J^rF_EX as a fiber bundle, associated with the bundle of $(r+1)$-

frames $F^{r+1}X$. The projection of this fiber bundle is denoted by $\pi^r_{X,E}$, and its fiber is denoted by T^r_nE. Recall that T^r_nE has the structure of a left L^{r+1}_n-manifold; the Lie algebra of fundamental vector fields on this L^{r+1}_n-manifold will be denoted by $L(T^r_nE)$.

Theorem 6.2. *Let ξ be a vector field on X, $x \in X$ a point, (U, φ), $\varphi = (x^i)$, a chart on X such that $x \in U$. Suppose that $\xi(x) = 0$. If ξ is expressed by (6.2.8), then the r-jet prolongation of the lift $F_E\xi$, $J^rF_E\xi$, is along the fiber $(\pi^r_{X,E})^{-1}(x)$ expressed by*

$$J^rF_E\xi = \sum_{m=1}^{r+1} \sum \frac{\partial^m \xi^i}{\partial x^{k_1} \dots \partial x^{k_m}} \bar{\Theta}_i^{k_1 \dots k_m}, \tag{6.2.24}$$

where $\bar{\Theta}_i^{k_1 \dots k_m}$ are some uniquely determined vector fields on $(\pi^r_{X,E})^{-1}(x)$, and we sum over $k_1 \leqq \dots \leqq k_m$. These vector fields do not depend on ξ^i and the derivatives of ξ^i. For each $(r+1)$-frame $\zeta \in (\pi^r_X)^{-1}(x)$ the vector fields $\Theta_i^{k_1 \dots k_m}$ defined by

$$T\varkappa_\zeta \,.\, \bar{\Theta}_i^{k_1 \dots k_m} = \Theta_i^{k_1 \dots k_m} \circ \varkappa_\zeta, \tag{6.2.25}$$

are fundamental vector fields on T^r_nE, and span the vector space $L(T^r_nE)$.

Proof. Using (6.2.9) and Lemma 6.1 we easily see that the vector field $J^rF_E\xi$ must be of the form (6.2.24), and that the vector fields $\bar{\Theta}_i^{k_1 \dots k_m}$ do not depend on ξ^i and the derivatives of ξ^i. To prove the second part of Theorem 6.2 choose an $(r+1)$-frame $\zeta \in (\pi^r_X)^{-1}(x)$ and consider the vector field ξ. Using the structure of a fiber bundle on J^rF_EX, associated with $F^{r+1}X$, with fiber T^r_nE, and the corresponding framing we get by (6.2.25)

$$T\varkappa_\zeta \,.\, J^rF_E\xi = \sum_{m=1}^{r+1} \sum \frac{\partial^m \xi^i}{\partial x^{k_1} \dots \partial x^{k_m}} \Theta_i^{k_1 \dots k_m}, \tag{6.2.26}$$

where the derivatives of ξ^i are considered at the point x, and we sum ovei non-decreasing sequences $(k_1, k_2, \dots, k_m)$. By Theorem 3.2, the vector field on the right is a fundamental vector field on T^r_nE, i.e., an element of $L(T^r_nE)$. Obviously, (6.2.26) holds for any ξ such that $\xi(x) = 0$, and the derivatives of ξ^i in (6.2.26) may be considered as independent parameters. This implies that each of the vector fields $\Theta_i^{k_1 \dots k_m}$ is a fundamental vectoi field on T^r_nE. Using Theorem 3.2 again we see that these vector fields must span the vector space $L(T^r_nE)$, and the proof is complete.

Remark 6.2. It should be pointed out that the vector fields $\Theta_i^{k_1 \dots k_m}$ from Theorem 6.2 need not be, in general, linearly independent.

Remark 6.3. We may use the $(r+1)$-frame $\zeta = J_0^{r+1}(\varphi^{-1}t_{-\varphi(x)})$ is definition (6.2.25). Then the corresponding framing is expressed by (6.2.7) which implies that

$$(6.2.27) \qquad Tx_\zeta \cdot \left(\frac{\partial}{\partial \bar{q}^\sigma_{k_1 \ldots k_m}}\right)_x = \frac{\partial}{\partial q^\sigma_{k_1 \ldots k_m}}.$$

In this case the fundamental vector fields $\Theta_i^{k_1 \ldots k_m}$ are obtained by simply replacing the coordinates $\bar{q}^\sigma, \bar{q}^\sigma_{k_1}, \ldots, \bar{q}^\sigma_{k_1 \ldots k_r}$ in the vector fields $\bar{\Theta}_i^{k_1 \ldots k_m}$ by the coordinates q^σ, $q^\sigma_{k_1}, \ldots, q^\sigma_{k_1 \ldots k_r}$, respectively. This remark, together with Theorem 6.2 and Lemma 6.1 can be used for a direct computation of fundamental vector fields on the r-jet prolongation of any $GL_n(R)$-module.

6.3. Lie bracket of fundamental vector fields on prolongations of $GL_n(R)$-modules. Let X be an n-dimensional manifold, E a $GL_n(R)$-module. With the notation of Sec. 6.2, we shall consider the fiber bundle $F_E X$ with fiber E, associated with the bundle of frames FX, the fiber bundle $J^r F_E X$ with fiber $T_n^r E$, associated with the bundle of $(r+1)$-frames $F^{r+1}X$, and the r-jet prolongations $J^r F_E \xi$ of vector fields ξ on X (Theorem 6.2). Let ξ and η be two vector fields on X. By Lemma 6.2 and Theorem 6.1,

$$(6.3.1) \qquad [J^r F_E \xi, J^r F_E \eta] = J^r F_E [\xi, \eta].$$

Using this formula, we shall obtain some commutation relations for the fundamental vector fields $\Theta_i^{k_1 \ldots k_m}$ defined by Theorem 6.2.

To this purpose it will be convenient to rewrite formula (6.2.24) in a form where summation is taking place over all sequences $(k_1, k_2, \ldots, k_m)$, not only over non-decreasing ones. Let ξ be a vector field on X, (U, φ), $\varphi = (x^i)$, a chart on X, $x \in U$ a point. Suppose that $\xi(x) = 0$. Let $(k_1, k_2, \ldots, k_m)$ be any sequence of integers such that $1 \leqq k_1, \ldots, k_m \leqq n$. We denote by $N(k_1, \ldots, k_m)$ the number of all different sequences $(p_1, p_2, \ldots, p_m)$ arising by permuting the sequence $(k_1, k_2, \ldots, \ldots, k_m)$. Obviously

$$(6.3.2) \qquad N(k_1, \ldots, k_m) = \frac{m!}{i_1!\, i_2! \ldots i_n!},$$

where i_s is the number of integers s in the sequence $(k_1, k_2, \ldots, k_m)$. Then (6.2.24) can be written as

$$(6.3.3) \qquad J^r F_E \xi = \sum_{m=1}^{r+1} \left\{\frac{\partial^m \xi^i}{\partial x^{k_1} \ldots \partial x^{k_m}}\right\}_x \bar{\vartheta}_i^{k_1 \ldots k_m},$$

where we sum over $k_1, k_2, \ldots, k_m = 1, 2, \ldots, n$, and $\bar{\vartheta}_i^{k_1 \ldots k_m}$ are the unique vector fields, symmetric in the superscripts, such that

(6.3.4) $$\bar{\vartheta}_i^{k_1 \dots k_m} = \frac{1}{N(k_1, \dots, k_m)} \bar{\Theta}_i^{k_1 \dots k_m}, \qquad 1 \leqq k_1 \leqq \dots \leqq k_m \leqq n.$$

Let $\zeta \in (\pi_X^r)^{-1}(x)$ be an r-frame at the point x. If we now define vector fields $\vartheta_i^{k_1 \dots k_m}$ on $T_n^r E$ by

(6.3.5) $$T\varkappa_\zeta \, . \, \bar{\vartheta}_i^{k_1 \dots k_m} = \vartheta_i^{k_1 \dots k_m} \circ \varkappa_\zeta,$$

we again obtain fundamental vector fields, spanning the vector space $L(T_n^r E)$.

Before going on to the main theorem, we want to establish a lemma on differentiation of the product of functions of many independent variables.

Lemma 6.3. *Let f and g be two real functions defined on an open subset of R^n, let x^i, $1 \leqq i \leqq n$, be the canonical coordinates on R^n, $s \geqq 1$ an integer. Let λ^i, $1 \leqq i \leqq n$, be any system of real numbers. Then*

(6.3.6) $$\begin{aligned} &\frac{\partial^s (f \, . \, g)}{\partial x^{i_1} \dots \partial x^{i_s}} \lambda^{i_1} \dots \lambda^{i_s} = \\ &= \left(\frac{\partial^s f}{\partial x^{i_1} \dots \partial x^{i_s}} g + \binom{s}{1} \frac{\partial^{s-1} f}{\partial x^{i_1} \dots \partial x^{i_{s-1}}} \frac{\partial g}{\partial x^{i_s}} + \dots + \right. \\ &+ \binom{s}{r} \frac{\partial^{s-r} f}{\partial x^{i_1} \dots \partial x^{i_{s-r}}} \frac{\partial^r g}{\partial x^{i_{s-r+1}} \dots \partial x^{i_s}} + \dots + \\ &+ \left. f \frac{\partial^s g}{\partial x^{i_1} \dots \partial x^{i_s}} \right) \lambda^{i_1} \dots \lambda^{i_s}. \end{aligned}$$

Proof. Choose a point $x = (x^1, x^2, \dots, x^n)$ belonging to the domain of definition of the functions f and g, and any real numbers $\lambda^1, \dots, \lambda^n$. Put $\Phi(t) = f(x^1 + \lambda^1 t, \dots, x^n + \lambda^n t)$, $\Psi(t) = g(x^1 + \lambda^1 t, \dots, x^n + \lambda^n t)$. The functions Φ and Ψ are defined on an open interval in R, containing the origin 0, and are differentiable provided f and g are differentiable. By the well-known "Leibniz rule" for differentiating of the product of two functions of a real variable

(6.3.7) $$\begin{aligned} D^s(\Phi \, . \, \Psi) &= D^s\Phi \, . \, \Psi + \binom{s}{1} D^{s-1}\Phi \, . \, D\Psi + \dots + \\ &+ \binom{s}{r} D^{s-r}\Phi \, . \, D^r\Psi + \dots + \Phi \, . \, D^s\Psi. \end{aligned}$$

Computing the expressions on both sides at the point $t = 0$ we easily obtain formula (6.3.6) at the point x. Since x is arbitrary, this proves Lemma 6.3

Remark 6.4. Formula (6.3.6) can be rewritten without the auxiliary system λ^i; to this aim one should symmetrize the coefficient at $\lambda^{i_1} \dots \lambda^{i_s}$ on the right in the subscripts $i_1, \ldots, i_s$, and then omit the product of the variables $\lambda^{i_1} \ldots \lambda^{i_s}$ on both sides.

In the following theorem the fundamental vector fields $\vartheta_i^{k_1 \ldots k_m}$ on $T_n^r E$ are defined by (6.3.3) and (6.3.5).

Theorem 6.3. *The fundamental vector fields* $\vartheta_i^{k_1 \ldots k_m}$, $1 \leqq m \leqq r + 1$, *obey the commutation relations*

$$\begin{aligned} &[\vartheta_i^{k_1 \ldots k_s}, \vartheta_j^{p_1 \ldots p_t}] = \frac{(s+t-1)!}{s!\,t!} (\delta_i^{p_1} \vartheta_j^{k_1 \ldots k_s p_2 \ldots p_t} + \\ &+ \ldots + \delta_i^{p_t} \vartheta_j^{k_1 \ldots k_s p_1 \ldots p_{t-1}} - \delta_j^{k_1} \vartheta_i^{k_2 \ldots k_s p_1 \ldots p_t} - \\ &- \ldots - \delta_j^{k_s} \vartheta_i^{k_1 \ldots k_{s-1} p_1 \ldots p_t}), \quad 1 \leqq s, t \leqq r+1, \quad 2 \leqq s + t \leqq r + 2, \\ &[\vartheta_i^{k_1 \ldots k_s}, \vartheta_j^{p_1 \ldots p_t}] = 0, \quad 1 \leqq s, t \leqq r + 1, \\ &r + 3 \leqq s + t \leqq 2(r+1). \end{aligned} \tag{6.3.8}$$

Proof. Let us take $X = R^n$, and denote by x^i the canonical coordinates on R^n Consider two vector fields ξ and η on R^n such that $\xi(0) = 0$, $\eta(0) = 0$, and theri Lie bracket $\zeta = [\xi, \eta]$. If ξ and η are expressed by

$$\xi = \xi^i \frac{\partial}{\partial x^i}, \qquad \eta = \eta^i \frac{\partial}{\partial x^i}, \tag{6.3.9}$$

then

$$\zeta = \zeta^i \frac{\partial}{\partial x^i}, \qquad \zeta^i = \xi^k \frac{\partial \eta^i}{\partial x^k} - \eta^k \frac{\partial \xi^i}{\partial x^k}, \tag{6.3.10}$$

and obviously $\zeta(0) = 0$. The lifts $J^r F_E \xi$, $J^r F_E \eta$, and $J^r F_E \zeta$ satisfy (6.3.1),

$$J^r F_E \zeta = [J^r F_E \xi, J^r F_E \eta]. \tag{6.3.11}$$

Each of the vector fields $J^r F_E \xi$, $J^r F_E \eta$, and $J^r F_E \zeta$ is tangent to the fiber $(\pi^r_{R^n, E})^{-1}(0)$ since it projects onto the zero vector at $0 \in R^n$; consequently, relation (6.3.11) holds when we replace these vector fields by their restrictions to $(\pi^r_{R^n, E})^{-1}(0)$. We shall express both sides of (6.3.11) in terms of the coordinates x^i in the same way as in (6.3.3).

Let us consider the vector field $J^r F_E \zeta$. We have on $(\pi^r_{R^n, E})^{-1}(0)$

$$J^r F_E \zeta = \sum_{s=1}^{r+1} \left\{ \frac{\partial^s \zeta^i}{\partial x^{k_1} \ldots \partial x^{k_s}} \right\}_0 \vartheta_i^{k_1 \ldots k_s}. \tag{6.3.12}$$

By (6.3.10) and (6.3.6)

$$(6.3.13)\qquad J^rF_E\zeta = \sum_{s=1}^{r+1}\left(\sum_{m=0}^{s}\binom{s}{m}\left(\frac{\partial^{s-m}\xi^k}{\partial x^{k_1}\dots\partial x^{k_{s-m}}}\frac{\partial^{m+1}\eta^l}{\partial x^{k_{s-m+1}}\dots\partial x^{k_s}\partial x^k} - \frac{\partial^{s-m}\eta^k}{\partial x^{k_1}\dots\partial x^{k_{s-m}}}\frac{\partial^{m+1}\xi^l}{\partial x^{k_{s-m+1}}\dots\partial x^{k_s}\partial x^k}\right)\right)\vartheta_l^{k_1\dots k_s},$$

where all the derivatives are considered at the point 0. We set in the first term $s' = s - m$, $t = m + 1$, substitute these indices in (6.3.13), and then replace s' back by s. Then the first term gives the expression

$$(6.3.14)\qquad \sum_{2\leq s+t\leq r+2}\binom{s+t-1}{t-1}\frac{\partial^s\xi^l}{\partial x^{k_1}\dots\partial x^{k_s}}\frac{\partial^t\eta^j}{\partial x^{p_1}\dots\partial x^{p_t}}\delta_l^{p_t}\bar\vartheta_j^{k_1\dots k_s p_1\dots p_{t-1}} = \sum_{2\leq s+t\leq r+2}\binom{s+t-1}{t-1}\frac{1}{t}\frac{\partial^s\xi^l}{\partial x^{k_1}\dots\partial x^{k_s}}\frac{\partial^t\eta^j}{\partial x^{p_1}\dots\partial x^{p_t}}\cdot(\delta_l^{p_1}\bar\vartheta_j^{k_1\dots k_s p_2\dots p_t} + \dots + \delta_l^{p_t}\bar\vartheta_j^{k_1\dots k_s p_1\dots p_{t-1}}).$$

Notice that $\binom{s+t-1}{t-1}\frac{1}{t} = \frac{(s+t-1)!}{s!\,t!}$. In the second term we set $t = s - m$, $s' = m + 1$, substitute in (6.3.13), and then replace s' by s. The second term gives, up to the minus sign,

$$(6.3.15)\qquad \sum_{2\leq s+t\leq r+2}\binom{s+t-1}{s-1}\frac{\partial^s\xi^l}{\partial x^{k_1}\dots\partial x^{k_s}}\frac{\partial^t\eta^j}{\partial x^{p_1}\dots\partial x^{p_t}}\delta_j\bar\vartheta_l^{p_1\dots p_t k_1\dots k_{s-1}} = \sum_{2\leq s+t\leq r+2}\binom{s+t-1}{s-1}\frac{1}{s}\frac{\partial^s\xi^l}{\partial x^{k_1}\dots\partial x^{k_s}}\frac{\partial^t\eta^j}{\partial x^{p_1}\dots\partial x^{p_t}}\cdot(\delta_j^{k_1}\bar\vartheta_l^{k_2\dots k_s p_1\dots p_t} + \dots + \delta_j^{k_s}\bar\vartheta_l^{k_1\dots k_{s-1} p_1\dots p_t}),$$

where we used the symmetry of $\bar\vartheta_j^{i_1\dots i_m}$ in the superscripts. Notice that $\binom{s+t-1}{s-1}\frac{1}{s} = \frac{(s+t-1)!}{s!\,t!}$.

On the other hand, to compute the right-hand side expression in (6.3.11) we use the representations of the form (6.3.3) for both $J^rF_E\xi$ and $J^rF_E\eta$. We get

$$(6.3.16)\qquad [J^rF_L\xi, J^rF_L\eta] = \sum_{s=1}^{r+1}\sum_{t=1}^{r+1}\frac{\partial^s\xi^l}{\partial x^{k_1}\dots\partial x^{k_s}}\frac{\partial^t\eta^j}{\partial x^{p_1}\dots\partial x^{p_t}}\cdot[\bar\vartheta_l^{k_1\dots k_s}, \bar\vartheta_j^{p_1\dots p_t}],$$

where the derivatives are considered at the point $0 \in R^n$. We divide summation in this expression into two parts, according to whether s and t satisfy $2 \leqq s + t \leqq r + 2$, or $r + 3 \leqq s + t \leqq 2(r + 1)$. Then applying (6.3.11) to (6.3.13), (6.3.14), (6.3.15) and (6.3.16), and comparing the coefficients at different derivatives of ξ^i and η^j to zero, we obtain (6.3.8).

Example 6.1. To be explicit, we shall write down the commutation relations (6.3.8) for the case $r = 2$, i.e. for the fundamental vector fields ϑ_k^l, ϑ_k^{ij}, ϑ_k^{ijm}, spanning the Lie algebra $L(T_n^2E)$. In this case we have

$$
\begin{aligned}
&[\vartheta_i^k, \vartheta_j^p] = \delta_i^p\vartheta_j^k - \delta_j^k\vartheta_i^p,\\
&[\vartheta_i^{km}, \vartheta_j^p] = \delta_i^p\vartheta_j^{km} - \delta_j^k\vartheta_i^{mp} - \delta_j^m\vartheta_i^{kp},\\
&[\vartheta_i^{km}, \vartheta_j^{pq}] = \frac{3}{2}(\delta_i^p\vartheta_j^{kmq} + \delta_i^q\vartheta_j^{kmp} - \delta_j^k\vartheta_i^{mpq} - \delta_i^m\vartheta_j^{kpq}),\\
&[\vartheta_i^{kmq}, \vartheta_j^p] = \delta_i^p\vartheta_j^{kmq} - \delta_j^k\vartheta_i^{mqp} - \delta_j^m\vartheta_i^{kqp} - \delta_j^q\vartheta_i^{kmp},\\
&[\vartheta_i^{kmq}, \vartheta_j^{pr}] = 0,\\
&[\vartheta_i^{kmq}, \vartheta_j^{prs}] = 0.
\end{aligned}
\tag{6.3.17}
$$

7. THE STRUCTURE OF DIFFERENTIAL GROUPS

It has been shown in the previous sections that the most important notion for a systematic study of differential invariants is the notion of a differential group. In this chapter we determine the structure constants of a differential group, and consider the problem of generating its Lie algebra by its (minimal) vector subspaces. Further we determine all normal subgroups of the differential group which correspond to the well-known extent with all homomorphisms of this group into other Lie groups. Finally, we discuss a simple method of determining differential invariants with values in $GL_n(R)$ – manifolds by means of "absolute" invariants of some of its normal subgroups, the kernel of the canonical homomorphism of L_n^r onto L_n^1. Basic notions of this chapter are the following: Structure constants, differential group, fundamental vector fields, jet prolongation of a projectable vector field, Lie algebra of a differential group, normal Lie subgroup, ideal of a Lie algebra.

7.1. Structure constants of a differential group. We shall now apply the results of Sec. 6.3 to the problem of determining the structure constants of differential groups.

Let ξ be a vector field, defined on a neighborhood of $0 \in R^n$, such that $\xi(0) = 0$. ξ determines the $(r + 1)$-jet $J_0^{r+1}\xi$ which belongs to the Lie algebra $\Gamma_{(0,0)}^{r+1}TR^n$, introduced in Sec. 3.1, isomorphic with the Lie algebra $L(L_n^{r+1})$ of the group L_n^{r+1}; the bracket operation in $\Gamma_{(0,0)}^{r+1}TR^n$ is defined by (3.2.27), and an isomorphism $\nu : \Gamma_{(0,0)}^{r+1}TR^n \to L(L_n^{r+1})$ is defined by (3.2.11). Let P be a left L_n^1-manifold, $\Phi : L_n^{r+1} \times T_n^rP \to T_n^rP$ the induced left action of L_n^{r+1} on T_n^rP, and denote by $\Phi^r(J_0^{r+1}\xi)$ the fundamental vector field on T_n^rP, associated with the vector $\nu(J_0^{r+1}\xi) \in L(L_n^{r+1})$ (Sec. 1.3). Recall that for any $q \in T_n^rP$, $\Phi^r(J_0^{r+1}\xi)\,(q)$ is defined by

$$\Phi^r(J_0^{r+1}\xi)\,(q) = T_e\Phi_q \,.\, \nu(J_0^{r+1}\xi), \tag{7.1.1}$$

and is an element of the tangent space $T_qT_n^rP$.

Let X be an n-dimensional manifold, $x \in X$ a point, (U, φ) a chart on X such that $x \in U$, $\varphi(x) = 0$. Restricting U if necessary we may suppose that there exists a vector field ξ_φ on U such that ξ and ξ_φ are φ-related,

$$\xi_\varphi = (T\varphi^{-1} \,.\, \xi) \circ \varphi. \tag{7.1.2}$$

Put

$$\zeta = J_0^{r+1}\varphi^{-1}; \tag{7.1.3}$$

$\zeta \in F^{r+1}X$ is an $(r+1)$-frame at the point x. Fix a point $q \in T_n^rP$ and put $Z = [\zeta, q]$; Z is an r-jet from the space J^rF_PX over the point x. Let $J^rF_P\xi_\varphi$ be the r-jet prolongation of the lift $F_P\xi_\varphi$ of the vector field ξ_φ, and let $\varkappa_\zeta$ be the framing defined by the $(r+1)$-frame ζ.

Lemma 7.1. *The fundamental vector field* $\Phi^r(J_0^{r+1}\xi)$ *admits the representation*

$$\Phi^r(J_0^{r+1}\xi) = (T\varkappa_\zeta \,.\, J^rF_P\xi_\varphi) \circ \varkappa_\zeta^{-1}. \tag{7.1.4}$$

Proof. It is enough to show that

$$T_Z\varkappa_\zeta \,.\, J^rF_P\xi_\varphi(Z) - T_e\Phi_q \,.\, \nu(J_0^{r+1}\xi) = 0 \tag{7.1.5}$$

for any $q \in T_n^rP$ and $Z = [\zeta, q] = \varkappa^{-1}(q)$. Since $T_P\varkappa_\zeta$ is a linear isomorphism, this relation is equivalent to

$$J^rF_P\xi_\varphi(Z) - T_q\varkappa_\zeta^{-1} \circ T_e\Phi_q \,.\, \nu(J_0^{r+1}\xi) = 0. \tag{7.1.6}$$

Let α_t be the local one-parameter group of ξ. Then $J_0^{r+1}\xi = J_0^{r+1}(\mathrm{d}\alpha_t/\mathrm{d}t)_0$ and, by (3.2.11),

$$\nu(J_0^{r+1}\xi) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J_0^{r+1}\alpha_t\right\}_0. \tag{7.1.7}$$

This implies

$$T_q\varkappa_\zeta^{-1} \,.\, T_e\Phi_q \,.\, \nu(J_0^{r+1}\xi) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} \varkappa_\zeta^{-1} \circ \Phi_q(J_0^{r+1}\alpha_t)\right\}_0. \tag{7.1.8}$$

Denote $Q = T_n^rP$. Then

$$\varkappa_\zeta^{-1}\circ\Phi_q(J_0^{r+1}\alpha_t) = [\zeta, \Phi_q(J_0^{r+1}\alpha_t)] = [\zeta, J_0^{r+1}\alpha_t \,.\, q] =$$

(7.1.9) $$= [\zeta \,.\, J_0^{r+1}\alpha_{t,q}] = [J_x^{r+1}(\varphi^{-1}\alpha_t\varphi)\circ J_0^{r+1}\varphi^{-1}, q] =$$

$$= [F^{r+1}(\varphi^{-1}\alpha_t\varphi)(\zeta), q] = F_Q^{r+1}(\varphi^{-1}\alpha_t\varphi)(Z),$$

where $x = \varphi^{-1}(0)$. Since α_t is the one-parameter group of ξ, $\varphi^{-1}\alpha_t\varphi$ is the one-parameter group of the vector field ξ_φ. Using (5.5.16) we now get

(7.1.10) $$F_Q^{r+1}(\varphi^{-1}\alpha_t\varphi)(Z) = J^r F_P(\varphi^{-1}\alpha_t\varphi)(Z),$$

and

(7.1.11) $$\left\{\frac{\mathrm{d}}{\mathrm{d}t}\varkappa_\zeta^{-1}\circ\Phi_q(J_0^{r+1}\alpha_t)\right\}_0 = \left\{\frac{\mathrm{d}}{\mathrm{d}t}J^r F_P(\varphi^{-1}\alpha_t\varphi)(Z)\right\}_0 = J^r F_P \xi_\varphi(Z).$$

This proves (7.1.6) and at the same time Lemma 7.1.

Let us consider the tangent space $T_e L_n^{r+1}$ to the group L_n^{r+1} at the identity element. Let $a_{j_1}^i, a_{j_1 j_2}^i, \ldots, a_{j_1 j_2 \ldots j_{r+1}}^i$ be the canonical coordinates on L_n^{r+1}; the vectors $(\partial/\partial a_{j_1}^i)_e$, $(\partial/\partial a_{j_1 j_2}^i)_e$, $\ldots$, $(\partial/\partial a_{j_1 j_2 \ldots j_{r+1}}^i)_e$, where $1 \leqq i \leqq n$, $1 \leqq j_1 \leqq \ldots$ $\ldots, \leqq j_{r+1} \leqq n$, form a basis of $T_e L_n^{r+1}$. We put

(7.1.12) $$\lambda_i^{j_1 \ldots j_s} = \frac{1}{N(j_1 \ldots j_s)}\left(\frac{\partial}{\partial a_{j_1 \ldots j_s}^i}\right)_e.$$

The vectors $\lambda_i^{j_1 \ldots j_s}$, where $1 \leqq i \leqq n$, $1 \leqq s \leqq r+1$, $1 \leqq j_1 \leqq \ldots, \leqq j_s \leqq n$, form a basis of $T_e L_n^{r+1}$, called the *canonical basis.* One can easily find vector fields $\xi_i^{j_1 \ldots j_s}$ on R^n such that $\xi_i^{j_1 \ldots j_s}(0) = 0$, and

(7.1.13) $$\lambda_i^{j_1 \ldots j_s} = \nu(J_0^{r+1}\xi_i^{j_1 \ldots j_s}).$$

Clearly, it is sufficient to put in the canonical coordinates x^i on R^n

$$\xi_i^{j_1 \ldots j_s} = \xi^k \frac{\partial}{\partial x^k},$$

(7.1.14) $$\xi^i(x^1, \ldots, x^n) = \frac{1}{N(j_1 \ldots j_s)} x^{j_1} x^{j_2} \ldots x^{j_s},$$

$$\xi^k(x^1, \ldots, x^n) = 0, \qquad k \neq i$$

(no summation in the second formula). Then $\xi_i^{j_1 \ldots j_s} = (\mathrm{d}\chi_t/\mathrm{d}t)_0$, where

(7.1.15) $$x^i \circ \chi_t(x^1, \ldots, x^n) = x^i + t \,.\, \frac{1}{N(j_1 \ldots j_s)} x^{j_1} x^{j_2} \ldots x^{j_s},$$

$$x^k \circ \chi_t(x^1, \ldots, x^n) = x^k, \qquad k \neq i,$$

and the curve $t \to J_0^{r+1}\chi_t$ in L_n^{r+1} is given by the equations

$$a^j_j(J^{r+1}_0\chi_t) = \delta^i_j,$$

(7.1.16)
$$a^i_{j_1\dots j_s}(J^{r+1}_0\chi_t) = \frac{1}{N(j_1 \dots j_s)}\, t,$$

$$a^k_{p_1\dots p_m}(J^{r+1}_0\chi_t) = 0,$$

where the last equation holds in all cases when the ordered system of integers $(k; p_1, \dots, p_m)$ differs from $(i; j_1, \dots, j_s)$. Thus, by (3.2.11),

(7.1.17)
$$v(J^{r+1}_0\xi^{j_1\dots j_s}_i) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} J^{r+1}_0\chi_t\right\}_0 = \lambda^{j_1\dots j_s}_i$$

as required.

In particular, the vectors $J^{r+1}_0\xi^{j_1\dots j_s}_i$ form a basis of the vector space $\Gamma^{r+1}_{(0,0)}TR^n$.

Let E denote the vector space R^n with its canonical L^1_n-module structure. Recall that in the canonical coordinates a^i_j (resp. ξ^i) on L^1_n (resp. R^n) the left action $(A, p) \to$ $\to A\,.\,p$ of L^1_n on R^n is defined by the formula

(7.1.18)
$$\xi^i(A\,.\,p) = a^i_j(A)\,.\,\xi^j(q).$$

We shall apply Lemma 7.1 and the above remarks to the case when $X = R^n$, $P = E$, and $\zeta = J^{r+1}_0\varphi^{-1}$, where $\varphi = \mathrm{id}_{R^n}$. Since in this case $\xi_\varphi = \xi$, (7.1.4) gives

(7.1.19)
$$\Phi'(J^{r+1}_0\xi) \circ \varkappa_\zeta = T\varkappa_\zeta\,.\,J^rF_E\xi,$$

and we see that the vector fields $\Phi^r(J^{r+1}_0\xi)$ and $J^rF_E\xi$ are $\varkappa_\zeta$-related. In fact, (7.1.19) can be used as the natural identification of these vector fields. In the following lemma we successively choose for ξ the vector fields $\xi^{j_1\dots j_s}_i$ (7.1.14).

Lemma 7.2. *For any integers i, s, and any sequence of integers $(j_1, \dots, j_s)$ such that* $1 \leqq i \leqq n$, $1 \leqq s \leqq r + 1$, $1 \leqq j_1 \leqq \dots j_s \leqq n$

(7.20)
$$\Phi'(J^{r+1}_0\xi^{j_1\dots j_s}_i) = \vartheta^{j_1\dots j_s}_i.$$

Proof. Let ξ be a vector field on a neighborhood of $0 \in R^n$ such that $\xi(0) = 0$. By (7.3.3) and (7.3.5) we have along the fiber in $J^rF_ER^n$ over the point 0

(7.1.21)
$$T\varkappa_\zeta\,.\,J^rF_E\xi = \sum_{s=1}^{+1} \left\{\frac{\partial^s\xi^i}{\partial x^{j_1}\dots\partial x^{j_s}}\right\}_0 .\, \vartheta^{j_1\dots j_s}_i \circ \varkappa_\zeta.$$

Substituting in this relation $\xi = \xi^{j_1\dots j_s}_i$ and using (7.1.19) we get (7.1.20).

Let E be as above, an consider the prolongation T^r_nE of E. T^r_nE is endowed with the structure of a left $(L^1_n)^r_n$-manifold (Sec. 5.2), and can also be viewed as an L^{r+1}_n-module (Theorem 5.1). Let $\xi^i, \xi^i_{j_1}, \dots, \xi^{j_1\dots j_r}_i$ denote the coordinates on T^r_nE

associated with the canonical coordinates ξ^i on $E = R^n$, and let the mapping $(A, q) \to \Phi(A, q)$ denote the action of L_n^{r+1} on $T_n^r E$. It is directly seen that the left action of L_n^1 on E (7.1.18) is effective; we shall show that the same is true for Φ.

Lemma 7.3. *The action Φ of L_n^{r+1} on $T_n^r E$ is effective.*

Proof. Let us rewrite the action (7.1.18) of L_n^1 on E in the form

$$\bar{\xi}^i = a^i_j \xi^j, \tag{7.1.22}$$

and define functions b^i_j on L_n^{r+1} by the formula $a^i_p b^p_j = \delta^i_j$ (see (2.2.3)). Differentiating formally (7.1.22) we obtain the coordinate expressions for the action Φ in the form

$$\begin{aligned} \bar{\xi}^p &= a^p_s \xi^s, \\ \bar{\xi}^p_q &= a^p_{rs} b^r_q \xi^s + a^p_s b_q \xi^s, \\ \bar{\xi}^p_{qm} &= a^p_{rst} b^t_m b_q \xi^s - a^p_{rs} a^t_{uv} b^v_q b^u_m b_t \xi^s + \\ &- (a^p_{rs} b_q b^k_m + a^p_{us} b^u_m b^k_q - a^p_s a^t_{uv} b^v_q b^u_m b^k_t)\, \xi^s_k + a^p_s b^u_m b^k_q \xi^s_{uk}, \end{aligned} \tag{7.1.23}$$

etc. Requiring $\bar{\xi}^i = \xi^i$, $\bar{\xi}^i_j = \xi^i_j$, …, $\bar{\xi}^i_{j_1 \ldots j_r} = \xi^i_{j_1 \ldots j_r}$ for all points q of $T_n^r E$ we immediately obtain

$$a^i_{j_1} = \delta^i_{j_1}, \; a^i_{j_1 j_2} = 0, \; \ldots, \; a^i_{j_1 \ldots j_{r+1}} = 0. \tag{7.1.24}$$

Since the only point satisfying these equations is the identity element $J_0^{r+1} \mathrm{id}_{R^n} \in L_n^{r+1}$, the action Φ of L_n^{r+1} on $T_n^r E$ is effective.

In the following theorem we determine the structure constants of the Lie algebra $L(L_n^r)$ in the canonical basis (7.1.12).

Theorem 7.1. *Let $(\lambda_i^{j_1 \ldots j_s})$, $1 \leqq i \leqq n$, $1 \leqq s \leqq r$, $1 \leqq j_1 \leqq \ldots \leqq j_s \leqq n$, be the canonical basis of the Lie algebra $L(L_n^r)$. Then*

$$\begin{aligned} [\lambda_i^{k_1 \ldots k_s}, \lambda_j^{p_1 \ldots p_t}] &= \frac{(s+t-1)!}{s!\,t!} (\delta_j^{k_1} \lambda_i^{k_2 \ldots k_s p_1 \ldots p_t} + \\ &+ \ldots + \delta_j^{k_s} \lambda_i^{k_1 \ldots k_{s-1} p_1 \ldots p_t} - \delta_i^{p_1} \lambda_j^{k_1 \ldots k_s p_2 \ldots p_t} - \\ &- \ldots - \delta_i^{p_t} \lambda_j^{k_1 \ldots k_s p_1 \ldots p_{t-1}}), \quad 1 \leqq s, t \leqq r, \quad 2 \leqq s+t \leqq r+1, \\ [\lambda_i^{k_1 \ldots k_s}, \lambda_j^{p_1 \ldots p_t}] &= 0, \quad 1 \leqq s, t \leqq r, \quad r+2 \leqq s+t \leqq 2r. \end{aligned} \tag{7.1.25}$$

Proof. Since $\nu : \Gamma^r_{(0,0)} TR^n \to L(L_n^r)$ is a linear isomorphism (Lemma 3.2), the elements of the canonical basis of $L(L_n^r)$ obey the same commutation relations as the vectors $J_0^r \xi_i^{j_1 \ldots j_s}$ in $\Gamma^r_{(0,0)} TR^n$. Let us consider the L_n^1-module E and the action Φ

of L_n^r on $T_n^{r-1}E$ expressed by (7.1.23). By Lemma 7.3 this action is effective which implies that the mapping $J_0^r\xi \to -\Phi'(J_0^r\xi)$ of $\Gamma^r_{(0,0)}TR^n$ onto the Lie algebra of fundamental vector fields on $T_n^{r-1}E$ is an isomorphism of Lie algebras (Theorem 1.13, Corollary 2). Using (1.3.36) and Lemma 7.2 we get

$$\begin{aligned} &\Phi'(\{J_0^r\xi_i^{k_1\ldots k_s}, J_0^r\xi_j^{p_1\ldots p_t}\}) = \\ (7.1.26)\qquad &= -[\Phi'(J_0^r\xi_i^{k_1\ldots k_s}), \Phi'(J_0\xi_j^{p_1\ldots p_t})] = \\ &= -[\vartheta_i^{k_1\ldots k_s}, \vartheta_j^{p_1\ldots p_t}]. \end{aligned}$$

Now we apply Theorem 6.3, substitute $\Phi'(J_0^r\xi_i^{j_1\ldots j_s})$ instead of $\vartheta_i^{j_1\ldots j_s}$, and apply $(\Phi')^{-1}$ on both sides; we obtain some commutation relations for the elements $J_0^r\xi_i^{j_1\ldots j_s}$. Writing $\lambda_i^{j_1\ldots j_s}$ instead of $J_0^r\xi_i^{j_1\ldots j_s}$, which is possible because of the existence of the isomorphism ν, we obtain (7.1.25) as desired.

Example 7.1. If $r = 1$ then (7.1.25) are the well-known commutation relations for elements of the canonical basis of the Lie algebra $gl_n(R)$ of the general linear group $GL_n(R)$,

$$(7.1.27)\qquad [\lambda_i^k, \lambda_j^p] = \delta_j^k\lambda_i - \delta_i^p\lambda_j^k.$$

Example 7.2. Let us consider the case $n = 1$, and denote

$$(7,1,28)\qquad \lambda^{(p)} = \lambda_1^{11\ldots1}$$

(p superscripts equal to 1), Relations (7.1.25) become

$$\begin{aligned} &[\lambda^{(s)}, \lambda^{(t)}] = \frac{(s+t+1)!}{s!\,t!}(s-t)\,.\,\lambda^{(s+t-1)}, \qquad 1 \leqq s, t \leqq r, \\ (7.1.29)\qquad &2 \leqq s+t \leqq r+1, \\ &[\lambda^{(s)}, \lambda^{(t)}] = 0, \qquad 1 \leqq s, t \leqq r, \qquad r+2 \leqq s+t \leqq 2r. \end{aligned}$$

If, moreover, $r = 4$, we get

$$\begin{aligned} &[\lambda^{(1)}, \lambda^{(2)}] = -\lambda^{(2)}, \qquad [\lambda^{(1)}, \lambda^{(3)}] = -2\lambda^{(3)}, \\ (7.1.30)\qquad &[\lambda^{(1)}, \lambda^{(4)}] = -3\lambda^{(4)}, \qquad [\lambda^{(2)}, \lambda^{(3)}] = -2\lambda^{(4)}, \\ &[\lambda^{(2)}, \lambda^{(4)}] = 0, \qquad [\lambda^{(3)}, \lambda^{(4)}] = 0. \end{aligned}$$

Corollary 1. *Let $\pi_n^{r,s} : L_n^r \to L_n^s$, $1 \leqq s \leqq r$, be the canonical homomorphism of groups. For each s, the kernel $K_n^{r,s} = \ker \pi_n^{r,s}$ is a normal nilpotent subgroup L_n^r.*

Proof. Let $k_n^{r,s}$ denote the Lie algebra of $K_n^{r,s}$; it is sufficient to show that the Lie algebra $k_n^r = k_n^{r,0}$ is nilpotent, for then all its subalgebras are nilpotent. The Lie algebra k_n^r is spanned by the vectors $(\partial/\partial a^i_{j_1\ldots j_m})_e$, where $2 \leqq m \leqq r$, $1 \leqq j_1 \leqq \ldots$

$\ldots \leqq j_m \leqq n$. Using the vectors $\lambda_i^{j_1 \ldots j_m}$ (7.1.12) instead of these vectors, and Theorem 7.1, (7.1.25), we can see at once that

$$(7.1.31) \qquad [k_n^r, k_n^r] \subset k_n^{r,1}, [k_n^r, k_n^{r,1}] \subset k_n^{r,2}, \ldots, [k_n^r, k_n^{r,r-1}] \subset k_n^{r,r} = \{0\}.$$

This implies that the Lie algebra k_n^r is nilpotent.

7.2. Vector spaces generating the Lie algebra of a differential group. We shall apply the commutation relations from Theorem 7.1 to the problem of finding *minimal vector subspaces* of the Lie algebra $L(L_n^r)$ *generating* the whole Lie algebra.

Consider the canonical basis $\lambda_i^{k_1}, \lambda_i^{k_1 k_2}, \ldots, \lambda_i^{k_1 \ldots k_r}$ of the Lie algebra $L(L_n^r)$, and denote

$$(7.2.1) \qquad \lambda_i^{(p)} = \lambda_i^{ii \ldots i},$$

where p denotes the number of the superscripts i; for $n = 1$ we denote, as in Example 7.2,

$$(7.2.2) \qquad \lambda^{(p)} = \lambda_1^{(p)}.$$

We have the following result.

Theorem 7.2. (*a*) *Let $n = 1$. Then the Lie algebra $L(L_1^2)$ is generated by the vectors $\lambda^{(1)}, \lambda^{(2)}$. For $r \geqq 3$ the Lie algebra $L(L_1^r)$ is generated by the vectors $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$.*

(*b*) *Let $n \geqq 2$, and let s be any integer such that $1 \leqq s \leqq n$. Then for any integer $r \geqq 2$ the Lie algebra $L(L_n^r)$ is generated by the vectors λ_i^k, $1 \leqq i, k \leqq n$, and $\lambda_s^{(2)}$.*

Proof. (a) If $n = 1$, the vectors $\lambda^{(1)}, \ldots, \lambda^{(r)}$ obey (7.1.29). For $r = 2$ we get $[\lambda^{(1)}, \lambda^{(2)}] = -\lambda^{(2)}$ proving the first assertion. For $r \geqq 3$ we take in (7.1.29) $t = 2$ and obtain for any $s \geqq 3$, $s \leqq r - 1$,

$$(7.2.3) \qquad \lambda^{(s+1)} = \frac{2}{(s+1) \cdot (s-2)} [\lambda^{(s)}, \lambda^{(2)}]$$

proving the second assertion.

(b) The proof of part (b) of Theorem 7.2 is divided into four steps. Suppose that $n \geqq 2$.

1. Consider the case $r = 2$. In this case (7.1.25) gives, for $s = 2$ and $t = 1$,

$$(7.2.4) \qquad [\lambda_i^{kq}, \lambda_j^p\ ] = \delta_j^k \lambda_i^{qp} + \delta_j^q \lambda_i^{kp} - \delta_i^p \lambda_j^{kq}.$$

Fix an integer q, $1 \leqq q \leqq n$. Since $n \geqq 2$, there exists an integer $j \neq q$ such that $1 \leqq j \leqq n$. Take $i = k = p = q$. Then (7.2.4) gives $[\lambda_q^{qq}, \lambda_j^q] = -\lambda_j^{qq}$; we shall say that λ_j^{qq} is generated by λ_j^q and λ_q^{qq}. If $p \neq q$, $i = k = j = q$, we get $[\lambda_q^{qq}, \lambda_q^p] = 2\lambda_q^{qp}$, and λ_q^{qp}, $p \neq q$, is generated by λ_q^p and λ_q^{qq}. Now take $j = k = q$. Then (7.2.4) becomes $[\lambda_i^{qq}, \lambda_q^p] = 2\lambda_i^{qp} - \delta_i^p \lambda_q^{qq}$. In view of the above results this means that λ_i^{qp} is generated by λ_q^p, λ_q^{qq}, and λ_i^{qq} or, since λ_i^{qq} is generated by λ_i^q and λ_q^{qq}, by λ_q^p and λ_q^{qq}. Finally consider (7.2.4) with $j = q$. We obtain $\lambda_i^{kp} = [\lambda_i^{kq}, \lambda_q^p] - \delta_q^k \lambda_i^{qp} + \delta_i^p \lambda_q^{kq}$. Since all the vectors on the right contain the index q they are generated by λ_j^m and λ_q^{qq}, and so must be λ_i^{kd}. This proves that the vectors λ_j^m, $1 \leqq m, j \leqq n$, λ_q^{qq}, generate the Lie algebra $L(L_n^2)$.

2. Let $r \geqq 3$. We shall show that the Lie algebra $L(L_n^r)$ is generated by the vectors λ_i^j, λ_i^{jk}, and $\lambda_i^{(3)}$, where $1 \leqq i, j, k \leqq n$. Take in (7.1.25) $t = 2$. Then for any s, $3 \leqq s \leqq r - 1$, and any i, $1 \leqq i \leqq n$,

$$(7.2.5) \qquad [\lambda_i^{(s)}, \lambda_i^{(2)}] = \frac{(s+1)!}{2s!}(s-2)\,\lambda_i^{(s+1)}.$$

This implies that $\lambda_i^{(s+1)}$ is generated by $\lambda_i^{(2)}$ and $\lambda_i^{(3)}$.

Fix an integer i, $1 \leqq i \leqq n$. Since $n \geqq 2$, there exists an integer $p \neq i$, $1 \leqq p \leqq n$. Take in (7.1.25) $t = 2$, $k_1 = \ldots = k_s = p_1 = j = i$, $p_2 = p$. We get

$$(7.2.6) \qquad [\lambda_i^{(s)}, \lambda_i^{ip}] = \frac{(s+1)!}{2s!}(s-1)\,\lambda_i^{ii\ldots ip}$$

(s superscripts i on the right). Since by (7.2.5) $\lambda_i^{(s)}$ is generated by $\lambda_i^{(2)}$ and $\lambda_i^{(3)}$, the vector $\lambda_i^{ii\ldots ip}$, where $p \neq i$, is generated by λ_i^{ip}, $\lambda_i^{(2)}$, and $\lambda_i^{(3)}$ or, which is the same, by λ_i^{ik} and $\lambda_i^{(3)}$.

Similarly take in (7.1.25) $t = 2$, $p_1, p_2 \neq i$, and $k_s = j$. We obtain

$$(7.2.7) \qquad [\lambda_i^{k_1\ldots k_{s-1}j}, \lambda_j^{p_1p_2}] = \frac{(s+1)!}{2s!}(\delta_j^{k_1}\lambda_i^{k_2\ldots k_{s-1}jp_1p_2} + \ldots + \delta_j^{k_{s-1}}\lambda_i^{k_1\ldots k_{s-2}jp_1p_2} + \lambda_i^{k_1\ldots k_{s-1}p_1p_2}).$$

Summing over $j = 1, 2, \ldots, n$ on both sides we get

$$(7.2.8) \qquad \sum_{j=1}^{n} [\lambda^{k_1\ldots k_{s-1}j}, \lambda^{p_1p_2}] = \frac{(s+1)!}{2s!}(s-1+n)\,\lambda_i^{k_1\ldots k_{s-1}p_1p_2}.$$

This shows that the vectors $\lambda_i^{k_1\ldots k_{s-1}p_1p_2}$, where $p_1, p_2 \neq i$, are generated by the vectors $\lambda_j^{p_1p_2}$, $\lambda_i^{k_1\ldots k_s}$. In other words this says that whenever two of the superscripts in $\lambda_i^{k_1\ldots k_{s+1}}$ differ from the subscript i, $\lambda_i^{k_1\ldots k_{s+1}}$ is generated by $\lambda_j^{p_1p_2}$ and $\lambda_k^{q_1\ldots q_s}$. Now summarizing all the results of this part of the proof we see that the Lie algebra $L(L_n^r)$ is generated by the vectors λ_i^j, λ_i^{jk}, $\lambda_i^{(3)}$, $1 \leqq i, j, k \leqq n$.

3. Let us fix an integer q, $1 \leqq q \leqq n$. It is easily seen that the vectors λ_i^{jk} are generated by λ_i^k and $\lambda_q^{(2)}$. Taking $s = 2$ and $t = 1$ in (7.1.25) we obtain, as in the case $r = 2$, the commutators (7.2.4). Thus our assertion can be obtained in the same way as the analogous assertion proved in the first step of part (b) of this proof.

4. Let q be a fixed integer, $1 \leqq q \leqq n$. It remains to show that the vectors $\lambda_i^{(3)}$,

$1 \leqq i \leqq n$, are generated by λ_i^j and $\lambda_q^{(2)}$. Consider relation (7.1.25) for $s = 2$, $t = 2$, and $k_1 = k_2 = j = i$, $p_1 = p_2 = p \neq i$. We obtain $[\lambda_i^{(2)}, \lambda_i^{pp}] = 3\lambda_i^{ipp}$; hence by part 3 of this proof, λ_i^{ipp} is generated by $\lambda_k^i, \lambda_q^{(2)}$. Putting $k_1 = p_1 = p_2 = i$, $k_2 = j \neq i$ we obtain $[\lambda_i^{ij}, \lambda_j^{ii}] = (3/2)\,\lambda_i^{(3)} - 3\lambda_j^{iij}$. Since λ_j^{iij} is generated by $\lambda_k^i, \lambda_q^{(2)}$, the same is true for the vector $\lambda_i^{(3)}$. This completes the proof of Theorem 7.2.

7.3. The semi-direct product structure of a differential group and normal subgroups. Let $r \geqq 2$ be an integer, and consider the differential group L_n^r. As before, denote

(7.3.1) $$K_n^{r,s} = \ker \pi^{r,s},$$

where $\pi^{r,s} : L_n^r \to L_n^s$, $1 \leqq s \leqq r$, is the canonical jet projection. We also write $K_n^r = K_n^{r,1}$. Let $\iota^{s,r} : L_n^s \to L_n^r$ be a mapping defined, in the coordinates $b_{j_1}^i, b_{j_1 j_2}^i, \ldots, \ldots, b_{j_1 j_2 \ldots j_r}^i$ (2.2.3) on L_n^r by

(7.3.2) $$\iota^{s,r}(A) = (b_{j_1}^i(A), b_{j_1 j_2}^i(A), \ldots, b_{j_1 j_2 \ldots j_s}^i(A), 0, \ldots, 0).$$

If $s = 1$, we denote $\iota^{1,r} = \iota^r$.

Remark 7.1. If $s = 1$, $\iota^{1,r} = \iota^r$ is a homomorphism of groups. This immediately follows from the definition of ι^r and of the group multiplication in L_n^r. On the other hand, if $s \geqq 1$, $\iota^{r,sr}$ is not a homomorphism for by Theorem 7.2 the set $\iota^{s,r}(L_n^s) \subset L_n^r$ is not closed with respect to the group multiplication in L_n^r.

Theorem 7.3. *The differential group L_n^r is the interior semi-direct product of its Lie subgroups $\iota^r(L_n^1)$ and K_n^r.*

Proof. We apply Theorem 1.10 with $p = \pi^{r,1}$ and $s = \iota^r$.

Obviously, each of the groups $K_n^{r,s}$ is a normal Lie subgroup of L_n^r. Our aim now will be to describe all normal Lie subgroups of L_n^r. We begin by proving some lemmas.

Lemma 7.4. *Let $H \times_\varphi K$ be the exterior semi-direct product of Lie groups H and K, associated with a homomorphism $\varphi : H \to \operatorname{Aut} K$, let $K_0 \subset K$ be a subset. Then the set $\{e_H\} \times K_0 \times_\varphi K$ has the structure of a normal Lie subgroup of the Lie group $H \times_\varphi K$ if and only if the following two conditions are satisfied:*

(1) *K_0 is a normal Lie subgroup of K,*
(2) *for each $h \in H$ and $k_0 \in K_0$, $\varphi(h)\,(k_0) \in K_0$.*

Proof. 1. Let $\{e_H\} \times K_0$ be a normal Lie subgroup of $H \times_\varphi K$. Since for any $(h, k), (h_0, k_0) \in H \times_\varphi K$,

$$(7.3.3)\qquad \begin{aligned}&(h, k) . (h_0, k_0) . (h, k)^{-1} = \\ &= (hh_0h^{-1}, k . \varphi(h) (k_0) . (\varphi(hh_0h^{-1}) (k)^{-1}),\end{aligned}$$

Put $h_0 = e_H$ (the identity of H). Then $(h, k) . (e_H, k_0) . (h, k)^{-1} = (e_H, k . \varphi(h) (k_0) . . k^{-1})$ and by hypothesis, $k . \varphi(h) (k_0) . k^{-1} \in K_0$. If $h = e_H$ we get $k . k_0 . k^{-1} \in \in K_0$, and K_0 must be a normal subgroup of K. Since K_0 is a submanifold, condition (1) is satisfied. Now putting $h_0 = e_H$, $k = e_K$ (the identity of K) we get at once that $\varphi(h) (k_0) \in K_0$, and (2) also holds.

2. If conditions (1) and (2) hold, then for any $k_0 \in K_0$ and $(h, k) \in H \times K$ (7.3.3) gives $k . \varphi(h) (k_0) . k^{-1} \in K_0$. Thus $\{e_H\} \times K_0$ is a normal subgroup of $H \times_\varphi K$. Since K_0 is a submanifold of K, it is a normal Lie subgroup.

Let us return to the differential group L_n^r. This group can naturally be considered with the structure of the exterior semi-direct product of Lie groups L_n^1 and K_n^r associated with the homomorphism $\varphi : L_n^1 \to \operatorname{Aut} K_n^r$ defined by

$$(7.3.4)\qquad \varphi(A) (K) = \imath^r(A) \circ K \circ (\imath^r(A))^{-1},$$

(Theorem 1.6, Theorem 1.7), where $\circ$ means the composition of jets. Let $b^i_{j_1}$, $b^i_{j_1j_2}, \ldots, b^i_{j_1j_2\ldots j_r}$ be the global coordinates on L_n^r defined by (2.2.3). We shall determine the homomorphism φ in these coordinates. The subgroup K_n^r of L_n^r is defined by the equations

$$(7.3.5)\qquad b^i_j = \delta^i_j.$$

Notice that the expression on the right of (7.3.4) may be regarded as $J_0^r(\alpha\delta\alpha^{-1})$, where α is subject to the condition $J_0^r\alpha = \imath^r(J_0^1\alpha) = \imath^r(A)$, and $K = J_0^r\delta$. We get, using the definition of the coordinates $b^i_{j_1j_2}, \ldots, b^i_{j_1j_2\ldots j_r}$,

$$(7.3.6)\qquad \begin{aligned}&b^i_{j_1j_2}(\varphi(A)(K)) = b^i_q(A^{-1})\, b^q_{p_1p_2}(K)\, b^{p_1}_{j_1}(A)\, b^{p_2}_{j_2}(A),\\ &b^i_{j_1j_2j_3}(\varphi(A)(K)) = b^i_q(A^{-1})\, b^q_{p_1p_2p_3}(K)\, b^{p_1}_{j_1}(A)\, b^{p_2}_{j_2}(A)\, b^{p_3}_{j_3}(A),\\ &\ldots\\ &b^i_{j_1j_2\ldots j_r}(\varphi(A)(K)) = b^i_q(A^{-1})\, b^q_{p_1p_2\ldots p_r}(K)\, b^{p_1}_{j_1}(A)\, b^{p_2}_{j_2}(A) \ldots b^{p_r}_{j_r}(A)\end{aligned}$$

or, with the obvious convention,

$$(7.3.7)\qquad \begin{aligned}&\bar{b}^i_{j_1j_2} = a^i_q b^q_{p_1p_2} b^{p_1}_{j_1} b^{p_2}_{j_2},\\ &\bar{b}^i_{j_1j_2j_3} = a^i_q b^q_{p_1p_2p_3} b^{p_1}_{j_1} b^{p_2}_{j_2} b^{p_3}_{j_3},\\ &\ldots\\ &\bar{b}^i_{j_1j_2\ldots j_r} = a^i_q b^q_{p_1p_2\ldots p_r} b^{p_1}_{j_1} b^{p_2}_{j_2} \ldots b^{p_r}_{j_r}.\end{aligned}$$

Let $J_0^r\alpha \to \operatorname{Ad} J_0^r\alpha$ be the adjoint representation of the Lie group L_n^r in its Lie algebra $L(L_n^r)$. Recall that for each element $J_0^r\alpha \in L_n^r$, $\operatorname{Ad} J_0^r\alpha = T_e \operatorname{Int} J_0^r\alpha$, where e is the identity element of L_n^r, and $\operatorname{Int} J_0^n\alpha : L_n^r \to L_n^r$ is a mapping defined by

(7.3.8) $(\operatorname{Int} J_0^r\alpha)(J_0^r\beta) = J_0^r\alpha \circ J_0^r\beta \circ (J_0^r\alpha)^{-1} = J_0^r(\alpha\beta\alpha^{-1}).$

Since K_n^r is a normal subgroup of L_n^r, its Lie algebra k_n^r is invariant with respect to the adjoint representation: If $J_0^r\delta \in K_n^r$, then $(\operatorname{Int} J_0^r\alpha)(J_0^r\delta) \in K_n^r$ for any $J_0^r\alpha \in L_n^r$, hence $T_e \operatorname{Int} J_0^r\alpha$ is a mapping of k_n^r into itself. Hence there is the induced representation of L_n^r on k_n^r, and the induced representation of its subgroup $\imath^r(L_n^1)$ on k_n^r. Since the mapping $\varphi(A)$ (7.3.4) coincides with the restriction of Ing $J_0^r\alpha$ to K_n^r, under the condition that $J_0^r\alpha = \imath^r(J_0^r\alpha)$, the induced representation of L_n^1 on k_n^r, still denoted by Ad, may be obtained by differentiating the functions (7.3.6), or (7.3.7), with respect to $b^q_{p_1p_2}(K), \ldots, b^q_{p_1\ldots p_r}(K)$.

Let $\xi^i_{j_1j_2}, \ldots, \xi^i_{j_1j_2\ldots j_r}$, $1 \leq i \leq n$, $1 \leq j_1 \leq j_2 \leq \ldots \leq j_r \leq n$, be the coordinates on k_n^r, associated with the coordinates $b^i_{j_1}, b^i_{j_1j_2}, \ldots, b^i_{j_1j_2\ldots j_r}$ on L_n^r. In these coordinates

$$\begin{aligned} &\xi^i_{j_1j_2} \circ \operatorname{Ad} A = b^i_q(A)\, \xi^q_{p_1p_2} b^{p_1}_{j_1}(A)\, b^{p_2}_{j_2}(A), \\ (7.3.9)\qquad &\ldots \\ &\xi^i_{j_1\ldots j_r} \circ \operatorname{Ad} A = b^i_q(A^{-1})\, \xi^q_{0_1\ldots p_r} b^{p_1}_{j_1}(A) \ldots b^{p_r}_{j_r}(A) \end{aligned}$$

for any $A \in L_n^1$. It follows from these formulas that the Lie algebra k_n^r, considered as the L_n^1-module with respect to the adjoint representation, is isomorphic with the L_n^1-module

(7.3.10) $(R^n \otimes S^2R^{n*}) \oplus (R^n \otimes S^3R^{n*}) \oplus \ldots \oplus (R^n \otimes S^rR^{n*}),$

where $S^kR^{n*} \subset R^{n*} \otimes \ldots \otimes R^{n*}$ (k factors R^{n*}) denotes the submodule of symmetric tensors.

Lemma 7.5. *Let K_0 be a Lie subgroup of K_n^r, k_0 the Lie algebra of K_0. The following two conditions are equivalent.*

(1) *K_0 is a normal subgroup of K_n^r, and for each $A \in L_n^1$ and $K \in K_0$,* it holds $(\operatorname{Int} \imath^r(A))(K) \in K_0$.

(2) *k_0 is an ideal of k_n^r, and for each $A \in L_n^1$ and $\xi \in k_0$,* $(\operatorname{Ad} A)(\xi) \in k_0$.

Proof. Since K_n^r is simply connected, there exists a bijective correspondence between normal Lie subgroups of K_n^r and ideals of k_n^r. Suppose that (1) holds. Since $\operatorname{Ad} A = T_e \operatorname{Int} \imath^r(A)$ and by assumption $\operatorname{Int} \imath^r(A)$ maps K_0 into K_0, Ad A maps T_eK_0 into T_eK_0 which proves (2). Conversely, let Ad $\imath^r(A)$ maps k_0 into k_0. Since Ad $\imath^r(A)$ is an automorphism of k_0, there exists a unique homomorphism of Lie groups $\beta : K_0 \to K_0$ such that $T_e\beta = \operatorname{Ad} \imath^r(A)$; from the uniqueness arguments it follows that $\beta = \operatorname{Int} \imath^r(A)$.

We shall determine the Lie subalgebras k_0 of k_n^r satisfying the second condition of Lemma 7.5. We start with determining the vector subspaces k_0 invariant with respect to the action $(A, \xi) \to (\mathrm{Ad}\, A)(\xi)$ (7.3.9) of L_n^1 on k_n^r. Since this action coincides, in our coordinates, with the standard tensor representation of L_n^1 on the vector space (7.3.10), our problem consists in decomposing this tensor space into irreducible components which may be done by means of the classical Young–Kronecker theory of representations of the symmetric group. We shall apply, however, a different method based on invariant tensors and linear mappings associated with these tensors (Sec. 4.4).

Lemma 7.6. *Let E be a G-module, and let $\Phi : E \to E$ be a linear mapping. The following two conditions are equivalent:*

(1) *Φ is G-equivariant, i.e., $\Phi(g \,.\, q) = g \,.\, \Phi(q)$ for each $g \in G$ and $q \in E$.*

(2) *The vector subspace $\Phi(E) \subset E$, and its complementary vector subspace* $\ker \Phi$, *are G-invariant.*

Proof. 1. For each $q \in E$ define an element $\psi(q) \in E$ by the formula

$$q = \Phi(q) + \Psi(q). \tag{7.3.11}$$

Suppose that Φ is G-equivariant. Then for any $q' \in \Phi(E)$ and $g \in G$, $g \,.\, \Phi(q') = \Phi(g \,.\, q') \in \Phi(E)$, and $\Phi(E)$ is G-invariant. To show that the same is true for $\psi(E)$ it is enough to verify that Ψ is G-equivariant. For any $q \in E$ and $g \in G$, $\Psi(g \,.\, q) = g \,.\, q - \Phi(g \,.\, q) = g \,.\, (q - \Phi(q)) = g \,.\, \Psi(q)$ proving the G-equivariance of Ψ.

2. To prove the converse, take any element $q \in E$ and write it in the form (7.3.11). Then for any $g \in G$, $g \,.\, q = g \,.\, \Phi(q) + g \,.\, \Psi(q) = \Phi(g \,.\, q) + \Phi(g \,.\, q)$ and, since $g \,.\, \Phi(q) - \Phi(g \,.\, q) = \Psi(g \,.\, q) - g \,.\, \Psi(q)$ and this vector belongs to the complementary subspaces, it must be the zero vector. This proves that Φ is G-equivariant.

Our first aim is to investigate the *simple* (i.e. *irreducible*) L_n^1-submodules of the L_n^1-module k_n^r, and a decomposition of k_n^r into the direct sum of simple L_n^1-modules. Since an invariant decomposition is given by (7.3.10), we shall study each of its summands; we shall thus be looking for proper vector subspaces $k_0 \subset R^n \otimes S^i R^{n*}$, where $2 \leq i \leq r$, invariant with respect to the tensor representation of L_n^1, and such that there exists a complementary to k_0, L_n^1-invariant vector subspace of k_n^r. In fact, we shall be looking for all L_n^1-equivariant projections (Lemma 7.6).

Lemma 7.7. *Each L_n^1-equivariant projection $\pi : R^n \otimes S^k R^{n*} \to R^n \otimes S^k R^{n*}$ is expressed by the equations*

$$\bar{\xi}^m_{q_1 \ldots q_k} = t^m_{q_1 \ldots q_k} \xi_s^{p_1 \ldots p_k} \,.\, \xi^s_{p_1 \ldots p_k}, \tag{7.3.12}$$

where

$$\begin{aligned} t^m_{q_1 \ldots q_k} \delta_s^{p_1 \ldots p_k} = {} & A \delta_s^m \delta^p_{(q_1} \ldots \delta^p_{q_k)} + \\ & + B(\delta_s^{p_1} \delta^m_{(q_1} \delta^{p_2}_{q_2} \ldots \delta^{p_k}_{q_k)} + \ldots + \delta_s^{p_k} \delta^{p_1}_{(q_1} \ldots \delta^{p_{k-1}}_{q_{k-1}} \delta^m_{q_k)}), \end{aligned} \tag{7.3.13}$$

(*symmetrization in the indices in parentheses*), *and the coefficients* A, $B \in R$ *obey one of the following four possibilities:* (*a*) $A = 0$, $B = 0$, (*b*) $A = 0$, $B = 1/(n+k-1)$, (*c*) $A = 1$, $B = 0$, (*d*) $A = 1$, $B = -1/(n+k-1)$.

Proof. 1. Let $\pi : R^n \otimes S^k R^{n*} \to R^n \otimes S^k R^{n*}$ be an L^1_n-equivariant projection, (7.3.12) its expression in the canonical basis of $R^n \otimes S^k R^{n*}$. Then by the same arguments as in the proof of Theorem 4.5 $t = (t^{m}_{q_1 \ldots q_k s}{}^{p_1 \ldots p_k})$ is an absolute invariant tensor; we may suppose that this tensor is symmetric in the superscripts $p_1, \ldots, p_k$ and in the subscripts $q_1, \ldots, q_k$. Theorem 4.1 implies that t must have the form (7.3.13). We now apply the equality

$$t^{m}_{q_1 \ldots q_k}{}^{p_1 \ldots p_k}_{s} \cdot t^{s}_{p_1 \ldots p_k}{}^{i_1 \ldots i_k}_{j} = t^{m}_{q_1 \ldots q_k}{}^{i_1 \ldots i_k}_{j} \tag{7.3.14}$$

expressing that $\pi \circ \pi = \pi$, i.e., that π is a projection. This equality reads, since the symmetrization sign in one factor on the left may be omitted,

$$\begin{aligned} &[A\delta^m_s \delta^{p_1}_{(q_1} \ldots \delta^{p_k}_{q_k)} + B(\delta^{p_1}_s \delta^m_{(q_1} \delta^{p_2}_{q_2} \ldots \delta^{p_k}_{q_k)} + \ldots + \\ &+ \delta^{p_k}_s \delta^{p_1}_{(q_1} \ldots \delta^{p_{k-1}}_{q_{k-1}} \delta^m_{q_k)})] \cdot [A\delta^s_j \delta^{i_1}_{p_1} \ldots \delta^{i_k}_{p_k} + \\ &+ B(\delta^{i_1}_j \delta^s_{p_1} \delta^{i_2}_{p_2} \ldots \delta^{i_k}_{p_k} + \ldots + \delta^{i_k}_j \delta^{i_1}_{p_1} \ldots \delta^{i_{k-1}}_{p_{k-1}} \delta^s_{p_k}) = \\ &= A\delta^m_j \delta^{i_1}_{(q_1} \ldots \delta^{i_k}_{q_k)} + B(\delta^{i_1}_j \delta^m_{(q_1} \delta^{i_2}_{q_2} \ldots \delta^{i_k}_{q_k)} + \ldots + \\ &+ \delta^{i_k}_j \delta^{i_1}_{(q_1} \ldots \delta^{i_{k-1}}_{q_{k-1}} \delta^m_{q_k)}). \end{aligned} \tag{7.3.15}$$

After some calculation we obtain for the left-hand side the expression

$$\begin{aligned} &A^2 \delta^m_j \delta^{i_1}_{(q_1} \ldots \delta^{i_k}_{q_k)} + 2AB(\delta^{i_1}_j \delta^m_{(q_1} \delta^{i_2}_{q_2} \ldots \delta^{i_k}_{q_k)} + \ldots + \\ &+ \delta^{i_k}_j \delta^{i_1}_{(q_1} \ldots \delta^{i_{k-1}}_{q_{k-1}} \delta^m_{q_k)}) + B^2(n+k-1) \, . \\ &\cdot (\delta^{i_1}_j \delta^m_{(q_1} \delta^{i_2}_{q_2} \ldots \delta^{i_k}_{q_k)} + \ldots + \delta^{i_k}_j \delta^{i_1}_{(q_1} \ldots \delta^{i_{k-1}}_{q_{k-1}} \delta^m_{q_k)}). \end{aligned} \tag{7.3.16}$$

Comparing this expression with the right-hand side of (7.3.15) we get the equations

$$A^2 = A, \qquad 2AB + (n+k-1)B^2 - B = 0. \tag{7.3.17}$$

These equations have precisely the solutions (a)–(d) of Lemma 7.7.

2. To prove the converse we are to check that each of the mappings (7.3.15), where the coefficients are defined by (7.3.16) and A, B satisfy one of the possibilities (a)–(d), is an L^1_n-equivariant projection. By Theorem 4.1 and Theorem 7.5 all of these mappings are L^1_n-equivariant. Since the possibilities (a) and (c) obviously give projections and the sum of mappings (b) and (d) is the identity mapping of $R^n \otimes S^k R^{n*}$ it is sufficient to verify that (b) defines a projection. In this case we have the mapping expressed by

$$
(7.3.18)\qquad
\begin{aligned}
\bar{\xi}^m_{q_1\dots q_k} &= \frac{1}{n+k-1}(\delta^m_{(q_1}\delta^{p_2}_{q_2}\dots\delta^{p_k}_{q_k)}\xi^{p_1}_{p_1p_2\dots p_k} + \dots + \\
&+ \delta^{p_1}_{(q_1}\dots\delta^{p_k-1}_{q_k-1}\delta^m_{q_k)}\xi^{p_k}_{p_1\dots p_{k-1}p_k}) = \\
&= \frac{1}{n+k-1}(\delta^m_{q_1}\xi^j_{q_2\dots q_kj} + \dots + \delta^m_{q_k}\xi^j_{q_1\dots q_{k-1}j}).
\end{aligned}
$$

This implies that

$$
(7.3.19)\qquad \bar{\xi}^m_{q_1\dots q_{k-1}m} = \xi^m_{q_1\dots q_{k-1}m}.
$$

Now we apply this mapping to the point $\bar{\xi}^m_{q_1\dots q_k}$. Using (7.3.19) we get

$$
(7.3.20)\qquad
\begin{aligned}
\bar{\bar{\xi}}^i_{j_1\dots j_k} &= \frac{1}{n+k-1}(\delta^i_{j_1}\bar{\xi}^m_{j_2\dots j_km} + \dots + \delta^i_{j_k}\bar{\xi}_{j_1\dots j_{k-1}m}) = \\
&= \frac{1}{n+k-1}(\delta^i_{j_1}\xi^m_{j_2\dots j_km} + \dots + \delta^i_{j_k}\xi^m_{j_1\dots j_{k-1}m}) = \bar{\xi}^i_{j_1\dots j_k}.
\end{aligned}
$$

Thus the coefficients $A, B \in R$ satisfying (b) define a projection. This completes the proof.

Lemma 7.7 says that there are precisely four L^1_n-equivariant projections of $R^n \otimes S^kR^{n*}$ into itself. The first one, defined by the choice (a) $A = 0$, $B = 0$, is the projection onto the zero vector subspace, and the third one, (c) $A = 1$, $B = 0$, is the identity mapping of $R^n \otimes S^kR^{n*}$. We denote by π_1 (resp. π_2) the projection defined by the choice (d) $A = 1$, $B = -1/(n+k-1)$ (resp. (b) $A = 0$, $B = 1/(n+k-1)$), and we set

$$
(7.3.21)\qquad E^k_1 = \pi_1(R^n \otimes S^kR^{n*}), \qquad E^k_2 = \pi_2(R^n \otimes S^kR^{n*}).
$$

Corollary 1. *E^k_1 and E^k_2 are simple complementary submodules of the L^1_n-module $R^n \otimes S^kR^{n*}$, i.e.*

$$
7.3.22)\qquad R^n \otimes S^kR^{n*} = E^k_1 \oplus E^k_2, \qquad E^k_1 \cap E^k_2 = 0.
$$

Proof. This follows from Lemma 7.7.

We know that the Lie algebra k^r_n is isomorphic, as the L^1_n-module, with the L^1_n-module $(R^n \otimes S^2R^{n*}) \oplus \dots \oplus (R^n \otimes S^rR^{n*})$ (7.3.10). The linear isomorphism constructed above assigns to the vector $(\partial/\partial b^i_{j_1\dots j_k})_e \in k^r_n$ the element $(0, \dots, 0, e_i \otimes \left(\sum \frac{1}{k!} e^{j_{\sigma(1)}} \otimes \dots \otimes e^{j_{\sigma(k)}}\right), 0, \dots, 0)$ of $(R^n \otimes S^2R^{n*}) \oplus \dots \oplus (R^n \otimes S^rR^{n*})$, where e_i are vectors of the canonical basis of the vector space R^n, e^i are vectors of the dual basis or R^{n*}, and we sum over all permutations σ of the set $(1, 2, \dots, k)$. To simplify the notation we shall identify those two vectors and denote both of them by $(\partial/\partial b^l_{j_1\dots j_k})_e$. We shall also identify $(\partial/\partial b^l_{j_1\dots j_k})_e$ with an element of the vector space $R^n \otimes S^k R^{n*}$.

Let us consider the decomposition (7.3.22) of $R^n \otimes S^k R^{n*}$. Let $\xi \in R^n \otimes S^k R^{n*}$ be any vector. We have

$$
\begin{aligned}
\xi &= \sum_{q_1+\dots+q_k} \xi^m_{q_1\dots q_k}\left(\frac{\partial}{\partial b^m_{q_1\dots q_k}}\right)_e = \\
&= \sum_{q_1+\dots+q_k} \xi^m_{q_1\dots q_k} \sum_{j_1+\dots+j_k} \left\{\frac{\partial a^l_{j_1\dots j_k}}{\partial b^m_{q_1\dots q_k}}\right\}_e \left(\frac{\partial}{\partial a^l_{j_1\dots j_k}}\right)_e = \\
&= -\sum_{q_1+\dots+q_k} \xi^m_{q_1\dots q_k}\left(\frac{\partial}{\partial a^m_{q_1\dots q_k}}\right)_e = \\
&= -\sum_{q_1,\dots,q_k} \frac{1}{N(q_1\dots q_k)}\xi^m_{q_1\dots q_k}\left(\frac{\partial}{\partial a^m_{q_1\dots q_k}}\right)_e = \\
&= -\sum_{q_1,\dots,q_k} \xi^m_{q_1\dots q_k}\lambda^{q_1\dots q_k}_m = -\xi^m_{q_1\dots q_k}\lambda^{q_1\dots q_k}_m,
\end{aligned}
\tag{7.3.23}
$$

where we have used the transformation equations for the change of coordinates at the point $e \in L^r_n$ (compare with (2.2.5)), i.e. we have taken $a^l_j = \delta^l_j$, $b^l_j = \delta^l_j$, and the definition of the vectors $\lambda^{q_1\dots q_k}_m$ (7.1.12).

Corollary 2. (*a*) *The vector space E^k_1 is generated by the vectors*

$$
\lambda^{q_1\dots q_k}_i - \frac{1}{n+k-1}(\delta^{q_1}_i\lambda^{q_2\dots q_k j}_j + \dots + \delta^{q_k}_i\lambda^{q_1\dots q_{k-1} j}_j).
\tag{7.3.24}
$$

(b) *The vector space E^k_2 is generated by the vectors*

$$
\lambda^{q_1\dots q_{k-1} j}_j.
\tag{7.3.25}
$$

Proof. (a) By (7.3.22) a vector $\xi \in R^n \otimes S^k R^{n*}$ belongs to E^k_1 if and only if $\pi_2(\xi) = 0$, that is, by (7.3.18),

$$
\delta^m_{q_1}\xi^j_{q_2\dots q_k j} + \dots + \delta^m_{q_k}\xi^j_{q_1\dots q_{k-1} j} = 0.
\tag{7.3.26}
$$

Contracting the left side in m and q_k we see that this equality is equivalent to

$$
\xi^j_{q_1\dots q_{k-1} j} = 0.
\tag{7.3.27}
$$

Using this equality we obtain

$$
\begin{aligned}
\xi &= -\xi^m_{q_1\dots q_r}\lambda^{q_1\dots q_r}_m = \\
&= -\xi^m_{q_1\dots q_r}\Big[\lambda^{q_1\dots q_r}_m - \frac{1}{n+k-1}(\delta^{q_1}_m\lambda^{q_2\dots q_k j}_j + \dots + \delta^{q_k}_m\lambda^{q_1\dots q_{k-1} j}_j) + \\
&\quad + \frac{1}{n+k-1}(\delta^{q_1}_m\lambda^{q_2\dots q_k j}_j + \dots + \delta^{q_k}_m\lambda^{q_1\dots q_{k-1} j}_j)\Big] = \\
&= -\xi^m_{q_1\dots q_r}\Big[\lambda^{q_1\dots q_r}_m - \frac{1}{n+k-1}(\delta^{q_1}_m\lambda^{q_2\dots q_k j}_j + \dots + \delta^{q_k}_m\lambda^{q_1\dots q_{k-1} j}_j)\Big].
\end{aligned}
\tag{7.3.28}
$$

Thus if $\xi \in E_1^k$ then ξ is expressible as a linear combination of the vectors (7.3.24). Conversely, suppose that ξ is expressed by

$$\xi = -\zeta^m_{q_1 \dots q_k}\left[\lambda_m^{q_1 \dots q_k} - \frac{1}{n+k-1}(\delta_m^{q_1}\lambda_j^{q_2 \dots q_k j} + \dots + \delta_m^{q_k}\lambda_j^{q_1 \dots q_{k-1} j})\right]. \tag{7.3.29}$$

Then

$$\xi = -\xi^m_{q_1 \dots q_k}\lambda_m^{q_1 \dots q_k}, \tag{7.3.30}$$

where

$$\xi^m_{q_1 \dots q_k} = \zeta^m_{q_1 \dots q_k} - \frac{1}{n+k-1}(\delta^m_{q_1}\zeta^j_{j q_2 \dots q_k} + \dots + \delta^m_{q_k}\zeta^j_{q_1 \dots q_{k-1} j}). \tag{7.3.31}$$

This expression obviously satisfies (7.3.27) so that $\xi \in E_1^k$.

(b) By (7.3.18) a vector $\xi \in R^n \otimes S^k R^{n*}$ belongs to E_2^k if and only if its components satisfy

$$\xi^m_{q_1 \dots q_k} = \frac{1}{n+k-1}(\delta^m_{q_1}\xi^j_{q_2 \dots q_k j} + \dots + \delta^m_{k_k}\xi^j_{q_1 \dots q_{k-1} j}). \tag{7.3.32}$$

Substituting this expression into (7.3.23) we get

$$\xi = -\frac{k}{n+k-1}\,\xi^j_{q_1 \dots q_{k-1} j}\lambda^{q_1 \dots q_{k-1} l}. \tag{7.3.33}$$

Thus if $\xi \in E_2^k$ then ξ is expressible as a linear combination of $\lambda_i^{q_1 \dots q_{k-1} i}$. Conversely, suppose that ξ is expressed by

$$\xi = -\zeta_{q_1 \dots q_{k-1}}\lambda_i^{q_1 \dots q_{k-1} i}. \tag{7.3.34}$$

Then ξ is expressed by (7.3.30), where

$$\xi^m_{q_1 \dots q_k} = \frac{1}{k}(\delta^m_{q_1}\zeta_{q_2 \dots q_k} + \dots + \delta^m_{q_k}\zeta_{q_1 \dots q_{k-1}}). \tag{7.3.35}$$

We have to show that (7.3.32) holds. By a direct calculation

$$\begin{aligned}&\frac{1}{n+k-1}(\delta^m_{q_1}\xi^i_{q_2 \dots q_k i} + \dots + \delta^m_{q_k}\xi^i_{q_1 \dots q_{k-1} i}) = \\ &= \frac{1}{k}(\delta^m_{q_1}\zeta_{q_2 \dots q_k} + \dots + \delta^m_{q_k}\zeta_{q_1 \dots q_{k-1}}) = \xi^m_{q_1 \dots q_k}\end{aligned} \tag{7.3.36}$$

as desired.

The first main result of this section is formulated in the following theorem.

Theorem 7.4. (*a*) *Let $n = 1$. Then every ideal of the Lie algebra k_1^r, invariant with respect to the adjoing action of the group L_1^1 is one of the ideals 0, $k_1^{r,s}$, where $1 \leqq s \leqq \leqq r - 1$.*

(b) Let $n \geqq 2$. Then every ideal of the Lie algebra k_n^r, invariant with respect to the adjoint action of the group L_n^1, is one of the following:

(1) 0,
(2) $E_1^s \oplus (R^n \otimes S^{s+1}R^{n*}) \oplus \ldots \oplus (R^n \otimes S^r R^{n*}),\ 2 \leqq s \leqq r$,
(3) $E_2^s \oplus (R^n \otimes S^{s+1}R^{n*}) \oplus \ldots \oplus (R^n \otimes S^r R^{n*}),\ 2 \leqq s \leqq r$,
(4) $k_n^{r,s},\ 1 \leqq s \leqq r - 1$.

Proof. (a) Let us consider the commutation relations (7.1.29). These commutation relations imply that the ideal, generated by an element $\lambda^{(s)}$, is precisely the ideal $k_1^{r,s}$; to see this, one should consider (7.1.29) for $t = 2$. Since $k_1^{r,s}$ in an L_1^1-invariant ideal, this proves (a).

(b) The proof of the second assertion is divided into three steps.

1. Let s be any integer, $2 \leqq s \leqq r - 1$, and let k be the smallest ideal in k_n^r containing E_1^s. We shall show that k coincides with the ideal (2) of Theorem 7.4. To this purpose we use commutators (7.1.25) for $t = 2$,

$$(7.3.37)\qquad [\lambda_i^{k_1\ldots k_s}, \lambda_j^{p_1p_2}] = \frac{s+1}{2}(\delta_j^{k_1}\lambda_i^{k_2\ldots k_sp_1p_2} + \ldots + \delta_j^{k_s}\lambda_i^{k_1\ldots k_{s-1}p_1p_2} - \delta_i^{p_1}\lambda_j^{k_1\ldots k_sp_2} - \delta_i^{p_2}\lambda_j^{k_1\ldots k_sp_1}).$$

It is enough to verify that for any integer i, $1 \leqq i \leqq n$, and any sequence $(k_1, k_2, \ldots, \ldots, k_{s+1})$, where $1 \leqq k_1, k_2, \ldots, k_{s+1} \leqq n$, the vector $\lambda_i^{k_1\ldots k_{s+1}}$ belongs to k.

The vector space E_1^s is spanned by the elements

$$(7.3.38)\qquad \chi_i^{k_1\ldots k_s} = \lambda_i^{k_1\ldots k_s} - \frac{1}{n+s-1}(\delta_i^{k_1}\lambda_m^{k_2\ldots k_sm} + \ldots + \delta_i^{k_s}\lambda_m^{k_1\ldots k_{s-1}m}).$$

We easily obtain from (7.3.37)

$$(7.3.39)\qquad [\lambda_i^{k_1\ldots k_s}, \lambda_j^{k_{s+1}j}] = \frac{s+1}{2}((s-1)\lambda_i^{k_1\ldots k_{s+1}} - \delta_i^{k_{s+1}}\lambda_j^{k_1\ldots k_sj}).$$

Using this formula we get

$$(7.3.40)\qquad \begin{aligned}[\chi_i^{k_1\ldots k_s}, \lambda_j^{k_{s+1}j}] &= \frac{1}{2}(s+1)(s-1)\lambda_i^{k_1\ldots k_{s+1}} - \\ &- \frac{1}{2}(s+1)\delta_i^{k_{s+1}}\lambda_j^{k_1\ldots k_sj} - \\ &- \frac{1}{n+s-1}(\delta_i^{k_1}[\lambda_m^{k_2\ldots k_sm}, \lambda_j^{k_{s+1}j}] + \ldots + \\ &+ \delta_i^{k_s}[\lambda_m^{k_1\ldots k_{s-1}m}, \lambda_j^{k_{s+1}j}]).\end{aligned}$$

Contracting (7.3.39) and (7.3.40) we obtain

$$[\chi_i^{k_1 \ldots k_s}, \lambda_j^{ij}] = [\lambda_i^{k_1 \ldots k_s}, \lambda_j^{ij}] -$$

$$(7.3.41) \qquad - \frac{1}{n+s-1}([\lambda_m^{k_2 \ldots k_s m}, \lambda_j^{k_1 ij}] + \ldots +$$

$$+ [\lambda_m^{k_1 \ldots k_{s-1} m}, \lambda_j^{k_s ij}]),$$

$$(7.3.42) \qquad [\lambda_i^{k_1 \ldots k_{s-1} i}, \lambda_j^{k_s j}] = \frac{1}{2}(s+1)(s-2)\lambda_i^{k_1 \ldots k_s i}.$$

Substituting (7.3.42) into (7.3.41) we obtain

$$(7.3.43) \qquad [\chi_i^{k_1 \ldots k_s}, \lambda_j^{ij}] = \frac{(s+1)(1-n^2)}{2(s-1+n)}\lambda_j^{k_1 \ldots k_s j};$$

in particular, $\lambda_j^{k_1 \ldots k_s j} \in k$. Now relations (7.3.43) and (7.3.42) together with (7.3.40) imply that $\lambda_i^{k_1 \ldots k_{s+1}}$ belongs to the ideal k as required.

2. Let s be any integer, $2 \leqq s \leqq r-1$, and let k be the smallest ideal in k_n^r containing E_2^s. We shall show that k coincides with the ideal (3) of Theorem 7.4. It is sufficient to show that for any integer i, $1 \leqq i \leqq n$, and any sequence $(k_1, k_2, \ldots, \ldots, k_{s+1})$, where $1 \leqq k_1, k_2, \ldots, k_{s+1} \leqq n$, the vector $\lambda_j^{k_1 \ldots k_{s+1}}$ belongs to k.

The vector space E_2^s is spanned by the elements $\lambda_i^{p_1 \ldots p_{s-1} i}$. We easily obtain from (7.3.37)

$$(7.3.44) \qquad [\lambda_i^{k_1 \ldots k_{s-1} i}, \lambda_j^{k_s k_{s+1}}] = \frac{s+1}{2}(\delta_j^{k_1}\lambda_i^{k_2 \ldots k_{s+1} i} + \ldots +$$

$$+ \delta_j^{k_{s-1}}\lambda_i^{k_1 \ldots k_{s-2} k_s k_{s+1} i} - \lambda_j^{k_1 \ldots k_{s+1}}).$$

Expressing $\lambda_j^{k_1 \ldots k_{s+1}}$ from this equality and using the formula

$$(7.3.45) \qquad [\lambda_i^{k_1 \ldots k_{s-2} j i}, \lambda_j^{k_{s-1} k_s}] = \frac{1}{2}(s+1)(s+n-3)\lambda_i^{k_1 \ldots k_s i},$$

we can see at once that $\lambda_j^{k_1 \ldots k_{s+1}}$ belongs to the ideal k as required.

3. If $s = r$, the commutation relations (7.1.25) guarantee that both E_1^r and E_2^r are ideals of k_n^r.

This completes part (b) of the proof.

Corollary 1. *Let $K \subset K_n^r$ be a Lie subgroup, k the Lie algebra of K. Then K is a normal Lie subgroup of L_n^r if and only if k is equal to one of the ideals* (1)–(4) *of Theorem* 7.4.

Proof. This follows from Theorem 7.4 and Lemma 7.5.

Remark 7.2. Obviously, there exist ideals of k_n^r which are not invariant with respect to the adjoint action of the group L_n^1 on k_n^r. For example, if $n \geqq 2$ then any non-zero element $\xi \in k_n^r$ generates such an ideal.

Let us now consider the Lie algebra $L(L_n^r)$ of the differential group L_n^r. We shall write with the obvious convention

(7.3.46) $$L(L_n^r) = L(L_n^1) \oplus (R^n \otimes S^2R^{n*}) \oplus \dots \oplus (R^n \otimes S^rR^{n*}),$$

(identification of L_n^1-modules via the adjoint action). Notice that for $n = 1$, the module $R \otimes S^kR^*$ is canonically identified with R. The second main result of this section can now be formulated as follows.

Theorem 7.5. *(a) Every ideal of the Lie algebra $L(L_1^r)$ is one of the ideals 0, $k_1^{r,s}$, where $1 \leqq s \leqq r - 1$, and $L(L_1^r)$.*

(b) Let $n \geqq 2$. Every ideal of the Lie algebra $L(L_n^r)$ is one of the following:

(1) 0,

(2) $E_1^s \oplus (R^n \otimes S^{s+1}R^{n*}) \oplus \dots \oplus (R^n \otimes S^rR^{n*})$, $1 \leqq s \leqq r$,

(3) $E_2^s \oplus (R^n \otimes S^{s+1}R^{n*}) \oplus \dots \oplus (R^n \otimes S^rR^{n*})$, $1 \leqq s \leqq r$,

(4) $k_n^{r,s}$, $1 \leqq s \leqq r - 1$,

(5) $\eta \oplus k_n^r$,

where η is an ideal of the Lie algebra $L(L_n^1)$.

Proof. (a) By (7.1.29), if an ideal of $L(L_1^r)$ contains an element $\lambda^{(s)}$, then it also contains the element $\lambda^{(s+1)}$. This implies that (a) holds.

(b) Let $g \in L(L_n^r)$ be an ideal. We distinguish two cases.

1. Suppose that $g \subset k_n^r$. Since g is an ideal of $L(L_n^r)$, it is an ideal of k_n^r, which is invariant with respect to the adjoint action of the group L_n^1, and Theorem 7.4 implies that g is one of the ideals (1)–(4).

2. Consider $L(L_n^r)$ expressed by (7.3.46) and suppose that $g \in k_n^r$. Then there exists an element $\xi \in g \cap L(L_n^1)$, $\xi \neq 0$. Denote by $[\xi]$ the ideal in $L(L_n^r)$ generated by ξ, i.e., the smallest ideal in $L(L_n^r)$ containing ξ. We want to show that $[\xi] = \eta \oplus k_n^r$, where η is an ideal in $L_n^1(L)$. Commutation relations (7.1.25) ensure that $[\xi] = \eta \oplus k$, where η is the ideal in $L(L_n^1)$ generated by ξ. It is thus enough to show that to each integer s, $2 \leqq s \leqq r$, there exist non-zero vectors $\xi_1 \in E_1^s$, $\xi_2 \in E_2^s$ such that $\xi_1, \xi_2 \in [L(L_n^r), \xi]$, where $[L(L_n^r), \xi]$ denotes the vector subspace of $L(L_n^r)$ generated by vectors of the form $[\zeta, \xi]$, where $\zeta \in L(L_n^r)$: obviously, k must be L_n^1-invariant so that conditions $k \cap E_1^s \neq \emptyset$, $k \cap E_2^s \neq \emptyset$ imply $k = k_n^r$ (see (7.3.10), (7.3.22)).

Let $\xi \in L(L_n^1)$ be a non-zero vector. We express ξ by

(7.3.47) $$\xi = \xi_p^j \lambda_j^p.$$

Choose any integer s such that $2 \leqq s \leqq r$. We know that the vectors

(7.3.48) $$\chi_i^{k_1 \dots k_s} = \lambda_i^{k_1 \dots k_s} - \frac{1}{n + s - 1} (\delta_i^{k_1} \lambda_m^{k_2 \dots k_s m} + \dots + \delta_i^{k_s} \lambda_m^{k_1 \dots k_{s-1} m})$$

form a basis of the vector space E_1^s (Corollary 2 to Lemma 7.7). Using the relation

$$(7.3.49)\qquad \begin{aligned}[\lambda_i^{k_1\ldots k_s}, \xi] = \xi_p^j(\delta_j^{k_1}\lambda_i^{k_2\ldots k_s p} + \ldots + \\ + \delta_j^{k_s}\lambda_i^{k_1\ldots k_{s-1}p} - \delta_i^p\lambda_j^{k_1\ldots k_s}),\end{aligned}$$

(see (7.1.25)) we obtain by a straightforward calculation

$$(7.3.50)\qquad \begin{aligned}[\chi_i^{k_1\ldots k_s}, \xi] = \xi_p(\delta_j^{k_1}\chi_i^{k_2\ldots k_s p} + \ldots + \\ + \delta_j^{k_s}\chi_i^{k_1\ldots k_{s-1}p} - \delta_p^i\chi_j^{k_1\ldots k_s}) = \xi_p^{k_1}\chi_i^{k_2\ldots k_s p} + \\ + \ldots + \xi_p^{k_s}\chi_i^{k_1\ldots k_{s-1}p} - \xi_i^j\chi_j^{k_1\ldots k_s}.\end{aligned}$$

It is easily seen that this is a non-zero vector. For supposing it is zero and putting $i = k_s$ and contracting in i we get

$$(7.3.51)\qquad \xi_p^{k_1}\chi_i^{k_2\ldots k_{s-1}ip} + \ldots + \xi_p^{k_{s-1}}\chi_i^{k_1\ldots k_{s-2}ip} = 0.$$

Since there exist j and p such that $\xi_p^j \neq 0$, (7.3.51) means that the vectors $\chi_i^{k_1\ldots k_{s-1}}$ are linearly dependent, contradicting Corollary 2 to Lemma 7.7. Thus putting

$$(7.3.52)\qquad \xi_1 = [\chi_i^{k_1\ldots k_s}, \xi],$$

we obtain the desired non-zero vector belonging to E_1^s.

Similarly, the vectors $\lambda_m^{k_1\ldots k_{s-1}m}$ form a basis of E_2^s (Corollary 2 to Lemma 7.7). We get from (7.3.49)

$$(7.3.53)\qquad \begin{aligned}[\lambda_m^{k_1\ldots k_{s-1}m}, \xi] = \xi_p^j(\delta_j^{k_1}\lambda_i^{k_2\ldots k_{s-1}ip} + \ldots + \\ + \delta_j^{k_{s-1}}\lambda_i^{k_1\ldots k_{s-2}ip}) = \xi_p^{k_1}\lambda_i^{k_2\ldots k_{s-1}ip} + \ldots + \\ + \xi_p^{k_{s-1}}\lambda_i^{k_1\ldots k_{s-2}ip},\end{aligned}$$

so that $\xi_2 = [\lambda_m^{k_1\ldots k_{s-1}m}, \xi] \in E_2^s$, and we prove as above that $\xi_2 \neq 0$. This ends the proof of Theorem 7.5.

Theorem 7.5 classifies all connected normal Lie subgroups of the differential group L_n^r. These are precisely the subgroups whose Lie algebras are the Lie algebras (1)–(5) from Theorem 7.5.

Corollary 1. *The smallest ideal in $L(L_n^r)$ containing a non-zero vector $\xi \in L(L_n^1)$, is the ideal $[\xi] = \eta \oplus k_n^r$, where η is the smallest ideal in $L(L_n^1)$ containing ξ.*

Proof. This refers to case (5) in Theorem 7.5.

7.4. Differential invariants with values in $GL_n(R)$-manifolds. Let us consider the differential group L_n^r. Since this group can be regarded as the interior semi-direct product of its Lie subgroup $\iota^r(L_n^1)$, canonically isomorphic with the Lie group

$GL_n(R)$, and its normal Lie subgroup K_n^r (Theorem 7.3), we can apply Lemma 1.6 and Theorem 1.12 to it. In the following theorem we consider a left $GL_n(R)$-manifold as an L_n^r-manifold, with the canonically extended group action. This allows us to speak of differential invariants with values in left $GL_n(R)$-manifolds.

Theorem 7.6. *Let Q be a left L_n^r-manifold, $\pi : Q \to Q/K_n^r$ the canonical projection onto the orbit space, and let P be an L_n^1-manifold. Each differential invariant $F : Q \to$ $\to P$ is of the form*

(7.4.1) $\qquad F = f \circ \pi,$

where $f : Q/K_n^r \to P$ is a uniquely determined L_n^1-equivariant mapping. If Q/K_n^r has the structure of the orbit manifold and F is smooth, then f is also smooth.

Proof. This assertion is a direct consequence of Lemma 1.6 and Theorem 1.12.

Remark 7.3. Theorem 7.6 says that each differential invariant with values in a $GL_n(R)$-manifold is uniquely determined by a $GL_n(R)$-equivariant mapping, i.e., by a differential invariant of "order" zero. With the notation of this theorem, the correspondence $F \to f$ (reducing the "order" of F, equal to r, to the "order" of f, equal to 1) is prescribed by the canonical projection $\pi : Q \to Q/K_n^r$, and is independent of F. The problem of finding all differential invariants from a left L_n^r-manifold to a left $GL_n(R)$-manifold is thus reduced to the problem of finding all $GL_n(R)$-equivariant mappings between certain left $GL_n(R)$-manifolds. This method of finding differential invariants can be applied, in particular, to the case of differential invariants with values in $GL_n(R)$-modules (tensor spaces), in the bundles of linear frames, etc.

Remark 7.4. If in Theorem 7.6, $Q = T_n^r S$, where S is a left $GL_n(R)$-manifold, we obtain as a consequence of (7.4.1) that the "derivatives" among the coordinates on $T_n^r S$ enter each differential invariant with values in a $GL_n(R)$-manifold in a "canonical" way, independent of the differential invariant. It can be proved in this way that in certain cases, every such a differential invariant depends polynomially on the derivatives. For example, every invariant tensor, depending on a linear connection and its derivatives up to a certain order r, is a polynomial in the derivatives.

PART 2

NATURAL GEOMETRIC OPERATIONS: EXAMPLES

8. NATURAL DIFFERENTIAL OPERATORS BETWEEN TENSOR BUNDLES

In this chapter we deal with natural differential operators between tensor bundles. We describe the method allowing to find all polynomial natural differential operators which are in some cases all globally defined operators. Our method does not give general theorems but is very simple for concrete calculations.

In Section 8.1 we deal with globally defined homogeneous functions. Theorem 8.2 allows us to find polynomial solutions of systems of partial differential equations which appear with finding differential invariants.

In Section 8.2 we describe natural differential operators of order zero between tensor bundles. These operators are closely connected with invariant tensors.

In Section 8.3 we discuss general method of finding natural differential operators of order greater than or equal to one. This method is based on Theorem 7.6 and on the use of formal connections.

Using the method of Section 8.3 we shall prove in Section 8.4 the uniqueness of exterior derivative under some very weak assumptions.

In Section 8.5 we describe some bilinear natural differential operations with TX-valued forms which are of the same type as the Frölicher–Nijenhuis bracket. As a special case we obtain the uniqueness of the Lie bracket.

8.1. Globally defined homogeneous functions. Let (x^i), $i = 1, \dots, n$, be the canonical coordinates on R^n. By a *global solution* of the equation

$$\frac{\partial f}{\partial x^i} x^i = kf, \tag{8.1.1}$$

where $k \in R$, we understand a smooth function f which is defined on whole R^n and satisfies this equation. The function $f \equiv 0$ is the solution of (8.1.1) for arbitrary k. The solutions different from $f \equiv 0$ will be called *non-zero solutions*.

From the Euler theorem it follows, that a polynomial of n variables satisfies (8.1.1) if and only if it is a homogeneous polynomial of degree k.

First let $n = 1$. Then (8.1.1) reduces to the equation

$$x \frac{dy}{dx} = ky, \tag{8.1.2}$$

and using separation of variables we obtain

$$y = cx^k, \tag{8.1.3}$$

where $x = y(1)$. If $k < 0$ and $c \neq 0$, (8.1.3) is not defined at the origin. If $k \in (p-1, p)$ for some natural p then the derivatives of y of order $\geqq p$ are not defined at the origin (if $c \neq 0$). So we have proved

Lemma 8.1. *The equation* (8.1.2) *has non-zero global solutions only if k is a natural number.*

Now let us suppose that f is a solution of equation (8.1.1) and let us put

$$F(t, x^1, \ldots, x^n) = f(tx^1, \ldots, tx^n). \tag{8.1.4}$$

Then we have

$$t\frac{\partial F}{\partial t} = \frac{\partial f(tx^1, \ldots, tx^n)}{\partial x^i} tx^i = kf(tx^1, \ldots, tx^n) = kF. \tag{8.1.5}$$

For fixed $(x^i) \in R^n$ we have equation (8.1.2) and from (8.1.3)

$$f(tx^1, \ldots, tx^n) = F(t, x^1, \ldots, x^n) = t^k f(x^1, \ldots, x^n). \tag{8.1.6}$$

(hus f is a homogeneous function of degree k.

Lemma 8.2. *If $k < 0$ there does not exist a non-zero global solution of equation* T8.1.1).

Proof. Let $f(x_0^1, \ldots, x_0^n) \neq 0$. From (8.1.6)

$$f(0, \ldots, 0) = \lim_{t\to 0+} t^k f(x_0^1, \ldots, x_0^n) = f(x_0^1, \ldots, x_0^n) \lim_{t \to 0+} \frac{1}{t^{|k|}}, \tag{8.1.7}$$

but this limit is improper which is a contradiction.

Let us denote $g = \partial f/\partial x^j$. If we differentiate (8.1.1) with respect to x^j we obtain

$$\frac{\partial g}{\partial x^i} x^i = (k-1)\, g. \tag{8.1.8}$$

Consequently, g is a homogeneous function of degree $(k-1)$. Let p be a natural number such that $k < (p+1)$. Then every partial derivative of f of order greater than p satisfies equation (8.1.1) with a negative coefficient and has to be identically zero. From the Taylor's formula we obtain that f is a polynomial of degree at most p and from the Euler theorem f has to be a homogeneous polynomial of degree k. Thus we have the following assertion.

Theorem 8.1. *Equation* $(\partial f/\partial x^i) \, . \, x^i = kf$ *has a non-zero global solution only of* k *is a natural number. Every global solution of this equation is a homogeneous polynomial of degree* k.

Now let us suppose that we have several groups of variables $x^i, y^p, \dots, z^s$ of total number N. Let us consider an equation

$$(8.1.9) \qquad a \frac{\partial f}{\partial x^i} x^i + b \frac{\partial f}{\partial y^p} y^p + \dots + c \frac{\partial f}{\partial z^s} z^s = kf,$$

where $a, b, \dots, c$ are positive real numbers and k is an arbitrary real number. We want to find a non-zero function $f(x^i, y^p, \dots, z^s)$ defined on the whole R^N, satisfying (8.1.9).

Let us suppose that f is a solution of (8.1.9) and let us put

$$(8.1.10) \qquad F(t, x^i, y^p, \dots, z^s) = f(t^a x^i, t^b y^p, \dots, t^c z^s).$$

Then

$$(8.1.11) \qquad t \frac{\partial F}{\partial t} = a \frac{\partial f}{\partial x^i} t^a x^i + b \frac{\partial f}{\partial y^p} t^b y^p + \dots + c \frac{\partial f}{\partial z^s} t^c z^s = kF.$$

For fixed $(x^i, y^p, \dots, z^s) \in R^N$ we have from (8.1.3)

$$(8.1.12) \qquad f(t^a x^i, t^b y^p, \dots, t^c z^s) = F = t^k f(x^i, y^p, \dots, z^s).$$

Lemma 8.3. *If* $k < 0$ *there does not exist non-zero global solution of* (8.1.9).

P r o o f. Let $f(x_0^i, y_0^p, \dots, z_0^s) \neq 0$. From (8.1.12) we have

$$(8.1.13) \qquad \lim_{t \to 0+} f(t^a x_0^i, t^b y_0^p, \dots, t^c z_0^s) = f(x_0^i, y_0^p, \dots, z_0^s) \, . \lim_{t \to 0+} \frac{1}{t^{|k|}} \, .$$

Because of $a, b, \dots, c$ are positive numbers the limit on the right hand side is equal to $f(0, 0, \dots, 0)$ while the limit on the left hand side is improper; this is a contradiction.

If we differentiate (8.1.9) with respect to x^j we obtain for $g = \partial f/\partial x^j$

$$(8.1.14) \qquad a \frac{\partial g}{\partial x^i} x^i + b \frac{\partial g}{\partial y^p} y^p + \dots + c \frac{\partial g}{\partial z^s} z^s = (k - a) \, g.$$

Similarly for $\partial f/\partial y^p, \ldots, \partial f/\partial z^s$ we obtain the equations of the type (8.1.9) with coeficients $(k - b), \ldots, (k - c)$ on the right hand side. By Lemma 8.3 every partial derivative of a sufficiently high order vanishes, and this implies that f is a polynomial. Let f be a polynomial solution of (8.1.9). Then every its monomial is also a solution. Let us consider a monomial P which is of degrees α in variables x^i, β in variables y^p, ... and γ in variables z^s. Then $(\partial P/\partial x^i) \cdot x^i = \alpha P$, $(\partial P/\partial y^p) \cdot y^p = \beta P, \ldots, (\partial P/\partial z^s) \cdot z^s = \gamma P$ and P satisfies (8.1.9) if and only if

$$a\alpha + b\beta + \ldots + c\gamma = k. \tag{8.1.15}$$

So we have proved

Theorem 8.2. *Global solutions of the equation*

$$a \frac{\partial f}{\partial x^i} x^i + b \frac{\partial f}{\partial y^p} y^p + \ldots + c \frac{\partial f}{\partial z^s} z^s = kf, \tag{8.1.16}$$

where $a, b, \ldots, c$ are positive real numbers and k is an arbitrary real number, are sums of homogeneous polynomials of degrees $(\alpha, \beta, \ldots, \gamma)$ such that the equation $a\alpha + b\beta + \ldots + c\gamma = k$ is satisfied. If this equation has no solution $(\alpha, \beta, \ldots, \gamma)$, where $\alpha, \beta, \ldots, \gamma$ are natural numbers, then the system of partial differential equations (8.1.16) *has no non-zero global solution.*

Example 8.1. Let us consider the equation

$$\frac{\partial f}{\partial x^i} x^i + 2 \frac{\partial f}{\partial y^p} y^p = 3f. \tag{8.1.17}$$

Then condition (8.1.15) $\alpha + 2\beta = 3$ is satisfied for two values of (α, β), $(\alpha, \beta) = (3,0)$ or $(\alpha, \beta) = (1,1)$. Hence every global solution of (8.1.17) is of the form $f = c_{ijk} x^i x^j x^k + c_{ip} x^i y^p$, where $c_{ijk}, c_{ip} \in R$ and c_{ijk} is symmetric.

Remark 8.1. The assumption of Theorem 8.2, that $a, b, \ldots, c$ are positive numbers, is necessary. Namely if

$$\frac{\partial f}{\partial x} x - \frac{\partial f}{\partial y} y = 0, \tag{8.1.18}$$

that is $a = 1$, $b = -1$, $k = 0$, is an equation on R^2 and $h(t)$ is an arbitrary (not necessarily polynomial) smooth function of one variable defined on the whole R, the function $f(x, y) = h(xy)$ is the solution of (8.1.18).

8.2. Natural differential operators of order zero. Let $X \in \operatorname{Ob} \mathscr{D}_n$. Let $T^{(p,q)}X$ and $T^{(r,s)}X$ be the tensor bundles of the type (p, q) and (r, s) over X. $T^{(p,q)}$ is the

P-lifting F_P^1 with type fiber $P = \overset{p}{\otimes} R^n \otimes \overset{q}{\otimes} R^{n*}$, where $n = \dim X$, on which the tensor action of L_n^1 is given. In the canonical global coordinates $(t^{i_1 \dots i_p}_{j_1 \dots j_q})$ on P this action has the coordinate expression

(8.2.1) $$\bar{t}^{i_1 \dots i_p}_{j_1 \dots j_q} = a^{i_1}_{k_1} \dots a^{i_p}_{k_p} b^{m_1}_{j_1} \dots b^{m_q}_{j_q} t^{k_1 \dots k_p}_{m_1 \dots m_q}.$$

Similarly $T^{(r,s)}$ is the S-lifting F_S^1 with type fiber $S = \overset{r}{\otimes} R^n \otimes \overset{s}{\otimes} R^{n*}$, with the tensor action of L_n^1 given in the canonical global coordinates $(u^{i_1 \dots i_r}_{j_1 \dots j_s})$ on S by

(8.2.2) $$\bar{u}^{i_1 \dots i_r}_{j_1 \dots j_s} = a^{i_1}_{k_1} \dots a^{i_r}_{k_r} b^{m_1}_{j_1} \dots b^{m_s}_{j_s} u^{k_1 \dots k_r}_{m_1 \dots m_s}.$$

From Theorem 5.7 it follows that a natural differential operator of order zero of $T^{(p,q)}$ to $T^{(r,s)}$ determines a unique L_n^1-equivariant mapping (differential invariant of L_n^1) of P to S. Theorem 3.4 implies that a mapping $f: P \to S$, with the coordinate expression $u^{i_1 \dots i_r}_{j_1 \dots j_s} = f^{i_1 \dots i_r}_{j_1 \dots j_s}(t^{k_1 \dots k_p}_{m_1 \dots m_q})$, is a differential invariant of L_n^1 if and only if for any $\xi \in L(L_n^1)$ the Lie derivative $\partial_\xi f = 0$ and there exist $a_0 \in L_n^{1(-)}$ such that $f(a_0 t) = a_0 f(t)$, $t \in P$. We have

$$\partial_\xi f(t) = \xi_S(f(t)) - T_t f(\xi_P(t)),$$

where ξ_P and ξ_S are fundamental vector fields on P and S, respectively, relative to ξ. So if f is a differential invariant of L_n^1 then the fundamental vector fields ξ_P and ξ_S are f-related.

Let us denote by Φ or Ψ the action (8.2.1) or (8.2.2) of the group L_n^1 on P or S. We define $\Phi_t : L_n^1 \to P$ by $\Phi_t(a) = \Phi(a, t)$ for every $t \in P$, $a \in L_n^1$. Similarly we define $\Psi_u : L_n^1 \to S$, $u \in S$. Then from (1.3.31) the fundamental vector fields on P or S relative to $\xi \in L(L_n^1)$ are expressed by the equations

(8.2.3) $$\xi_P(t) = (T_e \Phi_t)(\xi),$$

where $t \in P$, or

(8.2.4) $$\xi_S(u) = (T_e \Psi_u)(\xi),$$

where $u \in S$ and $e \in L_n^1$ is the unity, i.e. $e = J_0^1 \operatorname{id}_{R^n}$.

A vector field $\xi \in L(L_n^1)$ has coordinate expression $\xi = \xi^i_j (\partial / \partial a^i_j)$, where (a^i_j) are the canonical global coordinates on L_n^1.

Then on P there exist vector fields Ξ^j_{iP} such that $\xi_P(t) = \Xi^j_{iP}(t)\, \xi^i_j$. Similarly on S there exist vector fields Ξ^j_{iS} such that $\xi_S(u) = \Xi^j_{iS}(u)\, \xi^i_j$. If $f: P \to S$ is a differential invariant of the group L_n^1 then the vector fields Ξ^j_{iP} and Ξ^j_{iS} are f-related, i.e.

(8.2.5) $$\Xi^j_{iS}(f(t)) = (T_t f)(\Xi^j_{iP}(t)),$$

where $t \in P$. From (8.2.3) and $\partial b^p_q / \partial a^i_j = -b^p_i b^j_q$, the vector field Ξ^j_{iP} has the coordinate expression

$$\Xi^j_{iP} = \left(\frac{\partial \bar t^{i_1\ldots i_p}_{j_1\ldots j_q}}{\partial a^i_j}\right)_e \frac{\partial}{\partial t^{i_1\ldots i_p}_{j_1\ldots j_q}} =$$

(8.2.6)
$$= (\delta^{i_1}_i t^{j i_2\ldots i_p}_{j_1\ldots j_p} + \ldots + \delta^{i_p}_i t^{i_1\ldots i_{p-1} j}_{j_1\ldots j_q} - \delta^j_{j_1} t^{i_1\ldots i_p}_{i j_2\ldots j_q} - \ldots -$$
$$- \delta^j_{j_q} t^{i_1\ldots i_p}_{j_1\ldots j_{q-1} i}) \frac{\partial}{\partial t^{i_1\ldots i_p}_{j_1\ldots j_q}}.$$

Similarly

$$\Xi^j_{ix} = \left(\frac{\partial \bar u^{i_1\ldots i_r}_{j_1\ldots j_s}}{\partial a^i_j}\right)_e \frac{\partial}{\partial u^{i_1\ldots i_r}_{j_1\ldots j_s}} =$$

(8.2.7)
$$= (\delta^{i_1}_i u^{j i_2\ldots i_r}_{j_1\ldots j_s} + \ldots - \delta^j_{j_s} u^{i_1\ldots i_r}_{j_1\ldots j_{s-1} i}) \frac{\partial}{\partial u^{i_1\ldots i_r}_{j_1\ldots j_s}}.$$

Then the coordinate expression of (8.2.5) is

(8.2.8)
$$(\delta^{k_1}_i t^{j k_2\ldots k_p}_{m_1\ldots m_q} + \ldots - \delta^j_{m_q} t^{k_1\ldots k_p}_{m_1\ldots m_{q-1} i}) \frac{\partial f^{i_1\ldots i_r}_{j_1\ldots j_s}}{\partial t^{k_1\ldots k_p}_{m_1\ldots m_q}} =$$
$$= \delta^{i_1}_i f^{j i_2\ldots i_r}_{j i\ldots j_s} + \ldots - \delta^{j_s}_j f^{i_1\ldots i_r}_{j_1\ldots j_{s-1} i}.$$

If $i = j$ (no summation) then

(8.2.9)
$$(p - q) \frac{\partial f^{i_1\ldots i_r}_{j_1\ldots j_s}}{\partial t^{k_1\ldots k_p}_{m_1\ldots m_q}} t^{k_1\ldots k_p}_{m_1\ldots m_q} = (r - s) f^{i_1\ldots i_r}_{j_1\ldots j_s}.$$

Every equation in the system of partial differential equations (8.2.9) is of the type (8.1.1), and we have

Theorem 8.3. *If $p = q$, $r \neq s$, then the unique natural differential operator of order zero of $T^{(p,q)}$ to $T^{(r,s)}$ is the zero operator. If $p \neq q$ then every zero-order natural differential operator of $T^{(p,q)}$ to $T^{(r,s)}$ is given by a homogeneous mapping of degree $(r - s)/(p - q)$ of P to S.*

Theorem 8.1 implies that non-zero global solutions of (8.2.9) exist only if $k = (r - s)/(p - q)$ is a natural number and all such solutions are homogeneous polynomials of degree k. So a global differential invariant has the coordinate expression

(8.2.10)
$$u^{i_1\ldots i_r}_{j_1\ldots j_s} = A^{i_1\ldots i_r b_{11}\ldots b_{q1}\ldots b_{1k}\ldots b_{qk}}_{j_1\ldots j_s c_{11}\ldots c_{p1}\ldots c_{1k}\ldots c_{pk}} t^{c_{11}\ldots c_{p1}}_{b_{11}\ldots b_{q1}} \ldots t^{c_{1k}\ldots c_{pk}}_{b_{1k}\ldots b_{qk}},$$

where $A^{i_1\ldots b_{qk}}_{j_1\ldots c_{pk}}$ are real coefficients symmetric with respect to the change of groups of indices $\binom{b_{1i}\ldots b_{qi}}{c_{1i}\ldots c_{pi}}$, $i = 1, \ldots, k$. L^1_n-equivariancy of (8.2.10) implies that the coefficients $A^{i_1\ldots b_{qk}}_{j_1\ldots c_{pk}}$ satisfy condition (4.1.7) for an absolute invariant tensor and from Theorem 4.1

(8.2.11)
$$A^{i_1\ldots b_{qk}}_{j_1\ldots c_{pk}} = \sum_\sigma c_\sigma \delta^{i_1}_{\sigma(j_1)} \ldots \delta^{b_{qk}}_{\sigma(c_{pk})},$$

where σ runs over all permutations of $(r + kq)$ indices and $c_\sigma \in R$. (4.1.11) implies that every polynomial solution (8.2.10) of (8.2.9) is also a solution of (8.2.8).

Now we have to prove the existence of $a_0 \in L_n^{1(-)}$ such that $f(a_0 t) = a_0 f(t)$, where f is given by (8.2.10). Let a_0 be given by $a_0 = \varkappa(i)\,\delta_j^i$, where $\varkappa(1) = -1$ and $\varkappa(i) = 1$ for $i > 1$. Then $a_0^{-1} = a_0$ and from (8.2.10)

$$f(a_0 t) = A_{j_1 \ldots c_{pk}}^{i_1 \ldots b_{qk}} \varkappa(b_{11}) \ldots \varkappa(b_{qk})\, \varkappa(c_{11}) \ldots \varkappa(c_{pk})\, t_{b_{11} \ldots b_{q1}}^{c_{11} \ldots c_{p1}} \ldots t_{b_{1k} \ldots b_{qk}}^{c_{1k} \ldots c_{pk}}$$

and

$$a_0 f(t) = \varkappa(i_1) \ldots \varkappa(i_r)\, \varkappa(j_1) \ldots \varkappa(j_s)\, A_{j_1 \ldots c_{pk}}^{i_1 \ldots b_{qk}} t_{b_{11} \ldots b_{q1}}^{c_{11} \ldots c_{p1}} \ldots t_{b_{1k} \ldots b_{qk}}^{c_{1k} \ldots c_{pk}}.$$

The required assertion now follows from (4.1.15).

Remark 8.2. From the symmetry of $A_{j_1 \ldots c_{pk}}^{i_1 \ldots b_{qk}}$ it follows that there are $(r + kq)!/k!$ independent coefficients in (8.2.11).

Remark 8.3. The degree of a linear natural differential operator has to be $k = 1$' i.e. $p - q = r - s$.

Remark 8.4. If $p = q$, $r = s$ then the system of equations (8.2.9) is satisfied identically. It is easy to see that the polynomial mapping

$$u_{j_1 \ldots j_r}^{i_1 \ldots i_r} = A_{j_1 \ldots j_r c_{11} \ldots c_{pk}}^{i_1 \ldots i_r b_{11} \ldots b_{pk}} t_{b_{11} \ldots b_{p1}}^{c_{11} \ldots c_{p1}} \ldots t_{b_{1k} \ldots b_{pk}}^{c_{1k} \ldots c_{pk}}$$

is a solution of (8.2.8) for any natural k if $A_{j_1 \ldots c_{pk}}^{i_1 \ldots b_{pk}}$ is an absolute invariant tensor.

Example 8.2. A global zero-order natural differential operator of $T^{(1,2)}$ to $T^{(1,3)}$ corresponds to a quadratic differential invariant of L_n^1 of $R^n \otimes \overset{2}{\otimes} R^{n*}$ to $R^n \otimes \overset{3}{\otimes} R^{n*}$ given in the canonical coordinates by

$$\begin{aligned} u_{j_1 j_2 j_3}^{i} &= A_{j_1 j_2 j_3 p r}^{i q_1 q_2 s_1 s_2} t_{q_1 q_2}^{p} t_{s_1 s_2}^{r} = \sum_{\sigma(5)} c_\sigma \delta_{\sigma(j_1)}^{i} \ldots \delta_{\sigma(r)}^{s_2} t_{q_1 q_2}^{p} t_{s_1 s_2}^{r} = \\ &= c_1 \delta_{j_1}^{i} t_{j_2 j_3}^{p} t_{pr}^{r} + \ldots + c_{60} t_{j_2 j_1}^{i} t_{p j_3}^{p}. \end{aligned} \tag{8.2.12}$$

Example 8.3. As an example of a natural differential operator of order zero we describe the generalized volume element depending on a tensor field. By a *generalized volume element* on a manifold $X \in \mathrm{Ob}\,\mathscr{D}_n$ we mean an n-form ω on X. An *invariant generalized volume element* depending on a tensor field of type (p, q) is a natural differential operator of order zero of $T^{(p,q)}$ to $\wedge^n T^*$. The type fiber of the functor $\wedge^n T^*$ is the set of real numbers R with the action of L_n^1 given by

$$\bar{t} = (\det a)^{-1}\, t, \tag{8.2.13}$$

where $t \in R$, $a \in L_n^1$. We have to find differential invariants of L_n^1 of $P = \overset{p}{\otimes} R^n \otimes \overset{q}{\otimes} R^{n*}$ to R equivariant with respect to the actions (8.2.1) and (8.2.13). Such an invariant is given by a mapping $\omega : P \to R$, $t = \omega(t^{i_1 \ldots i_p}_{j_1 \ldots j_q})$, such that the vector fields Ξ^j_{iP} and Ξ^j_{iR} are ω-related. The vector field Ξ^j_{iP} is given by (8.2.6) and from (8.2.13) we obtain

$$(8.2.14) \qquad \Xi^j_{iG} = \left(\frac{\partial \bar{t}}{\partial a^i_j}\right)_e \frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\partial}{\partial a^i_j}(\det a)^{-1}\right)_e t \frac{\mathrm{d}}{\mathrm{d}t}.$$

Since $\frac{\partial}{\partial a^i_j}(\det a) = b^j_i(\det a)$ we have $\frac{\partial}{\partial a^i_j}(\det a)^{-1} = -b^j_i(\det a)^{-1}$.

Hence

$$(8.2.15) \qquad \Xi^j_{iR} = -\delta^j_i t \frac{\mathrm{d}}{\mathrm{d}t}.$$

So for $i = j$ we have the system of partial differential equations

$$(8.2.16) \qquad (p - q) \frac{\partial \omega}{\partial t^{i_1 \ldots i_p}_{j_1 \ldots j_q}} t^{i_1 \ldots i_p}_{j_1 \ldots j_q} = -n\omega,$$

which is a special case of system (8.2.9).

Thus we have arrived to the following result: If $p = q$ there exists only zero generalized volume element depending invariantly on a vector field of type (p, p). If $p \neq q$ then invariant generalized volume element depending on a vector field of type (p, q) is given by an L_n^1-equivariant homogeneous mapping of degree $n/(q - p)$ of P to R.

Let us consider for instance an invariant generalized volume element depending on a tensor field of type (0,2). Then $\omega : \overset{2}{\otimes} R^{n*} \to R$ has to be a homogeneous mapping of degree $n/2$. Theorem 8.1 implies that (8.2.16) has global solutions only if n is an even natural number and such solutions are polynomials. Let us consider $n = 2$, then ω has a form

$$(8.2.17) \qquad t = A^{ij} t_{ij}$$

and the L_n^1-equivariancy condition implies that real coefficients A^{ij} satisfy the condition

$$(8.2.18) \qquad (\det a) A^{ij} = a^i_r a^j_s A^{rs}$$

for all $a \in L_n^1$. Hence A^{ij} is a relative invariant tensor of the weight $(\det a)$ (Section 4.3). So that $A^{ij} = c\varepsilon^{ij}$, where c is a real constant and ε^{ij} is the Levi–Civita tensor described in Example 4.2. This implies

$$(8.2.19) \qquad t = c(t_{12} - t_{21}).$$

It is easy to verify that (8.2.19) is really an invariant generalized volume element. As a consequence of Theorem 4.3 we obtain that other invariant generalized volume elements defined globally on $T^{(0,2)}X$ do not exist for any n.

Now let us consider zero-order natural differential operators from the Whitney's sum of m tensor bundles $T^{(p_1,q_1)}X \oplus \dots \oplus T^{(p_m,q_m)}X$ to $T^{(r,s)}X$. The type fiber of the functor $T^{(p_1,q_1)} \oplus \dots \oplus T^{(p_m,q_m)}$ is the Cartesian product $P = P_1 \times \dots \times P_m$, where $P_i = \overset{p_i}{\otimes} R^n \otimes \overset{q_i}{\otimes} R^{n*}$, with the tensor action of the group L^1_n on every component. An element $\xi \in L(L^1_n)$ generates on P the fundamental vector field

$$(8.2.20) \qquad \Xi^j_{iP} = \left(\frac{\partial \bar{t}^{I_1}_{J_1}}{\partial a^i_j}\right)_e \frac{\partial}{\partial t^{I_1}_{J_1}} + \dots + \left(\frac{\partial \bar{t}^{I_m}_{J_m}}{\partial a^i_j}\right)_e \frac{\partial}{\partial t^{I_m}_{J_m}},$$

where $I_\alpha = (i_1, \dots, i_{p_\alpha})$, $J_\alpha = (j_1, \dots, j_{q_\alpha})$, $\alpha = 1, \dots, m$, are the multiindices. If a mapping $f: P \to S$, $u^{i_1 \dots i_r}_{j_1 \dots j_s} = f^{i_1 \dots i_r}_{j_1 \dots j_s}(t^{K_1}_{L_1}, \dots, t^{K_m}_{L_m})$, $K_\alpha = (k_1, \dots, k_{p_\alpha})$, $L_\alpha = (l_1, \dots, l_{q_\alpha})$, $\alpha = 1, \dots, m$, is a differential invariant of L^1_n then the vector fields (8.2.20) and (8.2.7) are f-related. If $i = j$ (no summation over i) we obtain the system of partial differential equations

$$(8.2.21) \qquad (p_1 - q_1) \frac{\partial f^{i_1 \dots i_r}_{j_1 \dots j_s}}{\partial t^{K_1}_{L_1}} t^{K_1}_{L_1} + \dots + (p_m - q_m) \frac{\partial f^{i_1 \dots i_r}_{j_1 \dots j_s}}{\partial t^{K_m}_{L_m}} t^{K_m}_{L_m} = (r - s) f^{i_1 \dots i_r}_{j_1 \dots j_s}.$$

It is difficult to find all solutions of system (8.2.21). The following theorem enables us to find all polynomial solutions of (8.2.21).

Theorem 8.4. *A non-zero polynomial differential invariant of L^1_n from P to S is a sum of homogeneous polynomials of degrees a_i in variables from P_i such that*

$$(8.2.22) \qquad \sum_{i=1}^{m} a_i(p_i - q_i) = r - s.$$

Proof. It is obvious that every polynomial solution of (8.2.22) is a sum of homogeneous polynomials. First we prove

Lemma 8.4. *Every homogeneous polynomial differential invariant of L^1_n from P to S is expressible as a combination of tensor products of tensors from P_i and a linear differential invariant of L^1_n from some tensor space to S. The converse is also true.*

Proof of the lemma. Let

$$(8.2.23) \qquad u^{i_1 \dots i_r}_{j_1 \dots j_s} = A^{i_1 \dots i_r L^1_1(1) \dots L^1_1(a_1) \dots L^m_m(a_m)}_{j_1 \dots j_s K^1_1(1) \dots K^1_1(a_1) \dots K^m_m(a_m)} t^{K^1_1(1)}_{L^1_1(1)} \dots t^{K^1_1(a_1)}_{L^1_1(a_1)} \dots t^{K^m_m(a_m)}_{L^m_m(a_m)},$$

where $K^\alpha_\alpha(\beta) = (k^\alpha_{1\beta}, \dots, k^\alpha_{p_\alpha \beta})$, $L^\alpha_\alpha(\beta) = (l^\alpha_{1\beta}, \dots, l^\alpha_{q_\alpha \beta})$, $\alpha = 1, \dots, m$, $\beta = 1, \dots, a_\alpha$, be a homogeneous polynomial solution of (8.2.21) of degrees a_i in variables from P_i,

$i = 1, \ldots, m$. From the L_n^1-equivariancy of (8.2.23) it follows that $A_{j_1 \ldots K_m^m(a_m)}^{i_1 \ldots L_m^m(a_m)}$ is an absolute invariant tensor and (4.1.12) implies that (8.2.23) has the required form.

Conversely, it is easy to see that the combination of the tensor product and a linear differential invariant F

$$\overset{a_1}{\times} P_1 \times \ldots \times \overset{a_m}{\times} P_m \xrightarrow{\otimes} \overset{a_1}{\otimes} P_1 \otimes \ldots \otimes \overset{a_m}{\otimes} P_m \xrightarrow{F} S \tag{8.2.24}$$

is a homogeneous polynomial differential invariant of L_n^1 from P to S which is of degrees a_i in variables from P_i.

Lemma 8.4 and Remark 8.3 now imply Theorem 8.4.

The finding of an element $a_0 \in L_n^{1(-)}$ such that (8.2.23) is equivariant with respect to a_0 is technical and is the same as in the case of polynomial solutions of (8.2.9). Further if a differential invariant will be expressed via absolute invariant tensors we shall omit the proof of existence of an element $a_0 \in L_n^{1(-)}$ such that this differential invariant is equivariant with respect to a_0.

Corollary 1. *If* $(p_i - q_i)$ *have the same sign for all* $i = 1, \ldots, m$, *then all global solutions of* (8.2.21) *are polynomial.*

Proof. This corollary immediately follows from Theorem 8.2.

Therefore, if the condition of Corollary 1 is satisfied we are able to find all global differential invariants of L_n^1 from P to S.

Example 8.4. If $(p_i - q_i)$ have not the same sign, there exist global solutions of (8.2.21) which are not polynomial. Let us consider for instance differential invariants from $R^n \times R^{n*}$ to R, where R is considered as the space of (0,0)-tensors with the trivial action of L_n^1. Then $f: R^n \times R^{n*} \to R$ is a differential invariant of L_n^1 if f has a form $f(x^i, y_j) = h(x^i y_i)$, where (x^i) are the canonical global coordinates on R^n, (y_j) are the canonical global coordinates on R^{n*} and $h: R \to R$ is an arbitrary smooth globally defined mapping. The differential invariant $x^i y_i$ forms the functional basis of differential invariants from $R^n \times R^{n*}$ to R.

Remark 8.5. If $p_i = q_i$ for some $i = 1, \ldots, m$ the degree of a polynomial differential invariant of L_n^1 from P to S in variables from P_i is arbitrary.

Remark 8.6. If there does not exist any natural solution of equation (8.2.22) then there does not exist any non-zero polynomial differential invariant from P to S.

Remark 8.7. Homogeneous polynomial differential invariants of degrees a_i in variables from P_i are symmetric with respect to the change of variables from P_i,

$i = 1, \ldots, m$. Thus such differential invariant has $(r + \sum_{i=1}^{m} a_i q_i)!/a_1! \ldots a_m!$ independent monomials.

8.3. Natural differential operators of higher orders. Theorem 5.7 implies that every natural differential operator of order r from $T^{(p,q)}$ to $T^{(k,l)}$ determines a unique differential invariant of the group L_n^{r+1} from $T_n^r P$ to S, where P and S are the type fibers of the functors $T^{(p,q)}$ and $T^{(k,l)}$. The action (8.2.1) of L_n^1 on P implies the action of L_n^{r+1} on $T_n^r P$ given by (5.2.1) and by Lemma 5.1. In the canonical coordinates $(t^{i_1 \ldots i_p}_{j_1 \ldots j_q}, \ldots, t^{i_1 \ldots i_p}_{j_1 \ldots j_q, m_1 \ldots m_r})$ on $T_n^r P$ this action is given by the formal differentiation of (8.2.1) up to order r. For instance, if $r = 1$ we obtain, together with (8.2.1),

$$\begin{aligned} \bar{t}^{i_1 \ldots i_p}_{j_1 \ldots j_q, m} = {} & (a^{i_1}_{k_1 s} b^s_m a^{i_2}_{k_2} \ldots a^{i_p}_{k_p} b^{l_1}_{j_1} \ldots b^{l_q}_{j_q} + \ldots + \\ & + a^{i_1}_{k_1} \ldots a^{i_{p-1}}_{k_{p-1}} a^{i_p}_{k_p s} b^s_m b^{l_1}_{j_1} \ldots b^{l_q}_{j_q} + a^{i_1}_{k_1} \ldots a^{i_p}_{k_p} b^{l_1}_{j_1 m} \ldots b^{l_q}_{j_q} + \ldots + \\ & + a^{i_1}_{k_1} \ldots a^{i_p}_{k_p} b^{l_1}_{j_1} \ldots b^{l_q}_{j_q m}) t^{k_1 \ldots k_p}_{l_1 \ldots l_q} + a^{i_1}_{k_1} \ldots a^{i_p}_{k_p} b^{l_1}_{j_1} \ldots b^{l_q}_{j_q} b^s_m t^{k_1 \ldots k_p}_{l_1 \ldots l_q, s}. \end{aligned} \tag{8.3.1}$$

Differential invariants from $T_n^r P$ to S have values in an L_n^1-space. Using Theorem 7.6 the problem of finding differential invariants of the group L_n^{r+1} is reduced to the problem of finding differential invariants of the group L_n^1. Theorem 7.6 says that every differential invariant $f : T_n^r P \to S$ determines a unique differential invariant F: $T_n^r P / K_n^{r+1} \to S$ of the group L_n^1 such that the diagram

$$\begin{array}{ccc} T^r P & \xrightarrow{\quad f \quad} & S \\ {\scriptstyle \pi^r} \searrow & & \nearrow {\scriptstyle F} \\ & T_n^r P / K_n^{r+1} & \end{array} \tag{8.3.2}$$

commutes, where K_n^{r+1} is the kernel of the canonical homomorphism of L_n^{r+1} on L_n^1 and π^r is the projection on the quotient space. Usually it is difficult to use Theorem 7.6 in the direct form because the projection π^r has not a simple coordinate expression. But this obstruction can be cancelled using formal connections.

Let us consider the vector space $Q_S = R^n \otimes \overset{2}{\odot} R^{n*}$, where $\odot$ denotes the symmetric tensor product. Let (Γ^i_{jk}), $1 \leqq i \leqq n$, $1 \leqq j \leqq k \leqq n$, denote the canonical global coordinates on Q_S. Let us consider on Q_S the action of the group L_n^2 given by

$$\bar{\Gamma}^i_{jk} = a^i_p (\Gamma^p_{qr} b^q_j b^r_k + b^p_{jk}). \tag{8.3.3}$$

It is easy to verify that (8.3.3) determines the left action of L_n^2 on Q_S. Then Q_S-lifting $F^2_{Q_S} X$ is the fiber bundle of linear symmetric connections on X. We shall call the elements of Q_S *formal connections*.

Let us consider the r-jet prolongation $T^r_n Q_S$ of Q_S. Let

$$(8.3.4) \qquad \Gamma_r = (\Gamma^i_{jk}, \Gamma^i_{jk,m_1}, \ldots, \Gamma^i_{jk,,m_1\ldots m_r}),$$

where $1 \leqq i \leqq n$, $1 \leqq j \leqq k \leqq n$, $1 \leqq m_1 \leqq \ldots \leqq m_s \leqq n$, $s = 1, \ldots, r$, be the canonical coordinates on $T^r_n Q_S$. The action of L^{r+2}_n on $T^r_n Q_S$ may be easily described in these canonical coordinates by the formal differentiation of (8.3.3) up to order r. For $r = 1$ we obtain, together with (8.3.3),

$$(8.3.5) \qquad \begin{aligned} \bar{\Gamma}^i_{jk,m} &= -a^i_s a^t_p b^s_{tm}(b^q_j b^r_k \Gamma^q_{qr} + b^p_{jk}) + \\ &+ a^i_p((b^q_{jm} b^r_k + b^q_j b^r_{km})\Gamma^p_{qr} + b^q_j b^r_k b^t_m \Gamma^p_{qr,t} + b^p_{jkm}). \end{aligned}$$

The action of L^4_n on $T^2_n Q_S$ is obtained by the formal differentiation of (8.3.5), etc.

We set in the canonical coordinates

$$(8.3.6) \qquad R^i_{jkm} = \Gamma^i_{jk,m} - \Gamma^i_{jm,k} + \Gamma^i_{pm}\Gamma^p_{jk} - \Gamma^i_{pk}\Gamma^p_{jm}.$$

The system of function R^i_{jkm} defined on $T^1_n Q_S$ will be called the *formal curvature tensor*. The *first formal covariant derivative* of the formal curvature tensor is the system of functions

$$(8.3.7) \qquad R^i_{jkl;m} = R^i_{jkl,m} + \Gamma^i_{pm}R^p_{jkl} - \Gamma^p_{mj}R^i_{pkl} - \Gamma^p_{mk}R^i_{jpl} - \Gamma^p_{ml}R^i_{jkp},$$

where

$$R^i_{jkl,m} = \sum_{r \leqq s}\left(\frac{\partial R^i_{jkl}}{\partial \Gamma^p_{rs}}\Gamma^p_{rs,m} + \frac{\partial R^i_{jkl}}{\partial \Gamma^p_{rs,q}}\Gamma^p_{rs,qm}\right).$$

Thus $R^i_{jkl;m}$ is a function defined on $T^2_n Q_S$. In general, the *s-th formal covariant derivative* of the formal curvature tensor is the system of functions $R^i_{jkl;m_1;\ldots;m_s}$, $s \leqq r - 1$, defined on $T^{s+1}_n Q_S$. These functions are induced formally in the same way as the s-th covariant derivative of the curvature tensor field on a manifold endowed with a linear connection and are transformed under the action of L^{s+3}_n as a $(1, s+3)$ tensor.

We put for every s, $1 \leqq s \leqq r$

$$(8.3.8) \qquad \Gamma^i_{jkm_1\ldots m_s} = \Gamma^i_{(jk,m_1\ldots m_s)},$$

where the symbol $(jk, m_1 \ldots m_s)$ on the right hand side denotes the symmetrization in the indices $j, k, m_1, \ldots, m_s$.

Lemma 8.5. *The system of functions*

$$(8.3.9) \qquad \begin{aligned} \bar{\Gamma}_r &= (\Gamma^i_{jk}, \Gamma^i_{jkm_1}, \ldots, \Gamma^i_{jkm_1\ldots m_r}), \\ R_{r-1} &= (R^i_{jkl}, R^i_{jkl;m_1}, \ldots, R_{jkl;m_1;\ldots;m_{r-1}}), \end{aligned}$$

contains a subsystem defining a global chart on $T^r_n Q_S$, $r \geqq 1$.

Proof. For each s, $1 \leqq s \leqq r$, consider the canonical coordinates Γ_r on $T_n^r Q_S$. We have the decomposition

$$\Gamma^i_{jk,m_1\dots m_s} = \Gamma^i_{jkm_1\dots m_s} + (\Gamma^i_{jk,m_1\dots m_s} - \Gamma^i_{jkm_1\dots m_s}). \tag{8.3.10}$$

The expression in the brackets may be written in a unique way as a linear combination (with real coefficients) of terms of the form

$$\Delta^i_{jklm_1\dots m_{s-1}} = \Gamma^i_{jk,lm_1\dots m_{s-1}} - \Gamma^i_{jl,km_1\dots m_{s-1}}.$$

Let us denote the systems $\Gamma_r, \overline{\Gamma}_r, \Delta_{r-1} = (\Delta^i_{jkl}, \Delta^i_{jklm_1}, \dots, \Delta^i_{jklm\dots m_{r-1}})$. Then (8.3.10) gives an injection of Γ_r into $(\overline{\Gamma}_r, \Delta_{r-1})$. If we denote this injective mapping by α, it is clear that the system of functions $\alpha\Gamma_r$ defines a global chart on $T_n^r Q_S$.

Now if we consider the s-th formal covariant derivative of R^i_{jkl} we have from definition

$$R^i_{jkl;m_1;\dots;m_{s-1}} = \Delta^i_{jklm_1\dots m_{s-1}} + P^i_{jklm_1\dots m_{s-1}},$$

where $P^i_{jklm_1\dots m_{s-1}}$ is a polynomial in the canonical coordinates on $T_n^{s-1}Q_S$. Hence replacing Δ_{r-1} in the global chart $\alpha\Gamma_r$ by R_{r-1} we obtain the subsystem of (8.3.9) required.

Each global chart on $T_n^r Q_S$ defined by Lemma 8.5 will be called an *adapted chart.* The functions (8.3.9) belonging to an adapted chart will be called *adapted coordinates.* Relations (8.3.10) imply that the functions $\Gamma^i_{jkm_1\dots m_s}$, $0 \leqq s \leqq r$, belong to each adapted chart.

It is easy to see that in the canonical coordinates on K_n^{r+2} and adapted coordinates on $T_n^r Q_S$, the action of K_n^{r+2} on $T_n^r Q_S$ is expressed by the formulas

$$\begin{aligned} &\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + b^i_{jk}, \\ &\overline{\Gamma}^i_{jkm_1\dots m_s} = \Gamma^i_{jkm_1\dots m_s} + S^i_{jkm_1\dots m_s} + b^i_{jkm_1\dots m_s}, \\ &\overline{R}^i_{jkl;m_1;\dots;m_{s-1}} = R^i_{jkl;m_1;\dots;m_{s-1}}, \end{aligned} \tag{8.3.11}$$

where $1 \leqq s \leqq r$, and $S^i_{jkm_1\dots m_s}$ is a polynomial in the canonical coordinates on K_n^{s+1} and in the adapted coordinates on $T_n^{s-1}Q_S$. (8.3.11) implies that the action of K_n^{r+2} on $T_n^r Q_S$ is free. Applying Theorem 1.11 we can see that the orbit space $T_n^r Q_S/K_n^{r+2}$ has the manifold structure such that the canonical projection π^r: $T_n^r Q_S \to T_n^r Q_S/K_n^{r+2}$ is a submersion. Then any system of independent functions $R^i_{jkl;m_1;\dots;m_{s-1}}$, $1 \leqq s \leqq r$, form a global chart on $T_n^r Q_S/K_n^{r+2}$. Using this fact and Theorem 7.6 we immediately obtain

Theorem 8.5. *Each L_n^{r+2}-equivariant mapping from $T_n^r Q_S$, $r \geqq 1$, to any L_n^1-manifold depends only on the formal curvature tensor and its formal covariant derivatives up to $(r-1)$-th order.*

In the sense of Theorem 8.5 the formal curvature tensor and its formal covariant derivatives form a base of differential invariants of a formal connection with values in an L_n^1-manifold.

Let E be a finite-dimensional vector space endowed with a linear representation of the group L_n^1. Let us consider some global coordinates (t_σ) on E. Then the functions $t_r = (t_\sigma, t_{\sigma, i_1}, \ldots, t_{\sigma, i_1 \ldots i_r})$, $1 \leqq i_1 \leqq \ldots \leqq i_s \leqq n$, $s = 1, \ldots, r$, form the canonical coordinates on $T_n^r E$. Using a formal connection we can define, in the same manner as for the formal curvature tensor, the formal covariant derivatives $\bar{t}_s = (t_{\sigma; i_1; \ldots; i_s})$, $s = 1, \ldots, r$.

Lemma 8.6. *The system of functions*

$$\begin{aligned} \bar{\Gamma}_{r-1} &= (\Gamma^i_{jk}, \Gamma^i_{jkm_1}, \ldots, \Gamma^i_{jkm_1 \ldots m_{r-1}}), \\ R_{r-2} &= (R^i_{jkl}, R^i_{jkl;m_1}, \ldots, R^i_{jkl;m_1;\ldots;m_{r-2}}), \\ \bar{t}_r &= (t_\sigma, t_{\sigma;m_1}, \ldots, t_{\sigma;m_1;\ldots;m_r}), \end{aligned} \tag{8.3.12}$$

contains a subsystem defining a global chart on $T_n^{r-1} Q_S \times T_n^r E$, $r \geqq 2$.

Proof. Using the same notation as in Lemma 8.5 we have the canonical global chart Γ_{r-1}, t_r on $T^{r-1} Q_S \times T_n^r E$. Then we can replace these coordinates by a new global chart $\alpha\Gamma_{r-1}$, $\bar{t}_r$ and Lemma 8.5 implies Lemma 8.6.

Remark 8.8. If $r = 1$ then $(\Gamma^i_{jk}, t_\sigma, t_{\sigma;m})$ form a global chart on $Q_S \times T_n^1 E$.

Each global chart on $T_n^{r-1} Q_S \times T_n^r E$ defined by Lemma 8.6 or by Remark 8.8 will be called an *adapted chart*. The function (8.3.12) belonging to an adapted chart will be called *adapted coordinates*. In the canonical coordinates on K_n^{r+1} and adapted coordinates on $T_n^{r-1} Q_S \times T_n^r E$ the action of K_n^{r+1} on $T_n^{r-1} Q_S \times T_n^r E$ is expressed by (8.3.11) if $s = 1, \ldots, r-1$ and by

$$\bar{t}_{\sigma; i_1; \ldots; i_s} = t_{\sigma; i_1; \ldots; i_s}, \tag{8.3.13}$$

where $s = 0, \ldots, r$.

This means that the system

$$R_{s-1} = (R^i_{jkl;m_1;\ldots;m_{s-2}}),\ \bar{t}_s = (t_\sigma, t_{\sigma;m_1}, \ldots, t_{\sigma;m_1;\ldots;m_s}),$$

where $s = 2, \ldots, r$, $r \geqq 2$, contains a subsystem which forms a global chart on the quotient space $(T_n^{r-1} Q_S \times T_n^r E)/K^{r+1}$. If $r = 1$ then $(t_\sigma, t_{\sigma;m})$ form a global chart on $(Q_S \times T_n^1 E)/K^2$. Consequently, we obtain from Theorem 7.6

Theorem 8.6. *Each L_n^{r+1}-equivariant mapping of $T_n^{r-1} Q_S \times T_n^r E$, $r \geqq 2$, to any L_n^1-manifold depends on the formal curvature tensor, its formal covariant derivatives up to $(r-2)$-nd order and on the formal covariant derivatives of elements of E up to order r. If $r = 1$ each L_n^2-equivariant mapping from $Q_S \times T_n^1 E$ to any L_n^1-manifold dependens on elements of E and its first-order formal covariant derivatives.*

Now let $E = P$ be a (p, q)-tensor space. We want to find L_n^{r+1}-equivariant mappings from $T_n^r P$ to a (k, l)-tensor space S. We replace $T_n^r P$ by $U = T_n^{r-1} Q_S \times T_n^r P$ and we shall find differential invariants of L_n^{r+1} from U to S. Then by Theorem 7.6 every differential invariant $g : U \to S$ of L_n^{r+1} determines a unique differential invariant $G : U/K_n^{r+1} \to S$ of L_n^1 such that $g = G \circ \pi^r$. If we consider only such differential invariants g which do not depend on the coordinates on $T_n^{r-1} Q_S$, i.e. $g = f \circ p_2$, where $p_2 : T_n^{r-1} Q_S \times T_n^r P \to T_n^r P$ is the canonical projection on the second component and $f : T_n^r P \to S$ is a differential invariant of L_n^{r+1}, we obtain the folowing commutative diagram

$$
\begin{array}{ccccc}
T_n^{r-1}Q_x \times T_n^r P & \xrightarrow{p_2} & T_n^r P & \xrightarrow{f} & S \\
 & {\scriptstyle \pi^r}\searrow & & \nearrow{\scriptstyle F} & \\
 & & (T_n^{r-1}Q_x \times T_n^r P)/K_n^{r+1} & &
\end{array}
\tag{8.3.14}
$$

where F is uniquely determined differential invariant of L_n^1. Because of Theorem 8.6 a differential invariant F has a coordinate expression

$$u_{j_1 \dots j_l}^{i_1 \dots i_k} = F_{j_1 \dots j_l}^{i_1 \dots i_k}(R_{jkl}^i, \dots, R_{jkl; m_1; \dots; m_{r-2}}^i, t_{j_1 \dots j_q}^{i_1 \dots i_p}, \dots, t_{j_1 \dots j_q; m_1; \dots; m_r}^{i_1 \dots i_p}). \tag{8.3.15}$$

Applying Theorem 8.4 we are able to find all polynomial differential invariants (8.3.15) (in many cases they will be all global differential invariants). Such an invariant has to be a sum of homogeneous polynomials of degrees a_r in $R_{jkl; m_1; \dots; m_s}^l$ and b_s in $t_{j_1 \dots j_q; m_1; \dots; m_s}^{i_1 \dots i_p}$ such that

$$\sum_{s=0}^{r-2} a_s(-2 - s) + \sum_{s=0}^{r} b_s(p - q - s) = k - l. \tag{8.3.16}$$

But we consider only differential invariants wich do not depend on the formal connection, i.e. $a_s = 0$ for all s. Then F is a polynomial in the formal covariant derivatives of $t_{j_1 \dots j_q}^{i_1 \dots i_p}$. If we put this part of the differential invariant which contains the formal connection equal to zero, we obtain some system of linear homogeneous equations for coefficients. Solving this system we obtain the needed differential invariant f.

Example 8.5. We shall describe all global natural differential operators of finite order from $T^{(1,2)}$ to $T^{(1,3)}$. Each natural differential operator of order r determines a differential invariant $f : T_n^r(R^n \otimes \overset{2}{\otimes} R^{n*}) \to R^n \otimes \overset{3}{\otimes} R^{n*}$. Using the above method we get that all global differential invariants F are sums of homogeneous polynomials of degrees b_r in $t_{j_1 j_2; m_1; \dots; m_s}^l$ such that

$$\sum_{s=0}^{r} b_s(-1 - s) = -2. \tag{18.3.17}$$

(8.3.17) has only two natural solutions: 1. $b_0 = 2$, $b_i = 0$, $i = 1, \dots, r$. 2. $b_0 = 0$, $b_1 = 1$, $b_i = 0$, $i = 2, \dots, r$. Then F has an expression

$$(8.3.18) \qquad u^i_{j_1j_2j_3} = A^{iq_1q_2s_1s_2}_{j_1j_1j_3p}{}^{r}_{p} t^p_{q_1q_2} t^r_{s_1s_2} + B^{iq_1q_2q_3}_{j_1j_2j_3p} t^p_{q_1q_2;q_3}.$$

The quadratic part of (8.3.18) we have determined in Example 8.2. The L^1_n-equivariancy implies that $B^{iq_1q_2q_3}_{j_1j_2j_3p}$ is an absolute invariant tensor, i.e.

$$B^{iq_1q_2q_3}_{j_1j_2j_3p} = \sum_{\sigma(4)} c_\sigma \delta^i_{\sigma(j_1)} \dots \delta^{q_3}_{\sigma(p)},$$

where σ runs over all permutations of four indices. F does not depend on the formal connection, i.e.

$$(8.3.19) \qquad B^{iq_1q_2q_3}_{j_1j_2j_3p}(t^p_{q_1q_2;q_3} - t^p_{q_1q_2,q_3}) = 0.$$

(8.3.19) gives for coefficients c_σ a system of homogeneous equations which has six independent variables. Denoting them by A_i, $i = 1, \dots, 6$, we obtain the linear part of (8.3.18) in the form

$$(8.3.20) \qquad \begin{aligned} u^i_{j_1j_2j_3} &= A_1\delta^i_{j_1}(t^p_{pj_2,j_3} - t^p_{pj_3,j_2}) + A_2\delta^i_{j_1}(t^p_{j_2p,j_3} - t^p_{j_3p,j_2}) + \\ &+ A_3\delta^i_{j_2}(t^p_{pj_1,j_3} - t^p_{pj_3,j_1}) + A_4\delta^i_{j_2}(t^p_{j_1p,j_3} - t^p_{j_3p,j_1}) + \\ &+ A_5\delta^i_{j_3}(t^p_{pj_1,j_2} - t^p_{pj_2,j_1}) + A_6\delta^i_{j_3}(t^p_{j_1p,j_2} - t^p_{j_2p,j_1}). \end{aligned}$$

Thus all global natural differential operators of a finite order from $T^{(1,2)}$ to $T^{(1,3)}$ are of order less than or equal to one. The operators of order one which are not of order zero are given by contractions, the exterior derivative and a linear natural differential operator of order zero of $\wedge^2 T^*$ to $T^{(1,3)}$.

8.4. The uniqueness of exterior derivative. It is well known that the *exterior derivative* d is a linear globally defined natural differential operator of order one from $\wedge^p T^*$ to $\wedge^{p+1} T^*$ which corresponds to differential invariant from $T^1_n(\wedge^p R^{n*})$ to $\wedge^{p+1} R^{n*}$ given in the canonical global coordinates $\omega_{i_1 \dots i_q}$, $1 \leqq i_1 < \dots < i_q \leqq n$ on $\wedge^q R^{n*}$ by

$$(8.4.1) \qquad \omega_{i_1 \dots i_{p+1}} = \omega_{[i_1 \dots i_p, i_{p+1}]},$$

where $[i_1 \dots i_p, i_{p+1}]$ means the antisymmetrization in the indices $i_1, \dots, i_{p+1}$.

Using the methods described in Section 8.3 we shall prove the uniqueness of exterior derivative. First we shall prove a more general

Theorem 8.7. (*i*) *All global natural differential operators* (*of any finite order*) *from* $T^{(0,1)}$ *to* $T^{(0,2)}$ *are linear combinations* (*with real coefficients*) *of the exterior derivative and the tensor product.*

(*ii*) *All global natural differential operators* (*of any finite order*) *from* $T^{(0,p)}$ *to*

$T^{(0,p+1)}$, *where* $2 \leqq p$, *are constant multiples of* $\mathrm{d} \circ \mathrm{Alt}$, *where* $\mathrm{Alt} : T^{(0,p)} \to \wedge^p T^*$ *is the linear natural differential operator (of order zero) given by the antisymmetrization of indices.*

Proof. Let f be a global natural differential operator of order r from $T^{(0,p)}$ to $T^{(0,p+1)}$. Then f is given by a global differential invariant from $T_n^r(\overset{p}{\otimes} R^{n*})$ to $\overset{p+1}{\otimes} R^{n*}$. Using Theorems 8.4. and 8.6 we obtain that such a global differential invariant is a sum of homogeneous polynomials of degrees a_i in i-th formal covariant derivatives of $\omega_{i_1 \dots i_p}$ such that

$$\sum_{i=0}^{r} a_i(p+i) = p+1. \tag{8.4.2}$$

(*i*) Let $p = 1$ then (8.4.2) has two possible natural solutions. 1. $a_0 = 2$, $a_i = 0$, $i = 1, \dots, r$. 2. $a_0 = 0$, $a_1 = 1$, $a_i = 0$, $i = 2, \dots, r$. Then the differential invariants are in the form

$$\omega_{i_1 i_2} = A_{i_1 i_2}^{j_1 j_2} \omega_{j_1} \omega_{j_2} + B_{i_1 i_2}^{j_1 j_2} \omega_{j_1; j_2},$$

where $A_{i_1 i_2}^{j_1 j_2}$ and $B_{i_1 i_2}^{j_1 j_2}$ are absolute invariant tensors. Thus $A_{i_1 i_2}^{j_1 j_2} = c_1 \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} + c_2 \delta_{i_1}^{j_2} \delta_{i_2}^{j_1}$, $B_{i_1 i_2}^{j_1 j_2} = \bar{c}_1 \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} + \bar{c}_2 \delta_{i_1}^{j_2} \delta_{i_2}^{j_1}$. $A_{i_1 i_2}^{j_1 j_2}$ has to satisfy the symmetry condition $A_{i_1 i_2}^{j_1 j_2} = A_{i_1 i_2}^{j_2 j_1}$ which implies $c_1 = c_2 = \beta/2$. $B_{i_1 i_2}^{j_1 j_2}$ has to satisfy the condition $B_{i_1 i_2}^{j_1 j_2}(\omega_{j_1; j_2} - \omega_{j_1, j_2}) = 0$ which implies $\bar{c}_1 = -\bar{c}_2 = \alpha/2$. Then

$$\omega_{i_1 i_2} = \beta \omega_{i_1} \omega_{i_2} + \alpha \omega_{[i_1, i_2]},$$

where α, β are real numbers, is the differential invariant corresponding to the natural differential operator $\omega \mapsto \beta \omega \otimes \omega + \alpha \, \mathrm{d}\omega$. This proves the first part of Theorem 8.7.

(*ii*) Let $1 < p$. Then (8.4.2) has the unique natural solution $a_0 = 0$, $a_1 = 1$, $a_i = 0$, $i = 2, \dots, r$. Then

$$\omega_{i_1 \dots i_{p+1}} = A_{i_1 \dots i_{p+1}}^{j_1 \dots j_{p+1}} \omega_{j_1 \dots j_p; j_{p+1}},$$

where $A_{i_1 \dots i_{p+1}}^{j_1 \dots j_{p+1}}$ is an absolute invariant tensor satisfying

$$A_{i_1 \dots i_{p+1}}^{j_1 \dots j_{p+1}} (\omega_{j_1 \dots j_p; j_{p+1}} - \omega_{j_1 \dots j_p, j_{p+1}}) = 0. \tag{8.4.3}$$

From (8.4.3) we get

$$A_{i_1 \dots i_{p+1}}^{j_1 \dots j_k \dots j_p j_{p+1}} + A_{i_1 \dots i_{p+1}}^{j_1 \dots j_{p+1} \dots j_p j_k} = 0,$$

for all $k = 1, \dots, p$. Because $A_{i_1 \dots i_{p+1}}^{j_1 \dots j_{p+1}} = \sum_{\sigma} c_\sigma \delta_{i_1}^{j_{\sigma(1)}} \dots \delta_{i_{p+1}}^{j_{\sigma(p+1)}}$ we obtain for any permutation σ of $(1, \dots, p+1)$

$$c_{(\sigma(1), \dots, \sigma(k), \dots, \sigma(p+1))} + c_{(\sigma(1), \dots, \sigma(p+1), \dots, \sigma(k))} = 0 \tag{8.4.4}$$

for any $k = 1, \dots, p$. Hence we have $(p+1)!\,p/2$ homogeneous equations of $(p+1)!$ variables. This system of homogeneous equations has a unique independent variable. Let us choose for this independent variable $c_{(1,\dots,p+1)} = \alpha/(p+1)!$, where $\alpha \in R$. Then from (8.4.4), $c_\sigma = (\text{sign}\,\sigma)\,\alpha/(p+1)!$ and

$$\omega_{i_1 \dots i_{p+1}} = \alpha\omega_{[i_1 \dots i_p, i_{p+1}]},$$

which proves *(ii)* in our theorem.

Now we immediately obtain

Theorem 8.8. *The exterior derivative is a unique, up to a multiplicative constant factor, global natural differential operator (of any finite order) from* $\wedge^p T^*$ *to* $\wedge^{p+1} T^*$, *where* $p \geqq 1$.

Proof. Theorem 8.8 is the direct consequence of Theorem 8.7 because of $\omega \wedge \omega = 0$.

Remark 8.9. The linearity condition is not necessary in the proof of Theorem 8.8. This condition follows from naturality but only if $p \geqq 1$. If $p = 0$ Theorem 8.8 is not true. It is easy to see that $f \mapsto f\,df$ is an example of a global natural differential operator of order one from $\wedge^0 T^*$ to $\wedge^1 T^*$ which is not a constant multiple of the exterior derivative.

8.5. Bilinear natural differential operators on vector valued forms. Let us denote the *space of TX-valued p-forms* on X by $\Omega^p(X; TX)$; an element of $\Omega^p(X; TX)$ is by definition a section of the vector bundle $TX \otimes \wedge^p T^*X$ over X. On $\Omega(X; TX) = \bigoplus_p \Omega^p(X; TX)$ we can define the so-called *Frölicher–Nijenhuis bracket* as follows: let $\varphi \in \Omega^r(X; TX)$ and $\psi \in \Omega^s(X; TX)$ then there is a unique $[\varphi, \psi] \in \Omega^{r+s}(X; TX)$ such that for any vector fields $\xi_1, \dots, \xi_{r+s}$ on X

$$\begin{aligned}
&[\varphi, \psi](\xi_1, \dots, \xi_{r+s}) = \\
&= \frac{1}{r!\,s!} \sum_\sigma \text{sign}\,\sigma\{[\varphi(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}), \psi(\xi_{\sigma(r+1)}, \dots, \xi_{\sigma(r+s)})] - \\
&- r\varphi(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r-1)}, [\xi_{\sigma(r)}, \psi(\xi_{\sigma(r+1)}, \dots, \xi_{\sigma(r+s)})]) - \\
&- s\psi([\varphi(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}), \xi_{\sigma(r+1)}], \xi_{\sigma(r+2)}, \dots, \xi_{\sigma(r+s)}) + \\
&+ \frac{rs}{2}\varphi(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r-1)}, \psi([\xi_{\sigma(r)}, \xi_{\sigma(r+1)}], \xi_{\sigma(r+2)}, \dots)) + \\
&+ \frac{rs}{2}\psi(\varphi(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r-1)}, [\xi_{\sigma(r)}, \xi_{\sigma(r+1)}]), \xi_{\sigma(r+2)}, \dots)\},
\end{aligned} \tag{8.5.1}$$

where σ runs over all permutations of the set $(1, \dots, r+s)$ and $[,]$ on the right hand side denotes the standard Lie bracket.

Let (U, Φ) be a chart on X, $\Phi = (x^i)$ the corresponding local coordinates. Then on $TX \otimes \wedge^p T^*X$ we have the induced coordinates $(x^i, \omega^i_{j_1 \dots j_p})$, $1 \leqq j_1 < \dots \dots < j_p \leqq n$. Recall that if $\omega \in T_x X \otimes \wedge^p T^*_x X$, where $x \in U$, then

$$\omega = \omega^i_{j_1 \dots j_p} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_p}. \tag{8.5.2}$$

Let $\varphi \in \Omega^r(X; TX)$, $\psi \in \Omega^s(X; TX)$ be two TX-valued forms, let $\varphi^i_{j_1 \dots j_r}(x)$ (resp. $\psi^i_{j_1 \dots j_s}(x)$) be the components of φ (resp. ψ) with respect to the chart (U, Φ). Then the TX-valued $(r + s)$-form $[\varphi, \psi]$ defined by (8.5.1) is expressed in the chart (U, Φ) by

$$\begin{aligned} [\varphi, \psi](x) = \Bigg(& \varphi^m_{j_1 \dots j_r}(x) \frac{\partial \psi^i_{j_{r+1} \dots j_{r+s}}(x)}{\partial x^m} - \\ & - \psi^m_{j_{r+1} \dots j_{r+s}}(x) \frac{\partial \varphi^i_{j_1 \dots j_r}(x)}{\partial x^m} - r \varphi^i_{j_1 \dots j_{r-1} m}(x) \frac{\partial \psi^m_{j_{r+1} \dots j_{r+s}}(x)}{\partial x^{j_r}} + \\ & + s \psi^i_{m j_{r+2} \dots j_{r+s}}(x) \frac{\partial \varphi^m_{j_1 \dots j_r}(x)}{\partial x^{j_{r+1}}} \Bigg) \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_{r+s}}. \end{aligned} \tag{8.5.3}$$

From the definition it immediately follows that the Frölicher – Nijenhuis bracket extends the Lie bracket of vector fields (TX-valued 0-forms) to arbitrary TX-valued forms.

From the coordinate expression (8.5.3) it follows that the Frölicher – Nijenhuis bracket $[,]$ defines a bilinear natural differential operator of order one of the functor $(T \otimes \wedge^r T^*) \oplus (T \otimes \wedge^s T^*)$ to the functor $T \otimes \wedge^{r+s} T^*$. The corresponding differential invariant of L^2_n from $T^1_n((R^n \otimes \wedge^r R^{n*}) \times (R^n \otimes \wedge^s R^{n*}))$ to $R^n \otimes \otimes \wedge^{r+s} R^{n*}$ is given by (8.5.3).

Let us consider more general case of finding bilinear natural differential operators of any finite order k from $T^{(1,r)} \oplus T^{(1,s)}$ to $T^{(1,r+s)}$.

Theorem 8.9. *All bilinear natural differential operators of any finite order k of $T^{(1,r)} \oplus T^{(1,s)}$ to $T^{(1,r+s)}$ are of order one.*

Proof. Let A_k be a bilinear natural differential operator of order k from $T^{(1,r)} \oplus T^{(1,s)}$ to $T^{(1,r+s)}$. Let $(\varphi^i_{j_1 \dots j_r}, \psi^i_{j_1 \dots j_s})$ be the canonical global coordinates on the type fiber $E = (R^n \otimes \otimes^r R^{n*}) \times (R^n \otimes \otimes^s R^{n*})$ of $T^{(1,r)} \oplus T^{(1,s)}$. The associated differential invariant of L^{k+1}_n is a bilinear mapping $\lambda_k : T^k_n E \to R^n \otimes \otimes \otimes^{r+s} R^{n*}$. Using Theorem 8.6 each such a differential invariant of L^{k+1}_n determines a unique differential invariant Λ_k of L^1_n depending on the formal covariant derivatives of $\varphi^i_{j_1 \dots j_r}$ and $\psi^i_{j_1 \dots j_s}$ up to order k. Λ_k has to be bilinear and hence polynomial. Theorem 8.4 implies that Λ_k is a sum of homogeneous polynomials of degrees a_l in $\varphi^i_{j_1 \dots j_r; m_1; \dots; m_l}$ and b_l in $\psi^i_{j_1 \dots j_s; m_1; \dots; m_l}$, $0 \leqq l \leqq k$, such that

(8.5.4) $$\sum_{l=0}^{k} (a_l(1 - r - l) + b_l(1 - s - l)) = 1 - r - s$$

is satisfied. Only two solutions of (8.5.4) correspond to bilinear mappings: *a*) $a_0 = 1$, $b_1 = 1$, and the other a_l, b_l are equal to zero; *b*) $a_1 = 1$, $b_0 = 1$, and the other a_l, b_l are equal to zero. This means that A_k is defined on $Q_S \times T^1_n E$ only and our theorem is proved.

The components $A^i_{j_1 \ldots j_{r+s}}$ of A_k are given by

(8.5.5) $$A^i_{j_1 \ldots j_{r+s}} = A^{iq_1 \ldots q_{r+1} u_1 \ldots u_s}_{j_1 \ldots j_{r+s} pt} \varphi^p_{q_1 \ldots q_r; q_{r+1}} \psi^t_{u_1 \ldots u_s} + \\ + B^{iq_1 \ldots q_r u_1 \ldots u_{s+1}}_{j_1 \ldots j_{r+s} pt} \varphi^p_{q_1 \ldots q_r} \psi^t_{u_1 \ldots u_s; u_{s+1}},$$

where $A^{i \ldots u_s}_{j_1 \ldots t}$, $B^{i \ldots u_{s+1}}_{j_1 \ldots t}$ are absolute invariant tensors, i.e. $A^{i \ldots u_s}_{j_1 \ldots t} = \sum_\sigma c_\sigma \delta^{\sigma(i)}_{j_1} \ldots \delta^{\sigma(u_s)}_t$, $B^{i \ldots u_{s+1}}_{j_1 \ldots t} = \sum_\sigma d_\sigma \delta^{\sigma(i)}_{j_1} \ldots \delta^{\sigma(u_{s+1})}_t$, where σ runs over all permutations of $(r + s + 2)$ indices and c_σ, d_σ are real numbers. We consider only such differential invariants (8.5.5) which do not depend on the formal connection, i.e.

(8.5.6) $$A^{i \ldots u_s}_{j_1 \ldots t} (\varphi^p_{q_1 \ldots q_r; q_{r+1}} - \varphi^p_{q_1 \ldots q_r, q_{r+1}}) \psi^t_{u_1 \ldots u_s} + \\ + B^{i \ldots u_{s+1}}_{j_1 \ldots t} \varphi^p_{q_1 \ldots q_r} (\psi^t_{u_1 \ldots u_s; u_{s+1}} - \psi^t_{u_1 \ldots u_s, u_{s+1}}) = 0,$$

(8.5.6) gives a system of homogeneous linear equations for c_σ and d_σ. Solving this system and substituting the solution into (8.5.5) where we replace the formal covariant derivatives by (ordinary) formal derivatives we obtain the required differential invariants of L^2_n from $T^1_n E$ to $R^n \otimes \otimes^{r+s} R^{n*}$.

Corollary 1. *All bilinear natural differential operators of finite order from* $(T \otimes \otimes \wedge^r T^*) \oplus (T \otimes \wedge^s T^*)$ *to* $T \otimes \wedge^{r+s} T^*$ *are of order one.*

Now we shall find all bilinear natural differential operators described above for same special cases.

A. Let $n = 1$. Let (t) be a local coordinate on $X \in \mathrm{Ob}\, \mathscr{D}_1$. Then a $(1, r)$-tensor field φ on X has the coordinate expression $\varphi(t)\, \mathrm{d}/\mathrm{d}t \otimes \mathrm{d}t \otimes \ldots \otimes \mathrm{d}t$, ($r$ factors $\mathrm{d}t$).

Theorem 8.10. *Let φ be a $(1, r)$-tensor field and ψ a $(1, s)$-tensor field on $X \in \in \mathrm{Ob}\, \mathscr{D}_1$. Then all bilinear natural differential operators from $T^{(1,r)} \oplus T^{(1,s)}$ to $T^{(1,r+s)}$ are given by*

(8.5.7) $$\left(A \frac{\mathrm{d}\varphi}{\mathrm{d}t} \psi + B \frac{\mathrm{d}\psi}{\mathrm{d}t} \varphi \right) \frac{\mathrm{d}}{\mathrm{d}t} \otimes \mathrm{d}t \otimes \ldots \otimes \mathrm{d}t,$$

$(r + s)$ factors $\mathrm{d}t$, where A, B are real numbers satisfying

(8.5.8) $$(r - 1)\, A + (s - 1)\, B = 0,$$

i.e. if $r = 1$, $s = 1$ there are two independent bilinear natural differential operators *and in the other cases there is a unique bilinear natural differential operator from* $T^{(1,r)} \oplus T^{(1,s)}$ *to* $T^{(1,r+s)}$.

Proof. Let us denote the type fiber of $T^{(1,r)}$ by $R^{(1,r)}$. Let (ω) be the canonical coordinate on $R^{(1,r)}$. A differential invariant corresponding to a bilinear natural differential operator from $T^{(1,r)} \oplus T^{(1,s)}$ to $T^{(1,r+s)}$ is given by (8.5.5) which reduces to

$$(8.5.9) \qquad \Lambda = A\varphi^{;}\psi + B\varphi\psi^{;},$$

where A, B are real numbers and $\varphi^{;}$ denotes the formal covariant derivative of φ. Then the condition (8.5.6) reduces to the equation (8.5.8).

If $r = 1$, $s = 1$ (8.5.8) is satisfied identically and there are two independent differential invariants and any differential invariant from $R^{(1,1)} \times R^{(1,1)}$ to $R^{(1,2)}$ can be expressed by

$$(8.5.10) \qquad \lambda = A\varphi'\psi + B\varphi\psi',$$

where φ' denotes the (ordinary) formal derivative.

If $r = 1$, $s \neq 1$ (or $r \neq 1$, $s = 1$) (8.5.8) implies $B = 0$ (or $A = 0$) and there is the unique bilinear differential invariant $A\varphi'\psi$ (or $B\varphi\psi'$) from $R^{(1,1)} \times R^{(1,s)}$ (or $R^{(1,r)} \times R^{(1,1)}$) to $R^{(1,s+1)}$ (or $R^{(1,r+1)}$).

If $r \neq 1$, $s \neq 1$ then from (8.5.8) $B = -A(r-1)/(s-1)$ and we get the unique differential invariant $A(\varphi'\psi - \varphi\psi'(r-1)/(s-1))$ from $R^{(1,r)} \times R^{(1,s)}$ to $R^{(1,r+s)}$.

Thus all bilinear differential invariants from $R^{(1,r)} \times R^{(1,s)}$ to $R^{(1,r+s)}$ are given by (8.5.10) where A, B are real numbers satisfying (8.5.8). We shall verify that (8.5.10) is really a differential invariant of L^2_1. On $T^1_1 R^{(1,r)}$ we have the canonical global coordinates (ω, ω'). The action of L^2_1 on $T^1_1 R^{(1,r)}$ is given by

$$(8.5.11) \qquad \bar{\omega} = \omega a^{1-r}, \qquad \bar{\omega}' = \omega' a^{-r} + \omega(1-r)\,\dot{a}a^{-1-r},$$

where $(a, \dot{a})$ are the canonical global coordinates on L^2_1, i.e. $a \neq 0$. Then (8.5.10) is a differential invariant of L^2_1 if

$$(8.5.12) \qquad \begin{aligned}(A\varphi'\psi + B\varphi\psi')\,a^{1-r-s} = A(\varphi' a^{-r} + \varphi(1-r)\,\dot{a}a^{-1-r})\,\psi a^{1-s} + \\ + B\varphi a^{1-r}(\psi' a^{-s} + \psi(1-s)\,\dot{a}a^{-1-s}).\end{aligned}$$

It is easy to see that (8.5 12) is satisfied if and only if (8.5.8) is satisfied.

B. Let $n \geqq 2$.

I. Let $r = 0$, $s = 0$. In this case the Frölicher – Nijenhuis bracket coincides with the Lie bracket.

Theorem 8.11. *The Lie bracket is a unique, up to a multiplicative constant, bilinear natural differential operator from* $T^{(1,0)} \oplus T^{(1,0)}$ *to* $T^{(1,0)}$.

Proof. Let (ξ^i, η^i) be the canonical global coordinates on the type fiber $R^n \times R^n$ of $T^{(1,0)} \oplus T^{(1,0)}$. All differential invariants corresponding to required operators have the expression (8.5.5), i.e.

(8.5.13) $\quad \Lambda^i = A^{iq}_{pt}\xi^p_{;q}\eta^t + B^{iq}_{pt}\xi^p\eta^t_{;q}.$

But A^{iq}_{pt}, B^{iq}_{pt} are absolute invariant tensors, i.e. $A^{iq}_{pt} = c_1\delta^i_p\delta^q_t + c_2\delta^i_t\delta^p_q$, $B^{iq}_{pt} = d_1\delta^i_p\delta^q_t + d_2\delta^i_t\delta^q_p$. (8.5.6) then implies $c_1 + d_2 = 0$, $c_2 = 0$ and $d_1 = 0$. Thus

(8.5.14) $\quad \lambda^i = c(\xi^i_{,m}\eta^m - \eta^i_{,m}\xi^m),$

which proves our theorem.

II. Let $r = 0$, $s = 1$.

Theorem 8.12. *There is a* 3-*parameter family of bilinear natural differential operators from* $T^{(1,0)} \oplus T^{(1,1)}$ *to* $T^{(1,1)}$. *This family associates to each vector field* ξ *and to each* (1,1)-*tensor field* φ *on* X *the* (1,1)-*tensor field*

(8.5.15) $\quad AI \otimes \mathrm{tr}\,(\xi \otimes \mathrm{d}(\mathrm{tr}\,\varphi)) + B\xi \otimes \mathrm{d}(\mathrm{tr}\,\varphi) + C[\xi, \varphi],$

where A, B, C *are real numbers,* [,] *is the Frölicher–Nijenhuis bracket,* tr *is the contraction of* (1,1)-*tensors,* d *is the exterior derivative, and* I *is the absolute invariant tensor field* $I = (\delta^i_j)$.

Proof. Let (ξ^i, φ^i_j) be the canonical global coordinates on the type fiber $P = R^n \times (R^n \otimes R^{n*})$ of $T^{(1,0)} \oplus T^{(1,1)}$. Theorem 8.9 implies that all bilinear natural differential operators from $T^{(1,0)} \oplus T^{(1,1)}$ to $T^{(1,1)}$ have corresponding bilinear differential invariants expressed by (8.5.5), i.e.

(8.5.16) $\quad \Lambda^i_j = A^{irs}_{jtp}\xi^t\varphi^p_{r;s} + B^{irs}_{jtp}\xi^t_{;r}\varphi^p_s,$

where A^{irs}_{jtp}, B^{irs}_{jtp} are absolute invariant tensors, i.e. $A^{irs}_{jtp} = c_1\delta^i_j\delta^r_t\delta^s_p + \dots + c_6\delta^s_j\delta^r_t\delta^i_p$, $B^{irs}_{jtp} = d_1\delta^i_j\delta^r_t\delta^s_p + \dots + d_6\delta^s_j\delta^r_t\delta^i_p$. From the condition (8.5.6) we obtain $c_1 = c_3 = c_6 = d_1 = d_2 = d_3 = d_6 = 0$; c_2, c_4 are arbitrary, and $d_4 + c_5 = 0$, $d_5 - c_5 = 0$. Then

(8.5.17) $\quad \lambda^i_j = A\delta^i_j\xi^q\varphi^p_{p,q} + B\xi^i\varphi^p_{p,j} + C(\xi^p\varphi^i_{j,p} - \xi^i_{,p}\varphi^p_j + \xi^p_{,j}\varphi^i_p).$

The expression at C is by (8.5.3) the differential invariant corresponding to the Frölicher–Nijenhuis bracket [,], and (8.5.17) is then the differential invariant corresponding to (8.5.15). This proves our theorem.

II. Let $r = 1$, $s = 1$.

Theorem 8.13. *There is a* 15-*parameter family of bilinear natural differential operators from* $T^{(1,1)} \oplus T^{(1,1)}$ *to* $T^{(1,2)}$. *All operators belonging to this family are* *linear combinations (with real coefficients) of the Frölicher–Nijenhuis bracket and operators which can be expressed by means of tensor product, the contraction of tensors, the exterior derivative and the tensor product with $I = (\delta^i_j)$.*

Proof. Let (φ^i_j, ψ^i_j) be the canonical global coordinates on the type fiber $(R^n \otimes R^{n*}) \times (R^n \otimes R^{n*})$ of $T^{(1,1)} \oplus T^{(1,1)}$. Theorem 8.9 implies that all bilinear natural differential operators from $T^{(1,1)} \oplus T^{(1,1)}$ to $T^{(1,2)}$ have corresponding differential invariants expressed by (8.5.5), i.e.

$$A^i_{jk} = A^{iq_1q_2u}_{jkpt}\varphi^p_{q_1;q_2}\psi^t_u + B^{iqu_1u_2}_{jkpt}\varphi^p_q\psi^t_{u_1;u_2}, \tag{8.5.18}$$

where $A^{iq_1q_2u}_{jkpt} = c_1\delta^i_j\delta^{q_1}_k\delta^{q_2}_p\delta^u_t + \ldots + c_{24}\delta^u_j\delta^{q_2}_k\delta^{q_1}_p\delta^i_t$, $B^{iqu_1u_2}_{jkpt} = d_1\delta^i_j\delta^q_k\delta^{u_1}_p\delta^{u_2}_t + \ldots + d_{24}\delta^{u_1}_j\delta^{u_2}_k\delta^q_p\delta^t_i$. (8.5.6) gives for the coefficients c_i, d_i, $i = 1, \ldots, 24$, a system of linear homogeneous equations which has 15 independent variables A_i. Then

$$\begin{aligned} \lambda^i_{jk} = {} & A_1\delta^i_j\varphi^p_{p,k}\psi^r_r + A_2\delta^i_j\psi^p_{p,k}\varphi^r_r + A_3\delta^i_k\varphi^p_{p,j}\psi^r_r + A_4\delta^i_k\psi^p_{p,j}\varphi^r_r + \\ & + A_5\delta^i_j(\varphi^p_{r,k}\psi^r_p + \psi^p_{r,k}\varphi^r_p) + A_6\delta^i_k(\varphi^p_{r,j}\psi^r_p + \psi^p_{r,j}\varphi^r_p) + \\ & + A_7\delta^i_j\varphi^p_{p,r}\psi^r_k + A_8\delta^i_j\psi^p_{p,r}\varphi^r_k + A_9\delta^i_k\varphi^p_{p,r}\psi^r_j + \\ & + A_{10}\delta^i_k\psi^p_{p,r}\varphi^r_j + A_{11}\varphi^p_{p,k}\psi^i_j + A_{12}\psi^p_{p,k}\varphi^i_j + A_{13}\varphi^p_{p,j}\psi^i_k + \\ & + A_{14}\psi^p_{p,j}\varphi^i_k + A_{15}(\varphi^i_p\psi^p_{j,k} - \psi^i_p\varphi^p_{j,k} - \varphi^i_p\psi^p_{k,j} + \\ & + \psi^i_p\varphi^p_{k,j} + \varphi^p_k\psi^i_{j,p} + \psi^p_k\varphi^i_{j,p} - \varphi^p_j\psi^i_{k,p} - \psi^p_j\varphi^i_{k,p}). \end{aligned} \tag{8.5.19}$$

The expression at A_{15} is by (8.5.3) the differential invariant corresponding to the Frölicher–Nijenhuis bracket of two TX-valued 1-forms. If $\varphi, \psi \in \Omega^1(X; TX)$ then the differential invariant (8.5.19) corresponds to the (1,2)-tensor field

$$\begin{aligned} & A_1 I \otimes \mathrm{d}(\mathrm{tr}\,\varphi) \otimes \mathrm{tr}\,\psi + A_2 I \otimes \mathrm{d}(\mathrm{tr}\,\psi) \otimes \mathrm{tr}\,\varphi + A_3\,\mathrm{d}(\mathrm{tr}\,\varphi) \otimes \mathrm{tr}\psi \otimes I + \\ & + A_4\,\mathrm{d}(\mathrm{tr}\,\psi) \otimes \mathrm{tr}\,\varphi \otimes I + A_5 I \otimes \mathrm{d}(\mathrm{tr}\,(C^1_2(\varphi \otimes \psi))) + \\ & + A_6\,\mathrm{d}(\mathrm{tr}\,(C^1_2(\varphi \otimes \psi)) \otimes I + A_7 I \otimes C^1_1(\mathrm{d}\,(\mathrm{tr}\,\varphi) \otimes \psi) + \\ & + A_8 I \otimes C^1_1(\mathrm{d}\,(\mathrm{tr}\,\psi) \otimes \varphi) + A_9 C^1_1(\mathrm{d}\,(\mathrm{tr}\,\varphi) \otimes \psi) \otimes I + \\ & + A_{10} C^1_1(\mathrm{d}\,(\mathrm{tr}\,\psi) \otimes \varphi) \otimes I + A_{11}\psi \otimes \mathrm{d}(\mathrm{tr}\,\varphi) + A_{12}\varphi \otimes \mathrm{d}(\mathrm{tr}\,\psi) + \\ & + A_{13}\,\mathrm{d}(\mathrm{tr}\,\varphi) \otimes \psi + A_{14}\,\mathrm{d}\,(\mathrm{tr}\,\psi) \otimes \varphi + A_{15}[\varphi, \psi], \end{aligned} \tag{8.5.20}$$

where C^i_j means the contraction of tensors in the i-th superscript and the j-th subscript. This proves our theorem.

The Frölicher–Nijenhuis bracket is the unique operator from the list of 15 operators given by (8.5.20) the values of which are in $TX \otimes \wedge^2 T^*X$. Using the antisymmetrization we get immediately

Theorem 8.14. *There is an 8-parameter family of bilinear natural differential operators from $T^{(1,1)} \oplus T^{(1,1)}$ to $T \otimes \wedge^2 T^*$. This family associates to any two TX-valued 1-forms φ and ψ the TX-valued 2-form*

$$\begin{aligned}(8.5.21)\qquad & B_1 I \wedge \mathrm{d}(\operatorname{tr}\varphi) \operatorname{tr}\psi + B_2 I \wedge \mathrm{d}(\operatorname{tr}\psi)\operatorname{tr}\varphi + B_3 I \wedge \mathrm{d}(\operatorname{tr}(C_2^1(\varphi\otimes\psi))) + \\ & + B_4 I \wedge C_1^1(\mathrm{d}(\operatorname{tr}\varphi)\otimes\psi) + B_5 I \wedge C_1^1(\mathrm{d}(\operatorname{tr}\psi)\otimes\varphi) + \\ & + B_6 \psi \wedge \mathrm{d}(\operatorname{tr}\varphi) + B_7 \varphi \wedge \mathrm{d}(\operatorname{tr}\psi) + B_8[\varphi, \psi].\end{aligned}$$

Remark 8.10. The finding of the complete list of bilinear natural differential operators from $T^{(1,r)} \oplus T^{(1,s)}$ to $T^{(1,r+s)}$, where $r \geqq 2$ or $s \geqq 2$, is similar but technically more complicated. However, in the case of TX-valued forms, if we find all bilinear natural differential operators of the type considered for $r, s = 2$, we get at the same time the complete list of bilinear natural differential operators for arbitrary $r, s \geqq 2$. This follows from the skew-symmetry of the TX-valued forms in the subscripts.

9. GEOMETRIC OBJECTS NATURALLY INDUCED FROM METRIC

In this chapter we describe several natural differential operators defined on the bundle of metrics on X. In Section 9.1 we prove the classical result that the Levi–Civita connection is a unique linear connection depending naturally on a metric tensor and its first order derivatives. In Section 9.2 we give an example of a linear connection depending naturally on a metric tensor and its 3rd-order derivatives.

In Section 9.3 we describe natural lifts of Riemannian metrics on X to metrics on the tangent bundle TX. This result generalizes the classical diagonal, horizontal and vertical lifts of Riemannian metrics.

9.1. The uniqueness of the Levi—Civita connection. Let Met X denotes the *bundle of metrics* on $X \in \mathrm{Ob}\,\mathscr{D}_k$, i.e. Met X is the subbundle in $T^{(0,2)}X$ formed by all symmetric regular tensors. The type fibre of Met X is $P = \overset{2}{\odot} R^{n*}$, $n = \dim X$, with the canonical global coordinates g_{ij}, $\det(g_{ij}) \neq 0$, $1 \leqq i \leqq j \leqq n$. The action of L_n^1 on P is given by the formula

$$(9.1.1)\qquad \bar{g}_{ij} = b_i^p b_j^q g_{pq}.$$

Let $CX \to X$ be the fiber bundle of linear connections on X. C is a Q-lifting F_Q^2 with the type fiber $Q = R^n \otimes \otimes R^{n*}$, with the action of L_n^2 given in the canonical global coordinates (Γ_{jk}^i), $1 \leqq i, j, k \leqq n$, on Q by the formula

(9.1.2) $$\bar{\Gamma}^i_{jk} = a^i_p \Gamma^p_{qr} b^q_j b^r_k + a^i_p b^p_{jk}.$$

A *natural connection of order r* is a natural differential operator of order r from Met to C. The *Levi–Civita connection* is a well known example of a natural connection of order one. The Lewi–Civita connection determines the differential invariant of L^2_n from $T^1_n P$ to Q given by

(9.1.3) $$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}),$$

where g^{ij} is the inverse matrix of g_{ij}, i.e. $g^{im}g_{mj} = \delta^i_j$.

Theorem 9.1. *The Levi–Civita connection is a unique natural connection of order one.*

Proof. To prove Theorem 9.1 we have to prove the uniqueness of the differential invariant (9.1.3). The action of L^2_n on $T^1_n P$ is expressed in the canonical coordinates $(g_{ij}, g_{ij,k})$ on $T^1_n P$ by (9.1.1) and by

(9.1.4) $$\bar{g}_{ij,m} = (b^p_{im} b^q_j + b^p_i b^q_{jm})\, g_{pq} + b^p_i b^q_j b^r_m g_{pq,r}.$$

The fundamental vector fields on $T^1_n P$, relative to this action, are

(9.1.5) $$\begin{aligned} \Xi^q_{pT^1_nP} &= \left(\frac{\partial \bar{g}_{ij}}{\partial b^p_q}\right)_e \frac{\partial}{\partial g_{ij}} + \left(\frac{\partial \bar{g}_{ij,m}}{\partial b^p_q}\right)_e \frac{\partial}{\partial g_{ij,m}} = \\ &= (\delta^q_i g_{pj} + \delta^q_j g_{ip}) \frac{\partial}{\partial g_{ij}} + (\delta^q_i g_{pj,m} + \delta^q_j g_{ip,m} + \delta^q_m g_{ij,p}) \frac{\partial}{\partial g_{ij,m}}, \\ \Xi^{qr}_{pT^1_nP} &= \left(\frac{\partial \bar{g}_{ij,m}}{\partial b^p_{qr}}\right)_e \frac{\partial}{\partial g_{ij,m}} = g_{ip}\left(\frac{\partial}{\partial g_{iq,r}} + \frac{\partial}{\partial g_{ir,q}}\right) \end{aligned}$$

and the fundamental vector fields on Q, relative to (9.1.2), are

(9.1.6) $$\begin{aligned} \Xi^q_{pQ} &= \left(\frac{\partial \bar{\Gamma}^i_{jk}}{\partial b^p_q}\right)_e \frac{\partial}{\partial \Gamma^i_{jk}} = (-\delta^i_p \Gamma^q_{jk} + \delta^q_j \Gamma^i_{pk} + \delta^q_k \Gamma^i_{jp}) \frac{\partial}{\partial \Gamma^i_{jk}}, \\ \Xi^{qr}_{pQ} &= \left(\frac{\partial \bar{\Gamma}^i_{jk}}{\partial b^p_{qr}}\right)_e \frac{\partial}{\partial \Gamma^i_{jk}} = \frac{1}{2}\left(\frac{\partial}{\partial \Gamma^p_{qr}} + \frac{\partial}{\partial \Gamma^p_{rq}}\right), \end{aligned}$$

where $e \in L^2_n$ is the unity, i.e. $e = J^2_0\, \mathrm{id}_{R^n}$. Theorem 3.4 implies that if a mapping $F : T^1_n P \to Q$ is an L^2_n-equivariant mapping then F satisfies the following system of partial differential equations

(9.1.7) $$\begin{aligned} &(\delta^q_a g_{pb} + \delta^q_b g_{ap}) \frac{\partial F^i_{jk}}{\partial g_{ab}} + (\delta^q_a g_{pb,m} + \delta^q_b g_{ap,m} + \delta^q_m g_{ab,p}) \frac{\partial F^i_{jk}}{\partial g_{ab,m}} = \\ &= -\delta^i_p F^q_{jk} + \delta^q_j F^i_{pk} + \delta^q_k F^i_{jp}, \end{aligned}$$

$$(9.1.8) \qquad g_{ap}\left(\frac{\partial F^i_{jk}}{\partial g_{aq,r}} + \frac{\partial F^i_{jk}}{\partial g_{ar,q}}\right) = \frac{1}{2}\,\delta^i_p(\delta^q_j\delta^r_k + \delta^r_j\delta^q_k).$$

Contracting both sides of (9.1.8) by g^{ps} and rewriting the system for cyclic permutation of the lower indices of the independent variables we get

$$(9.1.9) \qquad \begin{aligned} \frac{\partial F^i_{jk}}{\partial g_{sq,r}} + \frac{\partial F^i_{jk}}{\partial g_{sr,q}} &= \frac{1}{2}\,g^{is}(\delta^q_j\delta^r_k + \delta^q_k\delta^r_j),\\ \frac{\partial F^i_{jk}}{\partial g_{rs,q}} + \frac{\partial F^i_{jk}}{\partial g_{rq,s}} &= \frac{1}{2}\,g^{ir}(\delta^s_j\delta^q_k + \delta^s_k\delta^q_j),\\ \frac{\partial F^i_{jk}}{\partial g_{qr,s}} + \frac{\partial F^i_{jk}}{\partial g_{qs,r}} &= \frac{1}{2}\,g^{iq}(\delta^r_j\delta^s_k + \delta^r_k\delta^s_j) \end{aligned}$$

and from (9.1.9) we obtain

$$(9.1.10) \qquad \frac{\partial F^i_{jk}}{\partial g_{sr,q}} = \frac{1}{4}\,(g^{is}(\delta^q_j\delta^r_k + \delta^q_k\delta^r_j) + g^{ir}(\delta^s_j\delta^q_k + \delta^s_k\delta^q_j) - g^{iq}(\delta^r_j\delta^s_k + \delta^r_k\delta^s_j)).$$

Hence

$$(9.1.11) \qquad \begin{aligned} F^i_{jk} &= \frac{1}{4}\,(g^{is}(g_{sk,j} + g_{sj,k}) + g^{ir}(g_{jr,k} + g_{kr,j}) -\\ &- g^{iq}(g_{kj,q} + g_{jk,q})) + \gamma^i_{jk} = \frac{1}{2}\,g^{is}(g_{sj,k} + g_{sk,j} - g_{jk,s}) + \gamma^i_{jk}, \end{aligned}$$

where γ^i_{jk} does not depend on $g_{pq,r}$. Substituting (9.1.11) into (9.1.2) we obtain that γ^i_{jk} is a (1,2)-tensor and from the assumption or our theorem γ^i_{jk} is a differential invariant from P to Q. Then γ^i_{jk} has to satisfy the following system of differential equations

$$(9.1.12) \qquad 2g_{ps}\frac{\partial\gamma^i_{jk}}{\partial g_{ps}} = -\,\delta^i_p\gamma^q_{jk} + \delta^q_j\gamma^i_{pk} + \delta^q_k\gamma^i_{jp}.$$

Multiplying both sides of (9.1.12) by g^{pr} and using the symmetry of g_{rs} we get

$$(9.1.13) \qquad g^{pr}(-\,\delta^i_p\gamma^q_{jk} + \delta^q_j\gamma^i_{pk} + \delta^q_k\gamma^i_{jp}) = g^{pq}(-\,\delta^i_p\gamma^r_{jk} + \gamma^r_j\gamma^i_{pk} + \delta^r_k\gamma^i_{jp}).$$

Contracting (9.1.13) with respect to i, j and r, k we obtain

$$(9.1.14) \qquad g^{pq}\gamma^i_{ip} = ng^{pq}\gamma^i_{ip}.$$

Similarly using the contraction of (9.1.13) with respect to i, k and r, j we get

$$(9.1.15) \qquad g^{pq}\gamma^i_{pi} = ng^{pq}\gamma^i_{pi}.$$

A. Let $n > 1$. From (9.1.14) and (9.1.15) we have

$$(9.1.16) \qquad \gamma^i_{ip} = 0, \qquad \gamma^i_{pi} = 0.$$

Contracting (9.1.13) with respect to j, q and using (9.1.16) we get

$$(9.1.17)\qquad (n-1)\, g^{pr}\gamma^i_{pk} + g^{pr}\gamma^i_{kp} = -g^{iq}\gamma^r_{qk} + g^{pq}\delta^r_k\gamma^i_{qp}.$$

Multiplying both sides of (9.1.17) by g_{si} we obtain

$$(9.1.18)\qquad (n-1)\, g^{pr}g_{si}\gamma^i_{pk} + g^{pr}g_{si}\gamma^i_{kp} = -\gamma^i_{sk} + g^{pq}g_{si}\delta^r_k\gamma^i_{qp}.$$

Using the contraction of (9.1.18) with respect to r, s and (9.1.16) we obtain

$$(9.1.19)\qquad g^{pq}g_{ki}\gamma^i_{pq} = 0,$$

which implies $\gamma^i_{pq} = 0$.

B. Let $n = 1$. Then (9.1.12) has the form

$$(9.1.20)\qquad 2g\frac{\mathrm{d}\gamma}{dg} = \gamma$$

hence

$$(9.1.21)\qquad \gamma = c\sqrt{|g|}.$$

(9.1.21) is L^1_1-equivariant iff $c = 0$ and we have $\gamma = 0$.

Thus γ^i_{jk} is the zero tensor which proves our Theorem 9.1.

Remark 9.1. We note that the part of the proof beginning at (9.1.12) is a generalization of a proof given by Krupka to non-symmetric connections.

9.2. Natural connections of higher orders. Using the notation of Section 9.1, we describe some examples of natural connection, depending on a metric, of order higher than one. Such a natural connection of order r corresponds to a differential invariant

$$(9.2.1)\qquad \Gamma^i_{jk} = F^i_{jk}(g_{pq}, g_{pq,m}, \ldots, g_{pq,m_1\ldots m_r})$$

from T_nP to Q. The direct calculation of (9.2.1) leads to a system of partial differential equations which is difficult to solve. From (9.1.2) it immediately follows

Lemma 9.1. *Let Γ and Δ be two linear connections on X, i.e. Γ and Δ are two sections of CX, then the difference $\Delta - \Gamma$ is a* (1,2)*-tensor field on X.*

Now let Δ be a natural connection of order r given by (9.2.1) and Γ be the Levi–Civita connection given by (9.1.3). Then $\Delta - \Gamma$ is a natural differential operator of order r from Met to $T^{(1,2)}$. This implies

Lemma 9.2. *Every natural connection of order r is the sum of the Levi–Civita connection and a (1,2)-tensor field which is naturally induced from the metric tensor field and its derivatives up to order r.*

Lemma 9.2 restricts our problem of determining of natural connections of order r to determining of natural differential operators of order r from Met to $T^{(1,2)}$. Such operators correspond to differential invariants of L_n^{r+1} from $T_n^r P$ to $S = R^n \otimes \overset{2}{\otimes} R^{n*}$ which is considered with the tensor action of L_n^1.

If reg $\overset{2}{\otimes} R^n$ denotes the space of all regular (2,0)-tensors and reg $\overset{2}{\otimes} R^{n*}$ denotes the space of all regular (0,2)-tensors, we have

Lemma 9.3. *The inverse matrix t^{ij} of a matrix $t_{ij} \in$ reg $\overset{2}{\otimes} R^{n*}$ is a (rational) differential invariant of L_n^1 from* reg $\overset{2}{\otimes} R^{n*}$ *to* reg $\overset{2}{\otimes} R^n$.

Proof. Because of Theorem 8.3 a differential invariant h : reg $\overset{2}{\otimes} R^{n*} \to$ reg $\overset{2}{\otimes} R^n$ of L_n^1 has to satisfy

$$(9.2.2) \qquad -2\frac{\partial h^{ij}}{\partial t_{pq}} t_{pq} = 2h^{ij}.$$

If $h^{ij}(t_{pq}) = t^{ij}$, where t^{ij} is the inverse matrix of t_{ij}, i.e. $t_{im}t^{mj} = \delta_i^j$, then (9.2.2) is satisfied because of $\partial t^{ij}/\partial t_{pq} = -t^{ip}t^{qj}$.

Remark 9.2. Lemma 9.3 is true also for the opposite direction from reg $\overset{2}{\otimes} R^n$ to reg $\overset{2}{\otimes} R^{n*}$.

If we restrict Lemma 9.3 on symmetric tensors we obtain the rational differential invariant of L_n^1 from P to P^*, where P^* denotes the type fiber of the space of inverse metrics on X. Lemma 9.3 now implies that every differential invariant f of L_n^{r+1} from $T_n^r P$ to S determines a unique differential invariant g of L_n^{r+1} from $P^* \times T_n^r P$ to S such that $f = g \circ h$. If g is polynomial then the corresponding differential invariant f from $T_n^r P$ to S is rational.

Let us denote by $U_r = P^* \times T_n^r P$. We want to determine auxiliary differential invariants from U_r to S. Such differential invariants have coordinate expressions of the form

$$(9.2.3) \qquad t^i_{j_1 j_2} = f^i_{j_1 j_2}(g^{pq}, g_{pq}, \ldots, g_{pq, m_1 \ldots m_r}),$$

where

$$(9.2.4) \qquad (g^{pq}, g_{pq}, \ldots, g_{pq, m_1 \ldots m_r}),$$

$1 \leqq p \leqq q \leqq n$, $1 \leqq m_1 \leqq \ldots \leqq m_r \leqq n$, are the canonical coordinates on U_r.

Let us consider the following system of functions on U_r : $\Gamma^i_{jk, m_1 \ldots m_s}$, $s = 0, \ldots, \ldots, r-1$, where Γ^i_{jk} is given by (9.1.3), $R^i_{jkl; m_1; \ldots; m_s}$, $s = 0, \ldots, r-2$, where R^i_{jkl} s given by (8.3.6) and ";" denotes the formal covariant derivative with respect to Γ^i_{jk}. Hence Γ^i_{jk} determine the formal Levi–Civita connection and R^i_{jkl} is its formal curvature tensor. Let us denote $\Gamma^i_{jkm_1\ldots m_s} = \Gamma^i_{(jk,m_1\ldots m_s)}$, $s = 1, \ldots, r-1$, the symmetrization.

Lemma 9.4. *The system of functions*

$$
\begin{aligned}
& g^{ij}, g_{ij}, \\
(9.2.5) \qquad & \Gamma^i_{jk}, \ldots, \Gamma^i_{jkm_1\ldots m_{r-1}}, \\
& R^i_{jkl}, \ldots, R^i_{jkl;m_1;\ldots;m_{r-2}}
\end{aligned}
$$

contains a subsystem which forms a global chart on U_r, where $r \geqq 2$.

Proof. Let us consider the following system of functions on U_r

$$(9.2.6) \qquad g^{ij}, g_{ij}, \Gamma^i_{jk}, \ldots, \Gamma^i_{jk,m_1\ldots m_{r-1}}.$$

Then (9.2.6) forms a global chart on U_r. The transformation relation (9.2.4)–(9.2.6) is given by (9.1.3) and the formal derivatives of (9.1.3) up to the order $(r-1)$. The inverse transformation is given by

$$(9.2.7) \qquad g_{ij,k} = g_{jm}\Gamma^m_{ik} + g_{im}\Gamma^m_{jk}$$

and the formal derivatives of this relation up to the order ($r-1$.) Lemma 9.4 now follows from Lemma 8.5.

Every global chart on U_r defined by Lemma 9.4 will be called an *adapted chart.* The functions (9.2.5) belonging to an adapted chart will be called *adapted coordinates.* The action of K^{r+1}_n on U_r in the adapted coordinates has the expression

$$
\begin{aligned}
& \bar{g}^{ij} = g^{ij}, \qquad \bar{g}_{ij} = g_{ij}, \qquad \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + b^i_{jk}, \\
(9.2.8) \qquad & \bar{\Gamma}^i_{jkm_1\ldots m_s} = \Gamma^i_{jkm_1\ldots m_s} + S^i_{jkm_1\ldots m_s} + b^i_{jkm_1\ldots m_s}, \\
& \bar{R}^i_{jkl;m_1;\ldots;m_{s-1}} = R^i_{jkl;m_1;\ldots;m_{s-1}},
\end{aligned}
$$

where $s = 1, \ldots, r-1$ and $S^i_{jkm_1\ldots m_s}$ is a polynomial in the canonical coordinates on K^{s+1}_n and in the adapted coordinates on U_{s-1}. (9.2.8) implies that the action of $K_n{}^{+1}$ on U_r is free and applying Theorem 1.11 we can see that the orbit space U_r/K^{r+1}_n has the manifold structure such that the canonical projection $\pi^r : U_r \to$ $\to U_r/K^{r+1}_n$ is a submersion. Then any system of independent functions (g^{ij}, g_{ij}, $R^i_{jkl}, \ldots, R^i_{jkl;m_1;\ldots;m_{r-2}}$), $r \geqq 2$, forms a global chart on U_r/K^{r+1}_n.

According to Theorem 7.6 a differential invariant f from U_r to S determines a unique differential invariant $F : U_r/K^{r+1}_n \to S$ of L^1_n such that $f = F \circ \pi^r$. Hence we have

$$(9.2.9) \qquad t^i_{j_1j_2} = F^i_{j_1j_2}(g^{ij}, g_{ij}, R^i_{jkl}, \ldots, R^i_{jkl;m_1;\ldots;m_{r-2}}).$$

Our considerations do not depend on the type of the tensor space S. So we have by Theorem 7.6 and Lemma 9.3 a more general result

Theorem 9.2. *Every natural differential operator of finite order $r \geqq 2$ from* Met *to any S-lifting F^1_S depends naturally on the metric tensor, the inverse metric tensor and the formal covariant derivatives of the formal curvature tensor of the formal Levi–Civita connection up to the order $(r-2)$.*

If $r=1$, then on U_1 there are coordinates $(g^{ij}, g_{ij}, \Gamma^i_{jk})$ and on U_1/K^2_n coordinates (g^{ij}, g_{ij}). It implies

Theorem 9.3. *Every natural differential operator of order one from* Met *to any S-lifting F^1_S is either zero operator or is of order zero.*

According to Theorem 8.4 polynomial differential invariants (9.2.9) are sums of homogeneous polynomials of degrees a in g^{ij}, b in g_{ij} and c_s in $R^l_{jkl;m_1;\ldots;m_s}$, $s=0,\ldots,r-2$, $r \geqq 2$, such that

$$2a - 2b - \sum_{s=0}^{r-2} c_s(-2-s) = -1. \tag{9.2.10}$$

If $r=2$, (9.2.10) has no natural solution which implies

Corollary 2. *There is no polynomial (i.e. polynomial in the metric, the inverse metric and the formal curvature tensor of the formal Levi–Civita connection) natural connection of order 2 which is not of order 1.*

If $r=3$, (9.2.10) has the form $2a-2b-2c_0-3c_1=-1$ and this has, for instance, the solution $a=1$, $b=c_0=0$, $c_1=1$. Then (9.2.9) has the form

$$t^i_{jk} = A^{is_1s_2s_3s_4}_{jkpqr} g^{pq} R^r_{s_1s_2s_3;s_4},$$

where $A^{is_1s_2s_3s_4}_{jkpqr}$ is an absolute invariant tensor. Let, for example, $c\binom{is_1s_2s_3s_4}{pj\,k\,q\,r} = 1$ and let all the other coefficients vanish. Then we obtain an example of the 3rd-order natural connection

$$\Delta^i_{jk} = \Gamma^i_{jk} + g^{ip} R^q_{jkp;q}. \tag{9.2.11}$$

It is obvious that in this way we can obtain many examples of natural connections of order greater than or equal to 3. All such connections are rational because the dependence between the metric tensor and the inverse metric tensor is rational.

9.3. Natural prolongations of Riemannian metrics on manifolds to metrics on tangent bundles. Let $X \in \mathrm{Ob}\,\mathcal{D}_n$ and let g denote a Riemannian metric on X. Let (x^i), $1 \leqq i \leqq n$, be local coordinates on X and (x^i, u^i) the induced coordinates on the tangent bundle $p_X : TX \to X$. Let $\partial/\partial x^i$, $\partial/\partial u^i$ be the local coordinate vector fields on TX. Then passing to new local coordinates $(\bar{x}^i)$ we have the vector fields $\partial/\partial \bar{x}^i$, $\partial/\partial \bar{u}^i$, where

$$\begin{aligned} \frac{\partial}{\partial \bar{x}^i} &= b_i^p \frac{\partial}{\partial x^p} + b_{iq}^p a_r^q u^r \frac{\partial}{\partial u^p}, \\ \frac{\partial}{\partial \bar{u}^i} &= b_i^p \frac{\partial}{\partial u^p}. \end{aligned} \tag{9.3.1}$$

Let now

$$G_{ij}^1 = G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad G_{ij}^2 = G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u^j}\right), \quad G_{ij}^3 = G\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right), \tag{9.3.2}$$

be the local coordinates of a symmetric (0,2)-tensor field G on TX. Then using (9.3.1) we obtain the transformation relations

$$\begin{aligned} \bar{G}_{ij}^1 &= b_i^p b_j^q G_{pq}^1 + b_i^p b_{jm}^q a_r^m u^r G_{pq}^2 + b_j^q b_{im}^p a_r^m u^r G_{pq}^2 + b_{im}^p a_r^m u^r b_{js}^q a_t^s u^t G_{pq}^3, \\ \bar{G}_{ij}^2 &= b_i^p b_j^q G_{pq}^2 + b_{im}^p a_r^m u^r b_j^q G_{pq}^3, \\ \bar{G}_{ij}^3 &= b_i^p b_j^q G_{pq}^3. \end{aligned} \tag{9.3.3}$$

We shall describe shortly tree classical prolongations of a Riemannian metric g onto X to a metric (or pseudo-metric), i.e. to a symmetric (0,2)-tensor field, on TX. Let Γ be the Levi-Civita connection determined by g, i.e. the components Γ_{jk}^i of Γ are given by (9.1.3). Then the tangent space $T(TX)$, at every point u of TX, is splitted into the horizontal and the vertical subspaces with respect to Γ, i.e. $T_uTX = H_u \oplus V_u$. On the other hand T_uTX is isomorphic with $T_xX \oplus T_xX$, $x = p_X(u)$, and so $H_u \oplus V_u \approx T_xX \oplus T_xX$.

If $\xi = \xi^i(x)\, \partial/\partial x^i$ is a vector field on X we can define the *horizontal lift* ξ^H of ξ with respect to Γ by

$$\xi^H = \xi^i(x) \frac{\partial}{\partial x^i} - \Gamma_{pm}^i(x)\, u^p \xi^m(x) \frac{\partial}{\partial u^i}. \tag{9.3.4}$$

The mapping $\xi \mapsto \xi^H$ gives the linear isomorphism of T_xX and H_u for any u over x, i.e. $p_X(u) = x$.

Let η be a vector field on X and ω a 1-form on X. Then $u \mapsto \omega(u)$ is the function on TX. If we denote this function by $i\omega$ we can define the *vertical lift* η^V of η by $\eta^V(i\omega) = \omega(\eta) \circ p_X$. If $\eta = \eta^i(x)\, \partial/\partial x^i$, then

$$\eta^V = \eta^i(x) \frac{\partial}{\partial u^i}. \tag{9.3.5}$$

The mapping $\eta \mapsto \eta^V$ defines the linear isomorphism of T_xX and V_u, $u = p_X(u)$. In Section 10.2 we shall prove that the vertical lift is a natural operator.

Thus for any projectable vector field Ξ on TX there exist two vector fields ξ and η on X such that $\Xi(u) = \xi^H(u) + \eta^V(u)$.

Then we can define for any Riemannian metric g on X three classical lifts to the metrics on TX.

a) The *diagonal lift* g^d (called also the *Sasaki metric*) of g is a Riemannian metric on TX defined by

$$g^d_u(\xi^H, \eta^H) = g_x(\xi, \eta), \qquad g^d_u(\xi^H, \eta^V) = 0, \qquad g^d_u(\xi^V, \eta^V) = g_x(\xi, \eta),$$

where $x = p_X(u)$ and ξ, η are arbitrary vector fields on X. In coordinates we obtain from (9.3.1), (9.3.4) and (9.3.5) the coordinate expression of the diagonal lift in the form

$$\text{(9.3.6)} \qquad (g^d)^1_{ij} = u^p p^q \Gamma^r_{pi} \Gamma^s_{qj} g_{rs} + g_{ij}, \quad (g^d)^2_{ij} = u^p \Gamma^q_{pi} g_{qj}, \quad (g^d)^3_{ij} = g_{ij}.$$

b) The *horizontal lift* g^h (called also the *complete lift*) of g is a pseudo-Riemannian metric of signature (n, n) defined by

$$g^h_u(\xi^H, \eta^H) = 0, \qquad g^h_u(\xi^H, \eta^V) = g_x(\xi, \eta), \qquad g^h_u(\xi^V, \eta^V) = 0,$$

where $x = p_X(u)$ and ξ, η are arbitrary vector fields on X. Then the coordinate expression of the horizontal lift is

$$\text{(9.3.7)} \qquad (g^h)^1_{ij} = u^p \Gamma^q_{pi} g_{qj} + u^p \Gamma^q_{pj} g_{qi}, \qquad (g^h)^2_{ij} = g_{ij}, \qquad (g^h)^3_{ij} = 0.$$

c) The *vertical lift* g^v of g is a degenerate metric on TX defined by

$$g^v_u(\xi^H, \eta^H) = g_x(\xi, \eta), \qquad g^v_u(\xi^H, \eta^V) = 0, \qquad g^v_u(\zeta^V_G, \eta^V) = 0,$$

where $x = p_X(u)$ and ζ, η are arbitrary vector fields on X. Then the coordinate expression of the vertical lift is

$$\text{(9.3.8)} \qquad (g^v)^1_{ij} = g_{ij}, \qquad (g^v)^2_{ij} = 0, \qquad (g^v)^3_{ij} = 0.$$

All classical lifts are natural differential operators from $TX \oplus \mathrm{Met}_R X$ to the space of all symmetric (0,2)-tensors on TX which induce the identity on TX with respect to the projections $p_1 : TX \oplus \mathrm{Met}_R X \to TX$ and $T^{(0,2)}TX \to TX$, where $\mathrm{Met}_R X$ denotes the space of Riemannian metrics on X. From the coordinate expressions of these operators it follows that the diagonal and horizontal lifts are 1st-order operators and the vertical lift is a zero-order operator. Now we shall describe all first order natural operators of this type. Differential invariants corresponding to such operators are L^2_n-equivariant mappings from $P = R^n \times \times T^1_n(\overset{2}{\odot} R^n)$ to $S = R^n \times (\overset{2}{\odot} R^{n*} \oplus \overset{2}{\otimes} R^{n*} \oplus \overset{2}{\odot} R^{n*})$ given in global coordinates $(u^i, g_{ij}, g_{ij,k})$, $1 \leq i, j, k \leq n$, $g_{ij} = g_{ji}$, $\det(g_{ij}) \neq 0$, on P and $(v^i, G^1_{ij}, G^2_{ij}, G^3_{ij})$, $1 \leq i, j \leq n$, $G^1_{ij} = G^1_{ji}$, $G^3_{ij} = G^3_{ji}$, on S by

$$\text{(9.3.9)} \qquad v^i = u^i, \qquad G^\lambda_{ij} = G^\lambda_{ij}(u^p, g_{pq}, g_{pq,r}), \qquad \lambda = 1, 2, 3.$$

Let $\zeta \in L(L_n^2)$. Then the fundamental vector fields on P associated to ζ are given by

$$Z^q_{pP} = u^q \frac{\partial}{\partial u^p} - (\delta^q_i g_{pj} + \delta^q_j g_{ip}) \frac{\partial}{\partial g_{ij}} - \tag{9.3.10}$$
$$- (\delta^q_i g_{pj,m} + \delta^q_j g_{ip,m} + \delta^q_m g_{ij,p}) \frac{\partial}{\partial g_{ij,m}} .$$

$$Z^{qr}_{pP} = g_{ip} \left(\frac{\partial}{\partial g_{iq,r}} + \frac{\partial}{\partial g_{ir,q}} \right). \tag{9.3.11}$$

By (9.3.3), the fundamental vector fields on S associated to ζ are of the form

$$Z^q_{pS} = v^q \frac{\partial}{\partial v^p} - 2G^1_{ip} \frac{\partial}{\partial G^1_{iq}} - G^2_{pi} \frac{\partial}{\partial G^2_{qi}} - G^2_{ip} \frac{\partial}{\partial G^2_{iq}} - 2G^3_{ip} \frac{\partial}{\partial G^3_{iq}} , \tag{9.3.12}$$

$$2Z^{qr}_{pS} = (\delta^r_j v^q G^2_{ip} + \delta^q_j v^r G^2_{ip} + \delta^r_i v^q G^2_{jp} + \delta^q_i v^r G^2_{jp}) \frac{\partial}{\partial G^1_{ij}} + \tag{9.3.13}$$
$$+ (\delta^r_i v^q G^3_{pj} + \delta^q_i v^r G^3_{pj}) \frac{\partial}{\partial G^2_{ij}} .$$

Thus the induced system of differential equations for (9.3.9) is expressed by

$$2g_{ip} \frac{\partial G^\lambda_{ab}}{\partial g_{iq}} + (\delta^q_i g_{pj,k} + \delta^q_j g_{ip,k} + \delta^q_k g_{ij,p}) \frac{\partial G^\lambda_{ab}}{\partial g_{ij,k}} - \tag{9.3.14}$$
$$- u^q \frac{\partial G^\lambda_{ab}}{\partial u^p} = G^\lambda_{ap} \delta^q_b + G^\lambda_{pb} \delta^q_a ,$$

where $\lambda = 1, 2, 3$,

$$g_{ip} \left(\frac{\partial G^1_{ab}}{\partial g_{iq,r}} + \frac{\partial G^1_{ab}}{\partial g_{ir,q}} \right) = \frac{1}{2} (G^2_{ap} u^q \delta^r_b + G^2_{bp} u^r \delta^q_a + G^2_{ap} u^r \delta^q_b + G^2_{bp} u^q \delta^r_a), \tag{9.3.15}$$

$$g_{ip} \left(\frac{\partial G^2_{ab}}{\partial g_{iq,r}} + \frac{\partial G^2_{ab}}{\partial g_{ir,q}} \right) = \frac{1}{2} (G^3_{pb} u^q \delta^r_a + G^3_{pb} u^r \delta^q_a), \tag{9.3.16}$$

$$g_{ip} \left(\frac{\partial G^3_{ab}}{\partial g_{iq,r}} + \frac{\partial G^3_{ab}}{\partial g_{ir,q}} \right) = 0. \tag{9.3.17}$$

Let (g^{pq}) denotes the inverse matrix of (g_{pq}), i.e. $g_{pm} g^{mq} = \delta^q_p$. Contracting both sides of (9.3.15), (9.3.16) and (9.3.17) by g^{rk} and using cyclic permutations of the indices p, q, r we obtain

$$\frac{\partial G^1_{ab}}{\partial g_{pq,r}} = G^2_{as} u^m \frac{\partial \Gamma^s_{mb}}{\partial g_{pq,r}} + G^2_{bs} u^m \frac{\partial \Gamma^s_{ma}}{\partial g_{pq,r}} , \tag{9.3.18}$$

$$\frac{\partial G^2_{ab}}{\partial g_{pq,r}} = G^3_{sb} u^m \frac{\partial \Gamma^s_{ma}}{\partial g_{pq,r}} , \tag{9.3.19}$$

$$(9.3.20) \qquad \frac{\partial G^3_{ab}}{\partial g_{pq,r}} = 0.$$

To deduce (9.3.18) and (9.3.19) we have used (9.1.10). Integrating both sides of (9.3.19) with respect to $g_{pq,r}$ we obtain

$$(9.3.21) \qquad G^2_{ab} = G^3_{sb} u^m \Gamma^s_{ma} + F^2_{ab},$$

where F^2_{ab} do not depend on $g_{pq,r}$. Hence if we denote $G^3_{ab} = F^3_{ab}$ we have

$$(9.3.22) \qquad \frac{\partial F^\lambda_{ab}}{\partial g_{pq,r}} = 0,$$

where $\lambda = 2,3$. Substituting F^3_{ab} into (9.3.21) and (9.3.21) into (9.3.18) we obtain

$$(9.3.23) \qquad \begin{aligned} \frac{\partial G^1_{ab}}{\partial g_{pq,r}} &= u^m u^t \Gamma^p_{ma} \frac{\partial \Gamma^s_{tb}}{\partial g_{pq,r}} F^3_{ps} + u^m u^t \Gamma^p_{mb} \frac{\partial \Gamma^s_{ta}}{\partial g_{pq,r}} F^3_{ps} + \\ &+ u^t \frac{\partial \Gamma^s_{tb}}{\partial g_{pq,r}} F^2_{as} + u^t \frac{\partial \Gamma^s_{ta}}{\partial g_{pq,r}} F^2_{bs}. \end{aligned}$$

Integrating both sides of (9.3.23) with respect to $g_{pq,r}$ we obtain

$$(9.3.24) \qquad G^1_{ab} = u^m u^t \Gamma^p_{ma} \Gamma^s_{tb} F^3_{ps} + u^t \Gamma^s_{tb} F^2_{as} + u^t \Gamma^s_{ta} F^2_{bs} + F^1_{ab},$$

where F^1_{ab} do not depend on $g_{pq,r}$ and hence satisfy (9.3.22). After some routine calculations we obtain that F^λ_{ab}, $\lambda = 1, 2, 3$, satisfy

$$(9.3.25) \qquad 2g_{ip} \frac{\partial F^\lambda_{ab}}{\partial g_{iq}} - u^q \frac{\partial F^\lambda_{ab}}{\partial u^p} = F^\lambda_{ap} \delta^q_b + F^\lambda_{pb} \delta^q_a.$$

It is obvious that F^1_{ab} and F^3_{ab} are symmetric in the subscripts. So we have proved

Theorem 9.4. *Any differential invariant of L^2_n from P to S such that $v^i = u^i$ is expressed in the form*

$$(9.3.26) \qquad \begin{aligned} G^1_{ij} &= u^m u^t \Gamma^p_{mi} \Gamma^s_{tj} F^3_{ps} + u^t \Gamma^s_{tj} F^2_{is} + u^t \Gamma^s_{ti} F^2_{js} + F^1_{ij}, \\ G^2_{ij} &= u^m \Gamma^s_{mi} F^3_{sj} + F^2_{ij}, \\ G^3_{ij} &= F^3_{ij}, \end{aligned}$$

where Γ^i_{jk} are the formal Christoffel symbols and F^λ_{ij}, $\lambda = 1, 2, 3$, are functions on P satisfying the system of partial differential equations

$$(9.3.27) \qquad \begin{aligned} &\frac{\partial F^\lambda_{ij}}{\partial g_{pq,r}} = 0, \\ &2g_{mp} \frac{\partial F^\lambda_{ij}}{\partial g_{mq}} - u^q \frac{\partial F^\lambda_{ij}}{\partial u^p} = F^\lambda_{ip} \delta^q_j + F^\lambda_{pj} \delta^q_i. \end{aligned}$$

Moreover $F^1_{ij} = F^1_{ji}$, $F^3_{ij} = F^3_{ji}$.

Hence all 1st-order natural lifts of Riemannian metrics on a manifold X to metrics on the tangent bundle TX correspond to the differential invariants described by Theorem 9.4 and it implies that for the classification of natural liftings of a Riemannian metric it is sufficient to classify all solutions of (9.3.27). Now let us consider only polynomial 1st-order natural lifts of a Riemannian metric in the sense that a lifted metric on TX depends polynomially on a point $u \in TX$, the given Riemannian metric tensor field $g_{ij}(x)$, its first order derivatives and the inverse metric tensor field $g^{ij}(x)$.

Theorem 9.5. *All polynomial 1st-order natural lifts of a Riemannian metric on a manifold X to metrics on the tangent bundle TX are given by*

$$\begin{aligned} G_u(\xi^H, \eta^H) &= C_1(\| u \|^2_x)\, g_x(\xi, \eta) + D_1(\| u \|^2_x)\, g_x(\xi, u)\, g_x(\eta, u), \\ G_u(\xi^H, \eta^V) &= C_2(\| u \|^2_x)\, g_x(\xi, \eta) + D_2(\| u \|^2_x)\, g_x(\xi, u)\, g_x(\eta, u), \qquad (9.3.28) \\ G_u(\xi^V, \eta^V) &= C_3(\| u \|^2_x)\, g_x(\xi, \eta) + D_3(\| u \|^2_x)\, g_x(\xi, u)\, g_x(\eta, u), \end{aligned}$$

where $p_X(u) = x$, ξ, η are arbitrary vector fields on X, $\| u \|^2_x = g_x(u, u)$ and C_λ, D_λ, $\lambda = 1, 2, 3$, are arbitrary polynomials of one variable.

Proof. Differential invariants corresponding to polynomial natural lifts are given by Theorem 9.4 where F^λ_{ij}, $\lambda = 1, 2, 3$, are polynomial solutions of (9.3.27). Putting $p = q$ (no summation) we get because of Theorem 8.4 that such polynomial solutions of (9.3.27) have to be sums of homogeneous polynomials of degrees a in g_{ij} and b in u^i such that $2a - b = 2$, i.e. $b = 2(a-1)$ where $a \geqq 1$ is a natural number. Thus F^λ_{ij} are sums over all $a \geqq 1$ of the expressions

$$A^{p_1q_1\ldots p_aq_a}_{ijm_1\ldots m_{2(a-1)}} g_{p_1q_1} \cdots g_{p_aq_a} u^{m_1} \ldots u^{m_{2(a-1)}}, \qquad (9.3.29)$$

where $A^{p_1q_1\ldots p_aq_a}_{ijm_1\ldots m_{2a-1)}}$ are absolute invariant tensors. From the form of absolute invariant tensors we obtain that (9.3.29) has the form

$$c_a(g_{pq}u^pu^q)^{a-1} g_{ij} + d_a(g_{pq}u^pu^q)^{a-2}\, g_{im}u^m g_{jk}u^k, \qquad (9.3.30)$$

where c_a, d_a are real numbers and $d_1 = 0$. If we denote $g_{im}u^m = u_i$ and $g_{pq}u^pu^q = \| u \|^2$ then summing over $a \geqq 1$ we obtain differential invariants corresponding to polynomial natural lifts in the form

$$\begin{aligned} G^{1'}_{ij} = u^m u^t \Gamma^p_{mi} \Gamma^s_{tj} (C_3(\| u \|^2)\, g_{ps} + D_3(\| u \|^2)\, u_p u_s) + \\ + u^t \Gamma^s_{tj} (C_2(\| u \|^2)\, g_{is} + D_2(\| u \|^2)\, u_i u_s) + \\ + u^t \Gamma^s_{ti} (C_2(\| u \|^2)\, g_{sj} + D_2(\| u \|^2)\, u_s u_j) + \\ + C_1(\| u \|^2)\, g_{ij} + D_1(\| u \|^2)\, u_i u_j, \end{aligned} \qquad (9.3.31)$$

$$G_{ij}^2 = u^m \Gamma_{mi}^s (C_3(\| u \|^2)\, g_{sj} + D_3(\| u \|^2)\, u_s u_j) + \\ + C_2(\| u \|^2)\, g_{ij} + D_2(\| u \|^2)\, u_i u_j,$$

$$G_{ij}^3 = C_3(\| u \|^2)\, g_{ij} + D_3(\| u \|^2)\, u_i u_j,$$

where $C_\lambda, D_\lambda, \lambda = 1, 2, 3$, are arbitrary polynomials of one variable. It is easy to verify that (9.3.31) is just the differential invariant corresponding to (9.3.28).

Example 9.1. As an example of non-polynomial natural lift of a Riemannian metric we mention the *Cheeger–Gromoll metric* g^{CG} on TX defined by

$$(9.3.32)\qquad \begin{aligned} &g_u^{CG}(\xi^H, \eta^H) = g_x(\xi, \eta), \qquad g_u^{CG}(\xi^H, \eta^V) = 0, \\ &g_u^{CG}(\xi^V, \eta^V) = \frac{1}{1 + \| u \|_x^2}\, (g_x(\xi, \eta) + g_x(\xi, u)\, g_x(\eta, u)), \end{aligned}$$

where $p_X(u) = x$ and ξ, η are arbitrary vector fields on X. Then the coordinate expression of the Cheeger–Gromoll metric is given by

$$(9.3.33)\qquad \begin{aligned} &(g^{CG})_{ij}^1 = u^m u^t \Gamma_{mi}^p \Gamma_{tj}^s \frac{g_{ps} + u_p u_s}{1 + \| u \|^2} + g_{ij}, \\ &(g^{CG})_{ij}^2 = u^m \Gamma_{mi}^p \frac{g_{pj} + u_p u_j}{1 + \| u \|^2}, \\ &(g^{CG})_{ij}^3 = \frac{g_{ij} + u_i u_j}{1 + \| u \|^2}. \end{aligned}$$

Hence the differential invariant corresponding to the Cheeger–Gromoll metric is such differential invariant (9.3.26) where $F_{ij}^1 = g_{ij}$, $F_{ij}^2 = 0$ and $F_{ij}^3 = (g_{ij} + u_i u_j)/(1 + \| u \|^2)$. It is easy to verify that F_{ij}^3 satisfies the equation (9.3.27).

Remark 9.3. It is easy to see that all classical lifts are polynomial with $D_\lambda \equiv 0$, $\lambda = 1, 2, 3$. The diagonal lift is given by $C_3 \equiv 1$, $C_2 \equiv 0$ and $C_1 \equiv 1$. The horizontal lift is given by $C_1 \equiv 0$, $C_2 \equiv 1$ and $C_3 \equiv 0$. The vertical lift is given by $C_1 \equiv 1$, $C_2 \equiv 0$ and $C_3 \equiv 0$.

10. OTHER NATURAL DIFFERENTIAL OPERATORS

In Section 10.1 we describe all natural transformations of the second tangent functor TT into itself.

In Section 10.2 we consider natural lifts of vector fields on X to vector fields on TX or $T_k^1 X$. Our results generalize the classical complete and vertical lifts.

In auxiliary Section 10.3 we introduce principal connections on the semi-holonomic second order frame bundle which will be needed later. In Section 10.4 we describe all natural prolongations of linear connections to principal connections on the semi-holonomic second order frame bundle.

10.1. Natural transformations of the second tangent functor. Let $X \in \mathrm{Ob}\,\mathcal{D}_n$. We remind that the tangent bundle $p_X : TX \to X$ is defined as $TX = J_0^1(R, X)$. The second tangent bundle is defined by $TTX = T(TX)$. TT is an S-lifting F_S^2 with the type fiber $S = R^{3n}$. The global coordinates on S will be denoted by (u^i, v^i, w^i). The action of the group L_n^2 on S is then given by

$$\bar{u}^i = a_p^i u^p, \qquad \bar{v}^i = a_p^i v^p, \qquad \bar{w}^i = a_{pq}^i u^p v^q + a_p^i w^p. \tag{10.1.1}$$

On TTX there is a well-known *canonical involutive automorphism* $i_X : TTX \to \to TTX$. $i : TT \to TT$ is a natural differential operator of order zero (we shall call it *natural transformation of* TT). The coordinate expression of the associated differential invariant of L_n^2 is given by the interchange of coordinates u^i and v^i. Our aim now will be to determine all natural transformations of TT.

Theorem 10.1. *There are four families each of four real parameters of natural transformations of TT. The associated differential invariants of L_n^2 are given by*

$$\begin{aligned}
&a)\ u^i = k_1 u^i, \quad v^i = k_2 v^i, \quad w^i = k_3 u^i + k_4 v^i + k_1 k_2 w^i,\\
&b)\ u^i = l_1 v^i, \quad v^i = l_2 u^i, \quad w^i = l_3 u^i + l_4 v^i + l_1 l_2 w^i,\\
&c)\ u^i = 0, \quad v^i = m_1 u^i + m_2 v^i, \quad w^i = m_3 u^i + m_4 v^i,\\
&d)\ u^i = n_1 u^i + n_2 v^i, \quad v^i = 0, \quad w^i = n_3 u^i + n_4 v^i,
\end{aligned} \tag{10.1.2}$$

where k_i, l_i, m_i, n_i, $i = 1, \ldots, 4$, are arbitrary real numbers.

P r o o f. Let $\xi \in L(L_n^2)$. Then the fundamental vector fields on S associated to ξ are given by

$$\begin{aligned}
\Xi_p^q &= \left(\frac{\partial \bar{u}^i}{\partial a_q^p}\right)_e \frac{\partial}{\partial u^i} + \left(\frac{\partial \bar{v}^i}{\partial a_q^p}\right)_e \frac{\partial}{\partial v^i} + \left(\frac{\partial \bar{w}^i}{\partial a_q^p}\right)_e \frac{\partial}{\partial w^i} =\\
&= u^q \frac{\partial}{\partial u^p} + v^q \frac{\partial}{\partial v^p} + w^q \frac{\partial}{\partial w^p},
\end{aligned} \tag{10.1.3}$$

$$\Xi_p^{qr} = \left(\frac{\partial \bar{w}^i}{\partial a_{qr}^p}\right)_e \frac{\partial}{\partial w^i} = \frac{1}{2}(u^q v^r + u^r v^q) \frac{\partial}{\partial w^p} \tag{10.1.4}$$

where $e \in L_n^2$ is the unity. A differential invariant corresponding to a natural transformation of TT is a mapping $F : S \to S$ given by

$$u^i = f^i(u, v, w), \qquad v^i = g^i(u, v, w), \qquad w^i = h^i(u, v, w). \tag{10.1.5}$$

Theorem 3.4 implies that the mappings (10.1.5) have to satisfy the following systems of partial differential equations

$$u^q \frac{\partial f^i}{\partial u^p} + v^q \frac{\partial f^i}{\partial v^p} + w^q \frac{\partial f^i}{\partial w^p} = f^q \delta^i_p,$$

(10.1.6) $$u^q \frac{\partial g^i}{\partial u^p} + v^q \frac{\partial g^i}{\partial v^p} + w^q \frac{\partial g^i}{\partial w^p} = g^q \delta^i_p,$$

$$u^q \frac{\partial h^i}{\partial u^p} + v^q \frac{\partial h^i}{\partial v^p} + w^q \frac{\partial h^i}{\partial w^p} = h^q \delta^i_p,$$

(10.1.7) $$\frac{\partial f^i}{\partial w^p}(u^q v^r + u^r v^q) = 0,$$

(10.1.8) $$\frac{\partial g^i}{\partial w^p}(u^q v^r + u^r v^q) = 0,$$

(10.1.9) $$\frac{\partial h^i}{\partial w^p}(u^q v^r + u^r v^q) = \delta^i_p(f^q g^r + f^r g^q).$$

Putting $p = q$ (no summation) in (10.1.6) we obtain

(10.1.10) $$u^p \frac{\partial f^i}{\partial u^p} + v^p \frac{\partial f^i}{\partial v^p} + w^p \frac{\partial f^i}{\partial w^p} = f^i$$

and the same equation we obtain for g^i and h^i. Theorem 8.4 implies that global solutions of (10.1.10) are polynomials of degrees k in u^p, l in v^p and m in w^p such that

10.1.11) $$k + l + m = 1.$$

(10.1.11) has three possible solutions in natural numbers, $(k, l, m) = (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$, and hence f, g, h are linear in all variables. Thus

$$f^i = a_1 u^i + a_2 v^i + a_3 w^i,$$

(10.1.12) $$g^i = a_4 u^i + a_5 v^i + a_6 w^i,$$

$$h^i = a_7 u^i + a_8 v^i + a_9 w^i,$$

where a^i, $i = 1, \ldots, 9$ are real numbers.

Let us put $p = q = r$ in (10.1.7)–(10.1.9) (no summation). Then from (10.1.7) and (10.1.12) we have $a_3 = 0$; similarly from (10.1.8) and (10.1.12) we have $a_6 = 0$. From (10.1.9) and (10.1.12) we obtain $a_9 u^p v^p = (a_1 u^p + a_2 v^p)(a_4 u^p + a_5 v^p)$ which implies

$$a_1 a_4 = 0,$$

(10.1.13) $$a_2 a_5 = 0,$$

$$a_1 a_5 + a_2 a_4 = a_9 .$$

(10.1.13) has four different solutions

$$(10.1.14)\qquad \begin{aligned} &a)\ a_2 = a_4 = 0,\ a_9 = a_1 a_5,\\ &b)\ a_1 = a_5 = 0,\ a_9 = a_2 a_4,\\ &c)\ a_1 = a_2 = a_9 = 0,\\ &d)\ a_4 = a_5 = a_9 = 0. \end{aligned}$$

Denoting free variables in the case $a)$ as k_i, in the case $b)$ as l_i, in the case $c)$ as m_i and in the case $d)$ as n_i, $i = 1, \dots, 4$, we obtain (10.1.2). It is very easy to verify that (10.1.2) defines L^2_n-equivariant mappings and this ends our proof.

Remark 10.1. The canonical involution corresponds to the case $b)$ where $l_1 = = l_2 = 1$, $l_3 = l_4 = 0$.

10.2. Natural lifts of vector fields. Let $X \in \mathrm{Ob}\,\mathcal{D}_n$ and ξ be a vector field on X and α^ξ_t its local one-parameter group. Let T be the tangent functor. T is a 1st-order lifting, and hence we can construct the *tangent lift* $T\xi$ of ξ using the flow by

$$(10.2.1)\qquad \alpha^{T\xi}_t = T\alpha^\xi_t.$$

If (x^i) are some local coordinates on X and $\xi = \xi^i(x)\,\partial/\partial x^i$, then from (10.2.1) it follows that

$$(10.2.2)\qquad T\xi = \xi^i(x)\frac{\partial}{\partial x^i} + \frac{\partial \xi^i(x)}{\partial x^j}\,u^j\,\frac{\partial}{\partial u^i},$$

where (x^i, u^i) are the induced coordinates on the tangent bundle $p_X : TX \to X$. From (10.2.2) it is easy to see that the operator which associates to each vector field ξ on X and to any point u of TX the vector $T\xi(u)$ from T_uTX defines a 1st-order natural differential operator from $TX \oplus TX$ to TTX which induces the identity on TX with respect to the projections $p_1 : TX \oplus TX \to TX$ and $p_{TX} : TTX \to TX$. We shall describe all such 1st-order natural differential operators which give all 1st-order natural lifts of a vector field on X to vector fields on TX.

The functor $T \oplus T$ is a P-lifting F^1_P with the type fiber $P = R^n \times R^n$. The global coordinates on P will be denoted by (u^i, ξ^i). The action of L^1_n on P is given by

$$(10.2.3)\qquad \bar{u}^i = a^i_m u^m, \qquad \bar{\xi}^i = a^i_m \xi^m.$$

TT is an S-lifting F^2_S with the type fiber $S = R^{3n}$. The global coordinates on S will be denoted by (u^i, v^i, w^i). The action of L^2_n on S is then given by

$$(10.2.4)\qquad \bar{u}^i = a^i_m u^m, \qquad \bar{v}^i = a^i_m v^m, \qquad w^i = a^i_{km} u^k v^m + a^i_m w^m.$$

The differential invariant of L^2_n corresponding to the operator (10.2.2) is the mapping from $P_1 = R^n \times T^1_n R^n$ to S given by

$$(10.2.5)\qquad u^i = u^i, \qquad v^i = \xi^i; \qquad w^i = \xi^i_{,j} u^j.$$

Theorem 10.2. *There is a 3 parameter family of natural lifts of order one of vector fields on X to vector fields on TX. A vector field $\xi = \xi^i(x)\,\partial/\partial x^i$ on X is lifted to vector fields*

$$(10.2.6)\qquad b\left(\xi^i(x)\frac{\partial}{\partial x^i} + \frac{\partial \xi^i(x)}{\partial x^j}u^j\frac{\partial}{\partial u^i}\right) + c\xi^i(x)\frac{\partial}{\partial u^i_j} + du^{ij}\frac{\partial}{\partial u^i_j},$$

where b, c, d are real numbers.

Proof. Let $f: P \to S$ be a differential invariant of L^2_n corresponding to a required natural lift. Then its coordinate expression is given by

$$(10.2.7)\qquad u^i = u^i, \qquad v^i = f^i(u^p, \xi^p, \xi^p_{,q}), \qquad w^i = g^i(u^p, \xi^p, \xi^p_{,q}).$$

Let us recall that the action of L^2_n on $T^1_n R^n$ is expressed by the equations

$$(10.2.8)\qquad \bar{\xi}^i = a^i_m \xi^m, \qquad \bar{\xi}^i_{,j} = a^i_{km} b^k_j \xi^m + a^i_m \xi^m_{,k} b^k_j.$$

(10.2.7) has to be L^2_n-equivariant mapping with respect to the actions (10.2.3), (10.2.4) and (10.2.8).

Let $\zeta \in L(L^2_n)$. The fundamental vector fields on $P_1 = R^n \times T^1_n R^n$ relative to ζ are expressed by

$$(10.2.9)\qquad \begin{aligned} Z^q_p &= \left(\frac{\partial \bar{u}^i}{\partial a^p_q}\right)_e \frac{\partial}{\partial u^i} + \left(\frac{\partial \bar{\xi}^i}{\partial a^p_q}\right)_e \frac{\partial}{\partial \xi^i} + \left(\frac{\partial \bar{\xi}^i_{,j}}{\partial a^p_q}\right)_e \frac{\partial}{\partial \xi^i_{,j}} = \\ &= u^q \frac{\partial}{\partial u^p} + \xi^q \frac{\partial}{\partial \xi^p} + \xi^q_{,j} \frac{\partial}{\partial \xi^p_{,j}} - \xi^i_{,p} \frac{\partial}{\partial \xi^i_{,q}}. \end{aligned}$$

$$(10.2.10)\qquad Z^{qr}_p = \left(\frac{\partial \bar{\xi}^i_{,j}}{\partial a^p_{qr}}\right)_e \frac{\partial}{\partial \xi^i_{,j}} = \frac{1}{2}\left(\xi^r \frac{\partial}{\partial \xi^p_{,q}} + \xi^q \frac{\partial}{\partial \xi^p_{,r}}\right).$$

Similarly the fundamental vector fields on S relative to ζ are expressed by

$$(10.2.11)\qquad \begin{aligned} Z^q_{pS} &= \left(\frac{\partial \bar{u}^i}{\partial a^p_q}\right)_e \frac{\partial}{\partial u^i} + \left(\frac{\partial \bar{v}^i}{\partial a^p_q}\right)_e \frac{\partial}{\partial v^i} + \left(\frac{\partial \bar{w}^i}{\partial a^p_q}\right)_e \frac{\partial}{\partial w^i} = \\ &= u^q \frac{\partial}{\partial u^p} + v^q \frac{\partial}{\partial v^p} + w^q \frac{\partial}{\partial w^p}, \end{aligned}$$

$$(10.2.12)\qquad Z^{qr}_{pS} = \left(\frac{\partial w^i}{\partial a^p_{qr}}\right)_e \frac{\partial}{\partial w^i} = \frac{1}{2}(u^r v^q + u^q v^r)\frac{\partial}{\partial w^p},$$

where $e \in L^2_n$ is the unity.

Theorem 3.4 implies that the differential invariant (10.2.7) has to satisfy (if $p = q$) the following systems of partial differential equations

$$(10.2.13)\qquad \frac{\partial f^i}{\partial u^p} u^p + \frac{\partial f^i}{\partial \xi^p} \xi^p = f^i,$$

(10.2.14) $$\frac{\partial g^i}{\partial u^p} u^p + \frac{\partial g^i}{\partial \xi^p} \xi^p = g^i.$$

Theorem 8.4 implies that f^i and g^i are sums of linear functions of variables u and ξ^i where coeficients depend on $\xi^i_{,j}$, i.e.

(10.2.15) $$f^i(u^p, \xi^p, \xi^p_{,q}) = f^i_j(\xi^p_{,q})\, u^j + \bar{f}^i_j(\xi^p_{,q})\, \xi^j,$$

(10.2.16) $$g^i(u^p, \xi^p, \xi^p_{,q}) = g^i_j(\xi^p_{,q})\, u^j + \bar{g}^i_j(\xi^p_{,q})\, \xi^j.$$

The vector fields (10.2.10) and (10.2.12) are f-related which gives the system of partial differential equations for (10.2.15)

(10.2.17) $$\xi^r \left(\frac{\partial f^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} u^j + \frac{\partial \bar{f}^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} \xi^j \right) + \\ + \xi^q \left(\frac{\partial f^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} u^j + \frac{\partial \bar{f}^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} \xi^j \right) = 0.$$

This system has to be satisfied for all u^i, ξ^i and this implies

$$\frac{\partial f^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} = 0, \qquad \frac{\partial \bar{f}^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} = 0.$$

Hence f^i_j is constant and $\bar{f}^i_j$ is also constant. From the invariance condition $a^i_j(f^j_k u^k + \bar{f}^j_k \xi^k) = f^i_j a^j_k u^k + \bar{f}^i_j a^j_k \xi^k$ we have $a^i_j f^j_k = f^i_j a^j_k$, $a^i_j \bar{f}^j_k = \bar{f}^i_j a^j_k$ which means that f^i_j, $\bar{f}^i_j$ are absolute invariant tensors and then $f^i_j = a\delta^i_j$, $\bar{f}^i_j = b\delta^i_j$, where a, b are real numbers. Then

(10.2.18) $$v^i = au^i + b\xi^i.$$

Since (10.2.10) and (10.2.12) are g-related we get for (10.2.16)

(10.2.19) $$\xi^r \left(\frac{\partial g^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} u^j + \frac{\partial \bar{g}^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} \xi^j \right) + \xi^q \left(\frac{\partial g^i_j(\xi^p_{,q})}{\partial \xi^p_{,r}} u^j + \frac{\partial \bar{g}^i_j(\xi^p_{,q})}{\partial \xi^p_{,r}} \xi^j \right) = \\ = \delta^i_p((au^r + b\xi^r)\, u^q + (au^q + b\xi^q)\, u^r).$$

(10.2.19) has to be satisfied for all u^q, ξ^q and if we put $q = r$ (no summation) we obtain

(20.2.20) $$a = 0, \qquad \frac{\partial g^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} u^j = \delta^i_p b u^q, \qquad \frac{\partial \bar{g}^i_j(\xi^p_{,q})}{\partial \xi^p_{,q}} = 0.$$

(10.2.20) implies that $\bar{g}^i_j$ is constant and the invariance condition condition leads to $\bar{g}^i_j = c\delta^i_j$, where c is a real number. Further let us put $i = p$ and $j = q$. Then $\partial g^p_q / \partial \xi^p_{,q} = b$ and hence $g^p_q = b\xi^u_{,q} + d^p_q$, where d^p_q is a constant. Then

(10.2.21) $$w^i = (b\xi^i_{,j} + d^i_j)\, u^j + c\xi^i$$

and from the invariance condition we obtain $d_j^i = d\delta_j^i$. (10.2.18), together with 10.2.21), gives the differential invariant of L_n^2 from $R^n \times T_n^1 R^n$ to S expressed by

$$(10.2.22) \qquad u^i = u^i, \qquad v^i = b\zeta^i, \qquad w^i = b\xi^i_{,j} u^j + du^i + c\xi^i,$$

where b, c, d are real numbers. (10.2.22) corresponds to natural lifts given by (10.2.6) which ends the proof.

Remark 10.2. The first vector field in (10.2.6) is a multiple of the tangent lift $T\xi$ of ξ called also the *complete lift* of ξ. The second vector field is a multiple of the vertical lift of ξ defined in Section 9.3. The third vector field is a multiple of the *Liouville vector field*; notice that this vector field does not depend on the choice of the vector field ξ. The last two vector fields in (10.2.6) define zero order natural lifts of ξ.

Now let $T_k^1 X = J_0^1(R^k, X)$ be the space of k^1-velocities on X. Let (x^i) be some local coordinates on X. We have the induced coordinates (x^i, u_λ^i), $1 \leqq \lambda \leqq k$, on $T_k^1 X$. Via flow we can lift a vector field ξ on X into a vector field $T_k^1\xi$ on $T_k^1 X$ by

$$(10.2.23) \qquad \alpha_t^{T_k^1\xi} = T_k^1 \alpha_t^\xi.$$

In the induced coordinates we get

$$(10.2.24) \qquad T_k^1\xi = \xi^i(x)\frac{\partial}{\partial x^i} + \frac{\partial \xi^i(x)}{\partial x^j} u_\lambda^j \frac{\partial}{\partial u_\lambda^i}.$$

$T_k^1\xi$ is sometimes called the *complete lift* (or the *natural lift*) of ξ on $T_k^1 X$. This lift is a natural differential operator of order one from $T_k^1 \oplus T$ to TT_k^1 which induces the identity of $T_k^1 X$ with respect to the projections $p_1 : T_k^1 X \oplus TX \to T_k^1 X$ and $p_{T_k^1 X} : TT_k^1 X \to T_k^1 X$. The natural question is whether there exist another natural lifts of this type. We restrict ourselves only to the case of the first order operators.

Theorem 10.3. *Each vector field $\xi = \xi^i(x)\, \partial/\partial x^i$ on X can be naturally lifted to the system of vector fields on $T_k^1 X$*

$$(10.2.25) \qquad a\left(\xi^i(x)\frac{\partial}{\partial x^i} + \frac{\partial \xi^i(x)}{\partial x^j} u_\lambda^j \frac{\partial}{u_\lambda^i}\right) + c_\lambda \xi^i(x)\frac{\partial}{\partial u_\lambda^i} + d_\lambda^\mu u_\mu^i \frac{\partial}{\partial u_\lambda^i},$$

where $a, c_\lambda, d_\lambda^\mu, \lambda, \mu = 1, \ldots, k$, are real constants.

Proof. The type fiber of the functor $T_k^1 \oplus T$ is $(R^n \otimes R^{k*}) \times R^n$ with global coordinates (u_λ^i, ξ^i) and the action of L_n^1 given by

$$(10.2.26) \qquad \bar{u}_\lambda^i = a_m^i u_\lambda^m, \qquad \bar{\xi}^i = a_m^i \xi^m.$$

The type fiber of the functor TT_k^1 is $S = (R^n \otimes R^{k*}) \times R^n \times (R^n \otimes R^{k*})$, with global coordinates $(u_\lambda^i, v^i, v_\lambda^i)$ and the action of the group L_n^2 expressed by

(10.2.27) $\quad \bar{u}_\lambda^i = a_m^i u_\lambda^m, \qquad \bar{v}^i = a_m^i v^m, \qquad \bar{v}_\lambda^i = a_m^i v_\lambda^m + a_{mk}^i v^m u_\lambda^k.$

Differential invariants of L_n^2 corresponding to operators of our type are L_n^2-mappings from $P = (R^n \otimes R^{k*}) \times T_n^1 R^n$ to S given by

(10.2.28) $\quad u_\lambda^i = u_\lambda^i,$

(10.2.29) $\quad v^i = f^i(u_\lambda^p, \xi^p, \xi^p_{,q}),$

(10.2.30) $\quad v_\lambda^i = f_\lambda^i(u_\lambda^p, \xi^p, \xi^p_{,q}).$

Let $\zeta \in L(L_n^2)$. The fundamental vector fields on P relative to ζ are expressed by

(10.2.31) $$Z^q_{pP} = u_\lambda^q \frac{\partial}{\partial u_\lambda^p} + \xi^q \frac{\partial}{\partial \xi^p} + \xi^q_{,r} \frac{\partial}{\partial \xi^p_{,r}} - \xi^r_{,p} \frac{\partial}{\partial \xi^r_{,q}},$$

(10.2.32) $$Z^{qr}_{pP} = \frac{1}{2}\left(\xi^r \frac{\partial}{\partial \xi^p_{,q}} + \xi^q \frac{\partial}{\partial \xi^p_{,r}}\right).$$

Similarly the fundamental vector fields on S associated to ζ are expressed by

(10.2.33) $$Z^q_{pS} = u_\lambda^q \frac{\partial}{\partial u_\lambda^p} + v^q \frac{\partial}{\partial v^p} + v_\lambda^q \frac{\partial}{\partial v_\lambda^p},$$

(10.2.34) $$Z^{qr}_{pS} = \frac{1}{2}\left(v^q u_\lambda^r \frac{\partial}{\partial v_\lambda^p} + v^r u_\lambda^q \frac{\partial}{\partial v_\lambda^p}\right).$$

The differential invariant (10.2.29) has to satisfy the following systems of partial differential equations

(10.2.35) $$\frac{\partial f^i}{\partial u_\lambda^p} u_\lambda^q + \frac{\partial f^i}{\partial \xi^p} \xi^q + \frac{\partial f^i}{\partial \xi^p_{,r}} \xi^q_{,r} - \frac{\partial f^i}{\partial \xi^r_{,q}} \xi^r_{,p} = f^i,$$

(10.2.36) $$\xi^r \frac{\partial f^i}{\partial \xi^p_{,q}} + \xi^q \frac{\partial f^i}{\partial \xi^p_{,r}} = 0.$$

Putting $p = q$ (no summation) in (10.2.35) we obtain because of Theorem 8.2 that f^i has to be a sum of linear functions in variables u_λ^p and ξ^p where coefficients depend on $\xi^p_{,q}$, i.e.

(10.2.37) $$f^i(u^p, \xi^p, \xi^p_{,q}) = f^i_j(\xi^p_{,q})\, \xi^j + f^{i\lambda}_j(\xi^p_{,q})\, u^j_\lambda.$$

Substituting (10.2.37) into (10.2.36) we get $\partial f^i_j/\partial \xi^p_{,q} = 0$, $\partial f^{i\lambda}_j/\partial \xi^p_{,q} = 0$ which implies that f^i_j are constant and $f^{i\lambda}_j$ are constant for all $\lambda = 1, \ldots, k$. The invariance condition then implies $f^i_j = a\delta^i_j$, $f^{i\lambda}_j = b^\lambda \delta^i_j$, where a, b^λ, $\lambda = 1, \ldots, k$, are real numbers. Then

(10.2.38) $\quad v^i = a\xi^i + b^\lambda u^i_\lambda.$

Using similar methods for (10.2.30) we obtain $f^i_\lambda = f^i_{j\lambda}(\xi^p_{,q})\,\xi^j + f^{i\mu}_{j\lambda}(\xi^p_{,q})\,u^j_\mu$, where the following system of partial differential equations

(10.2.39) $$\frac{\partial f^i_{j\lambda}(\xi^p_{,q})}{\partial \xi^p_{,q}}\,\xi^q\xi^j + \frac{\partial f^{i\mu}_{j\lambda}(\xi^p_{,q})}{\partial \xi^p_{,q}}\,\xi^q u^j_\mu = \delta^i_p(a\xi^q u^q_\lambda + b^\mu u^q_\mu u^q_\lambda)$$

is satisfied. (10.2.39) implies that $b^\mu = 0$ for all $\mu = 1, \ldots, k$, and

(10.2.40) $$\frac{\partial f^i_{j\lambda}(\xi^p_{,q})}{\partial \xi^p_{,q}} = 0, \qquad \frac{\partial f^{i\mu}_{j\lambda}(\xi^p_{,q})}{\partial \xi^p_{,q}} = a\delta^i_p\delta^\mu_\lambda.$$

The first condition implies that $f^i_{j\lambda}$ are constant for all $\lambda = 1, \ldots, k$, and the second condition gives

(10.2.41) $\quad f^{i\mu}_{j\lambda} = a\delta^\mu_\lambda\xi^i_{,j} + d^{i\mu}_{j\lambda},$

where $d^{i\mu}_{j\lambda}$ are constant for all $\lambda, \mu = 1, \ldots, k$. Hence we have

(10.2.42) $\quad v^i_\lambda = f^i_{j\lambda}\xi^j + a\xi^i_{,j}u^j_\lambda + d^{i\mu}_{j\lambda}u^j_\mu.$

The invariance condition then implies $f^i_{j\lambda} = c_\lambda\delta^i_j$, $d^{i\mu}_{j\lambda} = d^\mu_\lambda\delta^i_j$, where c_λ, d^μ_λ, λ, $\mu = 1, \ldots, k$, are real numbers. This together with (10.2.38) give the L^2_n-equivariant mappings from P to S in the form

(10.2.43) $\quad u^i_\lambda = u^i_\lambda, \qquad v^i = a\xi^i, \qquad v^i_\lambda = c_\lambda\xi^i + a\xi^i_{,j}u^j_\lambda + d^\mu_\lambda u^i_\mu.$

(10.2.43) are just the differential invariants of L^2_n which correspond to natural lifts given by (10.2.25). This ends the proof.

Remark 10.3. The first vector field in (10.2.25) is a multiple of the complete lift $T^1_k\xi$ of ξ. The second vector field is some vertical lift and the third vector field is a constant vector field which does not depend on a given vector field ξ on X. The last two vector fields in (10.2.25) define zero order natural lifts of vector fields on X to vector fields on T^1_kX.

10.3. Principal connections on frame bundles. Let $\pi : Y \to X$ be a fibered manifold. A *(generalized) connection* on Y is a section $\Gamma : Y \to J^1Y$. Let $Y(X, G, \pi)$ be a principal fiber bundle with a structure group G. Then a connection $\Gamma : Y \to$ $\to J^1Y$ is called *principal connection* if Γ is G-invariant where the right action of G on J^1Y is given by $(J^1_x\gamma)\,g = J^1_x(\gamma g)$, $\gamma : X \to Y$ is a section. Let a connection Γ on Y be given by $\Gamma(y) = J^1_x\gamma(t)$ where $\gamma : X \to Y$ is a section such that $\gamma(x) = y$, $y \in Y_x$. Γ is a principal connection if $\Gamma(yg) = J^1_x(\gamma(t)\,g)$.

Let E be a fibered bundle with fiber F associated with the principal G-bundle Y, i.e. $E = Y \times_G F$. Then a connection Γ on Y, given by $\Gamma(y) = J^1_x \gamma$, $y \in Y_x$, determines a connection $\Delta : E \to J^1 E$ on E defined by $\Delta([y, u]) = J^1_x([\gamma(t), u])$, $u \in F$. The section Δ is well defined because $\Delta([yg, g^{-1}u]) = J^1_x([\gamma(t)\, g, g^{-1}u]) = J^1_x([\gamma(t), u]) = \Delta([y, u])$.

Now let Y be the principal bundle of linear frames on X. Let (U, φ) be a chart on X and $\varphi = (x^i)$ the corresponding local coordinates. Let $((\pi^1_X)^{-1}(U), \varphi^1)$ be the induced chart on $F^1 X$ and (x^i, u^i_j) the induced local coordinates. $F^1 X$ can be locally identified with the trivial principal bundle $U \times L^1_n$. Let $x \in U$ and $\gamma : U \to U \times L^1_n$ be a section such that $g(x) = (x, e)$ where $e \in L^1_n$ is the unity. In the coordinates γ is given by $(t^i) \to (t^i, \gamma^i_j(t))$ where $\gamma^i_j(x) = \delta^i_j$. Then the group multiplication in L^1_n implies that a principal connection Γ can be expressed by $\Gamma(x^i, u^i_j) = J^1_x(\gamma^i_m(t)\, u^m_j)$. If we denote $\Gamma^i_{jk}(x) = -(\partial \gamma^i_k(x))/\partial x^j$ the coordinate expression of the principal connection Γ is given by

$$u^i_{j,k} = -\Gamma^i_{km}(x)\, u^m_j, \tag{10.3.1}$$

where $(x^i, u^i_j, u^i_{j,k})$ are the induced local coordinates on $J^1 F^1 X$. The lifting (we shall denote it by the same symbol Γ) $\Gamma : F^1 X \oplus TX \to TF^1 X$ associated with the principal connection Γ has the coordinate expression

$$\mathrm{d}u^i_j = -\Gamma^i_{km}(x)\, u^m_j\, \mathrm{d}x^k. \tag{10.3.2}$$

The functions $\Gamma^i_{jk}(x)$ defined locally on X will be called components of principal connections on $F^1 X$. Let $(\bar{U}, \bar{\varphi})$ be another chart on X and $\bar{\varphi} = (\bar{x}^i)$ the corresponding local coordinates. Let $((\pi^1_X)^{-1}(\bar{U}), \bar{\varphi}^1)$ be the induced chart on $F^1 X$ and $(\bar{x}^i, \bar{u}^i_j)$ the induced local coordinates. Then the transformation law on $(\pi^1_X)^{-1}(U \cap \bar{U})$ is given by

$$\bar{x}^i = \bar{x}^i(x^p), \qquad \bar{u}^i_j = \frac{\partial \bar{x}^i}{\partial x^m} u^m_j, \tag{10.3.3}$$

which implies

$$\mathrm{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^m} \mathrm{d}x^m, \qquad \mathrm{d}\bar{u}^i_j = \frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^q} \mathrm{d}x^q u^p_j + \frac{\partial \bar{x}^i}{\partial x^p} \mathrm{d}u^p_j. \tag{10.3.4}$$

The connection Γ has in the coordinates $(\bar{x}^i, \bar{u}^i_j)$ expression

$$\mathrm{d}\bar{u}^i_j = -\bar{\Gamma}^i_{km}(x)\, \bar{u}^m_j\, \mathrm{d}\bar{x}^k. \tag{10.3.5}$$

Then substituting (10.3.3), (10.3.4) into (10.3.5) and using (10.3.2) we obtain the transformation relation

$$\bar{\Gamma}^i_{jk}(x) = \frac{\partial \bar{x}^i}{\partial x^p} \Gamma^p_{qr}(x) \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^r}{\partial \bar{x}^k} + \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}. \tag{10.3.6}$$

Denoting $\frac{\partial \bar{x}^i}{\partial x^j} = a^i_j$, $\frac{\partial x^i}{\partial \bar{x}^j} = b^i_j$, $\frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k} = b^i_{jk}$ we obtain the relation (9.1.2).

Let TX be the tangent bundle of X. Local coordinates (x^i) on X induce the local coordinates (x^i, v^i) on TX. By a *linear connection* on X we understand a linear section $\Lambda : TX \to J^1TX$. Let Λ be a connection on TX given by $v^i_j = \Lambda^i_j(x, v)$, where (x^i, v^i, v^i_j) are the induced coordinates on J^1TX. Then Λ is a linear connection if and only if $\Lambda^i_j(x, v)$ are linear function with respect to v, i.e. $\Lambda^i_j(x, v) = \Lambda^i_{jk}(x)\, v^k$.

Lemma 10.1. *The rule which associates to each principal connection Γ on F^1X, $\Gamma(u) = J^1_x\gamma(t)$, $u \in F^1_xX$, the connection Λ on TX, $\Lambda([u, v]) = J^1_x([\gamma(t), v])$, $v \in R^n$, $n = \dim X$, is a bijective mapping of principal connections on F^1X to linear connections on X.*

Proof. In the induced coordinates (x^i, u^i_j) on F^1X a principal connection Γ on F^1X is given by $\Gamma(x^i, u^i_j) = J^1(\gamma^i_m(t)\, u^m_j)$. A vector $v^i \in R^n$ is mapped by the section $\gamma^i_j(t)$ into $\gamma^i_j(t)\, v^j \in T_tX$. The connection Λ on TX is then given by $\Lambda(x^i, v^i) = J^1_x(\gamma^i_j(t)\, v^j)$. The coordinate expression of the corresponding lifting $\Lambda : TX \oplus TX \to T(TX)$ is given by

$$\text{(10.3.7)} \qquad dv^i = -\Gamma^i_{jk}(x)\, v^k\, dx^j,$$

which implies that Λ is a linear connection on X. On the other hand if a linear connection on X with equation (10.3.7) is given there is the unique principal connection on F^1X with the coordinate expression (10.3.2).

Further we shall identify linear connections on X with principal connections on F^1X and denote them by the same symbol. Components of principal connections on F^1X satisfy the same transformation law as components of linear connections ton X. Hence we can identify the space of components of principal connections on F^1X with $F^2_QX = CX$ defined in Section 9.1.

Let X, Y be two smooth manifolds. Let $J^1(X, Y)$ denotes the space of all 1-jets from X to Y. Let us denote the source projection by $\alpha : J^1(X, Y) \to X$ and the target projection by $\beta : J^1(X, Y) \to Y$. As a *non-holonomic 2-jet* with source in X and target in Y we define a 1-jet of a mapping $\varphi : X \to J^1(X, Y)$ which is a section with respect to α, i.e. $\alpha\varphi = \mathrm{id}_X$. The space of all non-holonomic 2-jets with source in X and target in Y will be denoted $\tilde{J}^2(X, Y)$. Let us remark that ordinary (holonomic) 2-jets $J^2(X, Y)$ form a subspace in $\tilde{J}^2(X, Y)$. The inclusion mapping is given by $J^2_xf \mapsto J^1_x(J^1f)$, $x \in X$. A non-holonomic 2-jet is *semi-holonomic* if it is given by $\varphi : X \to J^1(X, Y)$ satisfying $\varphi(x) = J^1_x(\beta\varphi)$. The space of all semi-holonomic 2-jets from X to Y will be denoted by $\bar{J}^2(X, Y)$.

Let $A \in \tilde{J}^2_{(x,y)}(X, Y)$, $A = J^1_x\varphi$, $\varphi : X \to J^1(X, Y)$, $\alpha(\varphi(x)) = x$, and $B \in \tilde{J}^2_y(Y, Z)$, $B = J^1\psi$, $\psi : Y \to J^1(Y, Z)$, $\alpha(\psi(y)) = y$. Then $\alpha\psi(\beta\varphi(x)) = \beta\varphi(x)$. Hence 1-jets

$\varphi(x)$ and $\psi(\beta\varphi(x))$ are composable and the mapping $x \mapsto \psi(\beta\varphi(x)) \circ \varphi(x)$ is a mapping from X to $J^1(X, Z)$ such that $\alpha(\psi(\beta\varphi(x)) \circ \varphi(x)) = x$. Then we can define the composition of A and B by $C = B \circ A = J^1_x(\psi(\beta\varphi(x) \circ \varphi(x)) \in \tilde{J}^2_x(X, Z)$.

Let (x^i) be local coordinates on X and (y^p) local coordinates on Y. On $J^1(X, Y)$ there are the induced coordinates (x^i, y^p, y^p_i) and on $\tilde{J}^2(X, Y)$ there are the induced coordinates $(x^i, y^p, y^p_i, y^p_{0i}, y^p_{ij})$ (no symmetry in subscripts). From the condition for semi-holonomic 2-jets it follows that $y^p_i = y^p_{0i}$. Thus on $\bar{J}^2(X, Y)$ there are the induced coordinates $(x^i, y^p, y^p_i, y^p_{ij})$. A semi-holonomic 2-jet is holonomic if and only if its 2nd-order coordinate is symmetric in subscripts, i.e. $y^p_{ij} = y^p_{ji}$.

Let (z^s) be local coordinates on Z. Then on $\tilde{J}^2(X, Z)$ there are the induced coordinates $(x^i, z^s, z^s_i, z^s_{0i}, z^s_{ij})$ and on $\tilde{J}^2(Y, Z)$ there are the induced coordinates $(y^p, z^s, z^s_p, z^s_{0p}, z^s_{pq})$. Let $A \in \tilde{J}^2_x(X, Y)$ with coordinate expression $A = (x^i, f^p(x), a^p_i, a^p_{0i}, a^p_{ij})$ and $B \in \tilde{J}^2_y(Y, Z)$ with the coordinate expression $B = (y^p, g^s(y), b^s_p, b^s_{0p}, b^s_{pq})$. A and B are composable if $y^p = f^p(x)$ and the composition $C = B \circ A = (x^i, g^s(f^p(x)), c^s_i, c^s_{0i}, c^s_{ij})$ where

$$(10.3.8) \qquad c^s_i = b^s_p a^p_i, \qquad c^s_{0i} = b^s_{0p} a^p_{0i}, \qquad c^s_{ij} = b^s_{pq} a^p_i a^q_{0j} + b^s_p a^p_{ij}.$$

Let $X \in \mathrm{Ob}\,\mathscr{D}_n$. The principal bundle of *non-holonomic second order frames* $\tilde{F}^2X$ on X is the space of all invertible non-holonomic 2-jets from R^n to X with source in the origin of R^n, i.e. $\tilde{F}^2X = \mathrm{inv}\,\tilde{J}^2_0(R^n, X)$. By definition of non-holonomic 2-jets, any local coordinates (x^i) on X induce the coordinates $(x^i, u^i_j, u^i_{0j}, u^i_{jk})$ (no symmetry in subscripts) on $\tilde{F}^2X$. The structure group of $\tilde{F}^2X$ is the group $\hat{L}^2_n = \mathrm{inv}\,\tilde{J}^2_{(0,0)}(R^n, R^n)$. The action of $\hat{L}^2_n$ on $\tilde{F}^2X$ is given by the composition of non-holonomic 2-jets (10.3.8).

The principal bundle of *semi-holonomic second order frames* on X is defined by $\bar{F}^2X = \mathrm{inv}\,\bar{J}^2_0(R^n, X)$. Local coordinates (x^i) on X induce the coordinates (x^i, u^i_j, u^i_{jk}) (no symmetry in subscripts) on $\bar{F}^2X$. The structure group of $\bar{F}^2X$ is the group $\bar{L}^2 = \mathrm{inv}\,\bar{J}^2_{(0,0)}(R^n, R^n)$. The group multiplication in $\bar{L}^2_n$ and the action of $\bar{L}^2_n$ on $\bar{F}^2X$ are given by the composition of semi-holonomic 2-jets. On $\bar{L}^2_n$ there given by the composition of semi-holonomic 2-jets. On $\bar{L}^2_n$ there are the canonical global coordinates (a^i_j, a^i_{jk}) (no symmetry in subscripts) where $\det(a^i_j) \neq 0$. If $a = (a^i_j, a^i_{jk})$ and $\bar{a} = (\bar{a}^i_j, \bar{a}^i_{jk})$ are two elements of $\bar{L}^2_n$ then from the composition of semi-holonomic 2-jets

$$(10.3.9) \qquad a \,.\, \bar{a} = (a^i_m \bar{a}^m_j,\ a^i_{pq} \bar{a}^p_j \bar{a}^q_k + a^i_p \bar{a}^p_{jk}).$$

Similarly the action of $\bar{L}^2_n$ on $\bar{F}^2X$ is given by the composition of semi-holonomic 2-jets. If $u = (x^i, u^i_j, u^i_{jk}) \in \bar{F}^2X$ then $u \,.\, a$ has the coordinate expression

$$(10.3.10) \qquad u \,.\, a = (x^i,\ u^i_p a^p_j,\ u^i_{pq} a^p_j a^q_k + u^i_p a^p_{jk}).$$

$\bar{F}^2X$ can be locally identified with the trivial principal bundle $U \times \bar{L}^2_n$. Let $x \in U$ and $\gamma : U \to U \times \bar{L}^2_n$ be a section such that $\gamma(x) = (x, e)$ where $e \in \bar{L}^2$ is the unity, i.e. γ is given by $t^i \mapsto (t^i, \gamma^i_j(t), \gamma^i_{jk}(t))$, $t \in U$, where $\gamma^i_j(x) = \delta^i_j$, $\gamma^i_{jk}(x) = 0$. By (10.3.9) a principal connection $\Gamma : \bar{F}^2X \to J^1\bar{F}^2X$ can be defined by $\Gamma(x^i, u^i_j, u^i_{jk}) = J^1_x(\gamma^i_m(t)\, u^m_j, \gamma^i_{pq}(t)\, u^p_j u^q_k + \gamma^i_p(t)\, u^p_{jk})$. If we put

$$(10.3.11) \quad \Gamma^i_{jk}(x) = -\frac{\partial \gamma^i_k(x)}{\partial x^j}, \qquad \Gamma^i_{jkl}(x) = -\frac{\partial \gamma^i_{kl}(x)}{\partial x^j},$$

we obtain the coordinate expression of the corresponding lifting $\Gamma : \bar{F}^2X \oplus TX \to T\bar{F}^2X$

$$(10.3.12) \quad \begin{aligned} du^i_j &= -\Gamma^i_{km}(x)\, u^m_j\, dx^k, \\ du^i_{jk} &= -(\Gamma^i_{plm}(x)\, u^l_j u^m_k + \Gamma^i_{pm}(x)\, u^m_{jk})\, dx^p. \end{aligned}$$

The functions $(\Gamma^i_{jk}(x), \Gamma^i_{jkl}(x))$ will be called the components of the principal connection Γ on $\bar{F}^2X$.

The space of ordinary (holonomic) second order frames on X is a subbundle in $\bar{F}^2X$. A semi-holonomic frame (x^i, u^i_j, u^i_{jk}) is holonomic iff $u^i_{jk} = u^i_{kj}$. Then a connection Γ on $\bar{F}^2X$ defines a connection on F^2X if its components $\Gamma^i_{jkl}(x)$ are symmetric with respect to the subscripts k, l, i.e. $\Gamma^i_{jkl}(x) = \Gamma^i_{jlk}(x)$.

Using transformation law for a change of local coordinates on X we can deduce after some routine calculation that components of principal connections on $\bar{F}^2X$ satisfy the transformation relations with respect to the group L^3_n given by

$$(10.3.13) \quad \begin{aligned} \bar{\Gamma}^i_{jk} &= a^i_p \Gamma^p_{qr} b^q_j b^r_k + a^i_p b^p_{jk}, \\ \bar{\Gamma}^i_{jkl} &= a^i_m \Gamma^m_{pqr} b^p_j b^q_k b^r_l + a^i_m \Gamma^m_{pq} b^p_j b^q_{kl} + a^i_{pm} \Gamma^m_{rq} b^r_j b^p_k b^q_l + \\ &\quad + a^i_{pm} \Gamma^p_{rq} b^r_j b^q_k b^m_l - a^i_{pqm} b^p_j b^q_k b^m_l - a^i_{mp} b^p_j b^m_{kl}. \end{aligned}$$

Thus the space of components of principal connections on $\bar{F}^2X$ can be identified with P-lift F^3_PX where $P = Q \times S$, $Q = R^n \otimes \overset{2}{\otimes} R^{n*}$, $S = R^n \otimes \overset{3}{\otimes} R^{n*}$, and the left action of L^3_n on P is given by (10.3.13).

Remark 10.4. The canonical projection $\bar{F}^2X \to F^1X$ induces the projection from the space of components of principal connections on $\bar{F}^2X$ to the space of components of principal connections on F^1X. Each principal connection on $\bar{F}^2X$ with components $(\Gamma^i_{jk}(x), \Gamma^i_{jkl}(x))$ induces the principal connection on F^1X with the components $\Gamma^i_{jk}(x)$.

10.4. Natural operations with linear connections. By Lemma 10.1 a linear connection on $X \in \mathrm{Ob}\, \mathscr{D}_n$ can be identified with a principal connection on F^1X. Such a connection is given as a section of the space of components of principal connections of F^1X. This space is the Q-lift F^2_QX defined in Section 9.1. The space of components of principal connections on the semi-holonomic second order frame bundle $\bar{F}^2X$ is the P-lift F^3_PX defined in Section 10.3.

In this section we shall solve the problem how to construct a principal connection on F^2X which depends naturally only on finite order derivatives of a given linear connection on X. Hence we shall solve the problem how to find all finite order natural differential operators from F_Q^2 to F_P^3.

There are two classical geometrical constructions of the prolongation of a linear connection on X to a principal connection on F^2X. We mention here only the corresponding differential invariants of these operators. The differential invariant associated to *Oproiu's prolongation operator* is an L_n^3-equivariant mapping from T_n^1Q to P given by $F^i_{jk}(\Gamma^p_{qr}, \Gamma^p_{qr,s})$, $F^i_{jkl}(\Gamma^p_{qr}, \Gamma^p_{qr,s})$ where

$$
(10.4.1)\qquad
\begin{aligned}
F^i_{jk} &= \Gamma^i_{jk} \text{ (the prolongation condition)},\\
F^i_{jkl} &= \Gamma^i_{jk,l} - \Gamma^i_{mk}\Gamma^m_{jl}.
\end{aligned}
$$

The differential invariant associated to *Kolář's prolongation operator* is an L_n^3-equivariant mapping $F : T_n^1Q \to P$ given by

$$
(10.4.2)\qquad
\begin{aligned}
F^i_{jk} &= \Gamma^i_{jk} \text{ (the prolongation condition)},\\
F^i_{jkl} &= \Gamma^i_{lk,j} + \Gamma^i_{jm}\Gamma^m_{lk} - \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk}.
\end{aligned}
$$

Thus both classical operators are of order one and they are prolongation operators in the sense that the prolonged connection on F^2X is over the given principal connection on F^1X.

Every natural differential operator of order r from F_Q^2 to F_P^3 determines a unique differential invariant $F : T_n^rQ \to P$ of the group L_n^s, $s = \max(2 + r, 3)$. Coordinate expressions of such a differential invariant are of the form

$$(10.4.3)\qquad \Gamma^i_{jk} = F^i_{jk}(\Gamma^p_{qt}, \dots, \Gamma^p_{qt,m_1\dots m_r}),$$

$$(10.4.4)\qquad \Gamma^i_{jkl} = F^i_{jkl}(\Gamma^p_{qt}, \dots, \Gamma^p_{qt,m_1\dots m_r}),$$

where F^i_{jk}, F^i_{jkl} are the components of F respective to the canonical global coordinates on P.

Lemma 10.2. *In the canonical coordinates* $(\Gamma^i_{jk}, \Gamma^i_{jk,m}, \dots, \Gamma^i_{jk,m_1\dots m_r})$ *on* T_n^rQ *the action of the group* L_n^{2+r} *is given by*

$$(10.4.5)\qquad \Gamma^i_{jk,m_1\dots m_s} = a^i_p\Gamma^p_{q_1q_2,q_3\dots q_{s+2}}b^{q_1}_j b^{q_2}_k b^{q_3}_{m_1} \dots b^{q_{s+2}}_{m_s} + S^i_{jkm_1\dots m_s},$$

where $s = 0, \dots, r$, *and* $S^i_{jkm_1\dots m_s}$ *is a polynomial of the coordinates on* $T_n^{s-1}Q$ *and of the canonical coordinates on* L_n^{s+2}, *and each monomial of* $S^i_{j\dots m_s}$ *contains non-trivially at least one of the coordinates* $a^i_{j_1j_2}, \dots, a^i_{j_1\dots j_{s+2}}$.

Proof. The coordinate expression of the action of L_n^{r+2} on T_n^rQ is given by the formal differentiation of the action (9.1.2) up to order r. It is easy to see that this action has the required form.

Now let us consider $\xi \in L(L_n^{n+2})$. The first fundamental vector fields on $T_n^r Q$ and P associated to ξ are expressed by

$$(10.4.6) \qquad \Xi^q_{pT_n^r Q} = \sum_{s=0}^{r} \left(\frac{\partial \overline{\Gamma}^i_{jk, m_1 \ldots m_s}}{\partial a^p_q} \right)_e \frac{\partial}{\partial \Gamma^i_{jk, m_1 \ldots m_s}}$$

and

$$\Xi^q_{pP} = \left(\frac{\partial \overline{\Gamma}^i_{jk}}{\partial a^p_q} \right)_e \frac{\partial}{\partial \Gamma^i_{jk}} + \left(\frac{\partial \overline{\Gamma}^i_{jkl}}{\partial a^p_q} \right)_e \frac{\partial}{\partial \Gamma^i_{jkl}} =$$

$$(10.4.7) \qquad = (\delta^i_p \Gamma^q_{jk} - \delta^q_j \Gamma^i_{pk} - \delta^q_k \Gamma^i_{jp}) \frac{\partial}{\partial \Gamma^i_{jk}} +$$

$$+ (\delta^i_p \Gamma^q_{jkl} - \delta^q_j \Gamma^i_{pkl} - \delta^q_k \Gamma^i_{jpl} - \delta^q_l \Gamma^i_{jkp}) \frac{\partial}{\partial \Gamma^i_{jkl}},$$

where $e \in L_n^{n+2}$ is the unity. Putting $p = q$ (no summation) and using Lemma 10.2 we get

$$(10.4.8) \qquad \Xi^p_{pT_n^r Q} = - \sum_{s=0}^{r} (s+1) \Gamma^i_{jk, m_1 \ldots m_s} \frac{\partial}{\partial \Gamma^i_{jk, m_1 \ldots m_s}}.$$

By the Theorem 3.4 a differential invariant (10.4.3) has to satisfy the following system of partial differential equations

$$(10.4.9) \qquad \sum_{s=0}^{r} (s+1) \Gamma^p_{qt, m_1 \ldots m_s} \frac{\partial F^i_{jk}}{\partial \Gamma^p_{qt, m_1 \ldots m_s}} = F^i_{jk}.$$

Because of Theorem 8.4 all global solutions of (10.4.9) are sums of homogeneous polynomials of degrees a_s in variables $\Gamma^p_{qt, m_1 \ldots m_s}$ such that

$$(10.4.10) \qquad \sum_{s=0}^{r} (s+1) a_s = 1.$$

(10.4.10) has a unique solution $a_0 = 1$, $a_i = 0$, $i = 1, \ldots, r$, in natural numbers. Hence F^i_{jk} is linear in Γ^p_{qt} and does not depend on $\Gamma^p_{qt, m_1 \ldots m_i}$, $i \geqq 1$, i.e.

$$(10.4.11) \qquad F^i_{jk} = A^{iqt}_{jkp} \Gamma^p_{qt},$$

where A^{iqt}_{jkp} is an absolute invariant tensor.

Corollary 1. *All global differential invariants of L_n^{r+2} from $T_n^r Q$ to Q depend only on coordinates on Q, i.e. all natural differential operators of finite order from F^2_Q into itself are of order zero.*

Similarly for (10.4.4) we obtain the system of partial differential equations

$$(10.4.12) \qquad \sum_{s=0}^{r} (s+1) \Gamma^p_{qt, m_1 \ldots m_s} \frac{\partial F^i_{jkl}}{\partial \Gamma^p_{qt, m_1 \ldots m_s}} = 2F^i_{jkl}.$$

All global solutions of (10.4.12) are sums of homogeneous polynomials of degrees a_s in variables $\Gamma^p_{qt,m_1\dots m_s}$ such that

(10.4.13) $$\sum_{s=0}^{r}(s+1)\,a_s = 2.$$

(10.4.13) has two possible solutions: a) $a_0 = 2$, $a_i = 0$, $i = 1, \dots, r$, $b)$ $a_0 = 0$, $a_1 = 1$, $a_i = 0$, $i = 2, \dots, r$. Hence F^i_{jkl} is a sum of a linear polynomial in $\Gamma^i_{jk,m}$ and a quadratic polynomial in Γ^i_{jk}, i.e.

(10.4.14) $$F^i_{jkl} = A^{iqtm}_{jklp}\Gamma^p_{qt,m} + B^{iq_1q_2s_1s_2}_{jklpr}\Gamma^p_{q_1q_2}\Gamma^r_{s_1s_2},$$

where A^{iqtm}_{jklp} and $B^{iq_1q_2s_1s_2}_{jklpr}$ are absolute invariant tensors and $B^{iq_1q_2s_1s_2}_{jklpr} = B^{is_1s_2q_1q_2}_{jklrp}$.

Corollary 2. *All global differential invariants of L^{r+2}_n from T^r_nQ to S depend only on the coordinates on T^1_nQ.*

Because of Corollaries 1 and 2 we get that all global natural differential operators of finite order from F^2_Q to F^3_P are of order less than or equal to one. Thus we can restrict ourselves only to the case $r = 1$.

Now we shall determine all differential invariants of L^3_n from T^1_nQ to $P = Q\times S$. The action of L^3_n on T^1_nQ is given by (9.2.1) and its formal derivative

(10.4.15) $$\begin{aligned}\bar\Gamma^i_{jk,l} &= a^i_p(\Gamma^p_{qt,r}b^q_jb^t_kb^r_l + \Gamma^p_{qt}b^q_{jl}b^t_k + \Gamma^p_{qt}b^q_jb^t_{kl} + b^p_{jkl}) +\\ &+ a^i_{pr}(b^r_l\Gamma^p_{qt}b^q_jb^t_k + b^r_lb^p_{jk}).\end{aligned}$$

The other fundamental vector fields on T^1_nQ and P related to $\xi \in L(L^3_n)$ are expressed by

(10.4.16) $$\begin{aligned}\Xi^{qt}_{pT^1_nQ} &= \left(\frac{\partial\bar\Gamma^i_{jk}}{\partial a^p_{qt}}\right)_e\frac{\partial}{\partial\Gamma^i_{jk}} + \left(\frac{\partial\bar\Gamma^i_{jk,l}}{\partial a^p_{qt}}\right)_e\frac{\partial}{\partial\Gamma^i_{jk,l}} =\\ &= \frac{1}{2}\left(-\frac{\partial}{\partial\Gamma^p_{qt}} - \frac{\partial}{\partial\Gamma^p_{tq}} + (\delta^i_p\Gamma^q_{jk}\delta^t_l + \delta^i_p\Gamma^t_{jk}\delta^q_l - \Gamma^i_{pk}\delta^q_j\delta^t_l -\right.\\ &\left. - \Gamma^i_{pk}\delta^t_j\delta^q_l - \Gamma^i_{jp}\delta^q_k\delta^t_l - \Gamma^i_{jp}\delta^t_k\delta^q_l)\frac{\partial}{\partial\Gamma^i_{jk,l}}\right),\end{aligned}$$

(10.4.17) $$\begin{aligned}\Xi^{qt}_{pP} &= \left(\frac{\partial\bar\Gamma^i_{jk}}{\partial a^p_{qt}}\right)_e\frac{\partial}{\partial\Gamma^i_{jk}} + \left(\frac{\partial\bar\Gamma^i_{jkl}}{\partial a^p_{qt}}\right)_e\frac{\partial}{\partial\Gamma^i_{jkl}} =\\ &= \frac{1}{2}\left(-\frac{\partial}{\partial\Gamma^p_{qt}} - \frac{\partial}{\partial\Gamma^p_{tq}} + (\delta^i_p\Gamma^q_{jl}\delta^t_k + \delta^i_p\Gamma^t_{jl}\delta^q_k + \delta^i_p\Gamma^q_{jk}\delta^t_l +\right.\\ &\left. + \delta^i_p\Gamma^t_{jk}\delta^q_l - \Gamma^i_{jp}\delta^q_k\delta^t_l - \Gamma^i_{jp}\delta^t_k\delta^q_l)\frac{\partial}{\partial\Gamma^i_{jkl}}\right),\end{aligned}$$

$$\Xi^{qtr}_{pT^1_nQ} = \left(\frac{\partial \overline{\Gamma}^i_{jk,l}}{\partial a^p_{qtr}}\right)_e \frac{\partial}{\partial \Gamma^i_{jk,l}} =$$

(10.4.18)

$$= -\frac{1}{6}\left(\frac{\partial}{\partial \Gamma^p_{qt,r}} + \frac{\partial}{\partial \Gamma^p_{qr,t}} + \frac{\partial}{\partial \Gamma^p_{tq,r}} + \frac{\partial}{\partial \Gamma^p_{tr,q}} + \frac{\partial}{\partial \Gamma^p_{rq,t}} + \frac{\partial}{\partial \Gamma^p_{rt,q}}\right),$$

$$\Xi^{qtr}_{pP} = \left(\frac{\partial \Gamma^i_{jkl}}{\partial a^p_{qtr}}\right)_e \frac{\partial}{\partial \Gamma^i_{jkl}} =$$

(10.4.19)

$$= -\frac{1}{6}\left(\frac{\partial}{\partial \Gamma^p_{qtr}} + \frac{\partial}{\partial \Gamma^p_{qrt}} + \frac{\partial}{\partial \Gamma^p_{tqr}} + \frac{\partial}{\partial \Gamma^p_{trq}} + \frac{\partial}{\partial \Gamma^p_{rqt}} + \frac{\partial}{\partial \Gamma^p_{rtq}}\right),$$

where $e \in L^3_n$ is the unity.

Now we have an L^2_n-equivariant mapping (10.4.11) from Q to Q. Then the fundamental vector fields (10.4.16) and (10.4.17) have to be related with respect to this mapping. It implies that (10.4.11) has to satisfy the system of partial differential equations

(10.4.20) $$\frac{\partial F^i_{jk}}{\partial \Gamma^p_{qt}} + \frac{\partial F^i_{jk}}{\partial \Gamma^p_{tq}} = \delta^i_p(\delta^q_j\delta^t_k + \delta^q_k\delta^t_j).$$

From the form of an absolute invariant tensor we obtain

10.4.21) $$F^i_{jk} = c_1\delta^i_j\Gamma^m_{km} + c_2\delta^i_j\Gamma^m_{mk} + c_3\delta^i_k\Gamma^m_{jm} + c_4\delta^i_k\Gamma^m_{mj} + c_5\Gamma^i_{jk} + c_6\Gamma^i_{kj}.$$

Substituting (10.4.21) into (10.4.20) we get the system of homogeneous equations $c_1 + c_2 = 0$, $c_3 + c_4 = 0$, $c_5 + c_6 = 0$. If we put $c_1 = A_2$, $c_3 = A_3$, $c_5 = A_1$ we get

(10.4.22) $$F^i_{jk} = A_1\Gamma^i_{jk} + (1 - A_1)\Gamma^i_{kj} + A_2\delta^i_j(\Gamma^m_{km} - \Gamma^m_{mk}) + A_3\delta^i_k(\Gamma^m_{jm} - \Gamma^m_{mj}).$$

Denoting $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$, (10.4.22) implies

Theorem 10.4. *All natural transformations of F^2_Q into itself form a three parameter family and every linear connection $\Gamma : X \to F^2_Q X$ on X with components $\Gamma^i_{jk}(x)$ is transformed into connections with components*

(10.4.23) $$A_1\Gamma^i_{jk}(x) + (1 - A_1)\Gamma^i_{kj}(x) + A_2\delta^i_j T^m_{km}(x) + A_3\delta^i_k T^m_{jm}(x),$$

where $T^i_{jk}(x)$ is the torsion tensor field of Γ and $A_i \in R$, $i = 1, 2, 3$.

Now let us consider the differential invariant from T^1_nQ to P given by (10.4.14) and (10.4.22). If (10.4.14) is a differential invariant of L^3_n then the fundamental vector fields (10.4.18) and (10.4.19) are F^i_{jkl}-related and we get the system of partial differential equations

$$(10.4.24)\quad \frac{\partial F^i_{jkl}}{\partial \Gamma^p_{qt,r}} + \frac{\partial F^i_{jkl}}{\partial \Gamma^p_{qr,t}} + \frac{\partial F^i_{jkl}}{\partial \Gamma^p_{tq,r}} + \frac{\partial F^i_{jkl}}{\partial \Gamma^p_{tr,q}} + \frac{\partial F^i_{jkl}}{\partial \Gamma^p_{rq,t}} + \frac{\partial F^i_{jkl}}{\partial \Gamma^p_{rt,q}} =$$
$$= \delta^i_p(\delta^q_j\delta^t_k\delta^r_l + \delta^q_j\delta^r_k\delta^t_l + \delta^t_j\delta^q_k\delta^r_l + \delta^t_j\delta^r_k\delta^q_l + \delta^r_j\delta^q_k\delta^t_l + \delta^r_j\delta^t_k\delta^q_l).$$

From the form of absolute invariant tensors we have

$$(10.4.25)\quad F^i_{jkl} = b_1\delta^i_j\Gamma^p_{kl,p} + \dots + b_{24}\Gamma^i_{lk,j} + c_1\delta^i_j\Gamma^p_{kl}\Gamma^r_{pr} + \dots + c_{60}\Gamma^i_{lk}\Gamma^p_{jp}.$$

Substituting (10.4.25) into (10.4.24) we obtain the system of linear equations

$$(10.4.26)\quad \sum_{i=1}^{6} b_i = 0,\ \sum_{i=7}^{12} b_i = 0,\ \sum_{i=13}^{18} b_i = 0,\ \sum_{i=19}^{24} b_i = 1.$$

Denoting free variables in (10.4.26) as B_i, $i = 1, \dots, 20$, we get the linear part of (10.4.14) in the form

$$(10.4.27)\quad \begin{aligned} &\delta^i_j(B_1\Gamma^m_{kl,m} + B_2\Gamma^m_{km,l} + B_3\Gamma^m_{lk,m} + B_4\Gamma^m_{lm,k} + B_5\Gamma^m_{mk,l} - \\ &- \sum_{i=1}^{5} B_i\Gamma^m_{ml,k}) + \delta^i_k(B_6\Gamma^m_{jl,m} + B_7\Gamma^m_{jm,l} + B_8\Gamma^m_{lj,m} + \\ &+ B_9\Gamma^m_{lk,m} + B_{10}\Gamma^m_{mj,l} - \sum_{i=6}^{10} B_i\Gamma^m_{ml,j}) + \delta^i_l(B_{11}\Gamma^m_{jk,m} + \\ &+ B_{12}\Gamma^m_{jm,k} + B_{13}\Gamma^m_{kj,m} + B_{14}\Gamma^m_{km,j} + B_{15}\Gamma^m_{mj,k} - \sum_{i=11}^{15} B_i\Gamma^m_{mk,j}) + \\ &+ B_{16}\Gamma^i_{jk,l} + B_{17}\Gamma^i_{jl,k} + B_{18}\Gamma^i_{kj,l} + B_{19}\Gamma^i_{kl,j} + B_{20}\Gamma^i_{lj,k} + \\ &+ (1 - \sum_{i=16}^{20} B_i)\,\Gamma^i_{lk,j}. \end{aligned}$$

Since the fundamental vector fields (10.4.16) and (10.4.17) are also F^i_{jkl}-related, and using (10.4.22), we obtain

$$(10.4.28)\quad \begin{aligned} &-\frac{\partial F^a_{bcd}}{\partial \Gamma^p_{qt}} - \frac{\partial F^a_{bcd}}{\partial \Gamma^p_{tq}} + (\delta^i_p\Gamma^q_{jk}\delta^t_l + \delta^i_p\Gamma^t_{jk}\delta^q_l - \Gamma^i_{pk}\delta^q_j\delta^t_l - \\ &- \Gamma^i_{pk}\delta^t_j\delta^q_l - \Gamma^i_{jp}\delta^q_k\delta^t_l - \Gamma^i_{jp}\delta^t_k\delta^q_l)\frac{\partial F^a_{bcd}}{\partial \Gamma^i_{jk,l}} = \\ &= \delta^a_p(\delta^t_c(A_1\Gamma^q_{bd} + (1 - A_1)\,\Gamma^q_{db} + A_2\delta^q_b(\Gamma^m_{dm} - \Gamma^m_{md})) + \\ &+ \delta^q_c(A_1\Gamma^t_{bd} + (1 - A_1)\,\Gamma^t_{db} + A_2\delta^t_b(\Gamma^m_{dm} - \Gamma^m_{md})) + \\ &+ \delta^t_d(A_1\Gamma^q_{bc} + (1 - A_1)\,\Gamma^q_{cb} + A_2\delta^q_b(\Gamma^m_{cm} - \Gamma^m_{mc}) + A_3\delta^q_c(\Gamma^m_{bm} - \Gamma^m_{mb})) + \\ &+ \delta^q_d(A_1\Gamma^t_{bc} + (1 - A_1)\,\Gamma^t_{cb} + A_2\delta^t_b(\Gamma^m_{cm} - \Gamma^m_{mc}) + A_3\delta^t_c(\Gamma^m_{bm} - \Gamma^m_{mb})) - \\ &- \delta^q_c\delta^t_d(A_1\Gamma^a_{bp} + (1 - A_1)\,\Gamma^a_{pb} + A_2\delta^a_b(\Gamma^m_{pm} - \Gamma^m_{mp})) - \\ &- \delta^t_c\delta^q_d(A_1\Gamma^a_{bp} + (1 - A_1)\,\Gamma^a_{pb} + A_2\delta^a_b(\Gamma^m_{pm} - \Gamma^m_{mp})). \end{aligned}$$

Substituting (10.4.25) to (10.4.29), where the linear part of (10.4.25) is given by (10.4.27), we obtain for c_i, $i = 1, \ldots, 60$, a system of linear equations with parameters B_i, $i = 1, \ldots, 20$, and A_i, $i = 1, 2, 3$. This system is divided into 15 systems each for four variables c_i and each of these systems has one independent variable. If we denote the independent variables as C_i, $i = 1, \ldots, 15$, we can write (10.4.25) in the form

$$
\begin{aligned}
F^i_{jkl} = {} & \delta^i_j(B_1\Gamma^m_{kl,m} + B_2\Gamma^m_{km,l} + B_3\Gamma^m_{lk,m} + B_4\Gamma^m_{lm,k} + \\
& + B_5\Gamma^m_{mk,l} - \sum_{i=1}^{5} B_i\Gamma^m_{ml,k} + C_1T^m_{kl}T^r_{mr} + (A_2 - B_2 - B_3 - B_4)\,\Gamma^m_{kl}\Gamma^r_{mr} + \\
& + (B_1 + B_2 + B_3 + B_4 - A_2)\,\Gamma^m_{kl}\Gamma^r_{rm} + B_3\Gamma^m_{lk}\Gamma^r_{mr} + C_2T^m_{km}T^r_{lr} + \\
& + C_3T^m_{kr}T^r_{lm} - B_1\Gamma^m_{kr}\Gamma^r_{ml} - B_3\Gamma^m_{rk}\Gamma^r_{lm}) + \\
& + \delta^i_k(B_6\Gamma^m_{jl,m} + B_7\Gamma^m_{jm,l} + B_8\Gamma^m_{lj,m} + B_9\Gamma^m_{lm,j} + B_{10}\Gamma^m_{mj,l} - \\
& - \sum_{i=6}^{10} B_i\Gamma^m_{ml,j} + C_4T^m_{jl}T^r_{mr} - (B_7 + B_8 + B_9)\,\Gamma^m_{jl}\Gamma^r_{mr} + \\
& + (B_6 + B_7 + B_8 + B_9)\,\Gamma^m_{jl}\Gamma^r_{rm} + B_8\Gamma^m_{lj}\Gamma^r_{mr} + C_5T^m_{jm}T^r_{lr} + C_6T^m_{jr}T^r_{lm} - \\
& - B_6\Gamma^m_{jr}\Gamma^r_{ml} - B_8\Gamma^m_{rj}\Gamma^r_{lm}) + \delta^i_l(B_{11}\Gamma^m_{jk,m} + B_{12}\Gamma^m_{jm,k} + \\
& + B_{13}\Gamma^m_{kj,m} + B_{14}\Gamma^m_{km,j} + B_{15}\Gamma^m_{mj,k} - \sum_{i=11}^{15} B_i\Gamma^m_{mk,j} + \\
& + C_7T^m_{jk}T^r_{mr} - (B_{12} + B_{13} + B_{14})\,\Gamma^m_{jk}\Gamma^r_{mr} + \\
& + (B_{11} + B_{12} + B_{13} + B_{14})\,\Gamma^m_{jk}\Gamma^r_{rm} + B_{13}\Gamma^m_{kj}\Gamma^r_{mr} + C_8T^m_{jm}T^r_{kr} + \\
& + C_9T^m_{jr}T^r_{km} - B_{11}\Gamma^m_{jr}\Gamma^r_{mk} - B_{13}\Gamma^m_{rj}\Gamma^r_{km}) + B_{16}\Gamma^i_{jk,l} + \\
& + B_{17}\Gamma^i_{kj,l} + B_{18}\Gamma^i_{kj,l} + B_{19}\Gamma^i_{kl,j} + B_{20}\Gamma^i_{lj,k} + \\
& + (1 - \sum_{i=16}^{20} B_i)\,\Gamma^i_{lk,j} + (A_1 - 1 + B_{18} + B_{19} + B_{20})\,\Gamma^i_{jm}\Gamma^m_{kl} + \\
& + (1 - \sum_{i=16}^{20} B_i)\,\Gamma^i_{jm}\Gamma^m_{lk} - (B_{18} + B_{20} + A_1 - 1)\,\Gamma^i_{mj}\Gamma^m_{kl} - \\
& - (B_{18} + B_{19} + B_{20} + A_1 - 1)\,\Gamma^i_{km}\Gamma^m_{jl} - (1 - A_1 - B_{20})\,\Gamma^i_{km}\Gamma^m_{lj} + \\
& + (B_{17} + B_{18} + B_{19} + B_{20} - 1)\,\Gamma^i_{mk}\Gamma^m_{jl} + (B_{16} + B_{17} + B_{19} - A_1)\,\cdot \\
& \cdot\,\Gamma^i_{lm}\Gamma^m_{jk} + (A_1 + B_{18} - 1)\,\Gamma^i_{lm}\Gamma^m_{kj} - (B_{17} + B_{19})\,\Gamma^i_{ml}\Gamma^m_{jk} + C_{10}T^i_{jk}T^m_{lm} + \\
& + C_{11}T^i_{jl}T^m_{km} + C_{12}T^i_{kl}T^m_{jm} + C_{13}T^i_{jm}T^m_{kl} + C_{14}T^i_{km}T^m_{jl} + \\
& + C_{15}T^i_{lm}T^m_{jk} + A_2(\Gamma^i_{jk}T^m_{ml} + \Gamma^i_{jl}T^m_{mk}) + A_3\Gamma^i_{kl}T^m_{mj},
\end{aligned}
\tag{10.4.29}
$$

where $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ and A_i, $i = 1, 2, 3$, B_i, $i = 1, \ldots, 20$, C_i, $i = 1, \ldots, 15$, are arbitrary real numbers. Thus we get

Theorem 10.5. *All natural differential operators of order one from F_Q^2 to F_P^3 form a 38-parameter family. The corresponding differential invariants have the coordinate expression given by* (10.4.22) *and* (10.4.29).

Remark 10.5. (10.4.23) implies that the prolonged connection is over a given connection if and only if $A_1 = 1$, $A_2 = A_3 = 0$. Thus we have a 35-parameter family of first order prolongation natural differential operators from F_Q^2 to F_P^3.

To find all natural prolongations of a given linear connection on X to principal connections on the holonomic second order frame bundle F^2X it is enough to put $F^i_{jkl} = F^i_{jlk}$ in (10.4.29). Then we get 18 independent coefficients and we have

Theorem 10.6. *There exist an 18-parameter family of first order natural differential operators which prolong a linear connection on X to principal connections on F^2X.*

Remark 10.6. It is easy to see that Oproiu's operator corresponds to the differential invariants (10.4.22) and (10.4.29) where $A_1 = 1$, $B_{16} = 1$ and the other coefficients vanish. Kolář's operator corresponds to the choice $A_1 = 1$ and the other coefficients vanish.

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PART 2

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