Demeter Krupka

Geometric Aspects of the Theory of Invariant Lagrange Structures [1]

1 Introduction

This work is devoted to the foundations of the geometric theory of invariant integral variational problems on fibred manifolds and to exposition of the results, published recently in the papers [1] and [2]. The subject belongs to the topics, studied by many authors, e.g. by Goldschmidt and Sternberg [3], Hermann [4, 5], Palais [6, 7], Sniatycki [8], Trautman [9, 10] and also Eells and Sampson [11], Kijowski [12, 13], Komorowski [14-16], Maurin [17]; it has also been motivated historically – apart from the beginnings of the variational calculus (see the Polak [18]), by the work of Hilbert, Noether, E. Cartan, Lepage and others (see e.g. [19-22]).

An integral variational problem is defined by a function of a section of a given fibred manifold, arising by integration of a differential form depending on the section (the *Lagrangian*); this function is usually called the *action function*. We are interested in the study of those sections whose prescribed "small deformations" do not change the value of the action function, and also in the transformations of the underlying fibred manifold that leave invariant the action function. All manifolds and mappings we consider belong to the category C^∞ of finite-dimensional real Haudorff paracompact manifolds. We are not interested in analytical aspects of the theory of variational problems, for instance in the existence

[1] This text is a translation of the original PhD thesis

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and differentiable properties of the minima of the action function (compare e.g. with [6, 7, 17]).

Our approach to the given class of variational problems is based on a systematic use of differential forms and vector fields. In this respect we follow more closely the classical work of Lepage and also the Hermann's books [4, 5], containing, however, in some respect a vague exposition. It should be pointed out that some authors prefer in formulation of the foundations of the calculus of variation the "classical" approach, based on the use of the so called *Lagrange function*, or, on the contrary, the approach using complicated morphisms of the corresponding fibre bundles (see e.g. [3, 7, 11, 17]); in the author's opinion, the use of differential forms is more adequate, is better adapted to the integral nature of the variational problems, and also provides a more precise exposition of the theory. Apart from the fact that there exist broad analytic tools for working with forms (integration, differentiation, Lie derivatives etc.), that can directly be applied and has a clear geometric meaning, further arguments for differential forms can also be found in the *field theory* (as a part of the calculus of variations). Here one should often consider integral variational functionals, composed of a unique, say a tensor, field, and the integrand cannot be split in an "integrated function" and a "volume element".

The second, third and fourth chapters of this work are devoted to the general variational theory on fibred manifolds. It includes fundamental definitions of geometric structures and notions, appearing in the vaariational theory, and also the results that can be derived without special hypotheses on the sets, on which the action function is considered. In chapter 5 we apply the general theory to a few concrete situations that appear in practical variational problems. To complete the text and the proofs we refer to the articles [1,2], closest to this dissertation.

All objects and morphisms we consider belong to the category C^∞. Concerning general differential-geometric terminology, we mainly follow Lang [23], and in some special cases (differential ideals and distributions, Lie derivatives, integration of forms) also Sternberg [24]. Our basic notation is the following: The *tangent space of a manifold X at a point x* is denoted by T_xX; T_xX is the *tangent bundle* of *X*. Tf denotes the *tangent mapping* of a morphism *f* and f^* the corresponding mapping, induced by *f* on differential forms (the *pull-back*). The following standard symbols are used: *d* (exterior derivative of differential forms), $i(\xi)$ (contraction of a form by a vector ξ), $\vartheta(\xi)$ (the Lie derivative with respect to a vector field ξ, and $\wedge$ (the exterior product). The field of real numbers ide denote by $\mathbf{R}$, and the real, *n*-dimensional Euclidean space by $\mathbf{R}^n$. Unless otherwise stated, we use in coordinate expressions the standard summation convention, where summation is always supposed when the same index appears in an expression twice. In the parts of the work where the theory of jets is used we mainly follow Ehresmann [25] and the lectures of Kolar [26]; the *r*-jet of a mapping *f* at a point *x* is denoted by $j_x^r f$.

2 Lagrange structures

Each surjective submersion in the considered category will be called a *fibred manifold*. If $\pi : Y \to X$ is a fibred manifold, and *V* an open subset of *Y*, then every isomorphism $\alpha : V \to \alpha(V) \subset Y$, such that there exists a isomorphism $\alpha_0 : \pi(V) \to \alpha_0(\pi(V)) \subset X$, such that

$$\pi\alpha = \alpha_0 \pi,$$

is called a *local automorphism* of the fibred manifold π. If an isomorphism α_0 exists, it is unique and is called the π *-projection* of the local automorphism α. A vector field Ξ on *Y*, whose local 1-parameter group is formed by local automorphisms of the fibred manifold π, is said to be π *-projectable*. A necessary and sufficient condition for Ξ to be π-projectable is that there exist a vector field ξ on *X*, such that for each $y \in Y$

$$T\pi \cdot \Xi = \xi \circ \pi.$$

If the vector field ξ exists, it is unique and is called the π *-projection* of the vector field Ξ. A π *-vertical vector field* is defined by the condition that its π-projection exists and is equal to the zero vector field.

The subject of this work is introduced by the following (cf. [9, 27]):

Definition 1 Each pair (π, λ), where $\pi : Y \to X$ is a fibred manifold over *n*-dimensional oriented base *X* and λ is an *n*-form on *Y*, is called a *Lagrange structure*. λ is called the *Lagrangian* of the Lagrange structure (π, λ).

In this work we suppose we are given a Lagrange structure (π, λ), where $\pi : Y \to X$ is a fibred manifold over an *n*-dimensional base *X*.

Let Ξ be a π-projectable vector field with π-projection, and denote by α_t^{Ξ} and α_t^{ξ} the local 1-parameter group of Ξ and ξ, respectively. With the help of the vector field Ξ we can assign to any section γ of the fibred manifold π a 1-parameter family of sections

$$\gamma_t = \alpha_t^{\Xi} \gamma \alpha_{-t}^{\xi}.$$

From the variational point of view γ_t can be regarded as a “small deformation” of the section γ. The 1-parameter family γ_t is called the *variation of* γ*, induced* by the vector field Ξ.

Choose in *X* a compact, *n*-dimensional submanifold Ω with boundary, and denote by $\Gamma_\Omega(\pi)$ the set of all sections of the fibred manifold π, each defined on a neighbourhood of Ω. Choose an orientation of *X* and consider Ω with the induced orientation. We get a real-valued function

$$\Gamma_\Omega(\pi) \ni \gamma \to \lambda_\Omega(\gamma) = \int_\Omega \gamma^* \lambda \in \mathbf{R},$$

called the *action function* of the Lagrange function (π,λ) (on the submanifold Ω). The main subject of the variational calculus is to study the behaviour of the action function, restricted to a given subset of the set $\Gamma_\Omega(\pi)$. The method consists in the study of the changes of the of the value $\lambda_\Omega(\gamma)$ of the action function under "small deformations" γ_t of every section γ, belonging to the subset. Choose a π-projectable vector field Ξ and a section $\gamma \in \Gamma_\Omega(\pi)$, and consider the variation γ_t, induced by the vector field Ξ. Using the same notation as above we see we get a function

$$(-\varepsilon,\varepsilon) \ni t \to \lambda_{\alpha_t^\xi(\Omega)}(\alpha_t^\Xi \gamma \alpha_{-t}^\xi) = \int_{\alpha_t^\xi(\Omega)} (\alpha_t^\Xi \gamma \alpha_{-t}^\xi)^* \lambda \in \mathbf{R},$$

defined for some $\varepsilon > 0$. We can suppose without loss of generality that all isomorphisms α_t^ξ preserve the orientation of the manifold *X*. Then the change of variatles theorem [24] together with basic properties of the pull-back of differential forms [23] give

$$\int_{\alpha_t^\xi(\Omega)} (\alpha_t^\Xi \gamma \alpha_{-t}^\xi)^* \lambda = \int_\Omega \gamma^* (\alpha_t^\Xi)^* \lambda.$$

Differentiating with respect to *t* at $t = 0$

$$\left\{ \frac{d}{dt} \lambda_{\alpha_t^\xi(\Omega)}(\alpha_t^\Xi \gamma \alpha_{-t}^\xi) \right\}_0 = \int_\Omega \gamma^* \vartheta(\Xi)\lambda.$$

This expression measures "sensibility" of the action function under the changes of γ, generated by the vector field Ξ. The arising function

$$\Gamma_\Omega(\pi) \ni \gamma \to (\vartheta(\Xi)\lambda)_\Omega(\gamma) = \int_\Omega \gamma^* \vartheta(\Xi)\lambda \in \mathbf{R}$$

is called the *first variation* of the action function, generated on Ω by the vector field Ξ. This formula leads to the following definition:

Definition 2 A section $\gamma \in \Gamma_\Omega(\pi)$ is said to be a Ξ*-stationary section* of the Lagrange structure (π,λ) over Ω, if it annihilates the first variation of the action function generated over Ω by the vector field Ξ, that is,

$$\int_{\Omega} \gamma^* \vartheta(\Xi)\lambda = 0.$$

It follows from Definition 2 that the theory of Lie derivatives will be an important tool in the study of stationary sections of Lagrange structures. Similar situation arises when we investigate symmetry properties of Lagrange structures. Nevertheless no work is known to the author in which this geometric theory would be systematically applied to the variational theory.

Main topic we wish to consider in this work is the study of sections of the fibred manifold π, satisfying a system of partial differential equations, stationary with respect to variations, permuting the solutions of these equations. We shall consider partial differential equations, admitting a direct geometric interpretation, namely equations, given in the form of a *differential ideal* – an ideal in the exterior algebra of differential forms (see e.g. [4, 5, 24]); we shall *not* suppose that this differential ideal is closed under exterior differentiation of forms.

Let $\mathscr{D}$ be a differential ideal on *Y*. A section γ of the fibred manifold π is said to be an *integral section* (*integral manifold*) of $\mathscr{D}$, if

$$\gamma^* \rho = 0$$

for all $\rho \in \mathscr{D}$. The set of integral sections of the differential ideal $\mathscr{D}$ will be denoted by $\Gamma_{\mathscr{D}}$. Our aim will be to study the action function of the Lagrange structure (π, λ), restricted to $\Gamma_{\mathscr{D}}$; this requires, in particular, that all variations of sections, belonging to the set $\Gamma_{\mathscr{D}}$, should again belong to this set.

Consider a local automorphism α of the fibred manifold π, defined on an open set *V*. α assigns to the differential ideal $\mathscr{D}$ a new differential ideal $\alpha^* \mathscr{D}$, constituted of all differential forms $\alpha^* \rho$, where $\rho \in \mathscr{D}$. We set $U = \pi(V)$.

Definition 3 A local automorphism α of the fibred manifold π is called $\mathscr{D}$*-admissible*, if every integral section γ of the differential ideal $\mathscr{D}$, defined on *U*, is an integral section of the differential ideal $\alpha^* \mathscr{D}$. We say that a π-projectable vector field Ξ *generates* $\mathscr{D}$*-admissible variations* of the fibred manifold π, or is $\mathscr{D}$*-admissible*, if its local 1-parameter group consists of $\mathscr{D}$-admissible local automorphisms of the fibred manifold π.

Denoting by α_0 the π-projection of the local automorphism α from Definition 3, then α is a $\mathscr{D}$-admissible local automorphism if and only if $\alpha\gamma\alpha_0^{-1} \in \Gamma_{\mathscr{D}}$ for any $\gamma \in \Gamma_{\mathscr{D}}$. In this sense $\mathscr{D}$-admissible local automorphisms permute sections of the differential ideal $\mathscr{D}$. The following proposition is an immediate consequence of definitions.

Proposition 1 *A π-projectable vector field Ξ generates $\mathscr{D}$-admissible variations of sections of the fibred manifold π if and only if for any $\gamma \in \Gamma_{\mathscr{D}}$,*

$$\gamma^* \vartheta(\Xi)\rho = 0.$$

for all $\rho \in \mathcal{D}$.

In a similar way we can introduce "compact" deformations of sections, i.e., the deformations, differing from the identity transformation only on compact subsets of the given fibred manifold. Recall that the *support* of a vector field is defined as the smallest closed set outside of which the vector field is equal to 0.

Definition 4 Let Ξ be a π-projectable vector field, α_t^{Ξ} its local 1-parameter group, Ω a compact *n*-dimensional submanifold of *X* with boundary. We say that Ξ generates $\mathcal{D}$*-admissible variations* of the fibred manifold π on Ω, if its support belongs to the set $\pi^{-1}(\Omega)$, and every $\gamma \in \Gamma_{\mathcal{D}} \cap \Gamma_{\Omega}(\pi)$ satisfies $(\alpha_t^{\Xi})^* \gamma \in \Gamma_{\mathcal{D}} \cap \Gamma_{\Omega}(\pi)$ for all sufficiently small *t*.

We can prove by a direct computation the following proposition, characterizing differential ideals with the same sets of $\mathcal{D}$-admissible vector fields.

Proposition 2 *Let* $\mathcal{D}$ *be a differential ideal on Y,* α *a local automorphism of the fibred manifold* π. *Then a* π*-projectable vector field* Ξ *is* $\mathcal{D}$*-admissible if and only if it is* $\alpha^* \mathcal{D}$*-admissible*.

We now give a basic definition, introducing stationary sections of Lagrange structures.

Definition 5 Let $\mathcal{D}$ be a differential ideal on *Y*, let Ω be a compact *n*-dimensional submanifold of *X* with boundary. We say that a section $\gamma \in \Gamma_{\mathcal{D}} \cap \Gamma_{\Omega}(\pi)$ is a $\mathcal{D}$*-critical section* of the Lagrange structure (π, λ) on Ω, if it is Ξ-stationary for every vector field Ξ, generating $\mathcal{D}$-admissible variations on the submanifold Ω. γ is said to be $\mathcal{D}$*-critical*, if it is $\mathcal{D}$-critical on every submanifold Ω lying in the domain of definition of γ.

Thus, $\mathcal{D}$-critical sections are characterized by the condition that they annihilate the first variation of the action function on every compact submanifold with boundary of the same dimension as the basis of the fibred manifold considered; the first variation is at the same time generated by vector fields, that in an "admissible" way (that is, inside of Ω) "deform" sections of the fibred manifold on compact subsets of the base.

The notion of an $\mathcal{D}$-admissible variation induces further concepts of significant transformations of the underlying Lagrange structure. Let $\mathcal{D}$ be a differential ideal on *Y*, α a $\mathcal{D}$-admissible local automorphism of the fibred manifold π, defined on an open set *V*, and let $U = \pi(V)$.

Definition 6 We say that α is a $\mathscr{D}$*-invariance transformation* of the Lagrange structure (π, λ), if

$$\gamma^* \alpha^* \lambda = \gamma^* \lambda$$

for every section $\gamma \in \Gamma_{\mathscr{D}}$, defined in *U*. We say that α is a *generalized* $\mathscr{D}$*-invariance transformation* of the Lagrange structure (π, λ), if for any compact *n*-dimensional submanifold with boundary $\Omega \subset U$

$$\int_\Omega \gamma^* \vartheta(\Xi) \alpha^* \lambda = \int_\Omega \gamma^* \vartheta(\Xi) \lambda$$

for all vector fields Ξ, generating $\mathscr{D}$-admissible variations on Ω, and for all sections $\gamma \in \Gamma_{\mathscr{D}} \cap \Gamma_\Omega(\pi)$. We say that a vector field *generates* $\mathscr{D}$*-invariance transformations* (resp. *generalized* $\mathscr{D}$*-invariance transformations*) of the Lagrange structure (π, λ), if its local 1-parameter group is formed by $\mathscr{D}$-invariance transformations (resp. generalized $\mathscr{D}$-invariance transformations).

The proofs of the following two propositions are immediate.

Proposition 3 *A* $\mathscr{D}$*-admissible vector field* Ξ *generates* $\mathscr{D}$*-invariance transformations of the Lagrange structure* (π, λ) *if and only if*

$$\gamma^* \vartheta(\Xi) \lambda = 0$$

for all sections $\gamma \in \Gamma_{\mathscr{D}}$.

Proposition 4 *A vector field* Θ *generates generalised* $\mathscr{D}$*-invariance transformations if and only if for every compact n-dimensional submanifold* $\Omega \subset X$ *with boundary*

$$\int_\Omega \gamma^* \partial_\Xi \partial_\Theta \lambda = 0$$

for all vector fields Ξ*, generating* $\mathscr{D}$*-admissible variations of sections of the fibred manifold* π *on* Ω *and for all sections* $\gamma \in \Gamma_{\mathscr{D}} \cap \Gamma_\Omega(\pi)$.

Consider any π-projectable vector field Θ and the associated Lagrange structure $(\pi, \partial_\Theta \lambda)$. If Θ generates generalised $\mathscr{D}$-invariance transformations of the Lagrange structure (π, λ), then every integral section of the differential ideal $\mathscr{D}$ is a $\mathscr{D}$-critical section of $(\pi, \partial_\Theta \lambda)$. It is therefore clear that existence of vector fields, generating generalised invariance transformations, will be an important characteristic of the properties of Lagrange structures.

We finally introduce a broad class of transformations of the Lagrange structure (π,λ), related to critical sections of (π,λ), contrary to the $\mathscr{D}$-invariance and generalised $\mathscr{D}$-invariance transformations. In what follows the symbols $\mathscr{D}$, V and U have the same meaning as above.

Definition 7 A local automorphism α is said to be a *$\mathscr{D}$-symmetry transformation*, or just a *$\mathscr{D}$-symmetry* of a $\mathscr{D}$-critical section γ of the Lagrange structure (π,λ), if for every compact n-dimensional submanifold with boundary $\Omega\subset X$, lying in the domain of definition of γ,

$$\int_\Omega \gamma * \vartheta(\Xi)\alpha * \lambda = 0$$

for all vector fields Ξ, generating $\mathscr{D}$-admissible variations of (π,λ) on Ω.

The following simple assertions hold.

Proposition 5 *Let γ be a $\mathscr{D}$-critical section of the Lagrange structure (π,λ). A $\mathscr{D}$-admissible vector field Ξ generates $\mathscr{D}$-symmetries of γ if and only if for every compact n-dimensional submanifold with boundary Ω, lying in the domain of definition of γ,*

$$\int_\Omega \gamma * \vartheta(\Theta)\vartheta(\Xi)\lambda = 0$$

for all vector fields Θ, generating $\mathscr{D}$-admissible variations on Ω.

Proposition 5 shows how to determine $\mathscr{D}$-critical sections of the given Lagrange structure with prescribed symmetry properties.

Proposition 6 *Let $\gamma\in\Gamma_{\mathscr{D}}$ and let Ξ be a $\mathscr{D}$-admissible vector field with projection π. Let α_t^{Ξ} and α_t^{ξ} be the corresponding local 1-parameter groups. Then the 1-parameter family of sections $\alpha_t^{\Xi}\gamma\alpha_{-t}^{\xi}$ is formed by $\mathscr{D}$-critical sections of the Lagrange structure (π,λ) if and only if γ satisfies the system*

$$\int_\Omega \gamma * \vartheta(\Theta)\lambda = 0, \quad \int_\Omega \gamma * \vartheta(\Theta)(\Xi)\lambda = 0$$

for every compact n-dimensional manifold $\Omega\subset X$ with boundary and all vector fields Θ, generating $\mathscr{D}$-admissible variations of (π,λ) on Ω.

The classes of transformations, associated with the given Lagrange structure, we have introduced, are not mutually independent.

Proposition 7 *Every $\mathscr{D}$-invariance transformation is a generalized $\mathscr{D}$-invariance transformation. Every generalized $\mathscr{D}$-invariance transformation is a $\mathscr{D}$-symmetry of each $\mathscr{D}$-critical section of the Lagrange structure* (π,λ) .

For simplicity, we call each of the transformations from Proposition 7 simply a $\mathscr{D}$-*symmetry* of (π,λ) .

In the next section we pass on a more detailed study of $\mathscr{D}$-critical sections and $\mathscr{D}$-symmetries of the given Lagrange structure.

3 First variation formula

Let $\pi: Y \to X$ be a fibred manifold. We shall denote by J^rY the set of *r*-jets of (local) sections of the fibred manifold π with natural differentiable structure, and by $\pi_r : J^rY \to X$ and $\pi_{rs} : J^rY \to J^sY$ ($0 \le s \le r$) the fibred manifolds defined by natural jet projections. The *r*-jet prolongation of a section γ of π is denoted by $j^r\gamma$; $j^r\gamma$ is a section of the fibred manifold π_r .

Recall that a differential form on *Y* is said t be π *-horizontal*, it it vanishes whenever at least one of its arguments is a π -vertical vector. A differential form ρ on J^rY is said to be *pseudovertical*, if

$$j^r\gamma * \rho = 0$$

for every section γ of the fibred manifold π . We denote by $\Omega^n(J^rY)$ the space of *n*-forms on J^rY , and $\Omega^n_X(J^rY)$ the space of π_r -horizontal *n*-forms.

Proposition 8 [1)] *For any n-form* $\lambda \in \Omega^n(Y)$ *there exists exactly one n-form* $h(\lambda) \in \Omega^n_X(J^1Y)$ *such that for all sections* γ *of the fibred manifold* π

$$j^1\gamma * h\lambda = \gamma * \lambda.$$

The mapping

$$\Omega^n(Y) \ni \lambda \to h(\lambda) \in \Omega^n_X(J^1Y)$$

is linear over the ring of functions and bijective.

For proofs and further properties of the mapping *h*, which can be defined for arbitrary *p*-forms, as well as for coordinate formulas in Proposition 9 and Propo-

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[1)] An error in the source document has been corrected: J^rY in Proposition 8 has been replaced with J^1Y .

sition 10, we refer to [1].

Consider a coordinate neighbourhood on Y with fibred coordinates (x_i, y_σ), $1 \le i \le n$, $1 \le \sigma \le m$, where $n = \dim X$, $m = \dim Y - \dim X$, and denote by $(x_i, y_\sigma, z_{i\sigma})$ the associated fibred coordinates on the manifold J^1Y. If the n-form λ is expressed by

$$\lambda = f_0 dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$$
$$+ \sum_{r=1}^{n} \sum_{s_1 < s_2 < \ldots < s_r} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_r} \frac{1}{r!} f^{s_1}_{\sigma_1}{}^{s_2}_{\sigma_2} \cdots {}^{s_r}_{\sigma_r} dx_1 \wedge \ldots \wedge dx_{s_1 - 1} \wedge dy_{\sigma_1} \wedge dx_{s_1+1}$$
$$\wedge \ldots \wedge dx_{s_r - 1} \wedge dy_{\sigma_r} \wedge dx_{s_r+1} \wedge \ldots \wedge dx_n,$$

then

$$h(\lambda) = \left(f_0 \quad + \sum_{r=1}^{n} \sum_{s_1 < s_2 < \ldots < s_r} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_r} \frac{1}{r!} f^{s_1}_{\sigma_1}{}^{s_2}_{\sigma_2} \cdots {}^{s_r}_{\sigma_r} z_{s_1\sigma_1} z_{s_2\sigma_3} \cdots z_{s_r\sigma_r} \right)$$
$$\cdot dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

In these formulas (and also everywhere in the following text) the Latin indices run through the values $1, 2, \ldots, n$, and the Greek indices through $1, 2, \ldots, m$. In many computations it is suitable to use the form $\pi^*_{10}\lambda$ instead of λ, where $\pi_{1,0}$ is the natural projection $J^1Y \to J^0Y = Y$. On the given coordinate neighbourhood $\pi^*_{10}\lambda$ can be represented as

$$\pi^*_{10}\lambda = \mathscr{L} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$$
$$+ \sum_{r=1}^{n} \sum_{s_1 < s_2 < \ldots < s_r} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_r} g^{s_1}_{\sigma_1}{}^{s_2}_{\sigma_2} \cdots {}^{s_r}_{\sigma_r} dx_1 \wedge \ldots \wedge dx_{s_1 - 1} \wedge \omega_{\sigma_1} \wedge dx_{s_1+1}$$
$$\wedge \ldots \wedge dx_{s_r - 1} \wedge \omega_{\sigma_r} \wedge dx_{s_r+1} \wedge \ldots \wedge dx_n,$$

where

$$\omega_\sigma = dy_\sigma - z_{k\sigma} dx_k$$

are pseudovertical 1-forms. When we substitute form this expression into the above coordinate representation of $\pi^*_{10}\lambda$, we get the following formula:

Proposition 9 *The functions* $\mathscr{L}$ *and* g^s_σ *satisfy*

$$\mathscr{L} = f_0 + \sum_{r=1}^{n} \sum_{s_1 < s_2 < \ldots < s_r} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_r} f^{s_1}_{\sigma_1}{}^{s_2}_{\sigma_2} \cdots {}^{s_r}_{\sigma_r} z_{s_1\sigma_1} z_{s_2\sigma_2} \cdots z_{s_r\sigma_r},$$

and

$$g_\sigma^s = \frac{\partial \mathscr{L}}{\partial z_{s\sigma}}.$$

It is now obvious that the *n*-form $\pi_{10}^*\lambda$ is decomposed, in an invariant way, as the sum of a π_1-horizontal and a pseudovertical forms. Note that a more general assertion holds:

Proposition 10 *To each n-form* $\lambda \in \Omega^n(J^rY)$ *there exists a unique n-form* $h(\lambda) \in \Omega_X^n(J^{r+1}Y)$ *and a unique pseudovertical n-form* $p(\lambda)$*, defined on* $J^{r+1}Y$ *, such that*

$$(\pi^{r+1,r})^*\lambda = h(\lambda) + p(\lambda).$$

Applying these considerations to the Lagrange structures we see that the action function of the Lagrange structure (π,λ) over any compact *n*-dimensional manifold with boundary $\Omega \subset X$ can be written as

$$\lambda_\Omega(\gamma) = \int_\Omega j^1\gamma^* \pi_{10}^*\lambda = \int_\Omega j^1\gamma^* h(\lambda).$$

We can now use this this expression in order to determine a formula for the first variation $(\partial_\Xi \lambda)_\Omega$ of the action function.

Consider the fibred coordinates (x_i, y_σ) and the associated fibred coordinates $(x_i, y_\sigma, z_{i\sigma}, z_{ij\sigma})$ on J^2Y ($i \le j$). Denote for any function *f* of the variables $(x_i, y_\sigma, z_{i\sigma})$

$$d_i f = \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial y_\sigma} z_{i\sigma} + \frac{\partial f}{\partial y_{j\sigma}} z_{ij\sigma}.$$

Expression $d_i f$ defines a function of the variables $(x_i, y_\sigma, z_{i\sigma}, z_{ij\sigma})$, called the *formal derivative* of the function *f* with respect to the variable x_i [28]. Let Ξ be any π-projectable vector field, ξ its π-projection, and α_t^Ξ and α_t^ξ the corresponding local 1-parameter groups. Then the formula

$$j^r\alpha_t^\Xi(j_x^r\gamma) = j_{\alpha_t^\xi(x)}^r \alpha_t^\Xi \gamma \alpha_{-t}^\xi$$

defines a local 1-parameter group $j^r\alpha_t^\Xi$ of automorphisms of the fibred manifold π_r. It is generated by the vector field

$$j^r\Xi(j^r_x\gamma)=\left\{\frac{d}{dt}j^r_{\alpha^\xi_t(x)}\alpha^\Xi_t\gamma\alpha^\xi_{-t}\right\}_0,$$

called the *r-jet prolongation* of the π -projectable vector field Ξ [1]. One can easily derive the chart representation of the vector field $j^r\Xi$. Restricting ourselves to the case $r=1$ we get

$$j^1\Xi=\xi_i\frac{\partial}{\partial x_i}+\Xi_\sigma\frac{\partial}{\partial y_\sigma}+\left(d_i\Xi_\sigma-z_{k\sigma}\frac{\partial\xi_k}{\partial x_i}\right)\frac{\partial}{\partial z_{i\sigma}}.$$

The first variation $(\vartheta(\Xi)\lambda)_\Omega$ of the action function will now be determined by the following proposition.

Proposition 11 *To each n-form* $\lambda\in\Omega^n(J^1Y)$ *and each* π *-projectable vector field* Ξ *,*

$$h(\vartheta(\Xi)\lambda)=\vartheta(j^1\Xi)h(\lambda).$$

Applying classical variational procedures one can easily determine the chart expression for the Lie derivative $\vartheta(j^1\Xi)h(\lambda)$. We introduce a function $\mathscr{L}$ in the local coordinates $(x_i,y_\sigma,z_{i\sigma})$ by

$$h(\lambda)=\mathscr{L}dx_1\wedge dx_2\wedge\ldots\wedge dx_n,$$

and functions $\mathscr{E}_\sigma(\mathscr{L})$, where $1\le\sigma\le m$, the *Euler expressions*, associated (in the given local coordinates) with the Lagrangian λ, by

$$\mathscr{E}_\sigma(\mathscr{L})=\frac{\partial\mathscr{L}}{\partial y_\sigma}-d_i\left(\frac{\partial\mathscr{L}}{\partial z_{i\sigma}}\right).$$

Using the function $\mathscr{L}$ and the Euler expressions $\mathscr{E}_\sigma(\mathscr{L})$, we get the following *first variation formula* in "infinitesimal" form [1]:

Proposition 12 *The n-form* $\pi^*_{21}\vartheta(j^1\Xi)h(\lambda)$ *has the chart representation*

$$\pi^*_{21}\vartheta(j^1\Xi)h(\lambda)=\mathscr{L}_\Xi dx_1\wedge dx_2\wedge\ldots\wedge dx_n,$$

where the function $\mathscr{L}_\Xi$ *is given by*

$$\mathscr{L}_\Xi=\mathscr{E}_\sigma(\mathscr{L})(\Xi_\sigma-z_{i\sigma}\xi_i)+d_k\left(\mathscr{L}\xi_k+\frac{\partial\mathscr{L}}{\partial z_{k\sigma}}(\Xi_\sigma-z_{i\sigma}\xi_i)\right).$$

The decomposition of the n-form $\pi_{21}^{*}\vartheta(j^1\Xi)h(\lambda)$ *in two summands is independent of the choice of fibred coordinates*.

In order to better understand the geometric meaning of the first variation formula (Proposition 12), consider a distribution Δ_2 on j^2Y, generated on the coordinate neighbourhood, covered by the coordinates $(x_i, y_\sigma, z_{i\sigma}, z_{ij\sigma})$, by the vector fields

$$\frac{\partial}{\partial x_i} + z_{i\sigma}\frac{\partial}{\partial y_\sigma}, \quad \frac{\partial}{\partial z_{i\sigma}}, \quad \frac{\partial}{\partial z_{ij\sigma}}.$$

Choose a point $j_x^2\gamma \in j^2Y$ in the given coordinate neighbourhood and an arbitrary tangent vector to j^2Y at this point,

$$\tilde{\Xi} = \xi_i \frac{\partial}{\partial x_i} + \Xi_\sigma \frac{\partial}{\partial y_\sigma} + \Xi_{i\sigma}\frac{\partial}{\partial z_{i\sigma}} + \sum_{i\leq j}\Xi_{ij\sigma}\frac{\partial}{\partial z_{ij\sigma}},$$

where the components ξ_i, Ξ_σ, $\Xi_{i\sigma}$, $\Xi_{ij\sigma}$ are real numbers. Then we have a unique decomposition

$$\tilde{\Xi} = h(\tilde{\Xi}) + v(\tilde{\Xi}),$$

where

$$h(\tilde{\Xi}) = \xi_i\left(\frac{\partial}{\partial x_i} + z_{i\sigma}\frac{\partial}{\partial y_\sigma}\right) + \Xi_{i\sigma}\frac{\partial}{\partial z_{i\sigma}} + \sum_{i\leq j}\Xi_{ij\sigma}\frac{\partial}{\partial z_{ij\sigma}}$$

belongs to the distribution Δ_2, and

$$v(\tilde{\Xi}) = (\Xi_\sigma - z_{i\sigma}\xi_i)\frac{\partial}{\partial y_\sigma}$$

to the complementary distribution. We introduce the *Euler-form*, associated with the Lagrangian λ, by

$$E(\lambda) = \mathscr{E}_\sigma(\mathscr{L})dy_\sigma \wedge dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

The independence of the expression on the right-hand side on the fibred charts can be proved by a direct computation. Applying this decomposition to the 2-jet prolongation of a π-projectable vector field, we can prove the first variation formula in the following form.

Theorem 1 *For any* π*-projectable vector field* Ξ *and any n-form* $\lambda \in \Omega^n(Y)$,

$$\pi_{21}^* h(\vartheta(\Xi)\lambda) = i(v(j^2\Xi)E(\lambda) + h(di(j^1\Xi)\pi_{10}^*\lambda).$$

One can prove Theorem 1 by a direct computation, with the help of Propositions 9, 11, and 12.

4 Critical sections

Choose in *X* a compact, *n*-dimensional submanifold Ω with boundary $\partial\Omega$. From Theorem 1 and from the Stokes' theorem on integration of differential forms on manifolds with boundary it follows that for any π-projectable vector field Ξ

$$\begin{aligned}(\vartheta(\Xi)\lambda)_\Omega &= \int_\Omega j^2\gamma * \pi_{21}^* h(\partial_\Xi \lambda) \\ &= \int_\Omega j^2\gamma * i(v(j^2\Xi))E(\lambda) + \int_{\partial\Omega} j^1\gamma * i(j^1\Xi)\pi_{10}^*\lambda.\end{aligned}$$

Restricting ourselves to π-projectable vector fields Ξ with support in $\pi^{-1}(\Omega)$, the boundary integral vanishes and we have:

Proposition 13 *A section* $\gamma \in \Gamma_\Omega(\pi)$ *is* Ξ*-stationary if and only if*

$$\int_\Omega j^2\gamma * i(v(j^2\Xi))E(\lambda) = 0.$$

Let now $\mathscr{D}$ be a differential ideal on *Y*. It is easily seen that for every π-vertical vector field Ξ the following identity holds

$$i(v(j^2\Xi))E(\lambda) = i(j^2\Xi)E(\lambda).$$

In view of the definition of $\mathscr{D}$-critical sections we can therefore give a basic consequence of the first variation formula as follows.

Theorem 2 *A section* $\gamma \in \Gamma_{\mathscr{D}} \cap \Gamma_\Omega(\pi)$ *is a* $\mathscr{D}$*-critical section of the Lagrange structure* (π, λ) *on the submanifold* Ω *if and only if*

$$\int_\Omega j^2\gamma * i(j^2\Xi)E(\lambda) = 0$$

for all vector fields Ξ *, generating* $\mathcal{D}$*-admissible variations of the Lagrange structure* (π,λ) *on* Ω *.* γ *is* $\mathcal{D}$*-critical if and only if this condition holds independently of the choice of the submanifold* Ω *.*

5 Invariance

Consider our Lagrange structure (π,λ) and the differential ideal $\mathcal{D}$ on *Y*. Having introduced the mapping *h*, assigning to *n*-forms on *Y* π_1-horizontal *n*-forms on J^1Y , and the Euler form $E(\lambda)$, appearing in the first variation formula (Theorem 1), we can further specify characteristics of local 1-parameter groups of $\mathcal{D}$-symmetries of (π,λ) .

Theorem 3 (Noether's equation) *A* $\mathcal{D}$*-admissible vector field* Ξ *generates* $\mathcal{D}$*-nvariance transformations of the Lagrange structure* (π,λ) *if and only if*

$$\vartheta(j^1\Xi)h(\lambda)=0.$$

This assertion follows from Proposition 11.

Theorem 4 *A* π*-projectable vector field* Θ *generates generalised* $\mathcal{D}$*-invariance transformations of the Lagrange structure* (π,λ) *if and only if for every n-dimensional compact submanifold with boundary* $\Omega\subset X$

$$\int_\Omega j^2\gamma * i(j^2\Xi)E(\vartheta(\Theta)\lambda)=0$$

for all vector fields Ξ *, generating* $\mathcal{D}$*-admissible variations of the Lagrange structure* (π,λ) *on* Ω *and all sections* $\gamma\in\Gamma_{\mathcal{D}}\cap\Gamma_\Omega(\pi)$ *.*

This assertion follows from Proposition 4 and Theorem 2, in which we replace the *n*-form λ by the *n*-form $\vartheta(\Theta)\lambda$.

The following modification of Theorem 4 gives us a relation between $\mathcal{D}$-critical sections and their local 1-parameter groups of symmetries.

Theorem 5 *Let* γ *be a* $\mathcal{D}$*-critical section of the Lagrange structure* (π,λ) *. A* $\mathcal{D}$*-admissible vector field* Θ *generates* $\mathcal{D}$*-symmetries of the section* γ *if and only if for every compact, n-dimensional submanifold with boundary* $\Omega\subset X$ *, lying in the domain of definition of* γ *,*

$$\int_\Omega j^2\gamma * i(j^2\Xi)E(\vartheta(\Theta)\lambda) = 0$$

for all vector fields Ξ *, generating* $\mathcal{D}$*-admissible variations of the Lagrange structure* (π,λ) *on* Ω *.*

6 Examples

Many applications of the calculus of variations and field theory consist in finding $\mathcal{D}$-critical sections and $\mathcal{D}$-invariance of a given Lagrange structure. In case when the differential ideals $\mathcal{D}$ induce sufficiently rich spaces of $\mathcal{D}$-admissible vector fields (that is, sufficiently large sets of "admissible deformations" of sections) one can characterize $\mathcal{D}$-critical sections in terms of partial differential equations. The same is true for the vector fields, generating $\mathcal{D}$-symmetry transformations.

We now give examples of Lagrange structures and differential ideals that obey this property.

A) Ordinary first order variational problems. Suppose we are given a Lagrange structure (π,λ), where $\pi : Y \to X$ is a fibred manifold with *n*-dimensional base *X*. We describe the theory of $\mathcal{D}$-critical sections and $\mathcal{D}$-invariance for the case of the trivial differential ideal on *Y*, $\mathcal{D} = \{0\}$. In this case every section of the fibred manifold *Y* belongs to the set Γ_Ω . Thus we can speak of *critical sections* of the Lagrange structure (π,λ) instead of $\mathcal{D}$-critical sections, and of *invariance* instead of $\mathcal{D}$-invariance. The results given below are contained, with minor modifications, in the papers [1,2].

Theorem *A section* γ *of the fibred manifold* π *, defined on an open set* $U \subset X$ *, is a critical section of the Lagrange structure* (π,λ) *on a compact, n-dimensional submanifold with boundary* $\Omega \subset U$ *if and only if the Euler form* $E(\lambda)$ *vanishes on the submanifold* $j^2\gamma(\Omega) \subset J^2Y$ *,*

$$E(\lambda) \circ j^2\gamma = 0$$

on Ω *.* γ *is a critical section of* (π,λ) *if and only if condition*

$$E(\lambda) \circ j^2\gamma = 0$$

holds on U.

There exists a natural equivalence relation on the set of Lagrange structures on a given fibred manifold. We say that two Lagrange structures (π,λ_1) , (π,λ_2)

are *equivalent*, if the corresponding Euler forms coincide,

$$E(\lambda_1) = E(\lambda_2).$$

It is a trivial consequence of the definition of the Euler form that two Lagrangians λ_1, λ_2, satisfying

$$h(\lambda_1) = h(\lambda_2),$$

define equivalent Lagrange structures. Since $E(\lambda)$ depends on λ $\mathbf{R}$-linearly, the equivalence problem is solved by the following assertion.

Theorem $E(\lambda) = 0$ *if and only if* $d\lambda = 0$.

This theorem states more precisely some classical assertions on the structure of the Lagrangians annihilated by the Euler form; some of these results appearing in the literature are not complete or correct. It can also serve for adequate description of generators of generalised invariance transformations.

Theorem *Let* Ξ *be a* π*-projectable vector field. The following four conditions are equivalent:*

(1) Ξ *generates generalised invariance transformations of the Lagrange structure* (π, λ_1).

(2) *The Lie derivative of the Euler form* $E(\lambda)$ *with respect to the* 2*-jet prolongation* $j^2\Xi$ *of the vector field* Ξ *vanishes,*

$$\vartheta(\Xi)E(\lambda) = 0.$$

(3) *There exists an n-form* $\rho \in \Omega^n(Y)$, *such that the generalised Noether-Bessel-hagen equation*

$$h(\vartheta(\Xi)\lambda - \rho) = 0, \quad d\rho = 0$$

is satisfied.

(4) *Condition*

$$E(\vartheta(\Xi)\lambda) = 0$$

holds.

Theorem *The set of all* π*-projectable vector fields, generating generalised invariance transformations of the Lagrange structure* (π, λ), *has the Lie algebra structure*.

We get the following result on symmetry transformations of critical sections:

Theorem *Let γ be a section of the fibred manifold π, Ξ a π-projectable vector field. The variation of γ, generated by Ξ, is constituted by critical sections of the Lagrange structure (π,λ) if and only if*

$$E(\lambda)\circ j^2\gamma = 0, \quad E(\vartheta(\Xi)\lambda)\circ j^2\gamma = 0.$$

This theorem shows that the critical sections with prescribed symmetry properties are solutions to the system of the Euler-Lagrange equations, given by the Lagrangian λ and the Lagrangian $\vartheta(\Xi)\lambda$.

B) Ordinary second order problems. Critical sections. Let $\pi : Y \to X$ be a fibred manifold with *n*-dimensional orientable base *X*, $\pi_1 : J^1Y \to X$ its 1-jet prolongation. Suppose we have a Lagrange structure (π_1,λ) and consider the differential ideal $\mathscr{D}$ on J^1Y, generated by the pseudovertical 1-forms

$$\omega_\sigma = dy_\sigma - z_{k\sigma}dx_k.$$

We determine the set $\Gamma_{\mathscr{D}}$ and the vector fields, generating $\mathscr{D}$-admissible variations.

Proposition *A section δ of the fibred manifold π_1 is an integral section of the differential ideal $\mathscr{D}$ if and only if there exists a section γ of the fibred manifold Y, such that*

$$\delta = j^1\gamma.$$

Proposition *A π_1-projectable vector field $\tilde{\Xi}$ generates $\mathscr{D}$-admissible variations of the Lagrange structure (π_1,λ) if and only if it coincides with the* 1*-jet prolongation of some π-projectable vector filed Ξ,*

$$\tilde{\Xi} = j^1\Xi.$$

The general Lagrange theory shows that for a description of $\mathscr{D}$-critical sections it is necessary to consider the *n*-form $h(\lambda)$, defined on the manifold $J^1(J^1Y)$, and the corresponding Euler form $E(\lambda)$, defined on the manifold $J^2(J^1Y)$. Consider some fibred coordinates (x_i, y_σ) on *Y*, where $1 \le i \le n$, $n = \dim X$, $n = \dim X$, $m = \dim Y - \dim X$, the associated fibred coordinates $(x_i, y_\sigma, z_{i\sigma})$ on J^1Y and the fibred coordinates $(x_i, y_\sigma, z_{i\sigma}, w_{i\sigma}, w_{ij\sigma})$ on

$J^1(J^1Y)$. In these local coordinates

$$h(\lambda) = \mathscr{L} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n,$$

and

$$E(\lambda) = (\mathscr{E}_\sigma(\mathscr{L}) dy_\sigma + \mathscr{E}_{i\sigma}(\mathscr{L}) dz_{i\sigma}) \wedge dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n,$$

where

$$\mathscr{E}_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y_\sigma} - d_k\left(\frac{\partial \mathscr{L}}{\partial w_{k\sigma}}\right), \quad \mathscr{E}_{i\sigma}(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial z_{i\sigma}} - d_k\left(\frac{\partial \mathscr{L}}{\partial w_{ki\sigma}}\right).$$

Writing now Theorem 2 in our situation, we see that the section δ of the fibred manifold π_1 is a $\mathscr{D}$-critical section of the Lagrange structure (π_1, λ) if and only if there exists a section γ of the fibred manifold π such that $\delta = j^1\gamma$ and

$$\int_\Omega j^1\delta * i(j^1(j^1\Xi))E(\lambda) = 0$$

for each n-dimensional compact submanifold with boundary $\Omega \subset X$ and all π-vertical vector fields Ξ with support in $\pi^{-1}(\Omega)$.

Using condition $\delta = j^1\gamma$, expression under the integral can be further simplified. To this purpose we need a generalisation of the definition of formal derivative, introduced in Chapter 3. Considered the fibred coordinates (x_i, y_σ) on Y and the associated coordinates $(x_i, y_\sigma, z_{i\sigma}, \ldots, z_{i_1 i_2 \ldots i_{r-1} i_r \sigma}, z_{i_1 i_2 \ldots i_r i_{r+1} \sigma})$ on $J^{r+1}Y$. Let f be a function of the local coordinates $(x_i, y_\sigma, z_{i\sigma}, \ldots, z_{i_1 i_2 \ldots i_{r-1} i_r \sigma})$. By the *formal derivative* of the function f with respect to the variable x_i we mean the function

$$f = \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial y_\sigma} z_{i\sigma} + \frac{\partial f}{\partial z_{j\sigma}} z_{ij\sigma} + \ldots + \sum_{i_1 \le i_2 \le \ldots \le i_r} \frac{\partial f}{\partial z_{i_1 i_2 \ldots i_r \sigma}} z_{i_1 i_2 \ldots i_r i \sigma},$$

defined on the corresponding coordinate neighbourhood in $J^{r+1}Y$.

Next consider our Lagrange structure (π_1, λ), and denote by

$$\tilde{h}(\lambda) = \tilde{\mathscr{L}} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$$

the restriction of the n-form $h(\lambda)$ to the submanifold J^2Y of $J^1(J^1Y)$. Introduce an $(n+1)$-form $\tilde{\mathscr{E}}(\lambda)$ on J^4Y by

$$\tilde{\mathscr{E}}(\lambda) = \tilde{\mathscr{E}}_\sigma(\lambda) dy_\sigma \wedge dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n,$$

where

$$\tilde{\mathscr{E}}_\sigma(\lambda) = \frac{\partial \tilde{\mathscr{L}}}{\partial y_\sigma} - d_i\left(\frac{\partial \tilde{\mathscr{L}}}{\partial z_{i\sigma}}\right) + \sum_{i \le j} d_i d_j \left(\frac{\partial \tilde{\mathscr{L}}}{\partial z_{ij\sigma}}\right).$$

One can verify that the form $\tilde{\mathscr{E}}(\lambda)$ is defined independently of the fibred coordinates. With these concepts we can prove the following theorem on $\mathscr{D}$-critical sections of the Lagrange structure (π_1, λ), connecting the abstract theory of Lagrange structures with the classical variational approach.

Theorem *A section* δ *is a* $\mathscr{D}$*-critical section of the Lagrange structure* (π_1, λ) *if and only if there exists a section* γ *of the fibred manifold* π *, satisfying the system of partial differential equations*

$$\tilde{\mathscr{E}}_\sigma(\lambda) \circ j^4\gamma = 0$$

and $\delta = j^1\gamma$.

The proof is based on a quite tedious coordinate computation, based on equations of the submanifold J^2Y in $J^1(J^1Y)$.

Since the $\mathscr{D}$-critical sections are uniquely determined in the considered case by the *n*-form $\tilde{h}(\lambda)$ (or in coordinates by the function $\tilde{\mathscr{L}}$), we call the corresponding variational problems *ordinary second order variational problems*.

C) Many examples of Lagrange structures are provided by the literature on the calculus of variations. Concrete examples can also be found in mathematical foundations of the general relativity theory. Consider at least one typical example. The fibred manifold is in this case usually a vector bundle, the base is a 4-dimensional manifold, admitting the *hyperbolic structure*, that is, a global covariant tensor field of degree 2, defining at every point a regular symmetric bilinear form with signature $(1,3)$, a *spacetime manifold*. Denote by π the 2-jet prolongation of the bundle of covariant tensors over a spacetime, and choose some local coordinates x_i on this basis. A Lagrangian λ , defined in the induced coordinates by

$$\lambda = R \cdot \sqrt{|\det \tilde{g}|}\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n,$$

where $\tilde{g}$ is the matrix of the covariant tensor $g = (g_{ij})$, and

$$R = R\left(g_{ij}, \frac{\partial g_{ij}}{\partial x_k}, \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}\right)$$

is the *scalar curvature*, associated with every pseudoriemannian structure, defines a Lagrange structure (π,λ). Choosing the differential ideal $\mathscr{D}$ in a similar way as in part B) of this chapter, the $\mathscr{D}$-critical sections will become solutions of the vacuum *Einstein equations*; these are hyperbolic metric fields, extremizing the action function, associated with the Lagrange structure (π,λ).

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