17th International Summer School

on

Global Analysis and its Applications

EXTENDED ABSTRACT BOOK

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Levoča, Slovakia

Satellite meeting of 6th European Congress of Mathematics

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17th International Summer School on Global Analysis and its Applications
August 17-22, 2012, Levoča, Slovakia

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Preface

The tradition of the Summer School on Global Analysis and its Applications was founded by prof. D. Krupka in 1996, in his birthplace, beautiful medieval town of Levoča, written in the UNESCO list (http://whc.unesco.org/en/list/620). Since 1996, the Summer School has been organized every year by prof. Krupka and his collaborators in several Slovak and Czech places, and has become a well-known scientific event all over the world.

The 2012 Summer School has again been held in Levoča, at the Juraj Páleš Institute of the Catholic University in Ružomberok, 17-22 August, as a Satellite Meeting of the 6th European Congress of Mathematics. In past, the 11th Summer School was a Satellite Conference of the International Congress of Mathematicians.

The programme of the school consisted of two courses, delivered by recognized specialists in the field. Moreover, to support the young participants in their scientific activity, the commented poster session and short presentations were organized.

The abstracts of main lecture series and short communications, contained in this book, are devoted to selected recent results and methods of geometric mechanics and control, and in global variational geometry. They summarize theoretical and applied aspects of the topics, considered at the school, and provide the reader with basic information on research interest of participants and further orientation of their work in these fields.

The organizers would like to thank the Faculty of Humanities and Natural Sciences, University of Prešov, Slovakia, the University of Ostrava, Czech Republic, and the Juraj Páleš Institute, Catholic University in Ružomberok, Slovakia, for their support to the Summer School. The organizers also appreciate the help of Ms. Jana Verešpejová during the Summer School.

Levoča, 21 August 2012

Ján Brajerčík

Chairman of the Organizing Committee

Programme

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INTRODUCTION TO GLOBAL VARIATIONAL GEOMETRY
Demeter Krupka

A BRIEF INTRODUCTION TO GEOMETRIC OPTIMAL CONTROL
Donghua Shi

GEOMETRIC MECHANICS AND CONTROL
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THE HELMHOLTZ CONDITIONS FOR SYSTEMS OF SECOND ORDER HOMOGENOUS DIFFERENTIAL EQUATIONS
Zbyněk Urban, Demeter Krupka

ON FINSLER GEOMETRY AND SOME OF ITS APPLICATIONS TO PHYSICS
Nicoleta Voicu

INTRODUCTION TO GLOBAL VARIATIONAL GEOMETRY

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Abstract

The aim of the lectures is to give an introduction to advanced calculus of variations on smooth manifolds and to the research topics in this field. The following topics will be covered:

- Fundamentals of the variational theory on fibred manifolds

  This part of the lectures is available on request in electronic form; the book was written as an introduction with broad coverage of the subject, containing complete proofs. Contents:

1 Jet prolongations of fibred manifolds

1.1 The rank theorem

1.2 Fibred manifolds

1.3 The contact of differentiable mappings

1.4 Jet prolongations of fibred manifolds

1.5 The horizontalisation

1.6 Jet prolongations of automorphisms of fibred manifolds

1.7 Jet prolongations of vector fields

2 Differential forms on jet prolongations of fibred manifolds

2.1 The first canonical decomposition

2.2 Contact components and geometric operations

2.3 The second canonical decomposition

2.4 Contact forms

2.5 Fibred homotopy operators

3 Formal divergence equations

3.1 Formal divergence equations

3.2 The cut formal derivative

3.3 Projectable extensions of differential forms

3.4 Fibred homotopy operators on jet prolongations of fibred manifolds

3.5 Integrability of formal divergence equations

4 Variational structures

4.1 Variational structures on fibred manifolds

4.2 Variational derivatives

4.3 Lepage forms

4.4 Euler-Lagrange forms

4.5 Lepage equivalents and the Euler-Lagrange mapping

4.6 The first variation formula

4.7 Extremals

4.8 Trivial Lagrangians

4.9 Source forms and the Veinberg-Tonti Lagrangians

4.10 The inverse problem of the calculus of variations

5 Invariant variational structures

5.1 Invariant differential forms

5.2 Invariant Lagrangians and conservation equations

5.3 Invariant Euler-Lagrange form

5.4 Symmetries of extremals and Jacobi vector fields

Appendix

Analysis on Euclidean spaces and manifolds

1 Jets of mappings of Euclidean spaces

2 Differentiation of linear mappings with respect to a parameter

3 The rank theorem

4 Differential equations

5 Integration of continuous functions on compact sets

6 Local flows of vector fields

7 Calculus on manifolds

8 The trace decomposition

9 The Levi-Civita symbol

- Additional topics for discussion
  Variational sequences and applications, interior Euler-Lagrange operators, variational energy-momentum tensors, variational problems in parametric form, homogeneity, inverse problem of the calculus of variations.

Keywords: Fibred manifold, Lagrangian, Euler-Lagrange form, Inverse variational problem, Symmetries, Noether currents, Conservation law.

A BRIEF INTRODUCTION TO GEOMETRIC OPTIMAL CONTROL

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Abstract

The purpose of the lecture is to give a brief introduction to basic notions and results in the area of geometric control by using some specific examples. It will focus on the controllability and optimal control of the system. Some aspects with geometric mechanics are also addressed.

Keywords: Controllability, Pontryagin maximum principle.

GEOMETRIC MECHANICS AND CONTROL

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Abstract

There are two primary expositions of mechanics. Lagrangian mechanics concentrates on variational principles. Hamiltonian mechanics, while being variational too, focuses on such features as symplectic (more generally, Poisson) maps defined by systems' phase flow and canonical transformations. These two formalisms are often, but not always, equivalent. We will mostly confine our attention to Lagrangian mechanics, studying the classical variational principle as well as some of their contemporary developments. The tentative plan is to cover

- Newton's laws, Euler-Lagrange equations, and Hamilton's principle
- The rigid body and the Euler-Poincare equations
- Moving frames and Hamel's equations
- Variational principles for Hamel's equations
- Symmetry and conservation laws
- Systems with velocity constraints.

Keywords: Hamilton's principle, Hamilton-Pontryagin's principle, Lagrange-d'Alembert principle.

SHEAF SPACES OF ORDERED QUASIGROUPS

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Abstract

There are known some characterizations of representable lattice ordered groups (i.e., lattice ordered groups, shortly l-groups, which are l-isomorphic to a subdirect product of totally ordered groups). One of these characterizations is based on the theory of sheaf spaces of l-groups. The central theorem which was used for this purpose gives the conditions (using ideals of l-groups) under which an l-group can be represented as sections of a sheaf space of l-groups (see Darnel, M.R., *Theory of latice-ordered groups*, Marcel Dekker, Inc. New York 1995). In the present contribution we generalize this result for ordered quasigroups.

A *quasigroup* is an algebra $(Q, \cdot, \backslash, /)$ with three binary operations $\cdot, \backslash, /$ satisfying the following identities: $y\backslash(y \cdot x) = x$; $(x \cdot y)/y = x$; $y \cdot (y\backslash x) = x$; $(x/y) \cdot y = x$.

These identities imply that, given $a, b \in Q$, the equations $b \cdot x = a$ and $y \cdot b = a$ have unique solutions $x = b\backslash a$ and $y = a/b$. Conversely, every groupoid such that the equations $b \cdot x = a$ and $y \cdot b = a$ have unique solutions x, y is a quasigroup, where $b\backslash a$ or a/b are defined as the solutions of equations $b \cdot x = a$ or $x \cdot b = a$, respectively. Clearly, every group is a quasigroup with $x/y = x \cdot y^{-1}$ and $y\backslash x = y^{-1} \cdot x$.

A quasigroup $(Q, \cdot, \backslash, /)$ with a binary relation $\leq$ is called a *partially ordered quasigroup* (*po-quasigroup*; in the notation $\mathcal{Q}$) if $(Q, \leq)$ is a partially ordered set and for all $x, y, a \in Q$, $x \leq y$ iff $ax \leq ay$ iff $xa \leq ya$.

We say that θ is a *convex congruence relation* (*directed congruence relation*) on $\mathcal{Q}$ if θ is a congruence relation on $(Q, \cdot, \backslash, /)$ and there exists $a \in Q$ such that the congruence class $[a]\theta$ is a convex subset (directed subset) of $\mathcal{Q}$. A po-quasigroup $\mathcal{Q}$ is called a *lattice ordered quasigroup* (shortly *l-quasigroup*), if it has a lattice-order.

Now, let E and X be topological spaces and let $\sigma : E \to X$ be a surjective local homeomorphism. A fibre over $x \in X$ is denoted by E_x. By $E\Delta E$ we denote the set $\bigcup_{x\in X}(E_x \times E_x)$ with the induced topology from $E \times E$. A triplet (E, X, σ) is called a *sheaf space of po-quasigroups* if each fibre E_x is a po-quasigroup and all the mappings $\cdot, \backslash, /$ defined by $(s,t) \mapsto s \cdot t$, $(s,t) \mapsto t\backslash s$ and $(s,t) \mapsto s/t$ are continuous from $E\Delta E$ to E. A sheaf space of po-quasigroups (E, X, σ) is said to be a *sheaf space of l-quasigroups* if each fibre E_x is an l-quasigroup and the mappings $(s,t) \mapsto s \vee t$, $(s,t) \mapsto s \wedge t$ are continuous from $E\Delta E$ into E.

Let (E, X, σ) be a sheaf space of po-quasigroups. Consider the following condition (S)

(S) *If f, g are continuous local sections over an open set $U \subseteq X$ such that $\mathtt{sup}\{f(u), g(u)\}$ exists for each $u \in U$, then $\{\mathtt{sup}\{f(u), g(u)\} : u \in U\}$ is an open set in E.*

Lemma 1 *Let (E, X, σ) be a sheaf space of po-quasigroups satisfying (S). If E_x is an l-group for each $x \in X$, then (E, X, σ) is a sheaf space of l-groups.*

The main result of our contribution is the following theorem.

Theorem 1 *Let $\mathcal{Q}$ be a po-quasigroup and let X be a topological space. Suppose that for each $x \in X$ there exists a convex congruence relation θ_x on $\mathcal{Q}$ such that the following conditions are satisfied*

(*i*) *if $[g]\theta_x \leq [h]\theta_x$ for each $x \in X$, then $g \leq h$,*

(*ii*) *for all $p, r \in Q$ the set $U_{pr} = \{x \in X : [p]\theta_x = [r]\theta_x\}$ is an open set of X.*

Then $\mathcal{Q}$ can be o-embedded into a po-quasigroup of continuous global sections of some sheaf space of po-quasigroups over X. Especially, if $\mathcal{Q}$ is an l-quasigroup and θ_x are directed convex congruence relations on $\mathcal{Q}$ satisfying (i) and (ii), then $\mathcal{Q}$ can be o-embedded into an l-quasigroup of continuous global sections of some sheaf space of l-quasigroups over X.

The last theorem generalizes the analogous result valid for l-groups.

MSC2010: 06F99, 20N05, 54B40
Keywords: Sheaf space, Partially ordered quasigroup.

MOMENTUM-CONSERVING INTEGRATOR FOR A ROTOR-STABILIZED SATELLITE

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Abstract

From the manual merry-go-rounds of our childhood that we could only make go round until our legs were tired to our office chairs and the orbiting satellites that send our phone calls and television shows to and for us, the phenomenon of rotation is ever so present in our lives. While many of these rotating objects are flexible, neglecting small deformations and idealizing the apparatus as a rigid body allows us to capture the essential dynamics of rotations.

Representing rotations using Euler's angles, being a useful theoretical technique, is not as practical in computer simulations because of ambiguities such as gimbal locking. 3D rotation modeling with rotation matrices or unit quaternions serves as a beneficial alternative, the latter offering the smallest redundancy for coding rotations.

In this talk, we utilize the unit quaternions to study the rigid body rotations in conjunction with angular momentum preserving computer simulation techniques. We then use these techniques to model the problem of stabilization of a satellite rotating about its intermediate inertia axis by coupling it to a controlled symmetric rotor. The discrete model for the satellite-rotor is derived from Hamilton's principle by approximating the action integral of the continuous time mechanics with an action sum and equating the sum to zero. The resulting discrete dynamics is known to be symplectic and, for systems with symmetry, momentum-preserving; the latter ensures that the numerical model is adequate in stopping the rotation of the satellite when the rotor stops.

This discrete model will be then compared to the original continuous system, revealing some unexpected discretization-related ramifications.

Though the satellite-rotor problem is solvable in the continuous-time setting, the development of a momentum-preserving discrete model is an important step towards simulating more complicated problems involving rotations.

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MSC2010: 70Q05, 70E50, 93D15, 34K35
Keywords: Structure-preservations, Rigid body, Feedback stabilization.

INTEGRABILITY OF THE RIGID BODY FOR THE SUSLOV PROBLEM

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Abstract

This paper is on the integration of the differential equations

$$\begin{aligned}
&\dot{\omega} = \mathcal{X}_\omega := \\
&\quad I^{-1}\left(\frac{\langle I\nu,\ \omega\rangle}{\langle I^{-1}\nu,\ \nu\rangle}\left(\omega\wedge I^{-1}\nu\right) + \frac{I^{-1}\nu}{\langle I^{-1}\nu,\ \nu\rangle}\wedge\left(\left(\gamma\wedge\frac{\partial U(\gamma)}{\partial\gamma}\right)\wedge\nu\right)\right), \qquad (1)\\
&\dot{\gamma} = \mathcal{X}_\gamma := \gamma\wedge\omega.
\end{aligned}$$

which describe the motion of a rigid body around a fixed point in a force field with potential $U(\gamma) = U(\gamma_1,\gamma_2,\gamma_3)$ and subject to the constraint $<\nu,\omega>=0$ with $\nu = \tilde{\mathbf{a}} =$ constant (Suslov's problem) and $\nu=\gamma$ (Veselova problem), where I the inertial tensor, $<,>$ and $\wedge$ are the inner and wedge product of $\mathbb{R}^3$, and $\dfrac{\partial U(\gamma)}{\partial\gamma}$ is the gradient of $U(\gamma)$ with respect to γ.

By considering that the differential system (1) has three independent first integrals $K_1 = \langle I\omega,\omega\rangle - U(\gamma)$, $K_2 = \langle\gamma,\gamma\rangle$, $K_3 = \langle\nu,\omega\rangle$, if exists an independent fourth first integral K_4 and the divergence of system (1) satisfies

$$\begin{aligned}
\mathrm{div}X_\gamma + \mathrm{div}X_\omega &= \frac{\left\langle I^{-1}\nu,(\nu\wedge\omega)\right\rangle}{\langle I^{-1}\nu,\nu\rangle} = -\frac{\mathrm{d}}{\mathrm{d}t}(\log M(\gamma)) \\
&= -\left\langle \omega, \frac{\partial M(\gamma)}{\partial\gamma}\wedge\gamma\right\rangle,
\end{aligned} \qquad (2)$$

then (1) is integrable by the Euler-Jacobi theorem. We provide the following new integrable case for the Suslov problem.

For the Suslov's problem under the condition that $\tilde{\mathbf{a}}$ is an eigenvector of I, we determine the conditions for the existence of a fourth independent first integral K_4 given implicitly by $I_1\,\omega_1 - \mu_2(\gamma, K_1, K_4) = 0$. This integrable problem contains the Suslov, Kharlamova-Zabelina's, Kozlov, Klebsh-Tisserand, Tisserand-Okunova and Dragović- Gajić-Jovanović first integrable problem as a particular cases.

MSC2010: 14P25, 34C05, 34A34
Keywords: Ordinary differential equation, Invariant measure, Integrability, Mechanical systems, Constraint, Rigid body, Suslov case, Veselova case.

VARIATIONAL NATURE OF QUIVER, SELF-INTERACTION, AND SPHERICAL TOP IN RELATIVITY

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Abstract

We intend to relate to each other certain equations of motion with higher derivatives that describe different physical phenomena in the framework of the relativistic analytical dynamics. Although two of these phenomena, the quiver (known as the 'Zitterbewegung') and the radiation friction, have been known only within the framework of Special Relativity, our approach allows to generalize the corresponding equations of motion to the curved spacetime. From the geometrical point of view these generalizations do not depend neither on the signature of the Riemannien space nor upon its dimension.

The other point is the system of the generalized canonical equations due to Grässer–Rund–Weyssenhoff, applicable together with the Legendre map $\ell\colon T^3M \to T^*(TM)$ to the case of the parameter-ambivalent variational problem satisfying the Zermelo conditions

$$\begin{cases} \mathcal{Z}_1 \overset{\text{def}}{=} u^n \dfrac{\partial L}{\partial \dot u^n} \equiv 0 \\ u^n \dfrac{\partial L}{\partial u^n} + 2\,\dot u^n \dfrac{\partial L}{\partial \dot u^n} - L \equiv 0\,. \end{cases} \tag{Z}$$

In terms of the Liouville form on $T^*(TM)$, $\Lambda = p_n dx^n + p^{(1)}{}_n du^n$, the system of the Grässer–Rund–Weyssenhoff canonical equations may be cast in the form of the following exterior differential equation:

$$\boxed{\ell^{-1} i_X d\Lambda = -\lambda \ell^{-1} dH - \mu \ell^{-1} dZ_1\,.} \tag{GRW}$$

Here ℓ^{-1} means taking the induced differential form along ℓ, λ and μ are the undetermined multipliers, and the well-defined function $Z_1 = u^n p^{(1)}{}_n$ is that giving rise to the first one of the Zermelo conditions (Z), $\ell^{-1} Z_1 = \mathcal{Z}_1$.

Consider the following change of variables in the configuration space:

$$\{x^n, u^n, \dot{u}^n\} \xrightarrow{\phi} \{x^n, u^n, u'^n\}, \qquad u'^n = \dot{u}^n + \Gamma^n{}_{lm} u^m u^l \,,$$

along with the change of the variables in the phase space $T^*(TM)$:

$$\varphi : \begin{cases} \pi^{(1)}{}_n = p^{(1)}{}_n \\ \pi_n \quad = p_n - \Gamma^l{}_{mn} u^m p^{(1)}{}_l \end{cases}$$

and denote:

$\mathcal{H} = H \circ \varphi^{-1}$, $\mathcal{L} = L \circ \phi^{-1}$, $\boldsymbol{\varpi}^1 = \ell^{-1}\boldsymbol{\pi}^{(1)}$, $\boldsymbol{\varpi} = \ell^{-1}\boldsymbol{\pi}$, $\widetilde{Le} = \varphi \circ \ell \circ \phi^{-1}$ and $\tilde{\mu} = \mu \circ \phi$. The prime $'$ will denote the covariant differentiation along a curve $x^n(\tau)$.

Proposition 1 *Let some Lagrange function $\mathcal{L}$ depend on all the variables exclusively through the differential invariants $\gamma = \boldsymbol{u}{\cdot}\boldsymbol{u}$, $\beta = \boldsymbol{u}{\cdot}\boldsymbol{u}'$, and $\alpha = \boldsymbol{u}'{\cdot}\boldsymbol{u}'$ only. In this case the Euler–Poisson expression is:*

$$\boxed{\mathcal{E}_n = -\varpi'{}_n - \varpi^{(1)}{}_l R_{nkm}{}^l u^m u^k}$$

Proposition 2 *Let $\mathcal{H}$ depend on $(x, u, \pi, \pi^{(1)})$ through the quantities $\gamma = \boldsymbol{u}{\cdot}\boldsymbol{u}$, $\eta = \boldsymbol{\pi}^{(1)} \cdot \boldsymbol{\pi}^{(1)}$, $\psi = \boldsymbol{\pi}.\,\boldsymbol{u}$, $\nu = \boldsymbol{\pi}^{(1)}.\,\boldsymbol{u}$. Than the canonical equations (GRW) read:*

$$\frac{dx^n}{d\tau} = u^n \tag{$\mathcal{H}$ 1}$$

$$u'^n = \left(\frac{\partial \mathcal{H}}{\partial \psi}\right)^{-1} \left[2\frac{\partial \mathcal{H}}{\partial \eta} g^{nk}\pi^{(1)}{}_k + \frac{\partial \mathcal{H}}{\partial \nu} u^n\right] \circ \widetilde{Le} + \tilde{\mu} u^n \tag{$\mathcal{H}$ 2}$$

$$\pi'{}_n \circ \widetilde{Le} = -R_{nkm}{}^l u^m u^k \varpi^1{}_l \tag{*}$$

$$\pi^{(1)\prime}{}_n \circ \widetilde{Le} = -\left(\frac{\partial \mathcal{H}}{\partial \psi}\right)^{-1} \left[2\frac{\partial \mathcal{H}}{\partial \gamma} u_n + \frac{\partial \mathcal{H}}{\partial \nu}\pi^{(1)}{}_n\right] \circ \widetilde{Le} - \varpi_n - \tilde{\mu}\varpi^1{}_n \tag{$\mathcal{H}$ 4}$$

Application to the relativistic mechanics. In the first approximation the self-interaction of the relativistic electron with its own electromagnetic radiation force in the flat spacetime of Special Relativity may be described, following Bopp, by a Lagrange function depending upon the first Frenet curvature k of the particle's world line:

$$L^k = (k^2 + A)\,\|\boldsymbol{u}\|\,. \tag{B}$$

Proposition 3 *In Riemannian space the Lagrange function (B) gives rise to a parameter-invariant variational equation that in fixed pameterization reads:*

$$\frac{D}{d\tau}\Big[\Big(-3\,\boldsymbol{u}'\cdot\boldsymbol{u}'+A\Big)\,u_n-2\,u''{}_n\Big]=-\varpi^1{}_l\,R_{nkm}{}^l u^m u^k\,,\qquad \boldsymbol{u}\cdot\boldsymbol{u}=1\quad(**)$$

Proposition 4 *In Riemannian space by means of the Legendre map the Lagrange function (B) gives birth to the following Hamilton function in the generalized canonical formalism of Grässer–Rund–Weyssenhoff:*

$$\mathcal{H}=\boldsymbol{\pi}.\,\boldsymbol{u}+\frac{\|\boldsymbol{u}\|^3}{4}\,\boldsymbol{\pi}^{(1)}\cdot\boldsymbol{\pi}^{(1)}-A\|\boldsymbol{u}\|+1.$$

In the theory of General Relativity the world line of a quasi-classical particle endowed with the inner angular momentum (said 'spin') $S_{nm}=-S_{mn}$, or the Relativistic Top, should obey a quite popular and fairly general Dixon system of 1st order ODEs

$$\begin{cases} P'{}_n &=& -\dfrac{1}{2}\,R_{nm}{}^{kl}\dot{x}^m S_{kl}\\ S'{}_{nm} &=& P_n\dot{x}_m-P_m\dot{x}_n\,,\end{cases}\qquad\text{(D)}$$

written in terms of the covariant derivatives. One possible way to make system (D) solvable is to add to it the Mathisson supplementary constraint

$$u^m S_{mn}=0,\qquad u^n=\dot{x}^n=\frac{dx^n}{d\tau}.\qquad\text{(M)}$$

The supplementary condition (M) allows us to eliminate in flat space-time of Special Relativity the variable S_{mn} from (D,M) by taking some differential prolongations. The resulting equation reads

$$\ddot{\boldsymbol{u}}+\left(k^2-\frac{m^2}{s^2}\right)\dot{\boldsymbol{u}}=0,\qquad \boldsymbol{u}\cdot\boldsymbol{u}=1\,,\qquad\text{(R)}$$

where $k^2=\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}$ is the first integral of (D,M) in flat space-time, and $s^2=S_{mn}S^{mn}$. Equation (R) was shown by Riewe to describe 'Zitterbewegung' (quiver) of a quasi-classical particle.

Proposition 5 *In flat space-time the equation (**) reduces to the Riewe–Costantelos equation (R) of the quasi-classical 'Zitterbewegung' on the constrained manifold $k=k_0$ by putting $A={k_0}^2+2\,\dfrac{m^2}{s^2}$.*

Proposition 6 *Equations (D) follow from (*) if we put* $S = \boldsymbol{u} \wedge \boldsymbol{\varpi}^1$. *The Mathisson supplementary condition (M) is satisfied automatically.*

MSC2000: 70H50, 83C10, 53B21, 53B50
Keywords: Relativistic top, Zitterbewegung, Homogeneous Ostrohrads'kyj mechanics.

THE ROLE OF RELATIVE CHARACTERISTIC COHOMOLOGY IN A COORDINATE-FREE APPROACH TO VARIATIONAL INTEGRALS

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Abstract

One purpose of global variational sequences [7] is to describe variational problems in a coordinate-free manner. The finite-order character of such a framework allows to keep track of the highest derivatives involved in the computations. On the other hand, non-integrability of the contact distribution on finite-order jets prevents one from deploying the powerful homological tools associated with involutive distributions. In the theory of infinite-order jet spaces [5], the order of derivatives that come into play becomes less relevant, but one gains a completely integrable distribution (though it sits on an infinite-dimensional manifold). This feature can give non-trivial benefits in several contexts, like integrable systems [4] and modern gauge theories [3], where this cohomological framework goes under the name of *characteristic*. In this talk I will show how variational integrals are nicely described in terms of characteristic cohomology, by firstly reviewing the cohomological theory of integration [1] and then adapting it to the "horizontal" or "total-derivative" context of the variational bicomplex.

In particular, I will shed light on the important role played by relative cohomology. Besides providing a nice link between a purely algebraic tool of homological algebra (firstly introduced in algebraic topology) and variational calculus, it allows to give (see [6]) a general invariant description of the natural boundary conditions [2].

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REGULARIZATION AND ENERGY ESTIMATION OF PENTAHEDRA (PYRAMIDS) USING GEOMETRIC ELEMENT TRANSFORMATION METHOD

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Abstract

By geometric element transformation method (GETMe) always we get a new element. It is based on geometric transformations, which, if applied iteratively, lead to the regularization of a pyramid (under condions). Energy function is a cost function for pentahedra which is applicable also for hexahedra, octahedra, decahera etc. is defined by a particular process, which we call as base diagonal apex method (BDAMe). In this paper, first, we define the characterization of energy function of a pyramid using base diagonal apex method (BDAMe). Then we have proved, the energy function

$$f_p\left(\sigma^n\right) = \frac{1}{N}\sum_{j=1}^{N} max_{v\in\,\sigma^n}\left|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}\right|,$$

where N is maximum total number of tetrahedron, of a 3D-figure (pentahedron, hexahedron, decahedron, octahedron, ..., etc) always lies between 0 and 1 and discussed regularization properties of a pyramid and try to regularize by using geometric element transformation method (GETMe). Finally we study the characterization of energy function of a particular type of pentahedron using GETMe and BDAMe. Here, we also try to investigate the characterization of different cost functions using BDAMe when we transform a pyramid by GETMe.

MSC2010: 97G50, 65M50

Keywords: Mesh quality, Iterative element regularization, Finite element mesh, Objective function, Cost function.

MATHEMATICAL DESCRIPTION OF THE ARTIFICIAL MUSCLE GEOMETRIC MODEL

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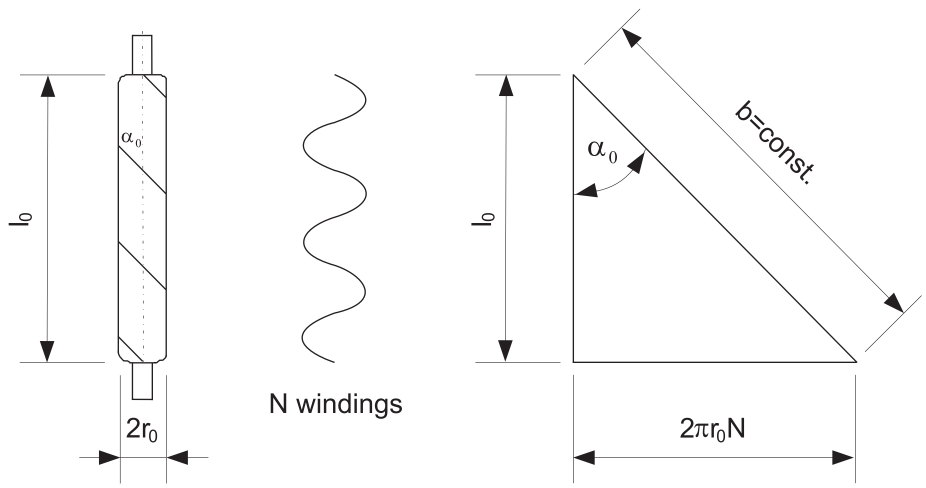
Abstract

Pneumatic artificial muscles (PAMs) are a progressive performance member of pneumatic and electropneumatic servo systems. Such systems have in specific applications (e.g. explosive environment) the irreplaceable position. The most common so far produced and used type of pneumatic artificial muscle is McKibben muscle and it is made commercially available by different companies (e.g. Fluidic Muscle manufactured by Festo Company).

The general behavior of pneumatic artificial muscle with regard to shape, contraction and tensile force when inflated depends on its geometry (Fig. 1), on physical effects ongoing inside the muscle and on the material used. Typical materials used for the membrane construction are latex and silicone rubber, while nylon is normally used in the fibers.



Mathematical relations can describe the dependence between variables of artificial muscle. The relation between tension, length, velocity, and activation are the major characteristics of actuators which vary greatly from type to type. The most often mentioned characteristic of PAMs is the force as a function of pressure and muscle contraction. The basic parameters of the pneumatic artificial muscle, which are modeled by an artificial muscle as a cylinder with thickness of non-zero h_m are length l_0, radius r_0 and the initial angle α_0 between the axis of the muscle and fibers. Fiber length b and number of wrapped of one fiber N are geometric constants of system assuming non expansion braiding fibers. Then the initial muscle volume is:

$$\begin{aligned} V_0 &= \pi \cdot (r_0 - h_m)^2 \cdot l_0 \\ &= \frac{b^3 \cdot \sin^2 \alpha_0 \cdot \cos \alpha_0}{4\pi \cdot N^2} - \frac{h_m \cdot b^2 \cdot \sin \alpha_0 \cdot \cos \alpha_0}{N} + h_m^2 \cdot \pi \cdot b \cdot \cos \alpha_0. \end{aligned} \tag{1}$$

One of the characteristics of pneumatic muscle is its natural suppleness which provides difficult frictional forces that complicate the muscle control. This "internal" friction has not already been sufficiently described. Very difficult characteristic which prevents the successful modeling of friction is the non-deterministic behavior. Stochastic component of frictional forces is probably caused by an accidental displacement braid of muscle after the inner membrane e.g. the full release of muscle. This action affects the environment variables such as temperature and pressure, which can change randomly.

Based on assumption that $\mathrm{d}V/\mathrm{d}p$ is equal to zero for artificial muscle with solid braid fibers, which are always in contact with the inner layer of muscle, the tensile force can be calculated by using law of conservation of energy as:

$$F = p \cdot \frac{\mathrm{d}V}{\mathrm{d}l} - V_0 \cdot \frac{\mathrm{d}W}{\mathrm{d}l} - F_f, \tag{2}$$

where p is the input actuation pressure, $\mathrm{d}V$ is the change in the actuator's interior volume, V_0 is the initial muscle volume, $\mathrm{d}W$ is the change in strain energy density, $\mathrm{d}l$ is the change in the actuator's length. F_f describes the lumped effects of friction arising from sources such as contact between the braid and the bladder and between the fibers of the braid itself.

For the dynamic behavior of the pneumatic muscle the knowledge of the actual pressure in the muscle is important too. It depends on the quotient of the amount of air in the muscle and the volume of the muscle. Dependence of pressure in the muscle on muscle volume can be found with the help of the equation for ideal gases, the Boyle-Mariotte law and Bernoulli equation:

$$\begin{aligned}\dot{P} &= \frac{1}{V}\cdot(P_a\cdot\dot{V}_a - P\cdot\dot{V})\\ &= \frac{1}{V}\cdot(P_a\cdot f_\nu\cdot C_a\cdot A_\nu\cdot\sqrt{p_{in}-p_{out}} - P\cdot\dot{V}),\end{aligned} \tag{3}$$

where P is absolute muscle air pressure, P_a is absolute ambient pressure, V is muscle volume, V_a is volume of air in the muscle under absolute ambient pressure, f_ν is coefficient describing the flow direction, C_a is aerodynamic correction coefficient, A_ν is smallest area for flow, p_{in} and p_{out} are pressures different for inflation and deflation.

By modeling the inner bladder as an incompressible Mooney-Rivlin material and estimating frictional losses, we are able to improve predictions of the actual output forces as a function of length and pressure. However, to eliminate the need to empirically determine frictional losses and any other factors not considered, further work is required.

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Keywords: Pneumatic Artificial Muscle, Geometric Model.

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PARAMETERISATION INVARIANT LAGRANGE FORMULATION BY KAWAGUCHI GEOMETRY

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Abstract

We introduce a construction of a geometrical structure which is appropriate for considering parameterisation invariant Lagrange formulation. We especially present here a first order field theory, which is set up on a bundle of multivectors endowed with a Kawaguchi structure.

Bellow are the basic structures we use for the formulation. Throughout the text we consider M as a n-dimensional C^∞-manifold, and take a chart $(U, \varphi), \varphi = (x^\mu)$.

Definition 1 *The triple $(\Lambda^k TM, \Lambda^k \tau_M, M)$ is the k-fold antisymmetric tensor product of the tangent bundle (TM, τ_M, M), and is a vector bundle with completely antisymmetric properties.*

Definition 2 *The first order Kawaguchi space (M, K) of k-dimensional parameter space is a pair of n-dimensional C^∞-differentiable manifold and a Kawaguchi function $K \in C^\infty(\Lambda^k TM)$, which for a induced chart on $\Lambda^k TM$, $(U; \varphi), \varphi = (x^\mu, y^{\mu_1 \cdots \mu_k})$, where $\mu, \mu_1, \cdots, \mu_k$ runs from $1, \cdots, n$ satisfies the homogeneity condition,*

$$K(x^\mu, \lambda y^{\mu_1 \cdots \mu_k}) = \lambda K(x^\mu, y^{\mu_1 \cdots \mu_k}), \quad \lambda > 0.$$

This condition is equivalent to the condition, $\frac{1}{k!} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} y^{\mu_1 \cdots \mu_k} = K$.

Definition 3 *The Kawaguchi form $\mathcal{K}$ is a k-form on $\Lambda^k TM \setminus \{0\}$, which in local coordinates are expressed by*

$$\mathcal{K} = \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$$

.

The structure for the calculus of Variation would be this Kawaguchi manifold, (M, K), where our Lagrangian is a Kawaguchi form.

Definition 4 *Let Σ a parameterisable submanifold on M, and $\sigma : P \to \Sigma \subset M$ its parameterisation, where P is a closed rectangle. There exists a map from P to $\mathbb{R}^k$ called a lift of parameterisation σ which is a C^1-map $\hat{\sigma} : P \to \hat{\Sigma} \subset \Lambda^k TM$ that preserves orientation, and defined by $\hat{\sigma}(t) = (\sigma(t), \dot{\sigma}(t))$, $\dot{\sigma}(t) = \frac{\partial(x^{\mu_1}\circ\sigma)}{\partial t^1}\Big|_t \cdots \frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^k}\Big|_t \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}}\right)_{\sigma(t)}$, $t \in \mathbb{R}^k$.*

Definition 5 *Taking the Kawaguchi form as a Lagrangian, we define the action with respect to the parameterisation by,*

$$\begin{aligned} S_P{}^{\mathcal{K}}[\sigma] &= \int_{\hat{\sigma}(P)} \mathcal{K} = \int_{\hat{\sigma}(P)} \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}} dx^{\mu_1\cdots\mu_k} \\ &= \int_{t_i^1}^{t_f^1} \cdots \int_{t_i^k}^{t_f^k} \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}} \circ \hat{\sigma}\, d(x^{\mu_1}\circ\hat{\sigma}) \wedge \cdots \wedge d(x^{\mu_k}\circ\hat{\sigma}) \\ &= \int_{t_i^1}^{t_f^1} \cdots \int_{t_i^k}^{t_f^k} K\left(x^{\mu}(\sigma(t)), \frac{\partial(x^{\mu_1}(\sigma(t))}{\partial t^1} \cdots \frac{\partial(x^{\mu_k}(\sigma(t))}{\partial t^k}\right) dt^1 \wedge \cdots \wedge dt^k. \end{aligned}$$

This action is independent of reparemeterisation, when the new parameterisation is related to the old by diffeomorphism, and preserves the orientation. We redefine our action by this equivalence class, $\tilde{S}_P^{\mathcal{K}} : Imm(P \to M)/\sim \to \mathbb{R}$.

Theorem 1 *Suppose we are given the action $\tilde{S}_P[[\sigma]]$. This gives an geometric k-area of the submanifold Σ. The condition which extremises this k-area with respect to small deformations of Σ is given in local coordinates as follows,*

$$\begin{cases} \left\{\dfrac{\partial^2 K}{\partial x^{\mu}\partial y^{\rho_1\cdots\rho_k}} dx^{\rho_1} - kd\left(\dfrac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}\right)\right\} \wedge dx^{\rho_2\cdots\rho_k} = 0, \\ y^{\nu\mu_2\cdots\mu_k}\left(\dfrac{\partial^2 K}{\partial y^{\mu_1\cdots\mu_k}\partial y^{\rho_1\cdots\rho_k}}\right) dx^{\rho_1\cdots\rho_k} = 0 \end{cases}$$

The formula will reduce to the standard Euler-Lagrange equation for fields, when pulled back to the parameter space.

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Keywords: Finsler geometry, Kawaguchi geometry, Lagrange formulation.

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THE HELMHOLTZ CONDITIONS FOR SYSTEMS OF SECOND ORDER HOMOGENOUS DIFFERENTIAL EQUATIONS

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Abstract

In this work we study variationality of systems of second order ordinary differential equations given by positive homogeneous functions. Euler-Lagrange equations associated with systems of this class admit positive homogeneous Lagrangian, and they have solutions which are set-solutions or in other words, integral curves which are independent of parametrization. From this point of view Lagrangians of the class of positive homogeneous and variational systems may represent fundamental functions for possible higher order generalizations of Finsler geometry.

Recently in [3], we have analysed by means of the geometric theory of jet differential groups the concept of positive homogeneity for functions depending on curves and their derivatives up to an arbitrary finite order. It appeared that this *higher order positive homogeneity* is equivalent with the well-known Zermelo conditions (see e.g. [4], [2], [1]), generalizing the standard Euler formula for positive homogeneous functions depending on curves and their first derivatives only. On this basis, every solution of a system of differential equations with left-hand sides given by positive homogeneous functions is a set-solution.

Our main results include: a) every positive homogeneous system of $m+1$ second order equations of $m+1$ dependent variables is linearly dependent, b) variationality of a system of $m+1$ second order differential equations, defined by positive homogenous functions, is equivalent with variationality of certain of its subsystem of m equations in sense of parametrized variational problems,

c) explicit relationship between Lagrangians of both of these systems is given. Finally, we give an example of two second order equations whose solution is a unit circle in $\mathbb{R}^2$, with analysis of variationality and positive homogeneity.

The methods can be extended to the theory of differential equations on manifolds, as well as to higher order systems. Examples in higher order dimension can be constructed analogously.

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Keywords: Variational differential equation, Helmholtz conditions, Lagrangian, Positive homogeneous function, Zermelo conditions, Set-solution of differential equation, Finsler geometry.

ON FINSLER GEOMETRY AND SOME OF ITS APPLICATIONS TO PHYSICS

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Abstract

Finsler geometry is a generalization of Riemannian geometry, which allows the metric tensor to depend on position and direction, in other words, on the coordinates on the tangent bundle of the considered manifold. The advantage is a greater possibility of modeling physical phenomena; still, some specific geometric structures (use of adapted frames to an Ehresmann connection instead of natural ones, some restrictions upon affine connections) are needed in order to simplify local computations, which would otherwise be very complicated. In the paper, we present in brief these - already classical - structures, together with our results, obtained in 2008-2011, regarding a Finslerian extension of classical electromagnetic field theory.

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Keywords: Finsler space, Tangent bundle, Electromagnetism.

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EXTENDED ABSTRACT BOOK

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