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Noether Theorems and Applications

1. Graded Lagrangian formalism. **2.** First Noether theorem. **3.** Gauge symmetries. **4.** Noether identities. **5.** Second Noether theorems. **6.** Lagrangian BRST theory. **7.** Applications. Topological BF theory.

Basic references[1]

G.Sardanashvily, **Noether's Theorems. Applications in Mechanics and Field Theory** (Springer, 2016).

G.Sardanashvily, Higher-stage Noether identities and second Noether theorems, **Adv. Math. Phys. 2015** (2015) 127481.

G.Sardanashvily, Graded Lagrangian formalism, **Int. J. Geom. Methods. Mod. Phys. 10** (2013) 1350016.

G.Giachetta, L.Mangiarotti, G.Sardanashvily, On the notion of gauge symmetries of generic Lagrangian field theory, **J. Math. Phys. 50** (2009) 012903.

D.Bashkirov, G.Giachetta, L.Mangiarotti, G.Sardanashvily, The KT-BRST complex of degenerate Lagrangian systems, **Lett. Math. Phys. 83** (2008) 237.

G.Giachetta, L.Mangiarotti, G.Sardanashvily, Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology, **Commun. Math. Phys. 259** (2005) 103.

[1] **http://www.g-sardanashvily.ru**

History

Classical Noether's theorems are well known to treat symmetries of Lagrangian systems.

• ***First Noether's theorem*** associates to a Lagrangian symmetry the conserved symmetry current whose total differential vanishes on-shell.

• ***Second Noether's theorems*** provide the correspondence between the gauge symmetries of a Lagrangian and the Noether identities which its Euler–Lagrange operator satisfies.

Let us refer for a rich history of Noether's theorems to the brilliant book:

Y.Kosmann-Schwarzbach, **The Noether Theorems. Invariance and the Conservation Laws in the Twentieth Century** (Springer, 2011).

However, ***one should go beyond classical Noether's theorems*** because they do not provide a complete analysis of the degeneracy of a generic Lagrangian system, namely, ***reducible degenerate*** Lagrangian systems.

A problem has come from ***Quantum Field Theory (QFT)*** where an analysis of the degeneracy of a field system is a preliminary step towards its quantization.

J.Fisch, M.Henneaux, Homological perturbation theory and algebraic structure of the antifield-antibracket formalism for gauge theories, **Commun. Math. Phys. 128** (1990) 627-640.

G.Barnich, F.Brandt, M.Henneaux, Local BRST cohomology in gauge theories. **Phys. Rep. 338** (2000) 439-569.

Basic Problem

A problem is that, ***for any Lagrangian***, its Euler–Lagrange operator satisfies Noether identities, which therefore must be separated into the trivial and non-trivial ones.

Moreover, these Noether identities obey ***first-stage*** Noether identities, which in turn are subject to the ***second-stage ones***, and so on.

Thus, there is a ***hierarchy of higher-stage Noether identities*** and, accordingly, gauge symmetries.

The corresponding ***higher-stage extension of Noether's theorems*** therefore should be formulated.

Main Theses

We aim to formulate the generalized Noether theorems in a very general setting of reducible degenerate Lagrangian systems of graded (even and odd) variables on graded bundles. This formulation is based on the following.

I.

Treating Noether identities in Lagrangian formalism, we follow the general notion of ***Noether identities of differential operators*** on fibre bundles.

G.Sardanashvily, Noether identities of a differential operator. The Koszul–Tate complex **Int. J. Geom. Methods Mod. Phys. 2** (2005) 873-886; **arXiv: math/0506103**.

• A key point is that ***any differential operator*** on a fibre bundle satisfies certain differential identities, called the Noether identities, which thus must be separated into the trivial and non-trivial ones.

• Furthermore, non-trivial Noether identities of a differential operator obey ***first stage*** Noether identities, which in turn are subject to the ***second-stage ones***, and so on. Thus, there is a ***hierarchy*** of higher-stage Noether identities of a differential operator.

• This hierarchy is described in terms of ***chain complexes*** whose boundaries are associated to trivial higher-stage Noether identities, but non-zero elements of their homology characterize non-trivial Noether identities modulo the trivial ones.

• As a result, if a certain homology condition holds, one constructs an exact chain complex, called the ***Koszul–Tate (KT) complex***, with a **KT** boundary operator whose nilpotentness is equivalent to all complete non-trivial Noether and higher-stage Noether identities of a differential operator.

• A differential operator is said to be ***degenerate*** if it admits non-trivial Noether identities, and ***reducible*** if there exist non-trivial higher-stage Noether identities.

• It should be noted that, though a differential operator is defined on a fibre bundle, the **KT** complex consists of graded (even and odd) elements, called ***antifields*** in accordance with the QFT terminology, and it is considered on graded manifolds and bundles.

II.

Given a Lagrangian system, we apply a general analysis of Noether identities of differential operators to the corresponding Euler–Lagrange operator. A result is the associated **KT** chain complex with the boundary operator whose nilpotency condition reproduces all non-trivial Noether and higher-stage Noether identities of an Euler–Lagrange operator. In the case of a ***variational*** Euler–Lagrange operator, we obtain something more.

• The ***extended inverse and direct second Noether theorems*** state the relations between higher-stage Noether identities and gauge symmetries of a Lagrangian system. Namely, these theorems associate to the above-mentioned **KT** complex a certain ***cochain sequence whose ascent operator consists of gauge and higher-order gauge symmetries of a Lagrangian system***. Therefore, it is called the ***gauge operator***.

• This cochain sequence, as like as the **KT** complex, consist of graded (even and odd) elements, called ***the ghosts*** in accordance with the QFT terminology. A problem is that the ***gauge operator, unlike the KT boundary operator, is not nilpotent***. Consequently, there is ***no self-consistent definition*** of non-trivial gauge symmetries, and therefore one has start just with Noether identities.

• Nevertheless, if gauge symmetries are algebraically closed, the gauge operator is extended to the nilpotent ***BRST operator*** which brings a cochain sequence into the ***BRST complex*** and provides a ***BRST extension*** of an original Lagrangian system by means of graded antifields and ghosts.

III.

Since the hierarchy of higher-stage Noether identities and gauge symmetries is described in the framework of graded homology and cohomology complexes, Lagrangian theory of graded even and odd variables is considered ***from the beginning***. In QFT, this is the case of fermion fields, ghosts in gauge theory and SUSY extensions of Standard Model.

• Lagrangian theory of even variables on a smooth manifold X conventionally is formulated in terms of ***fibre bundles and jet manifolds***. A key point is the ***classical Serre–Swan theorem*** which states the categorial equivalence between the projective modules of finite rank over a ring $C^\infty(X)$ of smooth real functions on X and the modules of global sections of vector bundles over X.

G.Giachetta, L.Mangiarotti, G.Sardanashvily, **Advanced Classical Field Theory** (World Scientific, 2009).
G.Sardanashvily, **Advanced Differential Geometry for Theoreticians. Fibre bundles, jet manifolds and Lagrangian theory** (Lambert Academic Publishing, 2013); **arXiv: 0908.1886**.

• However, different geometric models of odd variables either on graded manifolds or supermanifolds are discussed. It should be emphasized the ***difference*** between graded manifolds and supermanifolds. Both graded manifolds and supermanifolds are phrased in terms of sheaves of graded commutative algebras. ***Graded manifolds*** are characterized by sheaves on smooth manifolds, while ***supermanifolds*** are constructed by gluing sheaves on supervector spaces.

C.Bartocci, U.Bruzzo, D.Hernández Ruipérez, **The Geometry of Supermanifolds** (Kluwer, 1991).

• We follow the ***graded extension of the Serre–Swan theorem***. It states that, if a graded commutative $C^\infty(X)$-ring is generated by a projective $C^\infty(X)$-module of finite rank, it is isomorphic to the structure ring of graded functions on a graded manifold whose body is X. Therefore, we develop graded Lagrangian theory of even and odd variables in terms of ***graded manifolds and graded bundles***.

• A problem is that no conventional variational principle may be formulated for odd variables because there is no measure on graded manifolds. Nevertheless, a Lagrangian theory on a fibre bundle Y can be developed in ***algebraic*** terms of a ***variational bicomplex*** of differential forms on an infinite order jet manifold $J^\infty Y$ of Y, ***without appealing to a variational principle***. This technique has been extended to Lagrangian theory on graded bundles in terms of a ***graded variational bicomplex*** of graded differential forms on ***graded jet manifolds***.

IV.

Given a graded Lagrangian L, the cohomology of a variational bicomplex provides the global variational formula

$$dL = \mathcal{E}_L - d_H \Xi_L,$$

where $\mathcal{E}_L$ is the graded Euler–Lagrange operator and Ξ_L is a Lepage equivalent of a graded Lagrangian L.

• The extended ***first Noether theorem*** is a straightforward corollary of the above global variational formula in a general case of graded Lagrangians and their supersymmetries.

• It associates to a ***supersymmetry*** υ of a graded Lagrangian L the ***current*** $\mathcal{J}_\upsilon$ whose total differential $d_H \mathcal{J}_\upsilon$ vanishes on the shell $\delta L = 0$.

• If υ is a gauge supersymmetry of a graded Lagrangian L, the corresponding current $\mathcal{J}_\upsilon$ is a total differential on-shell. This statement sometimes is called the ***third Noether theorem***.

1 Graded Lagrangian formalism

• The ***differential calculus***, including ***formalism of linear differential operators and the Chevalley–Eilenberg complex of differential forms***, can be formulated over ***any ring***. A problem is that Euler–Lagrange operators ***need not be linear***.

• Theory of ***non-linear differential operators*** and, in particular, ***Lagrangian formalism*** conventionally are formulated on smooth fibre bundles over a smooth manifold X in terms of their jet manifolds.

• In the framework of the differential calculus over ***graded commutative rings***, Lagrangian formalism has been extended to graded manifolds and graded bundles over a smooth manifold X.

• In applications, this is the case both of ***classical field theory*** on bundles over X, $\dim X > 1$, and ***non-relativistic mechanics*** on fibre bundles over $X = \mathbb{R}$. Relativistic mechanics and classical string theory can be formulated as ***Lagrangian theory of submanifolds***.

G.Giachetta, L.Mangiarotti, G.Sardanashvily, **Advanced Classical Field Theory** (World Scientific, 2009).

1.1 Differential calculus over commutative rings

The differential calculus conventionally is defined over commutative rings. It straightforwardly is generalized to the case of graded commutative rings. However, this is ***not a particular case of the differential calculus over non-commutative rings***. A construction of the Chevalley–Eilenberg complex is generalized to an arbitrary ring, but an extension of the notion of differential operators to non-commutative rings meets ***difficulties***. A key point is that multiplication in a non-commutative ring is not a zero-order differential operator.

I.Krasil'shchik, V.Lychagin, A.Vinogradov, **Geometry of Jet Spaces and Nonlinear Partial Differential Equations** (Gordon and Breach, 1985).
G.Sardanashvily, **Lectures on Differential Geometry of Modules and Rings** (Lambert Academic Publishing, 2012); **arXiv: 0910.1515**.

In a case of graded commutative rings, one overcomes this difficulty by means of ***reformulating the notion of differential operators***. In particular, derivations both of commutative and non-commutative rings $\mathcal{A}$ obey the ***Leibniz rule***

$$\partial(ab) = \partial(a)b + a\partial(b), \qquad a, b \in \mathcal{A},$$

whereas the ***graded Leibniz rule*** for a graded commutative ring reads

$$\partial(ab) = \partial(a)b + (-1)^{[a][\partial]}a\partial(b), \qquad a, b \in \mathcal{A},$$

where $[a] = 1$, $[\partial] = 1$ for odd elements $a \in \mathcal{A}$ and derivations ∂.

Therefore, ***supergeometry is not particular non-commutative geometry***.

Remark 1.1: All algebras throughout are associative, unless they are Lie algebras and Lie superalgebras. By a ***ring*** is meant a unital algebra with a unit element $\mathbf{1} \neq 0$. Given a commutative ring $\mathcal{A}$, an additive group P is called the ***$\mathcal{A}$-module*** if it is provided with a distributive multiplication $\mathcal{A} \times P \to P$ by elements of $\mathcal{A}$ such that $ap = pa$ for all $a \in \mathcal{A}$, $p \in P$. A module over a field is called the ***vector space***. If a ring $\mathcal{A}$ is module over a commutative ring $\mathcal{K}$, it is said to be the $\mathcal{K}$-ring. A module P is called ***free*** if it admits a basis. A module is said to be of finite rank if it is the quotient of a free module with a finite basis. One says that a module P is ***projective*** if there exists a module Q such that $P \oplus Q$ is a free module. □

Let $\mathcal{K}$ be a commutative ring, $\mathcal{A}$ a commutative $\mathcal{K}$-ring, and let P and Q be $\mathcal{A}$-modules. A $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ can be endowed with two different $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \qquad (\Phi \bullet a)(p) = \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P.$$

Let us put $\delta_a\Phi = a\Phi - \Phi \bullet a$, $a \in \mathcal{A}$.

DEFINITION 1.1: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is called the ***linear s-order Q-valued differential operator*** on P if $(\delta_{a_0} \circ \cdots \circ \delta_{a_s})\Delta = 0$ for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. □

In particular, linear ***zero-order differential operators*** obey conditions

$$\delta_a\Delta(p) = a\Delta(p) - \Delta(ap) = 0, \qquad a \in \mathcal{A}, \qquad p \in P,$$

and, consequently, they coincide with $\mathcal{A}$-module morphisms $P \to Q$.

A linear ***first-order differential operator*** Δ satisfies a relation

$$(\delta_b \circ \delta_a)\Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \quad a, b \in \mathcal{A}.$$

For instance, a first-order differential operator Δ on $P = \mathcal{A}$ obeys a condition

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(\mathbf{1}), \qquad a, b \in \mathcal{A}.$$

DEFINITION 1.2: It is called a Q-valued ***derivation*** of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., it satisfies the Leibniz rule

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \qquad a, b \in \mathcal{A}.$$

□

If ∂ is a derivation of $\mathcal{A}$, then $a\partial$ is well for any $a \in \mathcal{A}$. Hence, derivations of $\mathcal{A}$ constitute an $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the ***derivation module*** of $\mathcal{A}$.

If $Q = \mathcal{A}$, the derivation module $\mathfrak{d}\mathcal{A} = \mathfrak{d}(\mathcal{A}, \mathcal{A})$ of $\mathcal{A}$ also is a ***Lie algebra*** over a ring $\mathcal{K}$ with respect to a ***Lie bracket***

$$[u, u'] = u \circ u' - u' \circ u, \qquad u, u' \in \mathfrak{d}\mathcal{A}.$$

A **fact** is that a linear s-order differential operator on an $\mathcal{A}$-module P is represented by a zero-order differential operator on a module of s-order jets of P (Theorem 1.1 below).

DEFINITION 1.3: Given an $\mathcal{A}$-module P, let $\mathcal{A}\otimes_{\mathcal{K}} P$ be a tensor product of $\mathcal{K}$-modules $\mathcal{A}$ and P. We put

$$\delta^b(a\otimes p)=(ba)\otimes p - a\otimes(bp), \qquad p\in P, \qquad a,b\in\mathcal{A}.$$

Let us denote by μ^{k+1} a submodule of $\mathcal{A}\otimes_{\mathcal{K}} P$ generated by elements

$$\delta^{b_0}\circ\cdots\circ\delta^{b_k}(a\otimes p)=a\delta^{b_0}\circ\cdots\circ\delta^{b_k}(\mathbf{1}\otimes p).$$

A k-order ***jet module*** $\mathcal{J}^k(P)$ of a module P is defined as the quotient of a $\mathcal{K}$-module $\mathcal{A}\otimes_{\mathcal{K}} P$ by μ^{k+1}. Its elements $c\otimes_k p$ are called the ***jets***. $\Box$

There exists a module morphism

$$J^k: P\ni p\to \mathbf{1}\otimes_k p\in\mathcal{J}^k(P) \tag{1.1}$$

such that $\mathcal{J}^k(P)$, seen as an $\mathcal{A}$-module, is generated by jets $J^k p$, $p\in P$.

THEOREM 1.1: Any linear k-order Q-valued differential operator Δ on an $\mathcal{A}$-module P uniquely factorizes as

$$\Delta: P\xrightarrow{J^k}\mathcal{J}^k(P)\xrightarrow{\mathfrak{f}^\Delta} Q$$

through the morphism J^k (1.1) and some $\mathcal{A}$-module homomorphism $\mathfrak{f}^\Delta: \mathcal{J}^k(P)\to Q$. $\Box$

In view of this fact, one says that jet modules $\mathcal{J}^k(P)$ of a module P play a role of the ***representative objects*** of linear differential operators on P.

Since the derivation module $\mathfrak{d}\mathcal{A}$ of a commutative $\mathcal{K}$-ring $\mathcal{A}$ is a Lie $\mathcal{K}$-algebra, one can associate to $\mathcal{A}$ the ***Chevalley–Eilenberg complex*** $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of differential forms over $\mathcal{A}$.

D.Fuks, **Cohomology of Infinite-Dimensional Lie Algebras** (Consultants Bureau, 1986)

It consists of $\mathcal{A}$-modules $\mathcal{O}^k[\mathfrak{d}\mathcal{A}]$ of $\mathcal{A}$-multilinear skew-symmetric maps

$$\mathcal{O}^k[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\overset{k}{\times}\mathfrak{d}\mathcal{A}, \mathcal{A}) \ni \phi : \overset{k}{\times}\mathfrak{d}\mathcal{A} \to \mathcal{A}, \tag{1.2}$$

provided both with the ***Chevalley–Eilenberg coboundary operator***

$$d\phi(u_0,\ldots,u_k) = \sum_{i=0}^{k}(-1)^i u_i(\phi(u_0,\ldots,\widehat{u_i},\ldots,u_k)) + \tag{1.3}$$
$$\sum_{i<j}(-1)^{i+j}\phi([u_i,u_j],u_0,\ldots,\widehat{u}_i,\ldots,\widehat{u}_j,\ldots,u_k),$$

and the exterior product

$$\phi\wedge\phi'(u_1,...,u_{r+s}) = \tag{1.4}$$
$$\sum_{i_1<\cdots<i_r;j_1<\cdots<j_s} \mathrm{sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi(u_{i_1},\ldots,u_{i_r})\phi'(u_{j_1},\ldots,u_{j_s}),$$
$$\phi\in\mathcal{O}^r[\mathfrak{d}\mathcal{A}], \qquad \phi'\in\mathcal{O}^s[\mathfrak{d}\mathcal{A}], \qquad u_k\in\mathfrak{d}\mathcal{A},$$

where $\mathrm{sgn}^{\cdots}_{\cdots}$ denotes the sign of a permutation. They obey relations

$$\phi\wedge\phi' = (-1)^{|\phi||\phi'|}\phi'\wedge\phi,$$
$$d(\phi\wedge\phi') = d(\phi)\wedge\phi' + (-1)^{|\phi|}\phi\wedge d(\phi'), \quad \phi,\phi'\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}], \tag{1.5}$$

and bring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a ***differential graded ring (DGR)***.

We also have the ***interior product*** $u\rfloor\phi = \phi(u)$, $u \in \mathfrak{d}\mathcal{A}$, $\phi \in \mathcal{O}^1[\mathfrak{d}\mathcal{A}]$. It is extended as

$$(u\rfloor\phi)(u_1, \ldots, u_{k-1}) = k\phi(u, u_1, \ldots, u_{k-1}), \quad u \in \mathfrak{d}\mathcal{A}, \quad \phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}], \quad (1.6)$$

to a DGR $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$, and obeys a relation

$$u\rfloor(\phi \wedge \sigma) = u\rfloor\phi \wedge \sigma + (-1)^{|\phi|}\phi \wedge u\rfloor\sigma.$$

With the interior product (1.6), one defines a derivation (a ***Lie derivative***)

$$\mathbf{L}_u(\phi) = d(u\rfloor\phi) + u\rfloor d\phi, \quad \phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}],$$
$$\mathbf{L}_u(\phi \wedge \sigma) = \mathbf{L}_u(\phi) \wedge \sigma + \phi \wedge \mathbf{L}_u\sigma,$$

of a graded ring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ for any $u \in \mathfrak{d}\mathcal{A}$. Then one can think of elements of $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ as being ***differential forms*** over $\mathcal{A}$.

The ***minimal*** Chevalley–Eilenberg complex $\mathcal{O}^*\mathcal{A}$ over a ring $\mathcal{A}$ consists of the monomials $a_0da_1 \wedge \cdots \wedge da_k$, $a_i \in \mathcal{A}$. It is called the ***de Rham complex*** of a $\mathcal{K}$-ring $\mathcal{A}$.

1.2 Lagrangian formalism on smooth fibre bundles

In order to formulate Lagrangian formalism on smooth fibre bundles, let us start with the differential calculus on smooth manifolds.

Let X be a smooth manifold and $C^\infty(X)$ an $\mathbb{R}$-ring of real smooth functions on X. The ***differential calculus on a smooth manifold*** X is defined as that over a real commutative ring $C^\infty(X)$.

Remark 1.2: A smooth manifold throughout is a finite-dimensional real manifold. It customarily is assumed to be a Hausdorff and second-countable topological space. Consequently, it is a locally compact countable at infinity space and ***paracompact*** space, which admits the partition of unity by smooth real functions. Let us emphasize that the paracompactness is very essential for a number of theorems in the sequel. □

In a general setting, we follow the conventional definition of manifolds as ***local-ringed spaces***, i.e., sheaves in local rings. This is the case both of smooth manifolds and graded manifolds in the sequel.

Let C^∞_X be a sheaf of germs of smooth real functions on a smooth manifold X, i.e., smooth functions are identified if they coincide on an open neighborhood of a point $x \in X$. Its stalk C^∞_x of germs at $x \in X$ has a unique maximal ideal of germs of functions vanishing at x. Therefore, (X, C^∞_X) is a local-ringed space. Though a sheaf C^∞_X exists on a topological space X, it fixes a unique smooth manifold structure on X as follows.

THEOREM 1.2: Let X be a paracompact topological space and $(X, \mathfrak{A})$ a local-ringed space. Let X admit an open cover $\{U_i\}$ such that a sheaf $\mathfrak{A}$ restricted to each U_i is isomorphic to a local-ringed space $(\mathbb{R}^n, C^\infty_{R^n})$. Then X is an n-dimensional smooth manifold together with a natural isomorphism of local-ringed spaces $(X, \mathfrak{A})$ and (X, C^∞_X). □

One can think of this result as being an equivalent definition of smooth real manifolds in terms of local-ringed spaces. A smooth manifold X also is algebraically reproduced as a certain subspace of the ***spectrum*** of a real ring $C^\infty(X)$ of smooth real functions on X.

Furthermore, the ***classical Serre–Swan theorem*** states the categorial equivalence between vector bundles over a smooth manifold X and projective modules of finite rank over the ring $C^\infty(X)$ of smooth real functions on X.

THEOREM 1.3: Let X be a smooth manifold. A $C^\infty(X)$-module P is a projective module of finite rank iff it is isomorphic to the ***structure module*** $Y(X)$ of global sections of some vector bundle $Y \to X$ over X. □

The following are COROLLARIES of this theorem.

• The derivation module of a real ring $C^\infty(X)$ coincides with a $C^\infty(X)$-module $\mathcal{T}_1(X)$ of ***vector fields*** on X.

• Its $C^\infty(X)$-dual $\mathcal{O}^1(X) = \mathcal{T}_1(X)^*$ is the structure module $\mathcal{O}^1(X)$ of the cotangent bundle T^*X of X, i.e., a module of one-forms on X and, conversely, $\mathcal{T}_1(X) = \mathcal{O}^1(X)^*$.

• It follows that the Chevalley–Eilenberg complex of a real ring $C^\infty(X)$ is exactly the ***de Rham complex*** $(\mathcal{O}^*(X), d)$ of ***exterior forms*** on X.

• Let $Y \to X$ be a vector bundle and $Y(X)$ its structure module. An r-order jet module $\mathcal{J}^r(Y(X))$ of a $C^\infty(X)$-module $Y(X)$ is the structure module $J^rY(X)$ of sections of an r-order ***jet bundle*** $J^rY \to X$.

• Then by virtue of Theorem 1.1, a liner k-order differential operator on a projective $C^\infty(X)$-module P of finite rank with values in a projective $C^\infty(X)$-module Q of finite rank is represented by a linear bundle morphism $J^kY \to E$ of a jet bundle $J^kY \to X$ to a vector bundle $E \to X$ where $Y \to X$ and $E \to X$ are smooth vector bundles with structure modules $Y(X) = P$ and $E(X) = Q$ in accordance with Serre–Swan Theorem 1.3.

Thus, the differential calculus on a smooth manifold X leads to a Lie algebra of vector fields on X, a DGR of exterior forms on X, jet manifolds of a vector bundle over X as representative objects of linear differential operators on this vector bundle.

A key point is that the construction of jet manifolds as representative objects of differential operators is ***generalized to a case of non-linear differential operators*** on fibre bundles.

Let $Y \to X$ be a smooth fibre bundle provided with bundle coordinates (x^λ, y^i). An r-order ***jet manifold*** J^rY of sections of a fibre bundle $Y \to X$ is defined as the disjoint union of equivalence classes $j^r_x s$ of sections s of $Y \to X$ which are identified by $r+1$ terms of their Taylor series at points of X.

D.Saunders, **The Geometry of Jet Bundles** (Cambridge Univ. Press, 1989).

G.Sardanashvily, **Advanced Differential Geometry for Theoreticians. Fibre bundles, jet manifolds and Lagrangian theory** (Lambert Academic Publishing, 2013); **arXiv: 0908.1886**.

A set J^rY is endowed with an atlas of adapted coordinates

$$(x^\lambda, y^i_\Lambda), \quad y^i_\Lambda \circ s = \partial_\Lambda s^i(x), \quad {y'}^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial' x^\lambda} d_\mu {y'}^i_\Lambda, \quad 0 \leq |\Lambda| \leq r, \qquad (1.7)$$

where the symbol d_λ stands for the higher order total derivative

$$d_\lambda = \partial_\lambda + \sum_{0\leq|\Lambda|\leq r-1} y^i_{\Lambda+\lambda}\partial^\Lambda_i, \qquad d'_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu.$$

We use the compact notation $d_\Lambda = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1}$, $\Lambda = (\lambda_r \ldots \lambda_1)$. The coordinates (1.7) ***brings a set J^rY into a smooth manifold***.

Given fibre bundles Y and Y' over X, every bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism f of X admits the r-order ***jet prolongation*** to a morphism of r-order jet manifolds

$$J^r\Phi : J^rY \ni j^r_x s \to j^r_{f(x)}(\Phi \circ s \circ f^{-1}) \in J^rY'.$$

Every section s of a fibre bundle $Y \to X$ has the r-order ***jet prolongation*** to a section $(J^rs)(x) = j^r_x s$ of a jet bundle $J^rY \to X$.

There are natural surjections of jet manifolds

$$\pi^r_{r-1} : J^rY \to J^{r-1}Y,$$

which form the ***inverse sequence*** of finite order jet manifolds

$$Y \overset{\pi}{\longleftarrow} J^1Y \longleftarrow \cdots J^{r-1}Y \overset{\pi^r_{r-1}}{\longleftarrow} J^rY \longleftarrow \cdots . \tag{1.8}$$

Its ***inverse limit*** $J^\infty Y$ is a minimal set so that there exist surjections

$$\pi^\infty : J^\infty Y \to X, \quad \pi^\infty_0 : J^\infty Y \to Y, \qquad \pi^\infty_k : J^\infty Y \to J^kY,$$

obeying the relations $\pi^\infty_r = \pi^k_r \circ \pi^\infty_k$, $r < k$. It consists of those elements

$$(\ldots, z_r, \ldots, z_k, \ldots), \qquad z_r \in J^rY, \qquad z_k \in J^kY,$$

of the Cartesian product $\prod\limits_k J^kY$ which satisfy the relations $z_r = \pi^k_r(z_k)$, $r < k$. One can think of elements of $J^\infty Y$ as being ***infinite order jets*** of sections of $Y \to X$ identified by their Taylor series at points of X.

A set $J^\infty Y$ is provided with the inverse limit topology. This is the coarsest topology such that the surjections π^∞_r are continuous. Its base consists of inverse images of open subsets of J^rY, $r = 0, \ldots,$ under the maps π^∞_r. With the inverse limit topology, $J^\infty Y$ is a paracompact Fréchet manifold. A bundle coordinate atlas $\{U_Y, (x^\lambda, y^i)\}$ of $Y \to X$ provides $J^\infty Y$ with a manifold coordinate atlas

$$\{(\pi^\infty_0)^{-1}(U_Y), (x^\lambda, y^i_\Lambda)\}_{0\leq|\Lambda|}, \qquad {y'}^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda.$$

One calls $J^\infty Y$ the ***infinite order jet manifold***.

F.Takens, A global version of the inverse problem of the calculus of variations, **J. Diff. Geom. 14** (1979) 543.

DEFINITION 1.4: Let $Y \to X$ and $E \to X$ be smooth fibre bundles. A bundle morphism $\Delta : J^kY \to E$ over X is called the E-valued k-order ***differential operator*** on Y. This differential operator sends each section s of $Y \to X$ to the section $\Delta \circ J^ks$ of $E \to X$. □

Jet manifolds J^kY of a fibre bundle $Y \to X$ constitute the inverse sequence (1.8) whose inverse limit is an infinite order jet manifold $J^\infty Y$. Then any k-order E-valued differential operator Δ on a fibre bundle Y is defined by a continuous bundle map

$$\Delta \circ \pi^\infty_r : J^\infty Y \underset{X}{\longrightarrow} E.$$

In particular, differential operators in Lagrangian theory on fibre bundles, e.g., Euler–Lagrange and Helmholtz–Sonin operators are represented by certain exterior forms on finite order jet manifolds and, consequently, on $J^\infty Y$.

The inverse sequence (1.8) of jet manifolds yields the direct sequence of DGRs $\mathcal{O}^*_r = \mathcal{O}^*(J^rY)$ of exterior forms on finite order jet manifolds

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi^1_0{}^*} \mathcal{O}^*_1 \longrightarrow \cdots \mathcal{O}^*_{r-1} \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_r \longrightarrow \cdots , \qquad (1.9)$$

where $\pi^r_{r-1}{}^*$ are the pull-back monomorphisms. Its ***direct limit*** $\mathcal{O}^*_\infty Y$ consists of all exterior forms on finite order jet manifolds modulo the pull-back identification. It is a DGR which inherits operations of the exterior differential d and the exterior product $\wedge$ of DGRs $\mathcal{O}^*_r$.

A DGR $\mathcal{O}^*_\infty Y$ is split into a variational bicomplex $\mathcal{O}^{*,*}_\infty Y$. Lagrangians L, Euler–Lagrange operators $\mathcal{E}_L = \delta L$ and the variational operator δ are defined as elements $L \in \mathcal{O}^{0,n}_\infty$, $\mathcal{E}_L \in \mathcal{O}^{1,n}_\infty Y$, $n = \dim X$, and the coboundary operator of this bicomplex. Its cohomology provides the global variational formula

$$dL = \mathcal{E}_L - d_H \Xi_L,$$

where $\mathcal{E}_L$ is an Euler–Lagrange operator and Ξ_L is a Lepage equivalent of L.

G.Sardanashvily, Cohomology of the variational complex in the class of exterior forms of finite jet order, **Int. J. Math. & Math. Sci. 30** (2002) 39.

We reproduce these results below in the framework of a graded Lagrangian formalism

It should be emphasized that we deal with a variational bicomplex of the DGR $\mathcal{O}^*_\infty Y$ of differential forms of ***bounded jet order*** on an infinite order jet manifold $J^\infty Y$. They are exterior forms on finite order jet manifolds $J^r Y$ modulo the pull-back identification. One also considers a variational bicomplex of a DGR $\mathcal{Q}^*_\infty Y$ of differential forms of ***locally finite jet order*** on $J^\infty Y$, which are differential forms on finite order jet manifolds only locally on an open neighborhood of each point of $J^\infty Y$.

I.Anderson, Introduction to the variational bicomplex. **Contemp. Math. 132**, 51-73 (1992)

A fact is that $J^\infty Y$ is a paracompact topological space which admits the partition of unity by elements of a ring $\mathcal{Q}^0_\infty Y$, but not $\mathcal{O}^0_\infty Y$. Therefore, one can apply the abstract de Rham theorem in order to find the cohomology of $\mathcal{Q}^*_\infty Y$. Then we proved that the cohomology of $\mathcal{O}^*_\infty Y$ equals that of $\mathcal{Q}^*_\infty Y$.

In a different way, variational sequences of finite jet order are considered.

D.Krupka, **Introduction to Global Variational Geometry** (Springer, 2015).

M.Palese, O.Rossi, E.Winterroth, J.Musilová, Variational Sequences, Representation Sequences and Applications in Physics, **SIGMA 12** (2016) 045.

1.3 Differential calculus over graded commutative rings

If there is no danger of confusion, by the term ***graded*** throughout is meant $\mathbb{Z}_2$***-graded***.

Let $\mathcal{K}$ be a commutative ring. A $\mathcal{K}$-module Q is called graded (i.e., $\mathbb{Z}_2$-graded) if it is decomposed into a direct sum $Q = Q_* = Q_0 \oplus Q_1$ of modules Q_0 and Q_1, called the ***even*** and ***odd*** parts of Q_*, respectively. A $\mathbb{Z}_2$-graded $\mathcal{K}$-module is said to be free if it has a basis composed by homogeneous elements.

A morphism $\Phi : P_* \to Q_*$ of graded $\mathcal{K}$-modules is said to be an ***even*** (resp. ***odd***) morphism if Φ preserves (resp. changes) the $\mathbb{Z}_2$-parity of all homogeneous elements. A morphism $\Phi : P_* \to Q_*$ of graded $\mathcal{K}$-modules is called ***graded*** if it is represented by a sum of even and odd morphisms. A set $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ of these graded morphisms is a graded $\mathcal{K}$-module.

DEFINITION 1.5: A $\mathcal{K}$-ring $\mathcal{A}$ is called ***graded*** (i.e., $\mathbb{Z}_2$-graded) if it is a graded $\mathcal{K}$-module $\mathcal{A}_*$, and a product of its homogeneous elements $\alpha\alpha'$ is a homogeneous element of degree $([a] + [a'])\mathrm{mod}\, 2$. In particular, $[\mathbf{1}] = 0$. Its even part $\mathcal{A}_0$ is a $\mathcal{K}$-ring, and the odd one $\mathcal{A}_1$ is an $\mathcal{A}_0$-module. □

DEFINITION 1.6: A graded ring $\mathcal{A}_*$ is called ***graded commutative*** if

$$aa' = (-1)^{[a][a']} a'a, \qquad a, a' \in \mathcal{A}_*.$$

□

A graded commutative ring can admit different graded commutative structures $\mathcal{A}_*$ in general.

By ***automorphisms*** of a graded commutative ring $\mathcal{A}_*$ are meant automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which are graded $\mathcal{K}$-module morphisms of $\mathcal{A}_*$. Obviously, they are even, and they preserve a graded structure of $\mathcal{A}$. However, there exist automorphisms ϕ of a $\mathcal{K}$-ring $\mathcal{A}$ which do not possess this property in general. Then $\mathcal{A}_*$ and $\phi(A_*)$ are isomorphic, but different graded commutative structures of a ring $\mathcal{A}$. Moreover, it may happen that a $\mathcal{K}$-ring $\mathcal{A}$ admits non-isomorphic graded commutative structures.

Example 1.3: Given a graded commutative ring $\mathcal{A}_*$ and its odd element κ, an automorphism

$$\phi : \mathcal{A}_0 \ni a \to a, \qquad \mathcal{A}_1 \ni a \to a(\mathbf{1}+\kappa),$$

of a $\mathcal{K}$-ring $\mathcal{A}$ does not preserve its original graded structure $\mathcal{A}_*$. □

Given a graded commutative ring $\mathcal{A}_*$, a ***graded*** $\mathcal{A}_*$-module Q_* is defined as an $(\mathcal{A}-\mathcal{A})$-bimodule which is a graded $\mathcal{K}$-module such that

$$[aq] = ([a]+[q]) \bmod 2, \qquad qa = (-1)^{[a][q]}aq, \qquad a \in \mathcal{A}_*, \quad q \in Q_*.$$

The following are constructions of new graded modules from the old ones.

• A ***direct sum*** of graded modules and a ***graded factor module*** are defined just as those of modules over a commutative ring.

• A ***tensor product*** $P_* \otimes Q_*$ of graded $\mathcal{A}_*$-modules P_* and Q_* is their tensor product as $\mathcal{A}$-modules such that

$$[p\otimes q] = ([p]+[q]) \mod 2, \qquad p \in P_*, \qquad q \in Q_*,$$
$$ap\otimes q = (-1)^{[p][a]}pa\otimes q = (-1)^{[p][a]}p\otimes aq, \qquad a \in \mathcal{A}_*.$$

• In particular, the ***tensor algebra***

$$\otimes P = \mathcal{A} \oplus P \oplus \cdots \oplus (\overset{k}{\underset{\mathcal{A}}{\otimes}} P) \oplus \cdots$$

of an $\mathcal{A}_*$-module P_* is defined just as that of a module over a commutative ring. Its quotient $\wedge P_*$ with respect to the ideal generated by elements

$$p \otimes p' + (-1)^{[p][p']} p' \otimes p, \qquad p, p' \in P_*,$$

is the ***exterior algebra*** of a graded module P_* with respect to the ***graded exterior product***

$$p \wedge p' = -(-1)^{[p][p']} p' \wedge p. \tag{1.10}$$

• A ***graded morphism*** $\Phi : P_* \to Q_*$ of graded $\mathcal{A}_*$-modules is their graded morphism as graded $\mathcal{K}$-modules which obeys the relations

$$\Phi(ap) = (-1)^{[\Phi][a]} a\Phi(p), \qquad p \in P_*, \qquad a \in \mathcal{A}_*. \tag{1.11}$$

These morphisms form a graded $\mathcal{A}_*$-module $\mathrm{Hom}_{\mathcal{A}}(P_*, Q_*)$. A graded $\mathcal{A}_*$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P_*, \mathcal{A}_*)$ is called the ***dual*** of a graded $\mathcal{A}_*$-module P_*.

In the sequel, we are concerned with graded manifolds. They are sheaves in Grassmann algebras, whose derivations form a Lie superalgebra. They are defined as follows.

A graded commutative $\mathcal{K}$-ring Λ_* is said to be the ***Grassmann algebra*** if the following hold.

• It is finitely generated in degree 1, i.e., it is a free $\mathcal{K}$-module of finite rank so that $\Lambda_0 = \mathcal{K} \oplus \Lambda_1^2$ and, consequently,

$$\Lambda = \mathcal{K} \oplus R, \qquad R = \Lambda_1 \oplus (\Lambda_1)^2,$$

where R is the ideal of ***nilpotents*** of a ring Λ. A surjection $\sigma : \Lambda \to \mathcal{K}$ is called the ***body map***.

• It is isomorphic to the exterior algebra $\wedge(R/R^2)$ of a $\mathcal{K}$-module R/R^2 where R is the ideal of nilpotents of Λ_*.

An exterior algebra $\wedge Q$ of a free $\mathcal{K}$-module Q of finite rank is a Grassmann algebra. Conversely, a Grassmann algebra admits a structure of an exterior algebra $\wedge Q$ by a choice of its minimal generating $\mathcal{K}$-module $Q \subset \Lambda_1$, and all these structures are mutually isomorphic if $\mathcal{K}$ is a field.

Let $\mathcal{A}_*$ be a graded commutative ring. A graded $\mathcal{A}_*$-algebra $\mathfrak{g}_*$ is called the ***Lie $\mathcal{A}_*$-superalgebra*** if its product $[.,.]$, called the ***Lie superbracket***, obeys the rules

$$[\varepsilon, \varepsilon'] = -(-1)^{[\varepsilon][\varepsilon']}[\varepsilon', \varepsilon],$$
$$(-1)^{[\varepsilon][\varepsilon'']}[\varepsilon, [\varepsilon', \varepsilon'']] + (-1)^{[\varepsilon'][\varepsilon]}[\varepsilon', [\varepsilon'', \varepsilon]] + (-1)^{[\varepsilon''][\varepsilon']}[\varepsilon'', [\varepsilon, \varepsilon']] = 0.$$

Clearly, an even part $\mathfrak{g}_0$ of a Lie superalgebra $\mathfrak{g}_*$ is a Lie $\mathcal{A}_0$-algebra. Given an $\mathcal{A}_*$-superalgebra, a graded $\mathcal{A}_*$-module P_* is called a $\mathfrak{g}_*$-module if it is provided with an $\mathcal{A}_*$-bilinear map

$$\mathfrak{g}_* \times P_* \ni (\varepsilon, p) \to \varepsilon p \in P_*, \qquad [\varepsilon p] = ([\varepsilon] + [p]) \bmod 2,$$
$$[\varepsilon, \varepsilon']p = (\varepsilon \circ \varepsilon' - (-1)^{[\varepsilon][\varepsilon']}\varepsilon' \circ \varepsilon)p.$$

The differential calculus over graded commutative rings is defined similarly to that over commutative rings, but it differs from the differential calculus over non-commutative rings.

G.Sardanashvily, **Advanced Differential Geometry for Theoreticians. Fibre bundles, jet manifolds and Lagrangian theory** (Lambert Academic Publishing, 2013); **arXiv: 0908.1886**.

Let $\mathcal{K}$ be a commutative ring and $\mathcal{A}$ a graded commutative $\mathcal{K}$-ring. Let P and Q be graded $\mathcal{A}$-modules. A graded $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of graded $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ admits two graded $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \qquad (\Phi \bullet a)(p) = \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P.$$

Let us put

$$\delta_a \Phi = a\Phi - (-1)^{[a][\Phi]}\Phi \bullet a, \qquad a \in \mathcal{A}.$$

DEFINITION 1.7: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is said to be the Q-valued ***graded differential operator*** of order s on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s} \Delta = 0$$

for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. A set $\mathrm{Diff}_s(P,Q)$ of these operators is a graded $\mathcal{A}$-module. □

In particular, ***zero-order graded differential operators*** coincide with graded $\mathcal{A}$-module morphisms $P \to Q$.

A ***first-order graded differential operator*** Δ satisfies a relation

$$\delta_a \circ \delta_b \Delta(p) = ab\Delta(p) - (-1)^{([b]+[\Delta])[a]} b\Delta(ap) - (-1)^{[b][\Delta]} a\Delta(bp) + (-1)^{[b][\Delta]+([\Delta]+[b])[a]} \Delta(bap) = 0, \qquad a, b \in \mathcal{A}, \quad p \in P.$$

For instance, a first-order graded differential operator Δ on $P = \mathcal{A}$ fulfils a condition

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]} a\Delta(b) - (-1)^{([b]+[a])[\Delta]} ab\Delta(\mathbf{1}), \qquad a, b \in \mathcal{A}.$$

DEFINITION 1.8: It is called the Q-valued ***graded derivation*** of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., if it obeys the ***graded Leibniz rule***

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]} a\Delta(b), \qquad a, b \in \mathcal{A}.$$

□

If ∂ is a graded derivation of $\mathcal{A}$, then $a\partial$ is so for any $a \in \mathcal{A}$. Hence, graded derivations of $\mathcal{A}$ constitute a graded $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the ***graded derivation module***.

If $Q = \mathcal{A}$, the graded derivation module $\mathfrak{d}\mathcal{A} = \mathfrak{d}(\mathcal{A}, \mathcal{A})$ also is a Lie $\mathcal{K}$-superalgebra with respect to the ***superbracket***

$$[u, u'] = u \circ u' - (-1)^{[u][u']} u' \circ u, \qquad u, u' \in \mathcal{A}.$$

Since the graded derivation module $\mathfrak{d}\mathcal{A}$ of a graded commutative ring $\mathcal{A}$ is a Lie $\mathcal{K}$-superalgebra, one can consider the Chevalley–Eilenberg complex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ whose cochains are graded $\mathcal{A}$-modules

$$C^k[\mathfrak{d}\mathcal{A};\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\overset{k}{\wedge}\mathfrak{d}\mathcal{A},\mathcal{A})$$

of $\mathcal{A}$-linear graded morphisms of graded exterior products $\overset{k}{\wedge}\mathfrak{d}\mathcal{A}$ of a graded $\mathcal{A}$-module $\mathfrak{d}\mathcal{A}$ to $\mathcal{A}$, seen as a $\mathfrak{d}\mathcal{A}$-module.

Let us bring homogeneous elements of $\overset{k}{\wedge}\mathfrak{d}\mathcal{A}$ into a form

$$\varepsilon_1\wedge\cdots\varepsilon_r\wedge\epsilon_{r+1}\wedge\cdots\wedge\epsilon_k, \qquad \varepsilon_i\in\mathfrak{d}\mathcal{A}_0, \quad \epsilon_j\in\mathfrak{d}\mathcal{A}_1.$$

Then a Chevalley–Eilenberg coboundary operator d of a complex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$, called the ***graded exterior differential*** reads

$$\begin{aligned}
& dc(\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s) = \\
& \sum_{i=1}^{r}(-1)^{i-1}\varepsilon_i c(\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\epsilon_s) + \\
& \sum_{j=1}^{s}(-1)^{r}\epsilon_j c(\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s) + \\
& \sum_{1\leq i<j\leq r}(-1)^{i+j}c([\varepsilon_i,\varepsilon_j]\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\widehat{\varepsilon}_j\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s) + \\
& \sum_{1\leq i<j\leq s}c([\epsilon_i,\epsilon_j]\wedge\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_i\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s) + \\
& \sum_{1\leq i<r,1\leq j\leq s}(-1)^{i+r+1}c([\varepsilon_i,\epsilon_j]\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s),
\end{aligned}$$

where the caret $\widehat{}$ denotes omission.

A graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the ***graded exterior product***

$$\phi\wedge\phi'(u_1,...,u_{r+s}) = \sum_{i_1<\cdots<i_r;j_1<\cdots<j_s} \mathrm{Sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi(u_{i_1},\ldots,u_{i_r})\phi'(u_{j_1},\ldots,u_{j_s}),$$
$$\phi\in\mathcal{O}^r[\mathfrak{d}\mathcal{A}], \qquad \phi'\in\mathcal{O}^s[\mathfrak{d}\mathcal{A}], \qquad u_k\in\mathfrak{d}\mathcal{A},$$

where $u_1,\ldots,u_{r+s}$ are graded-homogeneous elements of $\mathfrak{d}\mathcal{A}$ and

$$u_1\wedge\cdots\wedge u_{r+s} = \mathrm{Sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s} u_{i_1}\wedge\cdots\wedge u_{i_r}\wedge u_{j_1}\wedge\cdots\wedge u_{j_s}.$$

A graded differential d and a graded exterior product $\wedge$ bring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a ***differential bigraded ring*** (**DBGR**) whose elements obey relations

$$\phi\wedge\phi' = (-1)^{|\phi||\phi'|+[\phi][\phi']}\phi'\wedge\phi, \qquad d(\phi\wedge\phi') = d\phi\wedge\phi' + (-1)^{|\phi|}\phi\wedge d\phi'.$$

It is called the ***graded Chevalley–Eilenberg differential calculus*** over a graded commutative $\mathcal{K}$-ring $\mathcal{A}$.

In particular, $\mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A},\mathcal{A}) = \mathfrak{d}\mathcal{A}^*$ is the dual of the derivation module $\mathfrak{d}\mathcal{A}^*$. One can extend this duality relation to the graded interior product of $u\in\mathfrak{d}\mathcal{A}$ with any element $\phi\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ by the rules

$$u\rfloor(bda) = (-1)^{[u][b]}bu(a), \qquad a,b\in\mathcal{A},$$
$$u\rfloor(\phi\wedge\phi') = (u\rfloor\phi)\wedge\phi' + (-1)^{|\phi|+[\phi][u]}\phi\wedge(u\rfloor\phi').$$

As a consequence, a graded derivation $u\in\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ yields a graded derivation

$$\mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi), \qquad \phi\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}], \qquad u\in\mathfrak{d}\mathcal{A},$$
$$\mathbf{L}_u(\phi\wedge\phi') = \mathbf{L}_u(\phi)\wedge\phi' + (-1)^{[u][\phi]}\phi\wedge\mathbf{L}_u(\phi'),$$

termed the ***graded Lie derivative*** of a DBGR $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$.

The ***minimal graded Chevalley–Eilenberg differential calculus*** $\mathcal{O}^*\mathcal{A} \subset \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ over a graded commutative ring $\mathcal{A}$ consists of monomials $a_0 da_1 \wedge \cdots \wedge da_k$, $a_i \in \mathcal{A}$. The corresponding complex

$$0 \to \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{d} \mathcal{O}^1\mathcal{A} \xrightarrow{d} \cdots \mathcal{O}^k\mathcal{A} \xrightarrow{d} \cdots$$

is called the ***de Rham complex*** of a graded commutative $\mathcal{K}$-ring $\mathcal{A}$.

Let us note that, if $\mathcal{A} = \mathcal{A}^0$ is a commutative ring, graded differential operators and the graded Chevalley–Eilenberg differential calculus are reduced to the familiar commutative differential calculus.

1.4 Differential calculus on graded manifolds

As was mentioned above, we follow the ***Serre – Swan theorem*** extended to graded manifolds. It states that, if a graded commutative $C^\infty(X)$-ring is generated by a projective $C^\infty(X)$-module of finite rank, it is isomorphic to a ring of graded functions on a graded manifold whose body is X. Therefore, we aim to develop ***Lagrangian formalism of graded (even and odd) variables*** in terms just of graded manifolds.

Accordingly to a general concept of a manifold as a local-ringed space, a ***graded manifold*** has been defined as a sheaf on a smooth body manifold in local graded commutative algebras which are real Grassmann algebras.

C. Bartocci, U.Bruzzo, D.Hernández Ruipérez, **The Geometry of Supermanifolds** (Kluwer Academic Publ., 1991).

We start with the conventional notion of a $\mathbb{Z}_2$***-graded manifold***. It is a local-ringed space $(Z,\mathfrak{A})$ where Z is an n-dimensional smooth manifold, and $\mathfrak{A}=\mathfrak{A}_0\oplus\mathfrak{A}_1$ is a sheaf in real Grassmann algebras Λ such that:

- there is the exact sequence of sheaves

$$0\to\mathcal{R}\to\mathfrak{A}\xrightarrow{\sigma}C^\infty_Z\to 0, \qquad \mathcal{R}=\mathfrak{A}_1+(\mathfrak{A}_1)^2,$$

where C^∞_Z is the sheaf of smooth real functions on Z;

- $\mathcal{R}/\mathcal{R}^2$ is a locally free sheaf of C^∞_Z-modules of finite rank (with respect to pointwise operations), and the sheaf $\mathfrak{A}$ is locally isomorphic to the exterior product $\wedge_{C^\infty_Z}(\mathcal{R}/\mathcal{R}^2)$.

A sheaf $\mathfrak{A}$ is called the ***structure sheaf*** of a graded manifold $(Z,\mathfrak{A})$, its stalk (a fibre) at a point $z\in Z$ is the tensor product $C^\infty_z\underset{\mathbb{R}}{\otimes}\Lambda$. A manifold Z is said to be the ***body*** of $(Z,\mathfrak{A})$. Sections of the sheaf $\mathfrak{A}$ are termed the ***graded functions*** on a graded manifold $(Z,\mathfrak{A})$. They make up a graded commutative $C^\infty(Z)$-ring $\mathfrak{A}(Z)$ called the ***structure ring*** of $(Z,\mathfrak{A})$.

By virtue of the well-known ***Batchelor theorem***, graded manifolds possess the following structure.

THEOREM 1.4: Let $(Z,\mathfrak{A})$ be a graded manifold. There exists a vector bundle $E\to Z$ with an m-dimensional typical fibre V such that the structure sheaf $\mathfrak{A}$ of $(Z,\mathfrak{A})$ as a sheaf in real rings is isomorphic to the structure sheaf $\mathfrak{A}_E=\wedge E^*_Z$ of germs of sections of the exterior bundle

$$\wedge E^* = (Z\times\mathbb{R})\underset{Z}{\oplus} E^*\underset{Z}{\oplus}\wedge E^*\underset{Z}{\oplus}\overset{2}{\wedge}E^*\cdots,$$

whose typical fibre is the Grassmann algebra $\Lambda=\wedge V^*$. □

Note that ***Batchelor's isomorphism*** in Theorem 1.4 ***is not canonical***.

Combining Batchelor Theorem 1.4 and classical Serre–Swan Theorem, we come to the above mentioned ***Serre–Swan theorem for graded manifolds***.

THEOREM 1.5: Let Z be a smooth manifold. A graded commutative $C^\infty(Z)$-ring $\mathcal{A}$ is isomorphic to the structure ring of a graded manifold with a body Z if and only if it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank. □

In fact, the structure sheaf $\mathfrak{A}_E$ of a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$ in Theorem 1.4 is a sheaf in $\mathbb{N}$-graded commutative rings $\Lambda^* = \wedge V$ whose $\mathbb{N}$-graded structure is fixed.

Remark 1.4: Let $\mathcal{K}$ be a commutative ring. A direct sum of $\mathcal{K}$-modules

$$P = P^* = \bigoplus_{i\in\mathbb{N}} P^i \tag{1.12}$$

is called the $\mathbb{N}$***-graded*** $\mathcal{K}$***-module***. A $\mathcal{K}$-ring $\mathcal{A}$ is called $\mathbb{N}$***-graded*** if it is an $\mathbb{N}$-graded $\mathcal{K}$-module $\mathcal{A}^*$ (1.12) so that a product of homogeneous elements $\alpha\alpha'$ is a homogeneous element of degree $[\alpha]+[\alpha']$. Any $\mathbb{N}$-graded $\mathcal{K}$-module P (1.12) admits the ***associated*** $\mathbb{Z}_2$***-graded structure***

$$P = P_0 \oplus P_1, \qquad P_0 = \bigoplus_{i\in\mathbb{N}} P^{2i}, \qquad P_1 = \bigoplus_{i\in\mathbb{N}} P^{2i+1}.$$

Accordingly, an $\mathbb{N}$-graded ring $\mathcal{A}^*$ also is the $\mathbb{Z}_2$-graded one $\mathcal{A}_*$. ***The converse is not true***. An $\mathbb{N}$-graded $\mathcal{K}$-ring $\mathcal{A}^*$ is said to be ***graded commutative*** if

$$\alpha\beta = (-1)^{[\alpha][\beta]}\beta\alpha, \qquad \alpha, \beta \in \mathcal{A}^*.$$

An $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$ possesses an ***associated*** $\mathbb{Z}_2$***-graded commutative structure***

$$\mathcal{A}_0 = \bigoplus_k \mathcal{A}^{2k}, \qquad \mathcal{A}_1 = \bigoplus_k \mathcal{A}^{2k+1}, \qquad k \in \mathbb{N},$$
$$\alpha\beta = (-1)^{[\alpha][\beta]}\beta\alpha, \qquad \alpha, \beta \in \mathcal{A}_*.$$

The converse need not be true. However, a Grassmann algebra Λ possess an associated $\mathbb{N}$-graded structure of an exterior algebra $\Lambda = \Lambda^* = \wedge\Lambda^1$, ***which is not unique***. □

Hereafter, we consider a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$ when its Batchelor's isomorphism $(Z, \mathfrak{A} = \mathfrak{A}_E)$ holds. ***Why?***

• Though Batchelor's isomorphism is not canonical, in applications it ***is fixed from the beginning*** as a rule.

• From the mathematical viewpoint, we restrict our considerations to morphisms which preserve a fixed $\mathbb{N}$-graded structure.

Example 1.5: Let Λ be a real Grassmann algebra. Its associated $\mathbb{N}$-graded structure is defined by a choice of a minimal generating vector space $\Lambda^1 \subset \Lambda_1$. Given a basis $\{c^i\}$ for Λ^1, elements of a Grassmann algebra Λ take a form

$$a = \sum_{k=0,1,\ldots} \sum a_{i_1 \cdots i_k} c^{i_1} \cdots c^{i_k}, \qquad a_{i_1 \cdots i_k} c^{i_1} \in \mathbb{R}.$$

We call $\{c^i\}$ the ***generating basis*** for a Grassmann algebra Λ. Then one can show that any ring automorphism of Λ is a compositions of automorphisms

$$c^i \to c'^i = \rho^i_j c^j + b^i, \tag{1.13}$$

where ρ is an automorphism of a vector space Λ^1 and b^i are odd elements of $\Lambda^{>2}$, and of morphisms

$$c^i \to c'^i = c^i(\mathbf{1} + \kappa), \qquad \kappa \in \Lambda_1. \tag{1.14}$$

Automorphisms (1.13), where $b^i = 0$, preserve the $\mathbb{N}$-graded structure Λ^*. If $b^i \neq 0$, they keep a $\mathbb{Z}_2$-graded structure of Λ, but not the $\mathbb{N}$-graded one Λ^*. Automorphisms (1.14) preserve an even sector Λ_0 of Λ, but not the odd one Λ_1. However, one can show that different $\mathbb{N}$- and $\mathbb{Z}_2$-graded structures of a real Grassmann algebra are mutually isomorphic. $\Box$

Thus, we further deal with graded manifolds $(Z,\mathfrak{A}_E)$ ***which are*** $\mathbb{N}$-***graded***.

A graded manifold $(Z,\mathfrak{A}_E)$ is said to be ***modelled over a vector bundle*** $E\to Z$, and E is called its ***characteristic vector bundle***. Its ***structure ring*** $\mathcal{A}_E$ is the structure module $\mathcal{A}_E=\wedge E^*(Z)$ of sections of the exterior bundle $\wedge E^*$.

A ***key point*** is that automorphisms of an $\mathbb{N}$-graded manifold $(Z,\mathfrak{A}_E)$ are restricted to those induced by automorphisms of its characteristic vector bundle $E\to Z$.

Remark 1.6: One can treat a local-ringed space $(Z,\mathfrak{A}_0=C^\infty_Z)$ as a ***trivial graded manifold*** whose characteristic vector bundle is $E=Z\times\{0\}$. Its structure module is a ring $C^\infty(Z)$ of smooth real functions on Z. □

Given a graded manifold $(Z,\mathfrak{A}_E)$, every trivialization chart $(U;z^A,y^a)$ of its characteristic vector bundle $E\to Z$ yields a ***splitting domain*** $(U;z^A,c^a)$ of $(Z,\mathfrak{A}_E)$. ***Graded functions*** on such a chart are Λ-valued functions

$$f=\sum_{k=0}^{m}\frac{1}{k!}f_{a_1\ldots a_k}(z)c^{a_1}\cdots c^{a_k}, \qquad (1.15)$$

where $f_{a_1\cdots a_k}(z)$ are smooth functions on U and $\{c^a\}$ is the fibre basis for E^*. One calls $\{z^A,c^a\}$ the ***local basis*** for a graded manifold $(Z,\mathfrak{A}_E)$. Transition functions $y'^a=\rho^a_b(z^A)y^b$ of bundle coordinates on $E\to Z$ induce the corresponding transformation $c'^a=\rho^a_b(z^A)c^b$ of the associated local basis for a graded manifold $(Z,\mathfrak{A}_E)$ and the according coordinate transformation law of graded functions (1.15).

The following is an ***essential peculiarity*** of an $\mathbb{N}$-graded manifold $(Z,\mathfrak{A}_E)$ in comparison with the $\mathbb{Z}_2$-graded ones.

THEOREM 1.6: Derivations of the structure module $\mathcal{A}_E$ of a graded manifold $(Z,\mathfrak{A}_E)$ are represented by sections of a vector bundle

$$\mathcal{V}_E = \wedge E^* \underset{E}{\otimes} TE \to Z. \tag{1.16}$$

□

Due to the canonical splitting $VE = E\times E$, the vertical tangent bundle VE of $E \to Z$ can be provided with fibre bases $\{\partial_a\}$, which are the duals of bases $\{c^a\}$. Then ***graded derivations*** of $\mathcal{A}_E$ on a splitting domain $(U; z^A, c^a)$ of $(Z,\mathfrak{A}_E)$ read

$$u = u^A\partial_A + u^a\partial_a, \qquad [\partial_A] = 0, \qquad [\partial_a] = 1, \tag{1.17}$$
$$\partial_A\partial_B = \partial_B\partial_A, \qquad \partial_A\partial_a = \partial_a\partial_A, \qquad \partial_a\partial_b = -\partial_b\partial_a,$$

where u^λ, u^a are local graded functions on U possessing a transformation law

$$u'^A = u^A, \qquad u'^a = \rho^a_j u^j + u^A\partial_A(\rho^a_j)c^j.$$

The graded derivations (1.17) are called ***graded vector fields*** on a graded manifold $(Z,\mathfrak{A}_E)$. They act on graded functions $f \in \mathfrak{A}_E(U)$ by a rule

$$u(f_{a\ldots b}c^a\cdots c^b) = u^A\partial_A(f_{a\ldots b})c^a\cdots c^b + u^k f_{a\ldots b}\partial_k\rfloor(c^a\cdots c^b).$$

In accordance with Theorem 1.6, the ***graded derivation module*** $\mathfrak{d}\mathcal{A}_E$ is isomorphic to the structure module $\mathcal{V}_E(Z)$ of global sections of the vector bundle $\mathcal{V}_E \to Z$ (1.16). It is a real Lie superalgebra.

Given the structure ring $\mathcal{A}_E$ of graded functions on a graded manifold $(Z, \mathfrak{A}_E)$ and the real Lie superalgebra $\mathfrak{d}\mathcal{A}_E$ of its graded derivations, we consider the graded Chevalley–Eilenberg differential calculus $\mathcal{S}^*[E; Z] = \mathcal{O}^*[\mathfrak{d}\mathcal{A}_E]$:

$$0 \to \mathbb{R} \to \mathcal{A}_E \xrightarrow{d} \mathcal{S}^1[E; Z] \xrightarrow{d} \cdots \mathcal{S}^k[E; Z] \xrightarrow{d} \cdots , \tag{1.18}$$

over $\mathcal{S}^0[E; Z] = \mathcal{A}_E$. Since a graded derivation module $\mathfrak{d}\mathcal{A}_E$ is the structure module of sections of a vector bundle $\mathcal{V}_E \to Z$, elements of $\mathcal{S}^*[E; Z]$ are represented by sections of the exterior bundle $\wedge\overline{\mathcal{V}}_E$ of the $\mathcal{A}_E$-dual

$$\overline{\mathcal{V}}_E = \wedge E^* \underset{E}{\otimes} T^*E \to Z \tag{1.19}$$

of $\mathcal{V}_E$. With respect to the dual fibre bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for E^*, sections of $\overline{\mathcal{V}}_E$ (1.19) take a local form

$$\phi = \phi_A dz^A + \phi_a dc^a, \qquad \phi'_a = \rho^{-1}{}_a^b \phi_b, \qquad \phi'_A = \phi_A + \rho^{-1}{}_a^b \partial_A(\rho^a_j)\phi_b c^j.$$

The duality relation $\mathcal{S}^1[E; Z] = \mathfrak{d}\mathcal{A}_E^*$ is given by a ***graded interior product***

$$u\rfloor\phi = u^A\phi_A + (-1)^{[\phi_a]}u^a\phi_a.$$

The graded exterior differential reads

$$d\phi = dz^A \wedge \partial_A\phi + dc^a \wedge \partial_a\phi.$$

Elements of a DBGR $\mathcal{S}^*[E; Z]$ are called the ***graded differential forms*** on a graded manifold $(Z, \mathfrak{A}_E)$. Seen as an $\mathcal{A}_E$-ring, $\mathcal{S}^*[E; Z]$ on a splitting domain (z^A, c^a) is locally generated by even and odd one-forms dz^A and dc^i.

Cohomology of the DBGR $\mathcal{S}^*[E;Z]$ (1.18) is called the ***de Rham cohomology of a graded manifold*** $(Z,\mathfrak{A}_E)$. It equals the de Rham cohomology of its body Z. In particular, there exist both a cochain monomorphism $\mathcal{O}^*(Z) \to \mathcal{S}^*[E;Z]$ and a cochain body epimorphism $\mathcal{S}^*[E;Z] \to \mathcal{O}^*(Z)$.

G.Sardanashvily, Graded infinite order jet manifolds, **Int. J. Geom. Methods Mod. Phys. 4** (2007) 1335-1362.

A ***morphism of graded manifolds*** $(Z,\mathfrak{A}) \to (Z',\mathfrak{A}')$ is defined as that of local-ringed spaces $\phi : Z \to Z'$, $\widehat{\Phi} : \mathfrak{A}' \to \phi_*\mathfrak{A}$, where ϕ is a manifold morphism and $\widehat{\Phi}$ is a sheaf morphism of $\mathfrak{A}'$ to the direct image $\phi_*\mathfrak{A}$ of $\mathfrak{A}$ onto Z'. This morphism of graded manifolds is said to be:

- a ***monomorphism*** if ϕ is an injection and $\widehat{\Phi}$ is an epimorphism, and
- an ***epimorphism*** if ϕ is a surjection and $\widehat{\Phi}$ is a monomorphism.

An epimorphism of graded manifolds $(Z,\mathfrak{A}) \to (Z',\mathfrak{A}')$ where $Z \to Z'$ is a fibre bundle is called ***the graded bundle***.

D.Hernández Ruipérez, J.Muñoz Masqué, Global variational calculus on graded manifolds. **J. Math. Pures Appl. 63** (1984) 283-309.

T.Stavracou, Theory of connections on graded principal bundles, **Rev. Math. Phys. 10** (1998) 47-79.

In particular, let $(Y, \mathfrak{A})$ be a graded manifold whose body is a fibre bundle $Y \to X$. Let us consider a trivial graded manifold $(X, \mathfrak{A}_0 = C^\infty_X)$. Then we have a graded bundle

$$(Y, \mathfrak{A}) \to (X, C^\infty_X). \tag{1.20}$$

Let us denote it by $(X, Y, \mathfrak{A})$. Given a graded bundle $(X, Y, \mathfrak{A})$, the local basis for a graded manifold $(Y, \mathfrak{A})$ can take a form (x^λ, y^i, c^a) where (x^λ, y^i) are bundle coordinates of $Y \to X$. Therefore, we agree to call the graded bundle (1.20) over a trivial graded manifold (X, C^∞_X) the ***graded bundle over a smooth manifold***.

Note that a graded manifold $(X, \mathfrak{A})$ itself can be treated as the graded bundle $(X, X, \mathfrak{A})$ (1.20) associated to the identity smooth bundle $X \to X$.

Let $E \to Z$ and $E' \to Z'$ be vector bundles and $\Phi : E \to E'$ their bundle morphism over a morphism $\phi : Z \to Z'$. A bundle morphism (Φ, ϕ) induces a morphism of graded manifolds

$$(Z, \mathfrak{A}_E) \to (Z', \mathfrak{A}_{E'}). \tag{1.21}$$

It is a monomorphism (resp. epimorphism) if Φ is a bundle injection (resp. surjection). In particular, the graded manifold morphism (1.21) is a graded bundle if Φ is a fibre bundle. Let $\mathcal{A}_{E'} \to \mathcal{A}_E$ be the corresponding pull-back monomorphism of the structure rings. It yields a monomorphism of the DBGRs

$$\mathcal{S}^*[E'; Z'] \to \mathcal{S}^*[E; Z].$$

In order to formulate Lagrangian theory both of even and odd variables, let us consider graded manifolds whose body is a fibre bundle $Y \to X$.

Let $(Y, \mathfrak{A}_F)$ be a graded manifold modelled over a vector bundle $F \to Y$. This is a graded bundle $(X, Y, \mathfrak{A}_F)$:

$$(Y, \mathfrak{A}_F) \to (X, C^\infty_X), \tag{1.22}$$

modelled over a ***composite bundle***

$$F \to Y \to X. \tag{1.23}$$

The structure ring of graded functions on a graded manifold $(Y, \mathfrak{A}_F)$ is the graded commutative $C^\infty(X)$-ring $\mathcal{A}_F = \wedge F^*(Y)$. Let the composite bundle (1.23) be provided with adapted bundle coordinates (x^λ, y^i, q^a) possessing transition functions

$$x'^\lambda(x^\mu), \qquad y'^i(x^\mu, y^j), \qquad q'^a = \rho^a_b(x^\mu, y^j) q^b.$$

Then the corresponding basis for a graded manifold $(Y, \mathfrak{A}_F)$ is (x^λ, y^i, c^a) together with transition functions $c'^a = \rho^a_b(x^\mu, j^j) c^b$. We call it the ***local basis*** for a graded bundle $(X, Y, \mathfrak{A}_F)$ (1.22).

As was shown above, the differential calculus on a fibre bundle $Y \to X$ is formulated in terms of jet manifolds J^*Y of Y. Being fibre bundles over X, they can be regarded as trivial graded bundles $(X, J^kY, C^\infty_{J^kY})$. We describe their partners in a case of graded bundles as follows.

Let us note that, given a graded manifold $(X, \mathfrak{A}_E)$ and its structure ring $\mathcal{A}_E$, one can define the jet module $J^1\mathcal{A}_E$ of a $C^\infty(X)$-ring $\mathcal{A}_E$. It is a module of global sections of the jet bundle $J^1(\wedge E^*)$. A problem is that $J^1\mathcal{A}_E$ fails to be a structure ring of some graded manifold. By this reason, we have suggested a different construction of jets of graded manifolds (Definition 1.9), though it is applied only to $\mathbb{N}$-graded manifolds.

Let $(X, \mathcal{A}_E)$ be a graded manifold modelled over a vector bundle $E \to X$. Let us consider a k-order jet manifold J^kE of E. It is a vector bundle over X. Then let $(X, \mathcal{A}_{J^kE})$ be a graded manifold modelled over $J^kE \to X$.

DEFINITION 1.9: We call $(X, \mathcal{A}_{J^kE})$ the ***graded jet manifold*** of a graded manifold $(X, \mathcal{A}_E)$. $\Box$

Given a splitting domain $(U; x^\lambda, c^a)$ of a graded manifold $(Z, \mathcal{A}_E)$, the adapted splitting domain of a graded jet manifold $(X, \mathcal{A}_{J^kE})$ reads

$$(U; x^\lambda, c^a, c^a_\lambda, c^a_{\lambda_1\lambda_2}, \ldots c^a_{\lambda_1\ldots\lambda_k}), \quad c'^a_{\lambda\lambda_1\ldots\lambda_r} = \rho^a_b(x) c^a_{\lambda\lambda_1\ldots\lambda_r} + \partial_\lambda \rho^a_b(x) c^a_{\lambda_1\ldots\lambda_r}.$$

As was mentioned above, a graded manifold is a particular graded bundle over its body. Then Definition 1.9 of graded jet manifolds is generalized to graded bundles over smooth manifolds as follows.

Let $(X, Y, \mathfrak{A}_F)$ be the graded bundle (1.22) modelled over the composite bundle $F \to Y \to X$ (1.23). It is readily observed that the jet manifold J^rF of $F \to X$ is a vector bundle $J^rF \to J^rY$ coordinated by $(x^\lambda, y^i_\Lambda, q^a_\Lambda)$, $0 \leq |\Lambda| \leq r$. Let $(J^rY, \mathfrak{A}_{J^rF})$ be a graded manifold modelled over this vector bundle. Its local generating basis is $(x^\lambda, y^i_\Lambda, c^a_\Lambda)$, $0 \leq |\Lambda| \leq r$.

DEFINITION 1.10: We call $(J^rY, \mathfrak{A}_{J^rF})$ the ***graded jet manifold of a graded bundle*** $(X, Y, \mathfrak{A}_F)$. □

In particular, let $Y \to X$ be a smooth bundle seen as a trivial graded bundle (X, Y, C^∞_Y) modelled over a composite bundle $Y \times \{0\} \to Y \to X$. Then its graded jet manifold is a trivial graded bundle $(X, J^rY, C^\infty_{J^rY})$, i.e., the jet manifold J^rY of Y. Thus, Definition 1.10 of graded jet manifolds of graded bundles is compatible with the definition of jets of fibre bundles.

The affine bundles $J^{r+1}Y \to J^rY$ and the corresponding fibre bundles $J^{r+1}F \to J^rF$ also yield the graded bundles

$$(J^{r+1}Y, \mathfrak{A}_{J^{r+1}F}) \to (J^rY, \mathfrak{A}_{J^rF}).$$

As a consequence, we have the ***inverse sequence of graded manifolds***

$$(Y, \mathfrak{A}_F) \longleftarrow (J^1Y, \mathfrak{A}_{J^1F}) \longleftarrow \cdots (J^{r-1}Y, \mathfrak{A}_{J^{r-1}F}) \longleftarrow (J^rY, \mathfrak{A}_{J^rF}) \cdots . \quad (1.24)$$

One can think of its ***inverse limit*** $(J^\infty Y, \mathfrak{A}_{J^\infty F})$ as being the ***graded infinite order jet manifold*** whose body is an infinite order jet manifold $J^\infty Y$ and whose structure sheaf $\mathfrak{A}_{J^\infty F}$ is a sheaf of germs of graded functions on graded manifolds $(J^*Y, \mathfrak{A}_{J^*F})$. However $(J^\infty Y, \mathfrak{A}_{J^\infty F})$ is not a graded manifold in a strict sense because $J^\infty Y$ is not a smooth manifold.

The inverse system of graded jet manifolds (1.24) yields a direct system

$$\mathcal{S}^*[F;Y] \xrightarrow{\pi^*} \mathcal{S}^*_1[F;Y] \longrightarrow \cdots \mathcal{S}^*_{r-1}[F;Y] \xrightarrow{\pi^{r*}_{r-1}} \mathcal{S}^*_r[F;Y] \longrightarrow \cdots , \quad (1.25)$$

$$\pi^{r+1*}_r : \mathcal{S}^*_r[F;Y] \to \mathcal{S}^*_{r+1}[F;Y], \qquad \mathcal{S}^*_k[F;Y] = \mathcal{S}^*[J^kF; J^kY],$$

where π^{r+1*}_r are the pull-back monomorphisms of DBGRs.

The DBGR $\mathcal{S}^*_\infty[F;Y]$ associated to a graded bundle $(X,Y,\mathfrak{A}_F)$ is defined as the ***direct limit*** of the direct system (1.25). It include all graded differential forms $\phi\in\mathcal{S}^*_r[F;Y]$ on graded manifolds $(J^rY,\mathfrak{A}_{J^rF})$ modulo the monomorphisms π^{r+1*}_r.

THEOREM 1.7: There is an isomorphism $H^*(\mathcal{S}^*_\infty[F;Y])=H^*_{\rm DR}(Y)$ of the cohomology $H^*(\mathcal{S}^*_\infty[F;Y])$ of the de Rham complex

$$0\to\mathbb{R}\longrightarrow\mathcal{S}^0_\infty[F;Y]\stackrel{d}{\longrightarrow}\mathcal{S}^1_\infty[F;Y]\cdots\stackrel{d}{\longrightarrow}\mathcal{S}^k_\infty[F;Y]\longrightarrow\cdots$$

of a DBGR $\mathcal{S}^*_\infty[F;Y]$ to the de Rham cohomology $H^*_{\rm DR}(Y)$ of Y. $\Box$

In particular, monomorphisms $\mathcal{O}^*(J^rY)\to\mathcal{S}^*_r[F;Y]$ yield a cochain monomorphism of complexes $\mathcal{O}^*_\infty Y\to\mathcal{S}^*_\infty[F;Y]$, and body epimorphisms $\mathcal{S}^*_r[F;Y]\to\mathcal{O}^*_rY$ define a cochain epimorphism $\mathcal{S}^*_\infty[F;Y]\to\mathcal{O}^*_\infty Y$.

One can think of elements of $\mathcal{S}^*_\infty[F;Y]$ as being ***graded differential forms on an infinite order jet manifold*** $J^\infty Y$ in the sense that $\mathcal{S}^*_\infty[F;Y]$ is a submodule of the structure module of sections of some sheaf on $J^\infty Y$. In particular, one can restrict $\mathcal{S}^*_\infty[F;Y]$ to a coordinate chart of $J^\infty Y$ so that $\mathcal{S}^*_\infty[F;Y]$ as an $\mathcal{O}^0_\infty Y$-algebra is locally generated by the elements

$$(c^a_\Lambda, dx^\lambda,\theta^a_\Lambda=dc^a_\Lambda-c^a_{\lambda+\Lambda}dx^\lambda,\theta^i_\Lambda=dy^i_\Lambda-y^i_{\lambda+\Lambda}dx^\lambda),\qquad 0\leq|\Lambda|,$$

where c^a_Λ, θ^a_Λ are odd and dx^λ, θ^i_Λ are even. We agree to call (y^i,c^a) the local generating basis for $\mathcal{S}^*_\infty[F;Y]$. Let the collective symbol s^A stand for its elements. We further denote $[A]=[s^A]$.

1.5 Lagrangian theory of even and odd variables on graded bundles

Similarly to DGR $\mathcal{O}^*_\infty Y$ of differential forms on jet manifolds J^rY in Lagrangian formalism on a fibre bundle $Y\to X$, a DBGR $\mathcal{S}^*_\infty[F;Y]$ of graded differential forms is split into a graded variational bicomplex which provides Lagrangian theory of graded (even and odd) variables.

Let $(X,Y,\mathfrak{A}_F)$ be a graded bundle modelled over a composite bundle $F\to Y\to X$ over an n-dimensional smooth manifold X, and let $\mathcal{S}^*_\infty[F;Y]$ be the associated DBGA of graded exterior forms on graded jet manifolds of $(X,Y,\mathfrak{A}_F)$. A DBGA $\mathcal{S}^*_\infty[F;Y]$ is decomposed into $\mathcal{S}^0_\infty[F;Y]$-modules $\mathcal{S}^{k,r}_\infty[F;Y]$ of k-contact and r-horizontal graded forms together with the corresponding projections

$$h_k:\mathcal{S}^*_\infty[F;Y]\to\mathcal{S}^{k,*}_\infty[F;Y],\qquad h^m:\mathcal{S}^*_\infty[F;Y]\to\mathcal{S}^{*,m}_\infty[F;Y].$$

Accordingly, the graded exterior differential d on $\mathcal{S}^*_\infty[F;Y]$ falls into a sum $d=d_V+d_H$ of the ***vertical and total graded differentials***

$$d_V\circ h^m=h^m\circ d\circ h^m,\qquad d_V(\phi)=\theta^A_\Lambda\wedge\partial^\Lambda_A\phi,\qquad \phi\in\mathcal{S}^*_\infty[F;Y],$$
$$d_H\circ h_k=h_k\circ d\circ h_k,\qquad d_H\circ h_0=h_0\circ d,\qquad d_H(\phi)=dx^\lambda\wedge d_\lambda(\phi),$$
$$d_\lambda=\partial_\lambda+\sum s^A_{\lambda+\Lambda}\partial^\Lambda_A,$$

where d_λ are ***graded total derivatives***. These differentials obey the nilpotent relations

$$d_H\circ d_H=0,\qquad d_V\circ d_V=0,\qquad d_H\circ d_V+d_V\circ d_H=0.$$

A DBGA $\mathcal{S}^*_\infty[F;Y]$ also is provided with the graded projection morphism

$$\varrho = \sum_{k>0} \frac{1}{k}\overline{\varrho}\circ h_k \circ h^n : \mathcal{S}^{*>0,n}_\infty[F;Y] \to \mathcal{S}^{*>0,n}_\infty[F;Y],$$
$$\overline{\varrho}(\phi) = \sum (-1)^{|\Lambda|}\theta^A \wedge [d_\Lambda(\partial^\Lambda_A\rfloor\phi)], \qquad \phi\in \mathcal{S}^{>0,n}_\infty[F;Y],$$

such that $\varrho\circ d_H = 0$, and with the nilpotent ***graded variational operator***

$$\delta = \varrho\circ d : \mathcal{S}^{*,n}_\infty[F;Y] \to \mathcal{S}^{*+1,n}_\infty[F;Y].$$

With these operators a DBGA $\mathcal{S}^*_\infty[F;Y]$ is decomposed into the ***graded variational bicomplex***

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & {}_{d_V}\uparrow & & {}_{d_V}\uparrow & & {}_{d_V}\uparrow & & {}_{-\delta}\uparrow \\
 & 0\to & \mathcal{S}^{1,0}_\infty & \overset{d_H}{\longrightarrow} & \mathcal{S}^{1,1}_\infty & \overset{d_H}{\longrightarrow}\cdots & \mathcal{S}^{1,n}_\infty & \overset{\varrho}{\longrightarrow} & \mathbf{E}_1\to 0 \\
 & & {}_{d_V}\uparrow & & {}_{d_V}\uparrow & & {}_{d_V}\uparrow & & {}_{-\delta}\uparrow \\
0\to\mathbb{R}\to & & \mathcal{S}^0_\infty & \overset{d_H}{\longrightarrow} & \mathcal{S}^{0,1}_\infty & \overset{d_H}{\longrightarrow}\cdots & \mathcal{S}^{0,n}_\infty & \equiv & \mathcal{S}^{0,n}_\infty \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0\to\mathbb{R}\to & & \mathcal{O}^0(X) & \overset{d}{\longrightarrow} & \mathcal{O}^1(X) & \overset{d}{\longrightarrow}\cdots & \mathcal{O}^n(X) & \overset{d}{\longrightarrow} & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{S}^*_\infty = \mathcal{S}^*_\infty[F;Y]$ and $\mathbf{E}_k = \varrho(\mathcal{S}^{k,n}_\infty[F;Y])$.

We restrict our consideration to a ***short variational subcomplex***

$$0 \to \mathbb{R} \to \mathcal{S}^0_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{0,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\delta} \mathbf{E}_1 \qquad (1.26)$$

of this bicomplex and its subcomplex of one-contact graded forms

$$0 \to \mathcal{S}^{1,0}_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{1,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathcal{S}^{1,n}_\infty[F;Y] \xrightarrow{\varrho} \mathbf{E}_1 \to 0. \qquad (1.27)$$

They possess the following cohomology.

THEOREM 1.8: Cohomology of the complex (1.26) equals the de Rham cohomology of Y. The complex (1.27) is exact. □

Decomposed into a variational bicomplex, the DBGA $\mathcal{S}^*_\infty[F;Y]$ describes ***graded Lagrangian theory*** on a graded bundle $(X, Y, \mathfrak{A}_F)$. Its ***graded Lagrangian*** is defined as an element

$$L = \mathcal{L}\omega \in \mathcal{S}^{0,n}_\infty[F;Y], \qquad \omega = dx^1 \wedge \cdots \wedge dx^n, \qquad (1.28)$$

of the graded variational complex (1.26). Accordingly, a graded exterior form

$$\delta L = \theta^A \wedge \mathcal{E}_A\omega = \sum (-1)^{|\Lambda|}\theta^A \wedge d_\Lambda(\partial^\Lambda_A L)\omega \in \varrho(\mathcal{S}^{1,n}_\infty[F;Y]) \qquad (1.29)$$

is said to be its ***graded Euler–Lagrange operator***. Its kernel yields an ***Euler–Lagrange equation***

$$\delta L = 0, \qquad \mathcal{E}_A = \sum (-1)^{|\Lambda|}\theta^A \wedge d_\Lambda(\partial^\Lambda_A L) = 0. \qquad (1.30)$$

We call a pair $(\mathcal{S}^{0,n}_\infty[F;Y], L)$ the ***graded Lagrangian system*** and $\mathcal{S}^*_\infty[F;Y]$ its ***structure algebra***.

The following are corollaries of Theorem 1.8.

COROLLARY 1.9: Any δ-closed (i.e., ***variationally trivial***) graded Lagrangian $L\in \mathcal{S}^{0,n}_\infty[F;Y]$ is a sum

$$L=h_0\sigma + d_H\xi, \qquad \xi\in \mathcal{S}^{0,n-1}_\infty[F;Y],$$

where σ is a closed form on Y. $\Box$

COROLLARY 1.10: Given a graded Lagrangian L, there is the ***global variational formula***

$$dL=\delta L - d_H\Xi_L, \qquad \Xi\in \mathcal{S}^{n-1}_\infty[F;Y], \tag{1.31}$$

$$\Xi_L = L+\sum_{s=0}\theta^A_{\nu_s\ldots\nu_1}\wedge F^{\lambda\nu_s\ldots\nu_1}_A\omega_\lambda, \tag{1.32}$$

$$F^{\nu_k\ldots\nu_1}_A = \partial^{\nu_k\ldots\nu_1}_A\mathcal{L} - d_\lambda F^{\lambda\nu_k\ldots\nu_1}_A + \sigma^{\nu_k\ldots\nu_1}_A, \qquad k=1,2,\ldots,$$

where local graded functions σ obey relations $\sigma^\nu_A=0$, $\sigma^{(\nu_k\nu_{k-1})\ldots\nu_1}_A=0$. $\Box$

The form Ξ_L (1.32) provides a ***global Lepage equivalent*** of a graded Lagrangian L. In particular, one can locally choose Ξ_L (1.32) where all graded functions σ vanish.

2 First Noether theorem

Given a graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y],L)$, by its ***infinitesimal transformations*** are meant contact graded derivations of a real graded commutative ring $\mathcal{S}^0_\infty[F;Y]$. They constitute a $\mathcal{S}^0_\infty[F;Y]$-module $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ which is a real Lie superalgebra relative to a Lie superbracket. The following holds.

THEOREM 2.1: The derivation module $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ is isomorphic to the $\mathcal{S}^0_\infty[F;Y]$-dual $(\mathcal{S}^1_\infty[F;Y])^*$ of the module of graded one-forms $\mathcal{S}^1_\infty[F;Y]$. $\Box$

COROLLARY 2.2: A DBGA $\mathcal{S}^*_\infty[F;Y]$ is the Chevalley–Eilenberg minimal differential calculus over a real graded commutative ring $\mathcal{S}^0_\infty[F;Y]$. $\Box$

Let $\vartheta\rfloor\phi$, $\vartheta\in\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$, $\phi\in\mathcal{S}^1_\infty[F;Y]$, denote the corresponding ***interior product*** in accordance with Theorem 2.1. Extended to a DBGR $\mathcal{S}^*_\infty[F;Y]$, it obeys the rule

$$\vartheta\rfloor(\phi\wedge\sigma)=(\vartheta\rfloor\phi)\wedge\sigma+(-1)^{|\phi|+[\phi][\vartheta]}\phi\wedge(\vartheta\rfloor\sigma),\qquad \phi,\sigma\in\mathcal{S}^*_\infty[F;Y].$$

Restricted to a coordinate chart of $J^\infty Y$, a DBGR $\mathcal{S}^*_\infty[F;Y]$ is a free $\mathcal{S}^0_\infty[F;Y]$-module generated by graded one-forms dx^λ and θ^A_Λ, $[\theta^A_\Lambda]=[A]$. Due to the isomorphism stated in Theorem 2.1, any graded derivation $\vartheta\in\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ reads

$$\vartheta=\vartheta^\lambda\partial_\lambda+\vartheta^A\partial_A+\sum_{0<|\Lambda|}\vartheta^A_\Lambda\partial^\Lambda_A, \tag{2.1}$$

where the graded derivations ∂^Λ_A, $[\partial^\Lambda_A]=[A]$, obey the relations

$$\partial^\Lambda_A(s^B_\Sigma)=\partial^\Lambda_A\rfloor ds^B_\Sigma=\delta^B_A\delta^\Lambda_\Sigma.$$

Every graded derivation ϑ (2.1) of a graded commutative ring $\mathcal{S}^0_\infty[F;Y]$ yields a graded derivation (called the ***graded Lie derivative***) $\mathbf{L}_\vartheta$ of a DBGA $\mathcal{S}^*_\infty[F;Y]$ given by the relations

$$\mathbf{L}_\vartheta\phi = \vartheta\rfloor d\phi + d(\vartheta\rfloor\phi), \qquad \phi\in\mathcal{S}^*_\infty[F;Y],$$
$$\mathbf{L}_\vartheta(\phi\wedge\sigma) = \mathbf{L}_\vartheta(\phi)\wedge\sigma + (-1)^{[\vartheta][\phi]}\phi\wedge\mathbf{L}_\vartheta(\sigma).$$

The graded derivation ϑ (2.1) is called ***contact*** if the graded Lie derivative $\mathbf{L}_\vartheta$ preserves the ideal of contact graded forms of the DBGA $\mathcal{S}^*_\infty[F;Y]$ generated by contact graded one-forms θ^A.

THEOREM 2.3: With respect to the local generating basis (s^A) for a DBGA $\mathcal{S}^*_\infty[F;Y]$, any its ***contact graded derivation*** takes a form

$$\vartheta = \vartheta_H + \vartheta_V = \upsilon^\lambda d_\lambda + [\upsilon^A\partial_A + \sum_{|\Lambda|>0} d_\Lambda(\upsilon^A - s^A_\mu\upsilon^\mu)\partial_A^\Lambda], \tag{2.2}$$

where ϑ_H and ϑ_V denotes the horizontal and vertical parts of ϑ. □

A glance at the expression (2.2) shows that a contact graded derivation ϑ is the ***infinite order jet prolongation*** $\vartheta = J^\infty\upsilon$ of its restriction

$$\upsilon = \upsilon^\lambda\partial_\lambda + \upsilon^A\partial_A = \upsilon_H + \upsilon_V = \upsilon^\lambda d_\lambda + (u^A\partial_A - s^A_\lambda\partial^\lambda_A) \tag{2.3}$$

to a graded commutative ring $\mathcal{S}^0[F;Y]$. We call υ (2.3) the ***generalized graded vector field*** on a graded manifold $(Y,\mathfrak{A}_F)$. This fails to be a graded vector field on $(Y,\mathfrak{A}_F)$ because its component depends on jets of elements of the local generating basis for $(Y,\mathfrak{A}_F)$ in general. At the same time, any graded vector field u on $(Y,\mathfrak{A}_F)$ is the generalized graded vector field (2.3) generating the contact graded derivation $J^\infty u$.

In particular, the ***vertical contact graded derivation*** (2.3) reads

$$\vartheta = \upsilon^A \partial_A + \sum_{|\Lambda|>0} d_\Lambda \upsilon^A \partial_A^\Lambda. \qquad (2.4)$$

THEOREM 2.4: Any vertical contact graded derivation (2.4) obeys the relations

$$\vartheta\rfloor d_H\phi = -d_H(\vartheta\rfloor\phi), \qquad \mathbf{L}_\vartheta(d_H\phi) = d_H(\mathbf{L}_\vartheta\phi), \qquad \phi\in\mathcal{S}^*_\infty[F;Y].$$

□

The vertical contact graded derivation ϑ (2.4) is said to be ***nilpotent*** if

$$\mathbf{L}_\vartheta(\mathbf{L}_\vartheta\phi) = \sum_{0\leq|\Sigma|,0\leq|\Lambda|} (\upsilon^B_\Sigma \partial^\Sigma_B(\upsilon^A_\Lambda)\partial^\Lambda_A + (-1)^{[s^B][\upsilon^A]}\upsilon^B_\Sigma \upsilon^A_\Lambda \partial^\Sigma_B \partial^\Lambda_A)\phi = 0$$

for any horizontal graded form $\phi\in\mathcal{S}^{0,*}_\infty[F,Y]$.

THEOREM 2.5: The vertical contact graded derivation (2.4) is nilpotent only if it is odd. □

Remark 2.1: If there is no danger of confusion, the common symbol υ further stands for a generalized graded vector field υ (2.3), the contact graded derivation $\vartheta = J^\infty\upsilon$ determined by υ, and the Lie derivative $\mathbf{L}_\vartheta$. We call all these operators, in brief, a ***graded derivation*** of the structure algebra of a graded Lagrangian system. □

Remark 2.2: For the sake of convenience, ***right graded derivations*** $\overleftarrow{\upsilon} = \partial_A \upsilon^A$ also are considered. They act on graded differential forms ϕ on the right by the rules

$$\overleftarrow{\upsilon}(\phi) = d\phi\lfloor\overleftarrow{\upsilon} + d(\phi\lfloor\overleftarrow{\upsilon}), \qquad \theta_{\Lambda A}\lfloor\partial^{\Sigma B} = \delta^A_B\delta^\Sigma_\Lambda,$$
$$\overleftarrow{\upsilon}(\phi\wedge\phi') = (-1)^{[\phi']}\overleftarrow{\upsilon}(\phi)\wedge\phi' + \phi\wedge\overleftarrow{\upsilon}(\phi').$$

□

DEFINITION 2.1: Let $(\mathcal{S}^*_\infty[F;Y], L)$ be a graded Lagrangian system. A generalized graded vector field υ is called the ***supersymmetry*** (or, simply, ***symmetry***) of a graded Lagrangian L if a graded Lie derivative $\mathbf{L}_\vartheta L$ of L along the contact graded derivation $\vartheta = J^\infty\upsilon$ is d_H-exact, i.e., $\mathbf{L}_\vartheta L = d_H\sigma$. □

A corollary of the graded variational formula (1.31) is the ***graded first variational formula*** for a graded Lagrangian.

THEOREM 2.6: The graded Lie derivative of a graded Lagrangian along any contact graded derivation ϑ fulfils the graded first variational formula

$$\mathbf{L}_\vartheta L = \upsilon_V\rfloor\delta L + d_H(h_0(\vartheta\rfloor\Xi_L)) + d_V(\upsilon_H\rfloor\omega)\mathcal{L}, \tag{2.5}$$

where Ξ_L is a Lepage equivalent of a graded Lagrangian L. □

A glance at the expression (2.5) shows the following.

THEOREM 2.7: (i) A generalized graded vector field υ is a symmetry only if it is projected onto X.

(ii) Any projectable generalized graded vector field is a symmetry of a variationally trivial graded Lagrangian.

(iii) A generalized graded vector field υ is a symmetry if and only if its vertical part υ_V is well.

(iv) It is a symmetry if and only if the graded density $\upsilon_V\rfloor\delta L$ is d_H-exact. □

Symmetries of a graded Lagrangian L constitute a real vector subspace $\mathcal{SG}_L$ of the ***graded derivation module*** $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$. By virtue of item (ii) of Theorem 2.7, the Lie superbracket

$$\mathbf{L}_{[\vartheta,\vartheta']} = [\mathbf{L}_\vartheta, \mathbf{L}_{\vartheta'}]$$

of symmetries is a symmetry, and their vector space is a real Lie superalgebra.

An immediate corollary of the graded first variational formula (2.5) is the ***first Noether theorem***.

THEOREM 2.8: If the generalized graded vector field υ is a symmetry of a graded Lagrangian L, the first variational formula (2.5) leads to a ***weak conservation law***

$$0 \approx -d_H(-h_0(\vartheta\rfloor\Xi_L) + \sigma) \tag{2.6}$$

of the ***symmetry current***

$$\mathcal{J}_\upsilon = \mathcal{J}^\mu_\vartheta\omega_\mu = -h_0(\vartheta\rfloor\Xi_L) + \sigma \tag{2.7}$$

on-shell. □

3 Gauge symmetries

Treating gauge symmetries of Lagrangian theory, one traditionally is based on gauge theory of principal connections on principal bundles.

This notion of gauge symmetries has been generalized to Lagrangian theory on an arbitrary fibre bundle $Y \to X$. Gauge symmetry is defined as ***a differential operator*** on sections of some vector bundle $E \to X$ as gauge parameters with values in a space of symmetries of a Lagrangian L.

Let us generalize this construction of gauge symmetries to Lagrangian theory on graded bundles.

• Let $(\mathcal{S}^*_\infty[F;Y], L)$ be a graded Lagrangian system on a graded bundle $(X, Y, \mathfrak{A}_F)$ modelled over a composite bundle $F \to Y \to X$, and let (s^A) be its local generating basis. To define gauge symmetries of this Lagrangian system, one extends a graded manifold $(X, Y, \mathfrak{A}_F)$ as follows. Let us consider a composite bundle

$$E = E_1 \underset{X}{\oplus} E_0 \to E_0 \to X$$

and a graded bundle $(X, E^0, \mathfrak{A}_E)$ modelled over it, and let (c^r) be its local generating basis. Then we define the product $(X, E^0 \underset{X}{\times} Y, \mathfrak{A}_{E \underset{X}{\times} F})$ of graded bundles $(X, Y, \mathfrak{A}_F)$ and $(X, E^0, \mathfrak{A}_E)$ which is modelled over the product

$$F \underset{X}{\times} E \to Y \underset{X}{\times} E_0 \to X$$

of composite bundles E and F, and which possesses a generating basis (s^A, c^r).

• Let us consider the corresponding DBGR $\mathcal{S}^*_\infty[E\underset{X}{\times}F;E_0\underset{X}{\times}Y]$ together with the monomorphisms of DBGRs

$$\mathcal{S}^*_\infty[F;Y]\to\mathcal{S}^*_\infty[E\underset{X}{\times}F;E^0\underset{X}{\times}Y],\qquad \mathcal{S}^*_\infty[E;E^0]\to\mathcal{S}^*_\infty[E\underset{X}{\times}F;E^0\underset{X}{\times}Y].$$

Given a graded Lagrangian $L\in\mathcal{S}^{0,n}_\infty[F;Y]$, let us define its pull-back

$$L\in\mathcal{S}^{0,n}_\infty[F;Y]\subset\mathcal{S}^*_\infty[E\underset{X}{\times}F;E^0\underset{X}{\times}Y], \tag{3.1}$$

and consider an extended Lagrangian system

$$(\mathcal{S}^*_\infty[E\underset{X}{\times}F;E^0\underset{X}{\times}Y],L) \tag{3.2}$$

provided with the local generating basis (s^A,c^r).

DEFINITION 3.1: A ***gauge transformation*** of the Lagrangian L (3.1) is defined to be a contact graded derivation ϑ of the ring $\mathcal{S}^0_\infty[E\underset{X}{\times}F;E^0\underset{X}{\times}Y]$ such that a derivation ϑ equals zero on a subring

$$\mathcal{S}^0_\infty[E;E^0]\subset\mathcal{S}^0_\infty[E\underset{X}{\times}F;E^0\underset{X}{\times}Y].$$

A gauge transformation ϑ is called the ***gauge symmetry*** if it is a symmetry of the Lagrangian L (3.1). □

• In view of the condition in Definition 3.1, the variables c^r of the extended Lagrangian system (3.2) are treated as ***graded gauge parameters*** of a gauge symmetry ϑ. Furthermore, we additionally assume that a gauge symmetry is linear in gauge parameters c^r and their jets c^r_Λ. Then its generalized graded vector field υ reads

$$\upsilon=\left(\sum_{0\le|\Lambda|\le m}\upsilon^{\lambda\Lambda}_r(x^\mu)c^r_\Lambda\right)\partial_\lambda+\left(\sum_{0\le|\Lambda|\le m}\upsilon^{A\Lambda}_r(x^\mu,s^B_\Sigma)c^r_\Lambda\right)\partial_A. \tag{3.3}$$

Remark 3.1: In a general setting, one can define gauge symmetries which are non-linear in gauge parameters. However, the direct second Noether theorem is not relevant for them because Euler–Lagrange operator in this case satisfies the identities depending on gauge parameters. □

• If a gauge transformation υ (3.3) is a symmetry, it defines a weak conservation law in accordance with the first Noether theorem. The peculiarity of this conservation law is that the symmetry current $\mathcal{J}_\upsilon$ (3.4) is the total differential on-shell, i.e., it is reduced to a superpotential.

THEOREM 3.1: If u (3.3) is a gauge symmetry of a graded Lagrangian L, the corresponding symmetry current $\mathcal{J}_u$ (up to a d_H-closed term) takes a form

$$\mathcal{J}_u = W + d_H U = (W^\mu + d_\nu U^{\nu\mu})\omega_\mu, \tag{3.4}$$

where a term W vanishes on-shell and $U^{\nu\mu} = -U^{\mu\nu}$ is a ***superpotential*** which reads

$$\begin{aligned}
\mathcal{J}^\mu_u = (& \sum_{1<k\leq N} u^{i\ \ \mu\mu_k\ldots\mu_N}_{V r} c^r_{\mu_k\ldots\mu_N} + u^{A\mu}_{V r} c^r)\mathcal{E}_A - \\
&(\sum_{1<k\leq M} d_\nu J^{(\nu\mu)\mu_k\ldots\mu_M} c^r_{\mu_k\ldots\mu_M} + d_\nu J^{(\nu\mu)}_r c^r) - \\
&d_\nu(\sum_{1<k\leq M} J^{[\nu\mu]\mu_k\ldots\mu_M} c^r_{\mu_k\ldots\mu_M} + J^{[\nu\mu]}_r c^r).
\end{aligned}$$

□

G.Sardanashvily, Gauge conservation laws in a general setting. Superpotential, **Int. J. Geom. Methods Mod. Phys. 6**(2009) 1047-1056

One sometimes treats Theorem 3.1 as the ***third Noether theorem***.

4 Noether identities

We follow the general analysis of Noether identities (NI) and higher-stage NI of differential operators on fibre bundles when ***trivial and non-trivial NI are represented by boundaries and cycles of a chain complex***.

It should be noted that the notion of higher-stage Noether identities came from that of ***reducible constraints***. The Koszul–Tate complex of Noether identities has been invented similarly to that of constraints under the ***regularity condition*** that Noether identities are locally separated into the independent and dependent ones.

J.Fisch, M.Henneaux, Homological perturbation theory and algebraic structure of the antifield-antibracket formalism for gauge theories, **Commun. Math. Phys. 128** (1990) 627-640.

G.Barnich, F.Brandt, M.Henneaux, Local BRST cohomology in gauge theories. **Phys. Rep. 338** (2000) 439-569.

This condition is relevant for constraints, defined by ***a finite set of functions*** which the inverse mapping theorem is applied to. However, ***Noether identities of differential operators***, unlike constraints, are ***differential equations***. They are given by an infinite set of functions on a Fréchet manifold of infinite order jets where the inverse mapping theorem fails to be valid. Therefore, the regularity condition for the Koszul–Tate complex of constraints is replaced with a certain ***homology regularity condition***.

Let $(\mathcal{S}^*_\infty[F;Y],L)$ be a graded Lagrangian system on a graded bundle $(X,Y,\mathfrak{A}_F)$, modelled over a composite bundle $F\to Y\to X$.

• One can associate to a graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y],L)$ the chain complex (4.1) whose one-boundaries vanish on the shell $\delta L=0$. Let us consider the density-dual

$$\overline{VF}=V^*F\underset{F}{\otimes}\overset{n}{\wedge}T^*X\to F$$

of the vertical tangent bundle $VF\to F$, and let us enlarge an original DBGR $\mathcal{S}^*_\infty[F;Y]$ with the ***generating basis*** (s^A) to $\mathcal{P}^*_\infty[\overline{VF};Y]$ with the ***generating basis*** $(s^A,\overline{s}_A)$, $[\overline{s}_A]=[A]+1$. Following the terminology of BRST theory, we call its elements $\overline{s}_A$ the ***antifields*** of ***antifield number*** $\mathrm{Ant}[\overline{s}_A]=1$. A DBGR $\mathcal{P}^*_\infty[\overline{VF};Y]$ is endowed with the nilpotent right graded derivation $\overline{\delta}=\overleftarrow{\partial}^A\mathcal{E}_A$, where $\mathcal{E}_A=\delta_A L$ are the variational derivatives. Then we have a ***chain complex***

$$0\leftarrow\mathrm{Im}\,\overline{\delta}\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{VF};Y]_2 \qquad (4.1)$$

of graded densities of antifield number ≤ 2. Its one-boundaries $\overline{\delta}\Phi$, $\Phi\in\mathcal{P}^{0,n}_\infty[\overline{VF};Y]_2$, by very definition, vanish on-shell. Any one-cycle Φ of the complex (4.1) is a differential operator on a bundle $\overline{VF}$ such that its kernel contains the graded Euler–Lagrange operator δL, i.e.,

$$\overline{\delta}\Phi=0, \qquad \sum_{0\leq|\Lambda|}\Phi^{A,\Lambda}d_\Lambda\mathcal{E}_A\omega=0. \qquad (4.2)$$

Refereing to a notion of NI of a differential operator, we say that one-cycles Φ define the ***Noether identities*** (4.2) of an Euler–Lagrange operator δL.

• One-chains Φ are necessarily NI if they are boundaries. Therefore, these NI are called ***trivial***. They are of the form

$$\Phi = \sum_{0\leq|\Lambda|,|\Sigma|} T^{(A\Lambda)(B\Sigma)} d_\Sigma \mathcal{E}_B \overline{s}_{\Lambda A}\omega, \qquad T^{(A\Lambda)(B\Sigma)} = -(-1)^{[A][B]} T^{(B\Sigma)(A\Lambda)}.$$

Accordingly, ***non-trivial NI*** modulo trivial ones are associated to elements of the ***first homology*** $H_1(\overline{\delta})$ of the complex (4.1).

A Lagrangian L is called ***degenerate*** if there exist non-trivial NI.

• Non-trivial NI can obey ***first-stage*** NI. To describe them, let us assume that a module $H_1(\overline{\delta})$ is finitely generated. Namely, there exists a graded projective $C^\infty(X)$-module $\mathcal{C}_{(0)} \subset H_1(\overline{\delta})$ of finite rank possessing a local basis $\{\Delta_r\omega\}$ such that any element $\Phi \in H_1(\overline{\delta})$ factorizes as

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r,\Xi} d_\Xi \Delta_r \omega, \qquad \Phi^{r,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{4.3}$$

through elements $\Delta_r\omega$ of $\mathcal{C}_{(0)}$. Thus, all non-trivial NI (4.2) result from the NI

$$\overline{\delta}\Delta_r = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \tag{4.4}$$

called the ***complete NI***. Then the complex (4.1) can be extended to the chain complex (4.5) with a ***boundary operator whose nilpotency is equivalent to the complete NI*** (4.4) as follows.

By virtue of the Serre–Swan theorem, a graded module $\mathcal{C}_{(0)}$ is isomorphic to that of sections of the density-dual $\overline{E}_0$ of some graded vector bundle $E_0 \to X$. Let us enlarge $\mathcal{P}^*_\infty[\overline{VF}; Y]$ to a DBGR

$$\overline{\mathcal{P}}^*_\infty\{0\} = \mathcal{P}^*_\infty[\overline{VF} \underset{X}{\times} \overline{E}_0; Y]$$

with the ***generating basis*** $(s^A, \overline{s}_A, \overline{c}_r)$ where $\overline{c}_r$ are antifields such that $[\overline{c}_r] = [\Delta_r] + 1$ and $\mathrm{Ant}[\overline{c}_r] = 2$. This DBGR admits a derivation

$$\delta_0 = \overline{\delta} + \overleftarrow{\partial}^r \Delta_r$$

which is nilpotent if and only if the complete NI (4.4) hold. Then δ_0 is a ***boundary operator*** of a chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \overset{\overline{\delta}}{\longleftarrow} \mathcal{P}^{0,n}_\infty[\overline{VF}; Y]_1 \overset{\delta_0}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \overset{\delta_0}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_3 \qquad (4.5)$$

of graded densities of antifield number ≤ 3. One can show that its homology $H_1(\delta_0)$ vanishes, i.e., the complex (4.5) is one-exact.

• Let us consider the second homology $H_2(\delta_0)$ of the complex (4.5). Its two-cycles define the ***first-stage NI***

$$\delta_0 \Phi = 0, \qquad \sum_{0 \leq |\Lambda|} G^{r,\Lambda} d_\Lambda \Delta_r \omega = -\overline{\delta} H. \qquad (4.6)$$

Conversely, let the equality (4.6) hold. Then it is a cycle condition. The first-stage NI (4.6) are ***trivial*** either if the two-cycle Φ is a δ_0-boundary or its summand G vanishes on-shell. Therefore, ***non-trivial first-stage NI fails to exhaust the second homology*** $H_2(\delta_0)$ of the complex (4.5) in general. One can show that non-trivial first-stage NI modulo trivial ones are identified with elements of $H_2(\delta_0)$ iff any $\overline{\delta}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2$ is a δ_0-boundary.

A degenerate Lagrangian is called ***reducible*** if it admits non-trivial first-stage NI.

• Non-trivial first-stage NI can obey ***second-stage*** NI, and so on. Iterating the arguments, we say that a degenerate graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y],L)$ is N***-stage reducible*** if it admits non-trivial N-stage NI, but no non-trivial $(N+1)$-stage ones. It is characterized as follows.

(i) There are graded vector bundles $E_0,\ldots,E_N$ over X, and $\mathcal{P}^*_\infty[\overline{VF};Y]$ is enlarged to a DBGR

$$\overline{\mathcal{P}}^*_\infty\{N\}=\mathcal{P}^*_\infty[\overline{VF}\underset{X}{\times}\overline{E}_0\underset{X}{\times}\cdots\underset{X}{\times}\overline{E}_N;Y] \tag{4.7}$$

with a local ***generating basis*** $(s^A,\overline{s}_A,\overline{c}_r,\overline{c}_{r_1},\ldots,\overline{c}_{r_N})$ where $\overline{c}_{r_k}$ are k***-stage antifields*** of antifield number $\mathrm{Ant}[\overline{c}_{r_k}]=k+2$.

(ii) The DBGR (4.7) is provided with a nilpotent right graded derivation

$$\delta_{\rm KT}=\delta_N=\overline{\delta}+\sum_{0\leq|\Lambda|}\overleftarrow{\partial}^r\Delta^{A,\Lambda}_r\overline{s}_{\Lambda A}+\sum_{1\leq k\leq N}\overleftarrow{\partial}^{r_k}\Delta_{r_k}, \tag{4.8}$$

$$\Delta_{r_k}\omega=\sum_{0\leq|\Lambda|}\Delta^{r_{k-1},\Lambda}_{r_k}\overline{c}_{\Lambda r_{k-1}}\omega+ \tag{4.9}$$

$$\sum_{0\leq|\Sigma|,|\Xi|}(h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A}+...)\omega\in\overline{\mathcal{P}}^{0,n}_\infty\{k-1\}_{k+1},$$

of antifield number -1. The index $k=-1$ here stands for $\overline{s}_A$. The nilpotent derivation $\delta_{\rm KT}$ (4.8) is called the ***Koszul–Tate (KT) operator***.

(iii) With this **KT** operator, a module $\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{\leq N+3}$ of densities of antifield number $\leq (N+3)$ is split into the exact ***Koszul–Tate (KT) chain complex***

$$0\leftarrow \mathrm{Im}\,\overline{\delta} \overset{\overline{\delta}}{\longleftarrow} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \overset{\delta_0}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \overset{\delta_1}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{1\}_3 \cdots \tag{4.10}$$
$$\overset{\delta_{N-1}}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+1} \overset{\delta_{\mathrm{KT}}}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2} \overset{\delta_{\mathrm{KT}}}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+3}$$

which satisfies the following ***homology regularity condition***.

CONDITION 4.1: Any $\delta_{k<N}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{k\}_{k+3} \subset \overline{\mathcal{P}}^{0,n}_\infty\{k+1\}_{k+3}$ is a δ_{k+1}-boundary.

(iv) The nilpotentness of the **KT** operator (4.8) is ***equivalent*** to complete non-trivial NI (4.4) and complete non-trivial $(k\leq N)$-stage NI

$$\sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k} d_\Lambda(\sum_{0\leq|\Sigma|} \Delta^{r_{k-2},\Sigma}_{r_{k-1}} \overline{c}_{\Sigma r_{k-2}}) = -\overline{\delta}(\sum_{0\leq|\Sigma|,|\Xi|} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A}). \tag{4.11}$$

It may happen that a graded Lagrangian system possesses non-trivial NI of any stage. However, we restrict our consideration to N-reducible Lagrangians for a finite integer N. In this case, the **KT** operator (4.8) and the gauge operator (5.4) below contain a finite number of terms.

It also should be emphasized that, in order to describe a hierarchy of NI, we suppose that NI and higher-stage NI are finitely generated (i.e., they form projective modules of finite rank) and that homology Condition 3.1 is satisfied. These are not true for any Lagrangian.

5 Second Noether theorems

Different variants of the second Noether theorem have been suggested in order to relate reducible NI and gauge symmetries.

G.Barnich, F.Brandt, M.Henneaux, Local BRST cohomology in gauge theories, **Phys. Rep. 338** (2000) 439.

R.Fulp, T.Lada, J. Stasheff, Noether variational Theorem II and the BV formalism, **Rend. Circ. Mat. Palermo (2) Suppl.** No. 71 (2003) 115.

The ***extended inverse second Noether theorem***, that we formulate in ***homology terms***, associates to the **KT** complex (4.10) of non-trivial NI the cochain sequence (5.3) with the ascent operator **u** (5.4) whose components are gauge and higher-stage gauge symmetries of a Lagrangian system.

Given a DBGR $\overline{\mathcal{P}}^*_\infty\{N\}$ (4.7), let us consider the DBGRs

$$P^*_\infty\{N\} = P^*_\infty[F \underset{X}{\times} E_0 \underset{X}{\times} \cdots \underset{X}{\times} E_N; Y], \tag{5.1}$$

possessing the ***generating bases*** $(s^A, c^r, c^{r_1}, \ldots, c^{r_N})$, $[c^{r_k}] = [\overline{c}_{r_k}] + 1$, and the DBGRs

$$\mathcal{P}^*_\infty\{N\} = \mathcal{P}^*_\infty[\overline{VF} \underset{X}{\times} E_0 \underset{X}{\times} \cdots \underset{X}{\times} E_N \underset{X}{\times} \overline{E}_0 \underset{X}{\times} \cdots \underset{X}{\times} \overline{E}_N; Y] \tag{5.2}$$

with the ***generating bases*** $(s^A, \overline{s}_A, c^r, c^{r_1}, \ldots, c^{r_N}, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N})$. Their elements c^{r_k} are called the ***k-stage ghosts*** of ghost number $\mathrm{gh}[c^{r_k}] = k+1$ and antifield number $\mathrm{Ant}[c^{r_k}] = -(k+1)$. The **KT** operator δ_{KT} (4.8) naturally is extended to a graded derivation of a DBGR $\mathcal{P}^*_\infty\{N\}$.

THEOREM 5.1: Given the **KT** complex (4.10), a module of graded densities $P^{0,n}_\infty\{N\}$ is decomposed into a cochain sequence

$$0 \to \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\mathbf{u}} P^{0,n}_\infty\{N\}^1 \xrightarrow{\mathbf{u}} P^{0,n}_\infty\{N\}^2 \xrightarrow{\mathbf{u}} \cdots \qquad (5.3)$$

graded in ghost number. Its ***ascent operator***

$$\mathbf{u} = u + u^{(1)} + \cdots + u^{(N)} = u^A\frac{\partial}{\partial s^A} + u^r\frac{\partial}{\partial c^r} + \cdots + u^{r_{N-1}}\frac{\partial}{\partial c^{r_{N-1}}}, \qquad (5.4)$$

is an odd graded derivation of ghost number 1 where

$$u = u^A\frac{\partial}{\partial s^A}, \qquad u^A = \sum_{0\leq|\Lambda|} c^r_\Lambda \eta(\Delta^A_r)^\Lambda, \qquad (5.5)$$

$$\frac{\overleftarrow{\delta}(c^r\Delta_r)}{\delta\overline{s}_A}\mathcal{E}_A\omega = u^A\mathcal{E}_A\omega = d_H\sigma_0, \qquad (5.6)$$

is a symmetry of a graded Lagrangian L and the derivations

$$u^{(k)} = u^{r_{k-1}}\frac{\partial}{\partial c^{r_{k-1}}} = \sum_{0\leq|\Lambda|} c^{r_k}_\Lambda \eta(\Delta^{r_{k-1}}_{r_k})^\Lambda \frac{\partial}{\partial c^{r_{k-1}}}, \qquad k = 1,\ldots,N, \qquad (5.7)$$

obey the relations

$$\sum c^{r_k} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k}\overline{c}_{\Sigma r_{k-2}} d_\Xi \mathcal{E}_A\omega + u^{r_{k-1}}\sum \Delta^{r_{k-2},\Xi}_{r_{k-1}}\overline{c}_{\Xi r_{k-2}}\omega = d_H\sigma'_k. \qquad (5.8)$$

□

A glance at the symmetry u (5.5) shows that it is a derivation of a ring $P^0_\infty[0]$ which satisfies Definition 3.1 of gauge transformations. Consequently, u (5.5) is a ***gauge symmetry*** of a graded Lagrangian L associated to the complete non-trivial NI (4.4). Therefore, it is a ***non-trivial gauge symmetry***.

Turn now to the relation (5.8). The variational derivative of both its sides with respect to $\overline{c}_{r_{k-2}}$ leads to the equality

$$\sum d_\Sigma u^{r_{k-1}} \frac{\partial}{\partial c_\Sigma^{r_{k-1}}} u^{r_{k-2}} = \overline{\delta}(\alpha^{r_{k-2}}), \tag{5.9}$$
$$\alpha^{r_{k-2}} = -\sum \eta(h_{r_k}^{(r_{k-2})(A,\Xi)})^\Sigma d_\Sigma (c^{r_k} \overline{s}_{\Xi A}),$$

For $k = 1$, it takes a form

$$\sum d_\Sigma u^r \partial_r^\Sigma u^A = \overline{\delta}(\alpha^A)$$

of a ***first-stage gauge symmetry condition*** on-shell which the non-trivial gauge symmetry u (5.5) satisfies. Therefore, one can treat the odd graded derivation

$$u^{(1)} = u^r \partial_r, \qquad u^r = \sum c_\Lambda^{r_1} \eta(\Delta_{r_1}^r)^\Lambda,$$

as a ***first-stage gauge symmetry*** associated to a complete first-stage NI

$$\sum \Delta_{r_1}^{r,\Lambda} d_\Lambda (\sum \Delta_r^{A,\Sigma} \overline{s}_{\Sigma A}) = -\overline{\delta}(\sum h_{r_1}^{(B,\Sigma)(A,\Xi)} \overline{s}_{\Sigma B} \overline{s}_{\Xi A}).$$

Iterating the arguments, one comes to the relation (5.9) providing a ***k-stage gauge symmetry condition*** which is associated to the complete non-trivial k-stage NI (4.11). The odd graded derivation $u_{(k)}$ (5.7) is called the ***k-stage gauge symmetry***.

Thus, components of the ascent operator **u** (5.4) in Theorem 5.1 are non-trivial gauge and higher-stage gauge symmetries. Therefore, we agree to call it ***the gauge operator***.

The correspondence of gauge and higher-stage gauge symmetries to NI and higher-stage NI in Theorem 5.1 is unique due to the following ***direct second Noether theorem***.

THEOREM 5.2:

• If u (5.5) is a gauge symmetry, the variational derivative of the d_H-exact density $u^A\mathcal{E}_A\omega$ (5.6) with respect to ghosts c^r leads to the equality

$$\delta_r(u^A\mathcal{E}_A\omega) = \sum(-1)^{|\Lambda|}d_\Lambda[u_r^{A\Lambda}\mathcal{E}_A] = \\ \sum(-1)^{|\Lambda|}d_\Lambda(\eta(\Delta_r^A)^\Lambda\mathcal{E}_A) = \sum(-1)^{|\Lambda|}\eta(\eta(\Delta_r^A))^\Lambda d_\Lambda\mathcal{E}_A = 0, \tag{5.10}$$

which reproduces the complete NI (4.4).

• Given the k-stage gauge symmetry condition (5.9), the variational derivative of the equality (5.8) with respect to ghosts c^{r_k} leads to the equality, reproducing the k-stage NI (4.11). □

6 Lagrangian BRST theory

In contrast with the **KT** operator (4.8), the gauge operator $\mathbf{u}$ (5.3) ***need not be nilpotent***.

Let us study its extension to a nilpotent graded derivation

$$\mathbf{b} = \mathbf{u} + \gamma = \mathbf{u} + \sum_{1\leq k\leq N+1} \gamma^{(k)} = \mathbf{u} + \sum_{1\leq k\leq N+1} \gamma^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}} \tag{6.1}$$
$$= \left(u^A \frac{\partial}{\partial s^A} + \gamma^r \frac{\partial}{\partial c^r}\right) + \sum_{0\leq k\leq N-1} \left(u^{r_k} \frac{\partial}{\partial c^{r_k}} + \gamma^{r_{k+1}} \frac{\partial}{\partial c^{r_{k+1}}}\right)$$

of ghost number 1 by means of antifield-free terms $\gamma^{(k)}$ of higher polynomial degree in ghosts c^{r_i} and their jets $c^{r_i}_\Lambda$, $0 \leq i < k$.

One calls $\mathbf{b}$ (6.1) ***the BRST operator***, where k-stage gauge symmetries are extended to k-stage ***BRST transformations*** acting both on $(k-1)$-stage and k-stage ghosts.

If a BRST operator exists, the cochain sequence (5.3) is brought into a ***BRST complex***

$$0 \to \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\mathbf{b}} P^{0,n}_\infty\{N\}^1 \xrightarrow{\mathbf{b}} P^{0,n}_\infty\{N\}^2 \xrightarrow{\mathbf{b}} \cdots . \tag{6.2}$$

One can show the following.

• The gauge operator (5.3) admits the BRST extension (6.1) only if the gauge symmetry conditions (5.8) and the higher-stage NI (4.11) are satisfied off-shell.

• Gauge symmetries need not form an algebra. Therefore, we replace the notion of the algebra of gauge symmetries with some conditions on the gauge operator. Gauge symmetries are said to be ***algebraically closed*** if the gauge operator admits the nilpotent BRST extension (6.1).

J.Gomis, J.París, S.Samuel, Antibracket, antifields and gauge theory quantization, **Phys. Rep. 295** (1995) 1.

• A nilpotent BRST operator provides a ***BRST extension*** of an original Lagrangian system by means of graded antifields and ghosts as follows.

The DBGR $P^*_\infty\{N\}$ (5.2) is a particular field-antifield theory of the following type. Let us consider a pull-back composite bundle

$$W = Z \underset{X}{\times} Z' \to Z \to X$$

where $Z' \to X$ is a vector bundle. Let us regard it as an odd graded vector bundle over Z. The density-dual $\overline{VW}$ of the vertical tangent bundle VW of $W \to X$ is a graded vector bundle

$$\overline{VW} = ((\overline{Z}' \underset{Z}{\oplus} V^*Z) \underset{Z}{\otimes} \overset{n}{\wedge} T^*X) \underset{Y}{\oplus} Z'$$

over Z. Let us consider the DBGR $\mathcal{P}^*_\infty[\overline{VW}; Z]$ with the local generating basis $(z^a, \overline{z}_a)$, $[\overline{z}_a] = [z^a] + 1$. Its elements z^a and $\overline{z}_a$ are called fields and antifields, respectively. Graded densities of this DBGR are endowed with the antibracket

$$\{\mathfrak{L}\omega, \mathfrak{L}'\omega\} = \left[\frac{\overleftarrow{\delta}\mathfrak{L}}{\delta\overline{z}_a}\frac{\delta\mathfrak{L}'}{\delta z^a} + (-1)^{[\mathfrak{L}']([\mathfrak{L}']+1)}\frac{\overleftarrow{\delta}\mathfrak{L}'}{\delta\overline{z}_a}\frac{\delta\mathfrak{L}}{\delta z^a}\right]\omega. \tag{6.3}$$

Then one associates to any even Lagrangian $\mathfrak{L}\omega$ the odd vertical graded derivations

$$\upsilon_{\mathfrak{L}} = \overleftarrow{\mathcal{E}}^a \partial_a = \frac{\overleftarrow{\delta}\mathfrak{L}}{\delta \overline{z}_a}\frac{\partial}{\partial z^a}, \qquad \overline{\upsilon}_{\mathfrak{L}} = \overleftarrow{\partial}^a \mathcal{E}_a = \frac{\overleftarrow{\partial}}{\partial \overline{z}_a}\frac{\delta \mathfrak{L}}{\delta z^a}, \tag{6.4}$$

$$\vartheta_{\mathfrak{L}} = \upsilon_{\mathfrak{L}} + \overline{\upsilon}^l_{\mathfrak{L}} = (-1)^{[a]+1}\left(\frac{\delta \mathfrak{L}}{\delta \overline{z}^a}\frac{\partial}{\partial z_a} + \frac{\delta \mathfrak{L}}{\delta z^a}\frac{\partial}{\partial \overline{z}_a}\right), \tag{6.5}$$

such that $\vartheta_{\mathfrak{L}}(\mathfrak{L}'\omega) = \{\mathfrak{L}\omega, \mathfrak{L}'\omega\}$.

THEOREM 6.1: The following conditions are equivalent.

(i) The antibracket of a Lagrangian $\mathfrak{L}\omega$ is d_H-exact, i.e.,

$$\{\mathfrak{L}\omega, \mathfrak{L}\omega\} = 2\frac{\overleftarrow{\delta}\mathfrak{L}}{\delta \overline{z}_a}\frac{\delta \mathfrak{L}}{\delta z^a}\omega = d_H \sigma. \tag{6.6}$$

(ii) The graded derivation $\vartheta_{\mathfrak{L}}$ (6.5) is nilpotent. $\Box$

The equality (6.6) is called the ***classical master equation***. A solution of the master equation (6.6) is called ***non-trivial*** if both the derivations (6.4) do not vanish.

Being an element of the DBGA $\mathcal{P}^*_\infty\{N\}$ (5.2), an original Lagrangian L obeys the master equation (6.6) and yields the graded derivations $\upsilon_L = 0$, $\overline{\upsilon}_L = \overline{\delta}$ (6.4), i.e., it is a trivial solution of the master equation. Therefore, let us consider its extension

$$L_E = L + L_1 + L_2 + \cdots$$

by means of even densities L_i, $i \geq 2$, of zero antifield number and polynomial degree i in ghosts. Then the following is a corollary of Theorem 6.1.

COROLLARY 6.2: A Lagrangian L is extended to a non-trivial solution L_E of the master equation only if the gauge operator $\mathbf{u}$ (5.3) admits the nilpotent extension ϑ_E (6.5). □

However, one can say something more.

THEOREM 6.3: If the gauge operator $\mathbf{u}$ (5.3) can be extended to the BRST operator $\mathbf{b}$ (6.1), then the master equation has a non-trivial proper solution

$$L_E = L + \mathbf{b}\left(\sum_{0\leq k\leq N} c^{r_{k-1}}\overline{c}_{r_{k-1}}\right)\omega + d_H\sigma, \tag{6.7}$$

such that $\mathbf{b} = \upsilon_E$ is the graded derivation defined by the Lagrangian L_E (6.7). □

The Lagrangian L_E (6.7) is said to be the ***BRST extension*** of an original Lagrangian L.

This extension is a ***preliminary step*** towards quantization of reducible degenerate Lagrangian theories.

7 Applications. Topological BF theory

The most of basic Lagrangian models in field theory and mechanics are ***irreducible degenerate***.

G.Sardanashvily, **Noether's Theorems. Applications in Mechanics and Field Theory** (Springer, 2016).
G.Giachetta, L.Mangiarotti, G.Sardanashvily, **Advanced Classical Field Theory** (World Scientific, 2009).
G.Giachetta, L.Mangiarotti, G.Sardanashvily, **Geometric Formulation of Classical and Quantum Mechanics** (World Scientific, 2010).
G.Sardanashvily, Classical field theory. Advanced mathematical formulation, **Int. J. Geom. Methods Mod. Phys. 5** (2008) 1163-1189; **arXiv: 0811.0331**

• Gauge theory of principal connections on principal bundles (irreducible degenerate Lagrangian system)

• Gauge gravitation theory on natural bundles (irreducible degenerate Lagrangian system whose gauge symmetries are ***general covariant transformations***)

G.Sardanashvily, Gauge gravitation theory. Gravity as a Higgs field, **Int. J. Geom. Methods Mod. Phys. 13** (2016) 1650086

• Topological Chern–Simons gauge theory (irreducible degenerate Lagrangian system, whose gauge symmetries are variational, but ***not exact***)

• Topological BF theory (reducible degenerate Lagrangian system)

G.Sardanashvily, Higher-stage Noether identities and second Noether theorems, **Adv. Math. Phys. 2015** (2015) 127481

• SUSY gauge theory on principal graded bundles (irreducible degenerate graded Lagrangian system)

• Covariant (polysymplectic) Hamiltonian field theory, formulated as particular Lagrangian theory on a phase space

G.Sardanashvily, Polysymplectic Hamiltonian field theory, **arXiv: 1505.01444**

• Lagrangian and Hamiltonian non-autonomous mechanics on fibre bundles over $\mathbb{R}$

G.Sardanashvily, Noether's first theorem in Hamiltonian mechanics, **arXiv: 1510.03760**

• Relativistic mechanics as a Lagrangian theory of one-dimensional submanifolds

G.Sardanashvily, Lagrangian dynamics of submanifolds. Relativistic mechanics, **J. Geom. Mech. 4** (2012) 99-110

We address ***topological BF theory*** of two exterior forms A and B of form degree $|A|+|B|=\dim X-1$ on a smooth manifold X because it exemplifies reducible degenerate Lagrangian theory which satisfies homology regularity Condition 4.1.

D.Birmingham, M.Blau, Topological field theory, **Phys. Rep. 209** (1991) 129.

Its dynamic variables A and B are sections of a fibre bundle

$$Y=\overset{p}{\wedge}T^*X\oplus\overset{q}{\wedge}T^*X, \qquad p+q=n-1>1,$$

coordinated by $(x^\lambda, A_{\mu_1\ldots\mu_p}, B_{\nu_1\ldots\nu_q})$. Without a loss of generality, let q be even and $q\geq p$. The corresponding DGR is $\mathcal{O}^*_\infty Y$.

There are the canonical p- and q-forms

$$A=A_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}, \qquad B=B_{\nu_1\ldots\nu_q}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q}$$

on Y. A Lagrangian of topological BF theory reads

$$L_{\rm BF}=A\wedge d_HB=\epsilon^{\mu_1\cdots\mu_n}A_{\mu_1\ldots\mu_p}d_{\mu_{p+1}}B_{\mu_{p+2}\ldots\mu_n}\omega, \tag{7.1}$$

where ϵ is the Levi–Civita symbol. It is a reduced first order Lagrangian. Its first order Euler–Lagrange operator

$$\delta L=\mathcal{E}_A^{\mu_1\ldots\mu_p}dA_{\mu_1\ldots\mu_p}\wedge\omega+\mathcal{E}_B^{\nu_{p+2}\ldots\nu_n}dB_{\nu_{p+2}\ldots\nu_n}\wedge\omega,$$
$$\mathcal{E}_A^{\mu_1\ldots\mu_p}=\epsilon^{\mu_1\ldots\mu_n}d_{\mu_{p+1}}B_{\mu_{p+2}\ldots\mu_n}, \qquad \mathcal{E}_B^{\mu_{p+2}\ldots\mu_n}=-\epsilon^{\mu_1\ldots\mu_n}d_{\mu_{p+1}}A_{\mu_1\ldots\mu_p},$$

satisfies the Noether identities

$$d_{\mu_1}\mathcal{E}_A^{\mu_1\ldots\mu_p}=0, \qquad d_{\nu_1}\mathcal{E}_B^{\nu_1\ldots\nu_q}=0. \tag{7.2}$$

Given a family of vector bundles

$$E_k = \overset{p-k-1}{\wedge} T^*X \underset{X}{\times} \overset{q-k-1}{\wedge} T^*X, \qquad 0 \leq k < p-1,$$

$$E_k = \mathbb{R} \underset{X}{\times} \overset{q-p}{\wedge} T^*X, \qquad k = p-1,$$

$$E_k = \overset{q-k-1}{\wedge} T^*X, \quad p-1 < k < q-1,$$

$$E_{q-1} = X \times \mathbb{R},$$

let us enlarge an original DGR $\mathcal{O}^*_\infty$ to the DBGR $\mathcal{P}^*_\infty\{q-1\}$ (5.2) which is

$$\mathcal{P}^*_\infty\{q-1\} = \mathcal{P}^*_\infty[\overline{VY} \underset{Y}{\oplus} E_0 \oplus \cdots \underset{Y}{\oplus} E_{q-1} \underset{Y}{\oplus} \overline{E}_0 \underset{Y}{\oplus} \cdots \underset{Y}{\oplus} \overline{E}_{q-1}; Y]. \tag{7.3}$$

It possesses a local ***generating basis***

$$\{A_{\mu_1\ldots\mu_p}, B_{\nu_1\ldots\nu_q}, \varepsilon_{\mu_2\ldots\mu_p}, \ldots, \varepsilon_{\mu_p}, \varepsilon, \xi_{\nu_2\ldots\nu_q}, \ldots, \xi_{\nu_q}, \xi,$$
$$\overline{A}^{\mu_1\ldots\mu_p}, \overline{B}^{\nu_1\ldots\nu_q}, \overline{\varepsilon}^{\mu_2\ldots\mu_p}, \ldots, \overline{\varepsilon}^{\mu_p}, \overline{\varepsilon}, \overline{\xi}^{\nu_2\ldots\nu_q}, \ldots, \overline{\xi}^{\nu_q}, \overline{\xi}\}$$

of Grassmann parity

$$[\varepsilon_{\mu_k\ldots\mu_p}] = [\xi_{\nu_k\ldots\nu_q}] = (k+1)\mathrm{mod}\,2, \qquad [\varepsilon] = p \,\mathrm{mod}\, 2, \qquad [\xi] = 0,$$
$$[\overline{\varepsilon}^{\mu_k\ldots\mu_p}] = [\overline{\xi}^{\nu_k\ldots\nu_q}] = k \,\mathrm{mod}\, 2, \qquad [\overline{\varepsilon}] = (p+1)\mathrm{mod}\,2, \qquad [\overline{\xi}] = 1,$$

of ghost number

$$\mathrm{gh}[\varepsilon_{\mu_k\ldots\mu_p}] = \mathrm{gh}[\xi_{\nu_k\ldots\nu_q}] = k, \qquad \mathrm{gh}[\varepsilon] = p+1, \qquad \mathrm{gh}[\xi] = q+1,$$

and of antifield number

$$\mathrm{Ant}[\overline{A}^{\mu_1\ldots\mu_p}] = \mathrm{Ant}[\overline{B}^{\nu_{p+1}\ldots\nu_q}] = 1,$$
$$\mathrm{Ant}[\overline{\varepsilon}^{\mu_k\ldots\mu_p}] = \mathrm{Ant}[\overline{\xi}^{\nu_k\ldots\nu_q}] = k+1,$$
$$\mathrm{Ant}[\overline{\varepsilon}] = p, \qquad \mathrm{Ant}[\overline{\varepsilon}] = q.$$

One can show that homology regularity Condition 4.1 holds, and the DBGR $\mathcal{P}^*_\infty\{q-1\}$ (7.3) is endowed with the **KT** operator

$$\delta_{\rm KT} = \frac{\overleftarrow{\partial}}{\partial \overline{A}^{\mu_1\ldots\mu_p}}\mathcal{E}_A^{\mu_1\ldots\mu_p} + \frac{\overleftarrow{\partial}}{\partial \overline{B}^{\nu_1\ldots\nu_q}}\mathcal{E}_B^{\nu_1\ldots\nu_q} + \sum_{2\leq k\leq p}\frac{\overleftarrow{\partial}}{\partial \overline{\varepsilon}^{\mu_k\ldots\mu_p}}\Delta_A^{\mu_k\ldots\mu_p} +$$
$$\frac{\overleftarrow{\partial}}{\partial \overline{\varepsilon}} d_{\mu_p}\overline{\varepsilon}^{\mu_p} + \sum_{2\leq k\leq q}\frac{\overleftarrow{\partial}}{\partial \overline{\xi}^{\nu_k\ldots\nu_q}}\Delta_B^{\nu_k\ldots\nu_q} + \frac{\overleftarrow{\partial}}{\partial \overline{\xi}} d_{\nu_q}\overline{\xi}^{\nu_q},$$
$$\Delta_A^{\mu_2\ldots\mu_p} = d_{\mu_1}\overline{A}^{\mu_1\ldots\mu_p}, \qquad \Delta_A^{\mu_{k+1}\ldots\mu_p} = d_{\mu_k}\overline{\varepsilon}^{\mu_k\mu_{k+1}\ldots\mu_p}, \qquad 2\leq k<p,$$
$$\Delta_B^{\nu_2\ldots\nu_q} = d_{\nu_1}\overline{B}^{\nu_1\ldots\nu_q}, \quad \Delta_B^{\nu_{k+1}\ldots\nu_q} = d_{\nu_k}\overline{\xi}^{\nu_k\nu_{k+1}\ldots\nu_q}, \quad 2\leq k<q.$$

Its nilpotentness provides the complete Noether identities (7.2) and the $(k-1)$-stage ones

$$d_{\mu_k}\Delta_A^{\mu_k\ldots\mu_p} = 0, \qquad k=2,\ldots,p,$$
$$d_{\nu_k}\Delta_B^{\nu_k\ldots\nu_q} = 0, \qquad k=2,\ldots,q.$$

It follows that topological BF theory is $(q-1)$-***reducible***.

Applying inverse second Noether Theorem 5.1, one obtains the gauge operator (5.4) which reads

$$\mathbf{u} = d_{\mu_1}\varepsilon_{\mu_2\ldots\mu_p}\frac{\partial}{\partial A_{\mu_1\mu_2\ldots\mu_p}} + d_{\nu_1}\xi_{\nu_2\ldots\nu_q}\frac{\partial}{\partial B_{\nu_1\nu_2\ldots\nu_q}} + \tag{7.4}$$
$$\left[d_{\mu_2}\varepsilon_{\mu_3\ldots\mu_p}\frac{\partial}{\partial \varepsilon_{\mu_2\mu_3\ldots\mu_p}} + \cdots + d_{\mu_p}\varepsilon\frac{\partial}{\partial\varepsilon_{\mu_p}}\right] +$$
$$\left[d_{\nu_2}\xi_{\nu_3\ldots\nu_q}\frac{\partial}{\partial \xi_{\nu_2\nu_3\ldots\nu_q}} + \cdots + d_{\nu_q}\xi\frac{\partial}{\partial\xi_{\nu_q}}\right].$$

In particular, a ***gauge symmetry*** of the Lagrangian $L_{\rm BF}$ (7.1) is

$$u = d_{\mu_1}\varepsilon_{\mu_2\ldots\mu_p}\frac{\partial}{\partial A_{\mu_1\mu_2\ldots\mu_p}} + d_{\nu_1}\xi_{\nu_2\ldots\nu_q}\frac{\partial}{\partial B_{\nu_1\nu_2\ldots\nu_q}}.$$

It also is readily observed that the gauge operator $\mathbf{u}$ (7.4) is nilpotent. Thus, it is the ***BRST operator*** $\mathbf{b} = \mathbf{u}$. As a result, the Lagrangian $L_{\rm BF}$ is extended to the ***non-trivial solution of the master equation*** L_E (6.7) which reads

$$L_E = L_{\rm BF} + \varepsilon_{\mu_2\ldots\mu_p} d_{\mu_1} \overline{A}^{\mu_1\ldots\mu_p} + \sum_{1<k<p} \varepsilon_{\mu_{k+1}\ldots\mu_p} d_{\mu_k} \overline{\varepsilon}^{\mu_k\ldots\mu_p} + \varepsilon d_{\mu_p} \overline{\varepsilon}^{\mu_p} +$$
$$\xi_{\nu_2\ldots\nu_q} d_{\nu_1} \overline{B}^{\nu_1\ldots\nu_q} + \sum_{1<k<q} \xi_{\nu_{k+1}\ldots\nu_q} d_{\nu_k} \overline{\xi}^{\nu_k\ldots\mu_q} + \xi d_{\nu_q} \overline{\xi}^{\nu_q}.$$