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The Gurevich theorem on invariant tensors: Elementary proof

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Abstract A simple proof of a fundamental theorem of the classical algebraic invariant theory, characterizing the structure of invariant tensors (tensors with constant components), the *Gurevich theorem*, is given. The proof is based on the trace decomposition theorem.

Keywords Tensor, Invariant, Trace, Permutation group

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1 Introduction

The goal of this note is to give an elementary proof of a fundamental result of the classical algebraic invariant theory, the *Gurevich theorem* on the structure of invariant tensors (tensors with constant components).

The classification problem for invariant tensors was studied by several authors, and different approaches and proofs were presented (cf. Gurevich [2], Theorem 16.2, Sec. 16.5). The problem was also considered as a part of the theory of differential invariants and natural operations in differential geometry (Krupka and Janyska [6], Sec. 4.1, Kolar, Michor and Slovak [3], Sec. 24.3), with proofs based essentially on the original Gurevich approach. The proof given in this paper is based on application of the trace decomposition theory of tensors over real, finite-dimensional vector spaces (Krupka [4], [5]) to invariant tensor equations, found by Gurevich. We solve these equations by simple immediate analysis; the method avoids, in particular, the Gurevich's approach regarding linear dependences appearing in the equations for higher valency tensors.

For basic notions and terminology related to invariants, matrix groups and permutation groups we refer to H. Weyl [7] and Alperin and Bell [1].

In Section 2 we first recall the *trace decomposition theorem* for (r,r)-tensors, then we state its second version, the *complete trace decomposition theorem*. For proofs and further comments we refer to Krupka [4], [5]. Section 3 contains the theory of tensors on the vector space $\mathbf{R}^n$, *invariant* under the tensor action of the general linear group. The *Gurevich* theorem, stating that invariant tensors are essentially determined by real-valued functions on the permutation group of tensor indices, is proved as a corollary to the complete trace decomposition theorem.

In this paper $T_s^r\mathbf{R}^n$ is the vector space of tensors of type (r,s) on the vector space $\mathbf{R}^n$ ((r,s)-*tensors*). For short, we usually express tensors $U \in T_s^r\mathbf{R}^n$ in terms of components in the *canonical basis* of $\mathbf{R}^n$; we simply write $U = U^{k_1k_2\dots k_r}{}_{l_1l_2\dots l_s}$. $GL_n(\mathbf{R})$ is the general linear group, with elements invertible matrices $A = A_j^i$ of dimension *n*; the *inverse* A^{-1} is in components denoted by $A^{-1} = \tilde{A}_j^i$. Thus, $A_j^i\tilde{A}_k^j = \delta_k^i$, where δ_k^i is the *Kronecker symbol*. The *permutation group* of the set of *r* numbers $\{1,2,\dots,r\}$ is denoted by S_r. Standard summation convention over repeated indices is used unless otherwise stated.

2 Trace decomposition of tensor spaces

Recall that a tensor is said to be *traceless*, if all its traces vanish. A *Kronecker tupe tensor*, or just a *Kronecker tensor*, is any tensor, generated by the Kronecker δ-tensor $\delta = \delta_j^i$.

Lemma 1 *Every tensor* $U \in T_r^r\mathbf{R}^n$, $U = U^{i_1i_2\dots i_r}{}_{j_1j_2\dots j_r}$, *is expressible in the form*

$$
\begin{aligned}
U^{i_1i_2\dots i_r}{}_{j_1j_2\dots j_r} &= W^{i_1i_2\dots i_r}{}_{j_1j_2\dots j_r} \\
&+ \delta_{j_1}^{i_1}\,{}_1^1V^{i_2i_3\dots i_r}{}_{j_2j_3\dots j_r} + \delta_{j_2}^{i_1}\,{}_2^1V^{i_2i_3\dots i_r}{}_{j_1j_3\dots j_r} + \dots + \delta_{j_r}^{i_1}\,{}_r^1V^{i_2i_3\dots i_r}{}_{j_1j_2\dots j_{r-1}} \\
&+ \delta_{j_1}^{i_2}\,{}_1^2V^{i_1i_3\dots i_r}{}_{j_2j_3\dots j_r} + \delta_{j_2}^{i_2}\,{}_2^2V^{i_1i_3\dots i_r}{}_{j_1j_3\dots j_r} + \dots + \delta_{j_r}^{i_2}\,{}_r^2V^{i_1i_3\dots i_r}{}_{j_1j_2\dots j_{r-1}} \\
&+ \dots \\
&+ \delta_{j_1}^{i_r}\,{}_1^rV^{i_1i_2\dots i_{r-1}}{}_{j_2j_3\dots j_r} + \delta_{j_2}^{i_r}\,{}_2^rV^{i_1i_2\dots i_{r-1}}{}_{j_1j_3\dots j_r} + \dots + \delta_{j_r}^{i_r}\,{}_r^rV^{i_1i_2\dots i_{r-1}}{}_{j_1j_2\dots j_{r-1}},
\end{aligned}
\tag{1}
$$

where $W = W^{i_1i_2\dots i_r}{}_{j_1j_2\dots j_r}$ *is a uniquely defined traceless tensor, and for every p and q such that* $1 \le p,q \le r$, ${}_q^pV = {}_q^pV^{i_1i_2\dots i_{r-1}}{}_{j_1j_2\dots j_{r-1}}$ *is a tensor, belonging to the tensor space* $T_{r-1}^{r-1}\mathbf{R}^n$.

The *trace decomposition formula* (1) can be viewed as a system of linear equations for the components $W^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_r}$ and ${}^p_q V^{i_1 i_2 \dots i_{r-1}}{}_{j_1 j_2 \dots j_{r-1}}$ provided $U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_r}$ is given. From the uniqueness of the *traceless part* $W^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_r}$ it follows that also the complementary *Kronecker part* is unique. In general, this fact does not imply the uniqueness of the tensors ${}^p_q V^{i_1 i_2 \dots i_{r-1}}{}_{j_1 j_2 \dots j_{r-1}}$. However, if $2r \le n+1$, these tensors are also unique.

Clearly, tensors ${}^p_q V^{i_1 i_2 \dots i_{r-1}}{}_{j_1 j_2 \dots j_{r-1}}$ can also be decomposed by the trace decomposition formula; repeating this procedure r times, we can derive from formula (1) another expression for (r,r)-tensors on $\mathbf{R}^n$.

Let p be a positive integer. A tensor $U \in T^r_r \mathbf{R}^n$, $U = U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_r}$, is said to be $\delta^{(p)}$ *-generated*, if it admits an expression

$$U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_r} = \sum_{\kappa, \lambda \in S_r} \delta^{i_{\lambda(1)}}_{j_{\kappa(1)}} \delta^{i_{\lambda(2)}}_{j_{\kappa(2)}} \dots \delta^{i_{\lambda(p)}}_{j_{\kappa(p)}} \, {}^{\lambda(1),\lambda(2),\dots,\lambda(p)}_{\kappa(1),\kappa(2),\dots,\kappa(p)} V^{i_{\lambda(p+1)} i_{\lambda(p+2)} \dots i_{\lambda(r)}}{}_{j_{\kappa(p+1)} j_{\kappa(p+2)} \dots j_{\kappa(r)}}$$

for some tensors ${}^{\lambda(1),\lambda(2),\dots,\lambda(p)}_{\kappa(1),\kappa(2),\dots,\kappa(p)} V \in T^{r-p}_{r-p} \mathbf{R}^n$ (indexed on the left),

$${}^{\lambda(1),\lambda(2),\dots,\lambda(p)}_{\kappa(1),\kappa(2),\dots,\kappa(p)} V = {}^{\lambda(1),\lambda(2),\dots,\lambda(p)}_{\kappa(1),\kappa(2),\dots,\kappa(p)} V^{i_1 i_2 \dots i_{r-p}}{}_{j_1 j_2 \dots j_{r-p}}.$$

If in addition the tensors ${}^{\lambda(1),\lambda(2),\dots,\lambda(p)}_{\kappa(1),\kappa(2),\dots,\kappa(p)} V$ are *traceless*, then U is said to be *primitive*.

Lemma 2 *Every tensor* $U \in T^r_r \mathbf{R}^n$, $U = U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_r}$, *is expressible in the form*

(2) $$U = {}^0U + {}^1U + {}^2U + \dots + {}^rU,$$

where ${}^0U$ *is a traceless tensor, and for every* $p = 1,2,\dots,r$, ${}^pU$ *is a* $\delta^{(p)}$*-generated primitive tensor. Decomposition* (2) *is unique.*

Tensors ${}^0U, {}^1U, {}^2U, \dots, {}^rU$ are called *primitive parts* of U.

3 Invariant tensors

Recall that the tensor space $T^r_s \mathbf{R}^n$ is endowed with the canonical left *tensor action* $GL_n(\mathbf{R}) \times T^r_s \mathbf{R}^n \ni (A,U) \to A \cdot U \in T^r_s \mathbf{R}^n$ of the general linear group $GL_n(\mathbf{R})$. Expressing tensors U and $A \cdot U$ as $U = U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_s}$ and $A \cdot U = \bar{U}^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_s}$, *equations* of the tensor action are

(3) $$\bar{U}^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s} = A^{k_1}_{i_1} A^{k_2}_{i_2} \dots A^{k_r}_{i_r} \tilde{A}^{j_1}_{l_1} \tilde{A}^{j_2}_{l_2} \dots \tilde{A}^{j_s}_{l_s} U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_s}.$$

A tensor $U \in T_s^r \mathbf{R}^n$ is said to be *invariant*, if $A \cdot U = U$ for all elements $A \in GL_n(\mathbf{R})$, that is,

(4) $$U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s} = A^{k_1}_{i_1} A^{k_2}_{i_2} \dots A^{k_r}_{i_r} \tilde{A}^{j_1}_{l_1} \tilde{A}^{j_2}_{l_2} \dots \tilde{A}^{j_s}_{l_s} U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_s}.$$

Invariant tensors constitute a vector subspace of the tensor space $T_s^r \mathbf{R}^n$.

The classification problem for (r,s)-tensors such that $r \neq s$, has only trivial solution.

Lemma 1 *If $r \neq s$, then a tensor $U \in T_s^r \mathbf{R}^n$ is invariant if and only if $U = 0$.*

Proof Suppose for instance that $r > s$. If A is a matrix with components $A^k_i = c\delta^k_i$, where $c \neq 0$, then the components of the inverse matrix A^{-1} are $\tilde{A}^k_i = (1/c)\delta^k_i$. In this case equations (3) reduce to

$$U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s} = c^{r-s} U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s},$$

with arbitrary c. But $r - s \neq 0$ hence $U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s} = 0$.

For $r = s$ invariance condition (4) reads

(5) $$A^{k_1}_{i_1} A^{k_2}_{i_2} \dots A^{k_r}_{i_r} \tilde{A}^{j_1}_{l_1} \tilde{A}^{j_2}_{l_2} \dots \tilde{A}^{j_r}_{l_r} U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_r} = U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_r}$$

or, equivalently,

(6) $$A^{k_1}_{l_1} A^{k_2}_{l_2} \dots A^{k_r}_{l_r} U^{l_1 l_2 \dots l_r}{}_{j_1 j_2 \dots j_r} = A^{l_1}_{j_1} A^{l_2}_{j_2} \dots A^{l_r}_{j_r} U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_r}.$$

Lemma 2 *Let $U \in T_r^r \mathbf{R}^n$, $U = U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_r}$. The following two conditions are equivalent:*

(a) *U is an invariant tensor.*

(b) *The components $U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_r}$ satisfy*

(7) $$\begin{aligned} &\sum_{\tau \in S_r} \delta^{i_{\tau(1)}}_{j_1} \delta^{i_{\tau(2)}}_{j_2} \dots \delta^{i_{\tau(r)}}_{j_r} U^{k_1 k_2 \dots k_r}{}_{l_{\tau(1)} l_{\tau(2)} \dots l_{\tau(r)}} \\ &\quad = \sum_{\tau \in S_r} \delta^{k_1}_{l_{\tau(1)}} \delta^{k_2}_{l_{\tau(2)}} \dots \delta^{k_r}_{l_{\tau(r)}} U^{i_{\tau(1)} i_{\tau(2)} \dots i_{\tau(r)}}{}_{j_1 j_2 \dots j_r}. \end{aligned}$$

Proof 1. Suppose that invariance condition (a), that is, equation (6), is satisfied. Consider the difference

$$A^{k_1}_{l_1} A^{k_2}_{l_2} \dots A^{k_r}_{l_r} U^{l_1 l_2 \dots l_r}{}_{j_1 j_2 \dots j_r} - A^{l_1}_{j_1} A^{l_2}_{j_2} \dots A^{l_r}_{j_r} U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_r}.$$

In this expression A^i_j enters as an arbitrary element of the group $GL_n(\mathbf{R})$, which can be considered as an open set in a vector space $\mathbf{R}^n \times (\mathbf{R}^n)^*$ of all square matrices $\theta \in \mathbf{R}^n \times (\mathbf{R}^n)^*$, $\theta = \theta^i_j$. But $U^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_r}$ defines a mapping of the vector space $\mathbf{R}^n \times (\mathbf{R}^n)^*$ into the tensor space $T^r_r \mathbf{R}^n$ by the equations

$$V^{k_1 k_2 \ldots k_r}{}_{p_1 p_2 \ldots p_r} = \theta^{i_1}_{p_1} \theta^{i_2}_{p_2} \ldots \theta^{i_r}_{p_r} U^{k_1 k_2 \ldots k_r}{}_{i_1 i_2 \ldots i_r} - \theta^{k_1}_{i_1} \theta^{k_2}_{i_2} \ldots \theta^{k_r}_{i_r} U^{i_1 i_2 \ldots i_r}{}_{p_1 p_2 \ldots p_r}.$$

Writing $V^{k_1 k_2 \ldots k_r}{}_{p_1 p_2 \ldots p_r}$ in terms of coefficients, we have

$$\begin{aligned} &V^{k_1 k_2 \ldots k_r}{}_{p_1 p_2 \ldots p_r} \\ &= \theta^{\alpha_1}_{\beta_1} \theta^{\alpha_2}_{\beta_2} \ldots \theta^{\alpha_r}_{\beta_r} (\delta^{i_1}_{\alpha_1} \delta^{i_2}_{\alpha_2} \ldots \delta^{i_r}_{\alpha_r} \delta^{\beta_1}_{p_1} \delta^{\beta_2}_{p_2} \ldots \delta^{\beta_r}_{p_r} U^{k_1 k_2 \ldots k_r}{}_{i_1 i_2 \ldots i_r} \\ &- \delta^{k_1}_{\alpha_1} \delta^{k_2}_{\alpha_2} \ldots \delta^{k_r}_{\alpha_r} \delta^{\beta_1}_{i_1} \delta^{\beta_2}_{i_2} \ldots \delta^{\beta_r}_{i_r} U^{i_1 i_2 \ldots i_r}{}_{p_1 p_2 \ldots p_r}) \\ &= \theta^{\alpha_1}_{\beta_1} \theta^{\alpha_2}_{\beta_2} \ldots \theta^{\alpha_r}_{\beta_r} (\delta^{\beta_1}_{p_1} \delta^{\beta_2}_{p_2} \ldots \delta^{\beta_r}_{p_r} U^{k_1 k_2 \ldots k_r}{}_{\alpha_1 \alpha_2 \ldots \alpha_r} \\ &- \delta^{k_1}_{\alpha_1} \delta^{k_2}_{\alpha_2} \ldots \delta^{k_r}_{\alpha_r} U^{\beta_1 \beta_2 \ldots \beta_r}{}_{p_1 p_2 \ldots p_r}) \end{aligned}$$

or, with coefficients, symmetrized in the pairs of indices ${}^{\alpha_i}_{\beta_i}$,

$$\begin{aligned} &V^{k_1 k_2 \ldots k_r}{}_{p_1 p_2 \ldots p_r} \\ &= \theta^{\alpha_1}_{\beta_1} \theta^{\alpha_2}_{\beta_2} \ldots \theta^{\alpha_r}_{\beta_r} \sum_{\tau \in S_r} (\delta^{\beta_{\tau(1)}}_{p_1} \delta^{\beta_{\tau(2)}}_{p_2} \ldots \delta^{\beta_{\tau(r)}}_{p_r} U^{k_1 k_2 \ldots k_r}{}_{\alpha_{\tau(1)} \alpha_{\tau(2)} \ldots \alpha_{\tau(r)}} \\ &- \delta^{k_1}_{\alpha_{\tau(1)}} \delta^{k_2}_{\alpha_{\tau(2)}} \ldots \delta^{k_r}_{\alpha_{\tau(r)}} U^{\beta_{\tau(1)} \beta_{\tau(2)} \ldots \beta_{\tau(r)}}{}_{p_1 p_2 \ldots p_r}). \end{aligned} \tag{8}$$

Now if U is invariant, then $V^{k_1 k_2 \ldots k_r}{}_{p_1 p_2 \ldots p_r} = 0$ because $GL_n(\mathbf{R})$ is dense in $\mathbf{R}^n \times (\mathbf{R}^n)^*$ and the function $V^{k_1 k_2 \ldots k_r}{}_{p_1 p_2 \ldots p_r}$ is continuous; then, however, since θ^α_β are arbitrary,

$$\begin{aligned} &\sum_{\tau \in S_r} (\delta^{\beta_{\tau(1)}}_{p_1} \delta^{\beta_{\tau(2)}}_{p_2} \ldots \delta^{\beta_{\tau(r)}}_{p_r} U^{k_1 k_2 \ldots k_r}{}_{\alpha_{\tau(1)} \alpha_{\tau(2)} \ldots \alpha_{\tau(r)}} \\ &- \delta^{k_1}_{\alpha_{\tau(1)}} \delta^{k_2}_{\alpha_{\tau(2)}} \ldots \delta^{k_r}_{\alpha_{\tau(r)}} U^{\beta_{\tau(1)} \beta_{\tau(2)} \ldots \beta_{\tau(r)}}{}_{p_1 p_2 \ldots p_r}) = 0. \end{aligned} \tag{9}$$

Now formula (7) is obtained from (9) merely by changing the index notation $\beta \to i$, $p \to j$, and $\alpha \to l$.

2. Condition (b) implies (a) by means of (8).

Equivalent conditions (5), (6) and (7) are *equations of invariant tensors*. For every permutation $\sigma \in S_r$, set $\Delta_\sigma = \Delta_\sigma^{k_1 k_2 \ldots k_r}{}_{\alpha_1 \alpha_2 \ldots \alpha_r}$, where

$$\Delta_\sigma^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r} = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \ldots \delta^{i_r}_{j_{\sigma(r)}}.$$

Δ_σ can equivalently be defined by $\Delta_\sigma{}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r} = 1$ if $i_p = j_{\sigma(p)}$ for all $p = 1, 2, \ldots, n$, and $\Delta_\sigma{}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r} = 0$ otherwise.

A notable property of the tensors Δ_σ is that their components define the *stabilizers* of the *r*-tuples of indices $i_1, i_2, \ldots, i_r$ with respect to the action of the permutation group S_r. Equation of the stabilizer of $i_1, i_2, \ldots, i_r$ is

$$\Delta_\sigma{}^{i_1 i_2 \ldots i_r}{}_{i_1 i_2 \ldots i_r} = \delta^{i_1}_{i_{\sigma(1)}} \delta^{i_2}_{i_{\sigma(2)}} \ldots \delta^{i_r}_{i_{\sigma(r)}} = 1 .$$

Lemma 3 (a) $\Delta_\sigma = \Delta_\sigma{}^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_r}$ *is an invariant tensor*.

(b) *Any linear combination*

$$U = \sum_{\sigma \in S_r} c_\sigma \Delta_\sigma , \tag{10}$$

where $c_\sigma \in \mathbf{R}$ *, is an invariant tensor*.

Proof (a) To prove Lemma 3, we use equation (7). Setting

$$U^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_r} = \Delta_\sigma{}^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_r} = \delta^{k_1}_{l_{\sigma(1)}} \delta^{k_2}_{l_{\sigma(2)}} \ldots \delta^{k_r}_{l_{\sigma(r)}} ,$$

we have $U^{k_1 k_2 \ldots k_r}{}_{l_{\tau(1)} l_{\tau(2)} \ldots l_{\tau(r)}} = \delta^{k_1}_{l_{\sigma\tau(1)}} \delta^{k_2}_{l_{\sigma\tau(2)}} \ldots \delta^{k_r}_{l_{\sigma\tau(r)}}$ and, with substitution $l_{\tau(p)} = j_p$,

$$\begin{aligned} U^{k_1 k_2 \ldots k_r}{}_{l_{\tau(1)} l_{\tau(2)} \ldots l_{\tau(r)}} &= U^{k_1 k_2 \ldots k_r}{}_{j_1 j_2 \ldots j_r} = \delta^{k_1}_{j_{\sigma(1)}} \delta^{k_2}_{j_{\sigma(2)}} \ldots \delta^{k_r}_{j_{\sigma(r)}} \\ &= \delta^{k_1}_{l_{\tau\sigma(1)}} \delta^{k_2}_{l_{\tau\sigma(2)}} \ldots \delta^{k_r}_{l_{\tau\sigma(r)}} . \end{aligned}$$

Substituting these expressions into (7), we get for the left-hand side

$$\begin{aligned} &\sum_{\tau \in S_r} \delta^{i_{\tau(1)}}_{j_1} \delta^{i_{\tau(2)}}_{j_2} \ldots \delta^{i_{\tau(r)}}_{j_r} U^{k_1 k_2 \ldots k_r}{}_{l_{\tau(1)} l_{\tau(2)} \ldots l_{\tau(r)}} \\ &= \sum_{\tau \in S_r} \delta^{i_{\tau(1)}}_{j_1} \delta^{i_{\tau(2)}}_{j_2} \ldots \delta^{i_{\tau(r)}}_{j_r} \delta^{k_1}_{l_{\tau\sigma(1)}} \delta^{k_2}_{l_{\tau\sigma(2)}} \ldots \delta^{k_r}_{l_{\tau\sigma(r)}} . \end{aligned}$$

The right-hand side is

$$\begin{aligned} &\sum_{\tau \in S_r} \delta^{k_1}_{l_{\tau(1)}} \delta^{k_2}_{l_{\tau(2)}} \ldots \delta^{k_r}_{l_{\tau(r)}} U^{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(r)}}{}_{j_1 j_2 \ldots j_r} \\ &= \sum_{\tau \in S_r} \delta^{k_1}_{l_{\tau\sigma(1)}} \delta^{k_2}_{l_{\tau\sigma(2)}} \ldots \delta^{k_r}_{l_{\tau\sigma(r)}} \delta^{i_{\tau\sigma(1)}}_{j_{\sigma(1)}} \delta^{i_{\tau\sigma(2)}}_{j_{\sigma(2)}} \ldots \delta^{i_{\tau\sigma(r)}}_{j_{\sigma(r)}} \\ &= \sum_{\tau \in S_r} \delta^{k_1}_{l_{\tau\sigma(1)}} \delta^{k_2}_{l_{\tau\sigma(2)}} \ldots \delta^{k_r}_{l_{\tau\sigma(r)}} \delta^{i_{\tau(1)}}_{j_1} \delta^{i_{\tau(2)}}_{j_2} \ldots \delta^{i_{\tau(r)}}_{j_r} , \end{aligned}$$

proving invariance.

(b) Obvious.

Invariant tensors $\Delta_{(\sigma)}$ are called *elementary invariant tensors*.

Lemma 4 *If* $r \le n$ *, then every invariant tensor* $U \in T^r_r\mathbf{R}^n$ *has a unique expression* (10).

Proof Consider the subsystem of (7) defined by $i_p = j_p = p$,

$$\sum_{\tau \in S_r} \delta_1^{\tau(1)}\delta_2^{\tau(2)}\dots\delta_r^{\tau(r)} U^{k_1 k_2 \dots k_r}{}_{l_{\tau(1)} l_{\tau(2)} \dots l_{\tau(r)}}$$
$$= \sum_{\tau \in S_r} \delta^{k_1}_{l_{\tau(1)}}\delta^{k_2}_{l_{\tau(2)}}\dots\delta^{k_r}_{l_{\tau(r)}} U^{\tau(1)\tau(2)\dots\tau(r)}{}_{12\dots r}.$$

But $\delta_1^{\tau(1)}\delta_2^{\tau(2)}\dots\delta_r^{\tau(r)} \ne 0$ if and only if $\tau(1)=1, \tau(2)=2,\dots,\tau(r)=r$; hence τ must be the identity permutation, and we have

$$U^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_r} = \sum_{\tau \in S_r} \delta^{k_1}_{l_{\tau(1)}}\delta^{k_2}_{l_{\tau(2)}}\dots\delta^{k_r}_{l_{\tau(r)}} U^{\tau(1)\tau(2)\dots\tau(r)}{}_{12\dots r}.$$

Setting $c_\tau = U^{\tau(1)\tau(2)\dots\tau(r)}{}_{12\dots r}$ we get formula (10). To prove uniqueness of this decomposition, suppose that

$$\sum_{\sigma \in S_r} c_\sigma \Delta_\sigma = 0,$$

that is, in components,

$$\sum_{\tau \in S_r} c_\tau \delta^{k_1}_{l_{\tau(1)}}\delta^{k_2}_{l_{\tau(2)}}\dots\delta^{k_r}_{l_{\tau(r)}} = 0.$$

Then the component where $k_p = l_{\tau(p)} = p$ yields $c_\tau = 0$.

For $r \le n$, formula (10) clarifies the meaning of the coefficients:

$$U^{12\dots r}{}_{\nu(1)\nu(2)\dots\nu(r)} = \sum_{\sigma \in S_r} c_\sigma \Delta^{12\dots r}_{\sigma}{}_{\nu(1)\nu(2)\dots\nu(r)}$$
$$= \sum_{\sigma \in S_r} c_\sigma \delta^1_{\nu\sigma(1)}\delta^2_{\nu\sigma(2)}\dots\delta^r_{\nu\sigma(r)} = \sum_{\sigma \in S_r} c_\sigma \delta^{\sigma^{-1}(1)}_{\nu(1)}\delta^{\sigma^{-1}(2)}_{\nu(2)}\dots\delta^{\sigma^{-1}(r)}_{\nu(r)}$$
$$= c_{\nu^{-1}}.$$

We can also write $U^{\nu^{-1}(1)\nu^{-1}(2)\dots\nu^{-1}(r)}{}_{12\dots r} = c_{\nu^{-1}}$, thus, for every permutation τ ,

$$U^{\tau(1)\tau(2)\ldots\tau(r)}{}_{12\ldots r} = c_\tau .$$

Clearly, for any mutually different indices $k_1, k_2, \ldots, k_r$,

$$\begin{aligned} U^{k_1k_2\ldots k_r}{}_{k_{\tau(1)}k_{\tau(2)}\ldots k_{\tau(r)}} &= \sum_{\sigma\in S_r} c_\sigma \Delta_\sigma^{k_1k_2\ldots k_r}{}_{k_{\tau(1)}k_{\tau(2)}\ldots k_{\tau(r)}} \\ &= \sum_{\sigma\in S_r} c_\sigma \delta^{k_1}_{k_{\tau\sigma(1)}} \delta^{k_2}_{k_{\tau\sigma(2)}} \ldots \delta^{k_r}_{k_{\tau\sigma(r)}} = \sum_{\sigma\in S_r} c_\sigma \delta^1_{\tau\sigma(1)} \delta^2_{\tau\sigma(2)} \ldots \delta^r_{\tau\sigma(r)} \\ &= \sum_{\sigma\in S_r} c_\sigma \delta^{\sigma^{-1}(1)}_{\tau(1)} \delta^{\sigma^{-1}(2)}_{\tau(2)} \ldots \delta^{\sigma^{-1}(r)}_{\tau(r)} = c_{\tau^{-1}} . \end{aligned}$$

If $k_1, k_2, \ldots, k_r$ are *not* mutually different, then

$$\begin{aligned} U^{k_1k_2\ldots k_r}{}_{k_{\tau(1)}k_{\tau(2)}\ldots k_{\tau(r)}} &= \sum_{\sigma\in S_r} c_\sigma \Delta_\sigma^{k_1k_2\ldots k_r}{}_{k_{\tau(1)}k_{\tau(2)}\ldots k_{\tau(r)}} \\ &= \sum_{\sigma\in S_r} c_\sigma \delta^{k_1}_{k_{\tau\sigma(1)}} \delta^{k_2}_{k_{\tau\sigma(2)}} \ldots \delta^{k_r}_{k_{\tau\sigma(r)}} = \sum_{\sigma\in S_r} c_\sigma \delta^{k_{\sigma^{-1}(1)}}_{k_{\tau(1)}} \delta^{k_{\sigma^{-1}(2)}}_{k_{\tau(2)}} \ldots \delta^{k_{\sigma^{-1}(r)}}_{k_{\tau(r)}} \\ &= c_{\sigma_1} + c_{\sigma_2} + \ldots + c_{\sigma_N} , \end{aligned}$$

where $\sigma_1, \sigma_2, \ldots, \sigma_N$ are elements of the stabilizer of the *r*-tuple $k_1, k_2, \ldots, k_r$. In particular, this number does not depend on τ .

Note that equations $c_\sigma = c_{\sigma^{-1}}$ are *symmetry conditions* for the matrix $U^{l_1l_2\ldots l_r}{}_{k_1k_2\ldots k_r}$:

$$\begin{aligned} U^{l_1l_2\ldots l_r}{}_{k_1k_2\ldots k_r} &= \sum_{\sigma\in S_r} c_\sigma \Delta_\sigma^{l_1l_2\ldots l_r}{}_{k_1k_2\ldots k_r} = \sum_{\sigma\in S_r} c_{\sigma^{-1}} \Delta_{\sigma^{-1}}^{l_1l_2\ldots l_r}{}_{k_1k_2\ldots k_r} \\ &= \sum_{\sigma\in S_r} c_{\sigma^{-1}} \Delta_\sigma^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r} . \end{aligned}$$

System (7) splits in two autonomous subsystems:

Lemma 5 *Let* $U \in T_r^r \mathbf{R}^n$, $U = U^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r}$ *, be a tensor. The following two conditions are equivalent:*

(a) *U is an invariant tensor.*

(b) *The components* $U^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r}$ *, such that* $k_1, k_2, \ldots, k_r$ *is not a permutation of* $l_1, l_2, \ldots, l_r$ *, satisfy*

$$U^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r} = 0, \tag{11}$$

and, if the superscripts are permutations of the subscripts, then

$$\sum_{\tau\in S_r}\delta^{i_{\tau(1)}}_{i_{\nu(1)}}\delta^{i_{\tau(2)}}_{i_{\nu(2)}}\ldots\delta^{i_{\tau(r)}}_{i_{\nu(r)}}U^{l_{\mu(1)}l_{\mu(2)}\ldots l_{\mu(r)}}{}_{l_{\tau(1)}l_{\tau(2)}\ldots l_{\tau(r)}}$$
$$(12)\qquad =\sum_{\tau\in S_r}\delta^{l_{\mu(1)}}_{l_{\tau(1)}}\delta^{l_{\mu(2)}}_{l_{\tau(2)}}\ldots\delta^{l_{\mu(r)}}_{l_{\tau(r)}}U^{i_{\tau(1)}i_{\tau(2)}\ldots i_{\tau(r)}}{}_{i_{\nu(1)}i_{\nu(2)}\ldots i_{\nu(r)}}.$$

Proof 1. Suppose that $k_1,k_2,\ldots k_r$ is not a permutation of $l_1,l_2,\ldots,l_r$. Consider equation (7) determined by the choice $j_p=i_p=l_p$. Since the left-hand side becomes $N(l_1l_2\ldots l_r)U^{k_1k_2\ldots k_r}{}_{l_{\tau(1)}l_{\tau(2)}\ldots l_{\tau(r)}}$, where $N(l_1l_2\ldots l_r)$ is the dimension of the stabilizer of the r-tuple $l_1,l_2,\ldots,l_r$, and the right-hand side vanishes, we get (11).

2. Subsystem of (7), satisfying condition (b), is exactly the system (12).

In equations (12) the subscripts and superscripts run through the index set $1,2,\ldots,n$, and $\mu,\nu\in S_r$ are arbitrary permutations; thus, we have $(r!)^2n^{2r}$ equations for $r!n^r$ unknowns the components $U^{l_1l_2\ldots l_r}{}_{l_{\kappa(1)}l_{\kappa(2)}\ldots l_{\kappa(r)}}$.

Finally, note the following property of invariant tensors, arising from invariance with respect to suitable one-parameter subgroups of the general linear group.

Lemma 6 *Let $U\in T^r_r\mathbf{R}^n$, $U=U^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r}$, be an invariant tensor. Then for all indices p,q and $k_1,k_2,\ldots,k_r$, $l_1,l_2,\ldots,l_r$, the components $U^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r}$, satisfy*

$$\delta^{k_1}_pU^{qk_2k_3\ldots k_r}{}_{j_1j_2\ldots j_r}+\delta^{k_2}_pU^{k_1qk_3k_4\ldots k_r}{}_{j_1j_2\ldots j_r}+\ldots+\delta^{k_r}_pU^{k_1k_2\ldots k_{r-1}q}{}_{j_1j_2\ldots j_r}$$
$$=\delta^q_{j_1}U^{k_1k_2\ldots k_r}{}_{pj_2j_3\ldots j_r}+\delta^q_{j_2}U^{k_1k_2\ldots k_r}{}_{j_1pj_3j_4\ldots j_r}+\ldots+\delta^q_{j_r}U^{k_1k_2\ldots k_r}{}_{j_1j_2\ldots j_{r-1}p}.$$

Proof Consider equations of invariant tensors (6). Differentiating the left-hand side with respect to A^p_q at $A^p_q=\delta^p_q$ yields

$$\delta^{k_1}_p\delta^q_{l_1}\delta^{k_2}_{l_2}\delta^{k_3}_{l_3}\ldots\delta^{k_r}_{l_r}U^{l_1l_2l_3\ldots l_r}{}_{j_1j_2\ldots j_r}+\delta^{k_1}_{l_1}\delta^{k_2}_p\delta^q_{l_2}\delta^{k_3}_{l_3}\ldots\delta^{k_r}_{l_r}U^{l_1l_2l_3\ldots l_r}{}_{j_1j_2\ldots j_r}$$
$$+\ldots+\delta^{k_1}_{l_1}\delta^{k_2}_{l_2}\ldots\delta^{k_{r-1}}_{l_{r-1}}\delta^{k_r}_p\delta^q_{l_r}U^{l_1l_2\ldots l_{r-1}l_r}{}_{j_1j_2\ldots j_r}$$
$$=\delta^{k_1}_pU^{qk_2k_3\ldots k_r}{}_{j_1j_2\ldots j_r}+\delta^{k_2}_pU^{k_1qk_3k_4\ldots k_r}{}_{j_1j_2\ldots j_r}+\ldots+\delta^{k_r}_pU^{k_1k_2\ldots k_{r-1}q}{}_{j_1j_2\ldots j_r},$$

and similarly for the right-hand side

$$\delta^{l_1}_p\delta^q_{j_1}\delta^{l_2}_{j_2}\delta^{l_3}_{j_3}\ldots\delta^{l_r}_{j_r}U^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r}+\delta^{l_1}_{j_1}\delta^{l_2}_p\delta^q_{j_2}\delta^{l_3}_{j_3}\delta^{l_4}_{j_4}\ldots\delta^{l_r}_{j_r}U^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r}$$
$$+\ldots+\delta^{l_1}_{j_1}\delta^{l_2}_{j_2}\ldots\delta^{l_{r-1}}_{j_{r-1}}\delta^{l_r}_p\delta^q_{j_r}U^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_r}$$
$$=\delta^q_{j_1}U^{k_1k_2\ldots k_r}{}_{pj_2j_3\ldots j_r}+\delta^q_{j_2}U^{k_1k_2\ldots k_r}{}_{l_1pj_3j_4\ldots j_r}+\ldots+\delta^q_{j_r}U^{k_1k_2\ldots k_r}{}_{j_1j_2\ldots j_{r-1}p}.$$

Now we are in a position to state and prove the fundamental theorem on the structure of invariant tensors. The proof is based on the trace decomposition theory, particularized to invariant tensors.

Theorem (Gurevich) *Let* $U \in T_s^r \mathbf{R}^n$, $U = U^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_r}$, *be a tensor.*
(a) *If* $r \neq s$, *then* U *is invariant if and only if*

$$U = 0.$$

(b) *Suppose that* $r = s$. *Then* U *is invariant if and only if there exist numbers* $a_\sigma \in \mathbf{R}$, *where* $\sigma \in S_r$, *such that*

$$U = \sum_{\sigma \in S_r} a_\sigma \Delta_\sigma . \tag{13}$$

Proof Assertion (a), and also sufficiency of condition (13), have already been proved (Lemma 1, Lemma 3). Thus, only necessity of condition (13) needs proof.

1. For any matrix $A \in GL_n(\mathbf{R})$, $A = A_j^i$, and any tensor $U \in T_r^r \mathbf{R}^n$, $U = U^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$, denote for short $\tilde{U} = \tilde{U}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$, where

$$\tilde{U}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r} = A_{i_1}^{k_1} A_{i_2}^{k_2} \ldots A_{i_r}^{k_r} \tilde{A}_{l_1}^{j_1} \tilde{A}_{l_2}^{j_2} \ldots \tilde{A}_{l_r}^{j_r} U^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r} .$$

It is easily seen that if $U^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$ is $\delta^{(p)}$-generated, then also the tensor $\tilde{U}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$ is $\delta^{(p)}$-generated. Consider a $\delta^{(p)}$-generated tensor U of the form

$$\begin{aligned} &U^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r} \\ &= \sum_{\kappa, \lambda \in S_r} \delta^{i_{\lambda(1)}}_{j_{\kappa(1)}} \delta^{i_{\lambda(2)}}_{j_{\kappa(2)}} \ldots \delta^{i_{\lambda(p)}}_{j_{\kappa(p)}} {}^{\lambda(1), \lambda(2), \ldots, \lambda(p)}_{\kappa(1), \kappa(2), \ldots, \kappa(p)} V^{i_{\lambda(p+1)} i_{\lambda(p+2)} \ldots i_{\lambda(r)}}{}_{j_{\kappa(p+1)} j_{\kappa(p+2)} \ldots j_{\kappa(r)}} . \end{aligned}$$

The component $\tilde{U}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$ then includes a factor $A_{i_1}^{k_1} A_{i_2}^{k_2} \ldots A_{i_r}^{k_r} \tilde{A}_{l_1}^{j_1} \tilde{A}_{l_2}^{j_2} \ldots \tilde{A}_{l_r}^{j_r}$ and a factor

$$\begin{aligned} &A^{k_{\lambda(1)}}_{i_{\lambda(1)}} A^{k_{\lambda(2)}}_{i_{\lambda(2)}} \ldots A^{k_{\lambda(p)}}_{i_{\lambda(p)}} \tilde{A}^{j_{\kappa(1)}}_{l_{\kappa(1)}} \tilde{A}^{j_{\kappa(2)}}_{l_{\kappa(2)}} \ldots \tilde{A}^{j_{\kappa(p)}}_{l_{\kappa(p)}} \delta^{i_{\lambda(1)}}_{j_{\kappa(1)}} \delta^{i_{\lambda(2)}}_{j_{\kappa(2)}} \ldots \delta^{i_{\lambda(p)}}_{j_{\kappa(p)}} \\ &= A^{k_{\lambda(1)}}_{i_{\lambda(1)}} \tilde{A}^{j_{\kappa(1)}}_{l_{\kappa(1)}} \delta^{i_{\lambda(1)}}_{j_{\kappa(1)}} A^{k_{\lambda(2)}}_{i_{\lambda(2)}} \tilde{A}^{j_{\kappa(2)}}_{l_{\kappa(2)}} \delta^{i_{\lambda(2)}}_{j_{\kappa(2)}} \ldots A^{k_{\lambda(p)}}_{i_{\lambda(p)}} \tilde{A}^{j_{\kappa(p)}}_{l_{\kappa(p)}} \delta^{i_{\lambda(p)}}_{j_{\kappa(p)}} \\ &= A^{k_{\lambda(1)}}_{s_1} \tilde{A}^{s_1}_{l_{\kappa(1)}} A^{k_{\lambda(2)}}_{s_1} \tilde{A}^{s_1}_{l_{\kappa(2)}} \ldots A^{k_{\lambda(p)}}_{s_p} \tilde{A}^{s_p}_{l_{\kappa(p)}} \\ &= \delta^{k_{\lambda(1)}}_{l_{\kappa(1)}} \delta^{k_{\lambda(2)}}_{l_{\kappa(2)}} \ldots \delta^{k_{\lambda(p)}}_{l_{\kappa(p)}} . \end{aligned}$$

Thus $\tilde{U}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$ is $\delta^{(p)}$-generated.

2. Now let $U = U^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$ be an arbitrary *invariant* tensor. Consider the complete trace decomposition $U = {}^0U + {}^1U + {}^2U + \ldots + {}^rU$ (Lemma 2). Writing $\tilde{U}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r} = A^{k_1}_{i_1} A^{k_2}_{i_2} \ldots A^{k_r}_{i_r} \tilde{A}^{j_1}_{l_1} \tilde{A}^{j_2}_{l_2} \ldots \tilde{A}^{j_r}_{l_r} U^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$ as above, we have by hypothesis $U = \tilde{U}$ hence

$$ {}^0U + {}^1U + {}^2U + \ldots + {}^rU = {}^0\tilde{U} + {}^1\tilde{U} + {}^2\tilde{U} + \ldots + {}^r\tilde{U}. $$

Then, however, from Part 1 of this proof and Lemma 2, ${}^pU = {}^p\tilde{U}$ for all p. We wish to show that invariance of U implies

$$ (14) \qquad {}^0U = 0, \quad {}^2U = 0, \quad {}^2U = 0, \quad \ldots, \quad {}^{r-1}U = 0. $$

First we show that ${}^0U = 0$. Consider equations of invariant tensors (7)

$$ \sum_{\tau \in S_r} \delta^{i_{\tau(1)}}_{j_1} \delta^{i_{\tau(2)}}_{j_2} \ldots \delta^{i_{\tau(r)}}_{j_r} U^{k_1 k_2 \ldots k_r}{}_{l_{\tau(1)} l_{\tau(2)} \ldots l_{\tau(r)}} = \sum_{\tau \in S_r} \delta^{k_1}_{l_{\tau(1)}} \delta^{k_2}_{l_{\tau(2)}} \ldots \delta^{k_r}_{l_{\tau(r)}} U^{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(r)}}{}_{j_1 j_2 \ldots j_r}. $$

and apply this condition to the traceless tensor ${}^0U = W^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_r}$. Then expression on the left-hand side can be viewed, for any fixed indices $i_1, i_2, \ldots, i_r$, $j_1, j_2, \ldots, j_r$ as a tensor indexed with superscripts $k_1, k_2, \ldots, k_r$ and subscripts $l_1, l_2, \ldots, l_r$. But this tensor is both traceless and Kronecker, thus,

$$ (15) \qquad \sum_{\tau \in S_r} \delta^{i_{\tau(1)}}_{j_1} \delta^{i_{\tau(2)}}_{j_2} \ldots \delta^{i_{\tau(r)}}_{j_r} W^{k_1 k_2 \ldots k_r}{}_{l_{\tau(1)} l_{\tau(2)} \ldots l_{\tau(r)}} = 0. $$

Clearly, this equation holds for all values of the indices. Restricting this system to a subsystem defined by the indices $i_p = j_p = l_p$, we have

$$ \sum_{\tau \in S_r} \delta^{i_{\tau(1)}}_{i_1} \delta^{i_{\tau(2)}}_{i_2} \ldots \delta^{i_{\tau(r)}}_{i_r} W^{k_1 k_2 \ldots k_r}{}_{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(r)}} = 0. $$

Summation in this formula reduces to permutations $\tau \in S_r$ for which $\delta^{i_1}_{i_{\tau(1)}} \delta^{i_2}_{i_{\tau(2)}} \ldots \delta^{i_r}_{i_{\tau(r)}} = 1$, that is, to the stabilizer of the r-tuple $i_1, i_2, \ldots, i_r$. Then, however, $N(i_1 i_2 \ldots i_r) W^{k_1 k_2 \ldots k_r}{}_{i_1 i_2 \ldots i_r} = 0$, where $N(i_1 i_2 \ldots i_r)$ is the dimension of the stabilizer, hence $W^{k_1 k_2 \ldots k_r}{}_{i_1 i_2 \ldots i_r} = 0$, that is, ${}^0U = 0$.

The tensors ${}^pU$, where $p = 1, 2, \ldots, r-1$, can be considered in the same way. Setting ${}^pU = W^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_r}$ we get a tensor which is primitive and $\delta^{(p)}$-generated, but also $\delta^{(r)}$-generated. This is only possible when (15) holds, so we get $W^{k_1 k_2 \ldots k_r}{}_{i_1 i_2 \ldots i_r} = 0$, that is, ${}^pU = 0$.

3. Taking into account formulas (14), the complete trace decomposition formula for an invariant tensor $U = U^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r}$ yields

$$U = {}^{r}U.$$

Consequently, U must be of the form (13).

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