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Decompositions of covariant tensor spaces

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Abstract This paper is devoted to the problem of invariant decompositions of a tensor space into its subspaces. The method we apply consists in the construction of all invariant (natural) projection operators. The decomposability indicatrix is introduced as a simple tool to study decomposability equation for natural projectors. The theory gives a complete classification of natural projectors and their decomposability properties. It also provides a complete description of invariant partitions of the underlying tensor space.

Keywords Invariant tensor, Natural endomorphism, Projector, Partition of vector space

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1 Introduction

Our objective in this paper is the problem of invariant decompositions of a tensor space over a real, finite-dimensional vector space into its subspaces. This topic includes a classification of invariant (*natural*) projectors and invariant partitions of tensor spaces. As adequate means needed for the study of these questions we consider linear algebra of the projection operators together with the theory of invariant endomorphisms of tensor spaces.

In this paper we follow basic definitions and characteristics of natural projectors as introduced in Krupka [7]; we wish to complete the system of bilinear equations for natural projectors by the method how this system can effectively be solved.

To this purpose fundamentals of the theory of projectors in real, finite-dimensional vector spaces are needed; our basic references for elementary concepts are the books Halmos [4] and Kurosh [10] (Russian). Generalities on geometric invariants and natural operations as used in this work can be found in Krupka and Janyska [9]. For specific topics of the projector theory we refer to three sources, Yanai, Takeuchi, and Takane [12], Corporal and Regensburger [2], and Conrad [1]. Main classical source for new research in this field, based on the notion of a *tensor with constant components*, is Gurevich [3]. Additional remarks on the relationship between tensors with constant components and *invariant tensors*, including a proof of a basic Gurevich theorem, are presented in Krupka [8].

The problem of invariant decomposition of a tensor space was formulated by H. Weyl in 1938 (see Weyl [9], Preface) as a central part of the classical group representation theory. Since then, many aspects of this theory become a standard topic of research papers and monograph. It seems, however, that the original problem has not been reviewed completely: some aspects of the decomposition theory such as calculations of dimensions of the summands in the direct sum decompositions, or the treatment of mixed (r,s)-tensors still remain aside.

In Section 2 we recall for convenience the definitions and main properties of the projector theory in finite-dimensional vector spaces. Section 3 is devoted to partitions of vectors spaces; we introduce a *generalization* of this notion, covering the case when the projectors, entering a partition, should belong to a given family of projectors. This leads to an important concept of *decomposability indicatrix*, allowing us, on the basis of dimensions of underlying vector spaces, to find a class of necessary conditions for classifying decomposable projectors. Proofs in Section 1 and Section 2 are omitted.

Next two sections are devoted to natural projectors in covariant tensor spaces, the topic usually considered within the group representation theory. In Section 4 we derive *equations* of natural projectors; if the valency of tensors we consider is less or equal to the dimension of the underlying vector space, then the rank condition allows us to express these equations as a system of bilinear equations for the (finitely many) components of invariant projectors. The space of solutions represent *all* natural projectors for the given tensor valency. We also determine the *dimensions* of image spaces of the natural projectors. In Section 5 the decomposability theory of natural projectors is presented and applied to the partition problem of covariant tensor spaces. Given dimensions of the image spaces, main idea consists in the construction of the *decomposability indicatrix*, the set of pairs of positive integers, equal to the dimensions of the image spaces of the natural projectors. A simple comparison of dimensions then leads to *necessary* decomposability conditions for a given natual projector, and to the *decomposability equations* for the components of a natural projector. Using these equations, one can discover all *decomposable* natural projectors as well as the *primitive* natural projectors, which do not admit a non-trivial decomposition.

In two subsequent papers, we give a complete classification of natural projectors for $(0,3)$-tensors, and all decompositions and partitions of these tensors (Krupka [5]), and similarly for $(1,2)$-tensors ("torsions") (Krupka [6]). These two papers demonstrate, in particular, differences between the natural projector theory and the group representation projector theory.

2 Projectors

Throughout this section, E is a real vector space of dimension n; its identity endomorphism of E is denoted by Id_E.

An endomorphism P of the vector space E is said to be a *projector*, if

$$(1) \qquad P^2 = P.$$

If this condition is satisfied, then $(\mathrm{Id}_E - P)^2 = \mathrm{Id}_E - 2P + P^2 = \mathrm{Id}_E - P$ so the endomorphism $\mathrm{Id}_E - P$ is also a projector. The *zero endomorphism* 0 and the *identity endomorphism* Id_E are projectors. The zero projector is sometimes referred to as *trivial*.

If e_i is a basis of E and P^i_j is a matrix of a projector P in this basis, then $P\mathrm{e}_i = P^k_i \mathrm{e}_k$ and $P^2 \mathrm{e}_i = P^k_i P\mathrm{e}_k = P^k_i P^l_k \mathrm{e}_l$, thus, the matrix of P satisfies

$$(2) \qquad P^k_i P^l_k = P^l_i.$$

The following lemma describes *canonical forms* of projectors. Denote by $\mathrm{tr}\,P$ the *trace* of the endomorphism P; since the trace operation is invariant under similarity operations with matrices, $\mathrm{tr}\,P$ can be defined as the trace of *any* matrix of P.

Lemma 1 *Let $P: E \to E$ be a projector, and let $\mathrm{rank}\,P = r \geq 1$.*

(a) *There exists a basis e_i of E, such that the matrix P^i_j is of the form*

$$(3) \qquad \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

with r entries 1 *on the main diagonal*.

(b) *The vector space E has the direct sum decomposition*

$$E = \operatorname{Ker} P \oplus \operatorname{Im} P. \tag{4}$$

The dimensions of the vector subspaces $\operatorname{Ker} P$ *and* $\operatorname{Im} P$ *satisfy*

$$\dim \operatorname{Ker} P + \dim \operatorname{Im} P = n, \tag{5}$$

and

$$\dim \operatorname{Im} P = \operatorname{tr} P. \tag{6}$$

Proof (a) Choose a basis e_i such that the vectors e_1, e_2, $\dots$, e_r span the vector subspace $\operatorname{Im} P \subset E$, and e_{r+1}, e_{r+2}, $\dots$, e_n span $\operatorname{Ker} P \subset E$. Then the components in the expression $P\mathrm{e}_i = P_i^l \mathrm{e}_l$ must satisfy

$$P_i^l = \begin{cases} P_i^l, & i \le r, l > r, \\ 0, & i > r. \end{cases} \tag{7}$$

In this basis equations (2) reduce to $P_i^l P_l^k = P_i^k$, where $1 \le i,k,l \le r$. Since the matrix P_i^l, $1 \le i,l \le r$, is of rank *r*, using its inverse we conclude that P_i^l, for these values of the indices, is the identity matrix.

(b) Assertion (5) is the *rank-nullity theorem* for linear mappings, applied to *P*. Since $\dim \operatorname{Im} P = \operatorname{rank} P$, according to formula (3), the rank is equal to the trace of *P* proving (6).

The direct sum decomposition $E = \operatorname{Ker} P \oplus \operatorname{Im} P$ (4) is said to be *associated* with the projector *P*.

Further elementary properties of projectors are summarized in the following two lemmas.

Lemma 2 *Let* $P: E \to E$ *be a projector*.

(a) *If* $\alpha \in \mathbf{R}$ *and P is nontrivial, then* $Q = \alpha P$ *is a projector if and only if* $\alpha = 1$.

(b) *For any linear isomorphism* $S: E \to E$, *the endomorphism* SPS^{-1} *of E is a projector*.

(c) $\operatorname{Ker} P = \operatorname{Im}(\mathrm{Id}_E - P)$.

Proof (a) If *Q* is a projector, such that $Q = \alpha P$, then $Q^2 = \alpha^2 P^2 = \alpha^2 P$ and $Q^2 = Q = \alpha P$, proving the result.

(b) Obviously, $SPS^{-1}SPS^{-1} = SPPS^{-1} = SPS^{-1}$.

(c) If a vector $\xi \in E$ satisfies $P\xi = 0$, then $\xi = \xi - P\xi \in \operatorname{Im}(\mathrm{Id}_E - P)$ and $\operatorname{Ker} P \subset \operatorname{Im}(\mathrm{Id}_E - P)$. Conversely, if $\xi \in \operatorname{Im}(\mathrm{Id}_E - P)$, then $\xi = \zeta - P\zeta$

for some $\zeta \in E$ and $P\xi = P\zeta - P^2\zeta = 0$. Thus $\mathrm{Im}(\mathrm{Id}_E - P) \subset \mathrm{Ker}\, P$.

Lemma 3 *Let* $P, Q : E \to E$ *be two projectors.*
(a) $P + Q$ *is a projector if and only if* $QP = 0$ *and* $PQ = 0$.
(b) $P - Q$ *is a projector if and only if* $PQ = QP = Q$.
(c) *If* $P + Q$ *is a projector, then*

(8) $$\mathrm{Im}(P + Q) = \mathrm{Im}\, P \oplus \mathrm{Im}\, Q.$$

(d) *If* $P - Q$ *is a projector, then* $\mathrm{Im}\, Q \subset \mathrm{Im}\, P$.

Proof (a) We have $(P + Q)^2 = P + PQ + QP + Q$. If $P + Q$ is a projector, then $(P + Q)^2 = P + Q$ hence $PQ + QP = 0$. Then $PQ + PQP = 0$ and $PQP + QP = 0$ hence $PQ - QP = 0$. Therefore, $PQ = QP = 0$. The converse is evident.

(b) If $P - Q$ is a projector, then $(P - Q)^2 = P - PQ - QP + Q = P - Q$ and $PQ + QP = 2Q$. Thus $2Q^2 = QPQ + QP = PQ + QPQ$, hence $QP = PQ$. Therefore, $2PQ = 2QP = 2Q$. The converse is obvious.

(c) If $P + Q$ is a projector, then condition (a) implies $PQ = QP = 0$. Supposing that $\zeta \in \mathrm{Im}\, P \cap \mathrm{Im}\, Q$ we get $PQ\zeta = QP\zeta = P\zeta = Q\zeta = 0$ hence by (c), $\mathrm{Im}\, P \cap \mathrm{Im}\, Q = \{0\}$. Thus $\mathrm{Im}(P + Q)$ is the direct sum of its subspaces $\mathrm{Im}\, P$ and $\mathrm{Im}\, Q$.

(d) If $P - Q$ is a projector, then the projector $\mathrm{Id}_E - (P - Q)$ is the sum of two projectors $\mathrm{Id}_E - P$ and Q. Then by (c),

(9) $$\mathrm{Ker}(P - Q) = \mathrm{Im}(\mathrm{Id}_E - (P - Q)) = \mathrm{Im}(\mathrm{Id}_E - P) \oplus \mathrm{Im}\, Q.$$

Hence if $\zeta \in \mathrm{Im}\, Q$, then $\zeta \in \mathrm{Ker}(P - Q)$, that is, $(P - Q)\zeta = P\zeta - \zeta = 0$ hence $\zeta \in \mathrm{Im}\, P$.

Now we study *compositions* of projectors; in particular, we wish to determine when the composite PQ of two projectors is again a projector, and to find conditions, ensuring that P and Q *commute*, that is, $PQ = QP$.

Lemma 4 *Let* $P, Q : E \to E$ *be projectors. If* P *and* Q *commute, then the endomorphism*

(10) $$R = PQ = QP$$

is a projector, and

(11) $$\mathrm{Im}\, PQ = \mathrm{Im}\, P \cap \mathrm{Im}\, Q, \quad \mathrm{Ker}\, PQ = \mathrm{Ker}\, P + \mathrm{Ker}\, Q.$$

Proof If P and Q commute, then $R^2 = PQPQ = PPQQ = PQ = R$, thus R is a projector.

Further, if $\zeta \in \operatorname{Im} R$, then $PQ\zeta = QP\zeta = \zeta$ hence $\zeta \in \operatorname{Im} P \cap \operatorname{Im} Q$ and $\operatorname{Im} R \subset \operatorname{Im} P \cap \operatorname{Im} Q$. The converse $\operatorname{Im} R \supset \operatorname{Im} P \cap \operatorname{Im} Q$ is obvious.

To prove the second equality (11), suppose that we have a vector ξ such that $PQ\xi = 0$. Express ξ as $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \operatorname{Im} Q$ and $\xi_2 \in \operatorname{Ker} Q$. Then $PQ\xi = P\xi_1 = 0$. Thus, $\xi \in \operatorname{Ker} P + \operatorname{Ker} Q$, and we have the inclusion $\operatorname{Ker} PQ \subset \operatorname{Ker} P + \operatorname{Ker} Q$. Conversely, if $\xi \in \operatorname{Ker} P + \operatorname{Ker} Q$, then writing $\xi = \xi_1 + \xi_2$ for some vectors $\xi_1 \in \operatorname{Ker} P$ and $\xi_2 \in \operatorname{Ker} Q$, we get $R\xi = QP\xi_1 + PQ\xi_2 = 0$.

Lemma 5 *Let $P, Q : E \to E$ be projectors. The following two conditions are equivalent:*

(a) *The composite PQ is a projector.*

(b) *The kernels and images of P and Q satisfy*

(12) $$\operatorname{Im} Q \subset \operatorname{Im} P \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Ker} Q).$$

Proof 1. We prove that (a) implies (b). Clearly, for any vector $\zeta \in E$,

(13) $$\zeta = P\zeta + (\mathrm{Id}_E - P)\zeta = P\zeta + Q(\mathrm{Id}_E - P)\zeta + (\mathrm{Id}_E - Q)(\mathrm{Id}_E - P)\zeta.$$

We show that if $\zeta \in \operatorname{Im} Q$, that is, $\zeta = Q\zeta$, then the vector on the right-hand side belongs to the direct sum $\operatorname{Im} P \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Ker} Q)$. Applying *P* and *Q* to $(\mathrm{Id}_E - Q)(\mathrm{Id}_E - P)Q\zeta$,

$$P(\mathrm{Id}_E - Q)(\mathrm{Id}_E - P)Q\zeta = PQ\zeta - P^2Q\zeta - PQQ\zeta + PQPQ\zeta = 0$$

and

$$Q(\mathrm{Id}_E - Q)(\mathrm{Id}_E - P)Q\zeta = Q\zeta - QPQ\zeta - Q\zeta + QPQ\zeta = 0.$$

Consequently $(\mathrm{Id}_E - Q)(\mathrm{Id}_E - P)Q\zeta \in \operatorname{Ker} P \cap \operatorname{Ker} Q$. Next, since PQ is a projector, applying *P* to both sides of (13) we get

(13) $$\begin{aligned} 0 &= PQ(\mathrm{Id}_E - P)\zeta + P(\mathrm{Id}_E - Q)(\mathrm{Id}_E - P)Q\zeta \\ &= PQ(\mathrm{Id}_E - P)\zeta. \end{aligned}$$

Consequently, $Q(\mathrm{Id}_E - P)\zeta \in \operatorname{Ker} P \cap \operatorname{Im} Q$.

2. To show that (b) implies (a), choose a vector $\zeta \in E$. Since by hypothesis $Q\zeta$ admits a decomposition $Q\zeta = \zeta_1 + \zeta_2 + \zeta_3$, where $\zeta_1 \in \operatorname{Im} P$, $\zeta_2 \in \operatorname{Ker} P \cap \operatorname{Im} Q$, and $\zeta_3 \in \operatorname{Ker} P \cap \operatorname{Ker} Q$, then

(14) $$\begin{aligned} PQPQ\zeta &= PQP\zeta_1 + PQP\zeta_2 + PQP\zeta_3 = PQ\zeta_1 \\ &= PQ(\zeta_1 + \zeta_2 + \zeta_3) = PQ\zeta. \end{aligned}$$

In the following two theorems, one should distinguish between direct sums, associated with the projectors P or Q.

Theorem 1 *Let* $P,Q:E\to E$ *be projectors. Suppose that the kernels and images of P and Q satisfy*

(15) $$\operatorname{Im}Q=(\operatorname{Im}P\cap\operatorname{Im}Q)\oplus(\operatorname{Ker}P\cap\operatorname{Im}Q),$$

and

(16) $$\operatorname{Im}P=(\operatorname{Im}P\cap\operatorname{Im}Q)\oplus(\operatorname{Ker}Q\cap\operatorname{Im}P).$$

Then the endomorphisms PQ *and* QP *are projectors, and*

(17) $$PQ=QP.$$

Proof Suppose that formulas (15) and (16) hold. Then for every vector $\zeta\in E$, $Q\zeta$ can be decomposed as $Q\zeta=\zeta_1+\zeta_2$, where $\zeta_1\in\operatorname{Im}P\cap\operatorname{Im}Q$ and $\zeta_2\in\operatorname{Ker}P\cap\operatorname{Im}Q$. Since $PQ\zeta=P\zeta_1=\zeta_1$, we have $Q\zeta=PQ\zeta+\zeta_2$ hence

(18) $$PQ\zeta=Q\zeta-\zeta_2.$$

In particular,

(19) $$PQPQ\zeta=PQQ\zeta-PQ\zeta_2=PQ\zeta-P\zeta_2=PQ\zeta,$$

proving that PQ is a projector.

Analogously,

(20) $$QP\zeta=P\zeta-\zeta_2',$$

where $\zeta_2'\in\operatorname{Ker}Q\cap\operatorname{Im}P$, which implies that QP is a projector.

To prove that $PQ=QP$, it is sufficient to verify that the kernels and images of PQ and QP coincide.

First show that $\operatorname{Ker}PQ\subset\operatorname{Ker}Q+\operatorname{Ker}P$. Suppose that $Q\zeta-\zeta_2=0$. Then the formula $\zeta=\zeta-Q\zeta+\zeta_2$ decomposes ζ as the sum of two terms, $\zeta-Q\zeta\in\operatorname{Ker}Q$ and $\zeta_2\in\operatorname{Ker}P\cap\operatorname{Im}Q$, thus $\operatorname{Ker}PQ\subset\operatorname{Ker}Q+\operatorname{Ker}P$. Conversely, let $\zeta\in\operatorname{Ker}Q+\operatorname{Ker}P$, and write $\zeta=\zeta_1+\zeta_2$, where $\zeta_1\in\operatorname{Ker}Q$ and $\zeta_2\in\operatorname{Ker}P$. But since $\zeta_1\in\operatorname{Ker}Q$ and Q is a projector, $\zeta_2\in\operatorname{Im}Q$, thus $\zeta_2\in\operatorname{Ker}P\cap\operatorname{Im}Q$. But these properties of the vectors ζ_1 and ζ_2 imply $PQ\zeta=PQ\zeta_1+PQ\zeta_2=P\zeta_2=0$ hence $\zeta\in\operatorname{Ker}PQ$. In other words this means that $\operatorname{Ker}Q+\operatorname{Ker}P\subset\operatorname{Ker}PQ$. Consequently,

(21) $$\operatorname{Ker}PQ=\operatorname{Ker}Q+\operatorname{Ker}P.$$

The same applies to the projector QP, thus

(22) $\operatorname{Ker} QP = \operatorname{Ker} Q + \operatorname{Ker} P = \operatorname{Ker} PQ.$

Now we determine the image of the projector PQ. Clearly, for every vector $\zeta \in E$, $PQ\zeta \in \operatorname{Im} P$. But by formula (18), $PQ\zeta = Q\zeta - \zeta_2$, where $\zeta_2 \in \operatorname{Ker} P \cap \operatorname{Im} Q$ hence $PQ\zeta \in \operatorname{Im} Q$. Consequently, $\operatorname{Im} PQ \subset \operatorname{Im} P \cap \operatorname{Im} Q$. On the other hand, every vector $\xi \in \operatorname{Im} P \cap \operatorname{Im} Q$ satisfies $PQ\xi = P\xi = \xi$, proving that $\operatorname{Im} P \cap \operatorname{Im} Q \subset \operatorname{Im} PQ$. Thus,

(23) $\operatorname{Im} PQ = \operatorname{Im} P + \operatorname{Im} Q.$

Since the same proof applies to the projector QP, then

(24) $\operatorname{Im} QP = \operatorname{Im} Q + \operatorname{Im} P = \operatorname{Im} QP.$

Summarizing, formulas (15) and (16) imply

(25) $\operatorname{Im} PQ = \operatorname{Im} QP, \quad \operatorname{Ker} PQ = \operatorname{Ker} QP,$

and since both PQ and QP are projectors, we have $PQ = QP$.

Theorem 2 *Let $P, Q : E \to E$ be projectors, such that*

(26) $PQ = QP.$

Then the endomorphism $R = PQ = QP$ is a projector, and

(27) $\operatorname{Im} Q = (\operatorname{Im} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q),$

(28) $\operatorname{Im} P = (\operatorname{Im} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} Q \cap \operatorname{Im} P).$

Proof Condition (26) implies $PQPQ = PPQQ = PQ$, thus, $R = PQ$ is a projector.

To prove formula (27), note that $\operatorname{Im} Q \supset (\operatorname{Im} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q)$. The opposite inclusion can be shown as follows. Choose a vector $\zeta \in \operatorname{Im} Q$. ζ has an expression $\zeta = \zeta_1 + \zeta_2$ such that $\zeta_1 \in \operatorname{Im} P$ and $\zeta_2 \in \operatorname{Ker} P$. Since $\zeta = Q\zeta = Q\zeta_1 + Q\zeta_2$ and $\zeta_1 = P\zeta_1$, we have $\zeta = QP\zeta_1 + Q\zeta_2 = PQ\zeta_1 + Q\zeta_2$ by (26), and $PQ\zeta_1 \in \operatorname{Im} P \cap \operatorname{Im} Q$. The term $Q\zeta_2$ satisfies $PQ\zeta_2 = QP\zeta_2 = 0$, thus, $Q\zeta_2 \in \operatorname{Ker} P$, which implies $Q\zeta_2 \in \operatorname{Ker} P \cap \operatorname{Im} Q$. Consequently $\operatorname{Im} Q \subset (\operatorname{Im} P \cap \operatorname{Im} Q) + (\operatorname{Ker} P \cap \operatorname{Im} Q)$, proving (27). Formula (28) can be proved in the same way.

Remark 1 For Theorem 1 and Theorem 2 on commuting projectors we refer to Corporal and Regensburger [2].

3 Partitions of vector spaces and their generalizations

By a *partition* of the vector space *E* we mean a family $\{P_1,P_2,\dots,P_k\}$ of projectors $P_i:E\to E$ such that

(1) $$P_1+P_2+\dots+P_k=\mathrm{Id}_E.$$

If $k=2$, we have two projectors *P* and *Q* such $P+Q=I_E$ (the *complementary projectors*); every projector *P* defines a partition $\{P,Q\}$, where $Q=\mathrm{Id}_E-P$. In this case $P^2=P-PQ=P-QP$ hence $QP=PQ=0$. Also note that $\{P,Q\}$ defines a *direct sum decomposition* of *E*, namely

(2) $$E=\mathrm{Im}\,P\oplus\mathrm{Im}\,Q.$$

Indeed, for every vector $\xi\in E$, $\xi=P\xi+Q\xi$, and if $\xi\in\mathrm{Im}\,P\cap\mathrm{Im}\,Q$, then $\xi=\xi+\xi=2\xi=0$.

Similar properties of partitions are valid for arbitrary number of projectors $k\le n$ (cf. Yanai, Takeuchi, Takane [12]).

Theorem 3 *Let* $\{P_1,P_2,\dots,P_k\}$ *be a partition of E.*

(a) *The dimensions of image spaces* $\mathrm{Im}\,P_i$ *satisfy*

(3) $$\dim\mathrm{Im}\,P_1+\dim\mathrm{Im}\,P_2+\dots+\dim\mathrm{Im}\,P_k=n.$$

(b) *E is expressible as the direct sum*

(4) $$E=\mathrm{Im}\,P_1\oplus\mathrm{Im}\,P_2\oplus\dots\oplus\mathrm{Im}\,P_k.$$

(c) *For any i and j,* $i\ne j$,

(5) $$P_iP_j=0.$$

Proof (a) Calculating the trace on both sides of formula (1), we get

(6) $$\mathrm{tr}(P_1+P_2+\dots+P_k)=\mathrm{tr}\,P_1+\mathrm{tr}\,P_2+\dots+\mathrm{tr}\,P_k=n.$$

But $\mathrm{tr}\,P_i=\dim\mathrm{Im}\,P_i$, from the rank formula (Lemma 1, (6)), so (3) follows from (1).

(b) We have to show that the vector subspaces $\mathrm{Im}\,P_i$, $i=1,2,\dots,k$, generate *E*, and $\mathrm{Im}\,P_i\cap\mathrm{Im}\,P_j=\{0\}$ for all *i*, *j*, $i\ne j$. But by definition (1), every vector $\xi\in E$ has an expression $\xi=P_1\xi+P_2\xi+\dots+P_k\xi$, and condition

$\operatorname{Im} P_i \cap \operatorname{Im} P_j \neq \{0\}$ for some some $i, j,\ i \neq j$, contradicts equality (3).

(c) For any vector $\xi \in E$ and any j formula (1) yields

(7) $$\begin{aligned} &P_1 P_j \xi + P_2 P_j \xi + \ldots + P_{j-1} P_j \xi + P_j P_j \xi + P_{j+1} P_j \xi + \ldots + P_k P_j \xi \\ &= P_j \xi, \end{aligned}$$

hence

(8) $$P_1 P_j \xi + P_2 P_j \xi + \ldots + P_{j-1} P_j \xi + P_{j+1} P_j \xi + \ldots + P_k P_j \xi = 0.$$

But by (b), the vectors $P_1 P_j \xi$, $P_2 P_j \xi$, $\ldots$, $P_{j-1} P_j \xi$, $P_{j+1} P_j \xi$, $\ldots$, $P_k P_j \xi$ belong to different summands of the direct sum hence must vanish separately. This proves formula (5).

According to Theorem 3, every partition of the vector space *E* defines a direct sum decomposition of *E*. On the other hand, every direct sum decomposition $E = E_1 \oplus E_2 \oplus \ldots \oplus E_k$ defines a family of projectors $P_i : E \to E$, $i = 1, 2, \ldots, k$, by the condition

(9) $$P_i \xi = \xi_i,$$

where ξ_i is the component of ξ in E_i. Then $\{P_1, P_2, \ldots, P_k\}$ is a partition of *E*, *associated* with the direct sum decomposition $E = E_1 \oplus E_2 \oplus \ldots \oplus E_k$.

Remark 2 Property (5) of the partition $\{P_1, P_2, \ldots, P_k\}$ implies that for all *i* and *j*, the sum $P_i + P_j$ is a projector (Lemma 3 (a)).

Remark 3 Theorem 3 defines a one-to-one correspondence between the set of partitions of the vector space *E* and the set of *integer partitions* $(p_1, p_2, \ldots, p_k)$ of the positive integer $n = \dim E$, where $1 \leq k \leq n$, and $p_1, p_2, \ldots, p_k$ are positive integers (dimensions of subspaces) such that

(10) $$p_1 + p_2 + \ldots + p_k = n, \quad p_1 \geq p_2 \geq \ldots \geq p_k.$$

Now we study *simultaneous diagonalizability* of the projectors entering a partition $\{P_1, P_2, \ldots, P_k\}$ of the vector space *E* (Conrad [1], Sec. 5). Recall that a linear operator $Q : E \to E$ is *diagonalizable*, that is, has a diagonal matrix representation, if and only if it has a basis of eigenvectors in *E*. Indeed, the matrix of *Q* is defined in a basis e_i by $Q\mathrm{e}_i = Q_i^j \mathrm{e}_j$, so if Q_i^j is diagonal, that is $Q_i^j = \lambda_i \delta_i^j$, then $Q\mathrm{e}_i = \lambda_i \mathrm{e}_i$ so the basis consists of eigenvectors of *Q*; conversely, if $Q\mathrm{e}_i = \lambda_i \mathrm{e}_i$ for some basis e_i, then in this basis $Q_i^j = \lambda_i \delta_i^j$. We know that each of the projectors P_i has a diagonal matrix representation (Lemma 1).

Theorem 4 (Canonical representation of a partition) *For any partition* $\{P_1, P_2, \dots, P_k\}$ *of the vector space E there exists a basis of E, in which all projectors* P_i *are represented by diagonal matrices with entries* 1 *and* 0 *in the main diagonal.*

Proof 1. We prove Theorem 4 for $k = 2$. Let $\{Q_1, Q_2\}$ be a partition of a vector space F. Q_2 is diagonalizable with eigenvalues by $\lambda = 0,1$ (Lemma 1); denote by F_λ the corresponding eigenspaces F_0 and F_1. F has a basis formed by the eigenvectors of Q_2; for any eigenvector $u \in F_\lambda$ from this basis, $Q_2 u = \lambda u$. But Q_1 and Q_2 commute hence $Q_2 Q_1 u = Q_1 Q_2 u = \lambda Q_1 u$, thus, $Q_1 u$ is also an eigenvector of Q_2, belonging to the same eigenvalue λ. Consequently, the linear mapping $u \to Q_1 u$ restricts to a projector on the vector space $E_\lambda \subset E$. Choose a basis of the eigenspace E_λ in which $Q_1 : E_\lambda \to E_\lambda$ is diagonal (Lemma 1, (a)). All elements of this basis are also eigenvectors of Q_2, and are linearly independent. Since $E = E_0 \oplus E_1$, we get a basis of E, consisting of eigenvectors of Q_2; in this basis both Q_1 and Q_2 are diagonal.

2. Let m be an integer such that $2 \le m \le k-1$, and suppose that any family of pairwise commuting projectors $Q_1, Q_2, \dots, Q_{m-1}$ in a vector space F are simultaneously diagonalizable. We claim that then any family of pairwise commuting projectors $P_1, P_2, \dots, P_{m-1}, P_m$ in a vector space E are simultaneously diagonalizable. P_m is diagonalizable with eigenvalues $\lambda = 0,1$ (Lemma 1); denote by E_λ the corresponding eigenspaces of the projector P_m. Then for any vector $u \in E_\lambda$ and any integer $\alpha = 1,2,\dots,m-1$, we have $P_m P_\alpha u = P_\alpha P_m u = \lambda P_\alpha u$, thus, $P_\alpha u \in E_\lambda$. Consequently, the projectors $P_1, P_2, \dots, P_{m-1}$ restrict to a family of projectors $Q_\alpha = P_\alpha |_{E_\lambda}$ on the vector space E_λ. But by induction hypothesis, there exists a basis of E_λ in which all these projectors are diagonal. Since $E = E_0 \oplus E_1$, the corresponding bases in E_0 and E_1 define a basis of E, in which all projectors $P_1, P_2, \dots, P_{m-1}, P_m$ are diagonalizable.

3. It remains to verify that the diagonal elements of the matrices of the projectors $P_1, P_2, \dots, P_{m-1}, P_m$, constructed this way, are all equal to 1 or 0. Clearly, if any projector P is diagonal, $P = c_i \delta_i^j$, then

$$(11) \qquad P_i^j P_j^k = \sum_j c_i c_j \delta_i^j \delta_j^k = P_i^k = c_i \delta_i^k,$$

or, equivalently, $c_i c_1 \delta_i^1 \delta_1^k + c_i c_2 \delta_i^2 \delta_2^k + \dots + c_i c_n \delta_i^n \delta_n^k = c_i \delta_i^k$. If $k \ne i$, we get an identity; if $k = i$, then $c_i c_i = c_i$, proving that $c_i = 0,1$.

Theorem 4 determines all *canonical forms* of partitions of the vector space E. Let $\{P_1, P_2, \dots, P_k\}$ be a partition of E. We say that another partition $\{Q_1, Q_2, \dots, Q_m\}$ *refines* $\{P_1, P_2, \dots, P_k\}$, if for every i, $1 \le i \le k$, either $P_i = Q_\alpha$

for some α, $1 \le \alpha \le m$, or there exist α and β such that $1 \le \alpha, \beta \le m$, and $P_i = Q_\alpha + Q_\beta$. Any partition $\{Q_1, Q_2, \ldots, Q_m\}$ with these properties is called a *refinement* of $\{P_1, P_2, \ldots, P_k\}$. According to Theorem 4, if at least one of the vector subspaces $\operatorname{Im} P_i \subset E$ is of dimension ≥ 2, then $\{P_1, P_2, \ldots, P_k\}$ admits a nontrivial refinement.

4 $\mathscr{P}$-decomposability

We extend the concept of decomposability of projectors and a partition of a vector space by introducing *admissible* projectors, defined by a given set of projectors. This will result, in particular, to specifications of *decomposability* of projectors (in the given set), and to *primitive projectors* and *primitive partitions*.

Let $\mathscr{P} = \{P_\iota\}_{\iota \in J}$ be a family of projectors, finite or infinite, containing the zero projector, and such that for every $P \in \mathscr{P}$ also $\mathrm{Id}_E - P \in \mathscr{P}$. We shall say that a projector $P \in \mathscr{P}$ is *decomposable in* $\mathscr{P}$, if there exist two projectors $P_1, P_2 \in \mathscr{P}$, different from *P*, such that

(1) $$\operatorname{Im} P = \operatorname{Im} P_1 \oplus \operatorname{Im} P_2.$$

In this case we also say that P_1 *decomposes P* (in $\mathscr{P}$). A projector, which is *not* decomposable, is called *primitive*. The zero projector 0 is always primitive; if $\mathscr{P}$ includes at least one projector different from 0 and Id_E, then Id_E is decomposable.

The *decomposability problem* consists in finding conditions ensuring decomposability of the projectors $P \in \mathscr{P}$ in $\mathscr{P}$, and the methods how to determine P_1 and P_2. Given *P*, a *necessary* condition for *P* to be decomposable, the *dimension decomposability condition*, is the existence of projectors P_1 and P_2 in $\mathscr{P}$, different from *P*, such that

(2) $$\dim \operatorname{Im} P = \dim \operatorname{Im} P_1 + \dim \operatorname{Im} P_2.$$

We introduce the *decomposability indicatrix* $\mathscr{I} = \{I_{\iota\kappa}\}_{\iota,\kappa \in J}$ of $\mathscr{P}$ to be the family of *positive* integers

(3) $$I_{\iota\kappa} = \dim \operatorname{Im} P_\iota + \dim \operatorname{Im} P_\kappa.$$

For every projector $P \in \mathscr{P}$, we have the *decomposability equation*

(4) $$P_\iota + P_\kappa = P$$

for the unknowns $P_\iota, P_\kappa \in \mathscr{P}$. The meaning of the decomposability indicatrix, related to this equation, is formulated in the following lemma.

Lemma 6 *A necessary condition for a projector* $P \in \mathcal{P}$ *to be decomposable in* $\mathcal{P}$ *is that there exists* $I_{\iota\kappa} \in \mathcal{I}$ *such that* $\dim \operatorname{Im} P = I_{\iota\kappa}$.

Proof This is just a restating of the definition.

Since *P* is given and the dimension of is known, the left-hand side of equation (4) is determined by the decomposability indicatrix. Excluding trival eqations in which one of the projectors P_ι or P_κ is 0 or Id_E , we get a system of equations, which determines *all* decomposable projectors *P*.

By a $\mathcal{P}$ *-partition* of the vector space *E* we mean a partition, whose projectors belong to the family $\mathcal{P}$. A $\mathcal{P}$-partition is said to be *refinable*, if it has a refinement, which is a $\mathcal{P}$-partition. A $\mathcal{P}$-partition, which is *not* refinable, is called *primitive*.

Remark 4 Notice that the family $\{\dim P_\iota\}_{\iota \in J}$ of positive integers is always *finite*. Thus, if the dimensions $\dim P_\iota$ are known, then Lemma 6 represents a simple effective tool for calculating *all* decomposable projectors. In this case the decomposability indicatrix is a *finite set*, and equation

(5) $$\dim \operatorname{Im} P_\iota + \dim \operatorname{Im} P_\kappa = \dim P$$

with given right-hand side has at most finitely many solutions – the pairs of positive integers $(\dim \operatorname{Im} P_\iota, \dim \operatorname{Im} P_\kappa)$. The system (4) of decomposability equations is also finite.

5 Natural projectors in covariant tensor spaces

In this section $\mathbf{R}$ is the field of real numbers, and $\mathbf{R}^n$ is the real vector space of ordered *n*-tuples of real numbers. S_r denotes the *permutation group* of *r* numbers $\{1,2,\ldots,r\}$. The adjoint of a linear operator *P* with respect to the canonical scalar product is denoted by ${}^tP$.

By a *natural projector* in the tensor space $T_r^0\mathbf{R}^n$ we mean a natural endomorphism $P : T_r^0\mathbf{R}^n \to T_r^0\mathbf{R}^n$, which is a projector.

In the canonical basis,

(1) $$P = \sum_{\tau \in S_r} a_\tau \Delta_\tau ,$$

where Δ_τ are endomorphisms of the tensor space $T_r^0\mathbf{R}^n$, expressed as

(2) $$\Delta_\tau{}^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_r} = \delta^{i_1}_{j_{\tau(1)}} \delta^{i_2}_{j_{\tau(2)}} \ldots \delta^{i_r}_{j_{\tau(r)}}$$

(Gurevich [3], Krupka [8]), and the coefficients $a_\tau \in \mathbf{R}$ satisfy the *projector equation*

(3) $$P^2 = P.$$

Since

(4) $$P^2 = \sum_{\sigma,\nu \in S_r} a_\sigma a_\nu \Delta_\sigma \Delta_\nu = \sum_{\sigma,\nu \in S_r} a_\sigma a_\nu \Delta_{\sigma\nu} = \sum_{\tau \in S_r} \left(\sum_{\tau = \sigma\nu} a_\sigma a_\nu \right) \Delta_\tau,$$

a natural endomorphism P is a projector if and only if

(5) $$\sum_{\tau \in S_r} \left(a_\tau - \sum_{\tau = \sigma\nu} a_\sigma a_\nu \right) \Delta_\tau = 0.$$

We restrict our attention to the case when equation (5) determines uniquely the coefficients; this is the case $r \le n$, when the endomorphisms Δ_τ (2) are linearly independent. The following is our main result in this section.

Theorem 5 *If* $r \le n$, *then a natural endomorphism* $P : T_r^0 \mathbf{R}^n \to T_r^0 \mathbf{R}^n$ *is a projector if and only if it has an expression* (1) *such that the coefficients* a_τ *satisfy*

(6) $$a_\tau - \sum_{\sigma\nu = \tau} a_\sigma a_\nu = 0.$$

Proof Condition $r \le n$ implies that the endomorphisms Δ_τ are linearly independent; thus (5) is equivalent with (6).

Theorem 5 transforms the problem of finding natural projectors in a covariant tensor space to a system of $r!$ bilinear equations for $r!$ components a_τ of natural projectors or, which is the same, for an unknown real-valued function $\tau \to a_\tau$ on the permutation group S_r. These *equations of natural projectors* can be easily expressed explicitly and solved for any fixed valency r and dimension n such that $r \le n$.

Remark 5 It should be pointed out that Theorem 5 characterizes projectors in the tensor space $T_r^0 E$ for *any* underlying n-dimensional vector space E, not just for $E = \mathbf{R}^n$. Projectors in the tensor space $T_r^0 \mathbf{R}^n$ obtained by solving equations (6) can be transformed to $T_r^0 E$ in any basis of E; naturality property then ensures independence on the basis.

Equations (6) for the unknown function $\tau \to a_\tau$, *equations of natural projectors* in the tensor space $T_r^0\mathbf{R}^n$, can equivalently be written as the system, consisting of a quadratic equation

(7) $$a_\varepsilon(1-a_\varepsilon)-\sum_{\sigma\neq\varepsilon} a_\sigma a_{\sigma^{-1}}=0$$

when $\tau\neq\varepsilon$, and, a system of bilinear equations, when $\tau\neq\varepsilon$,

(8) $$a_\tau(1-2a_\varepsilon)-\sum_{\sigma\nu=\tau,\sigma,\nu\neq\varepsilon} a_\sigma a_\nu=0.$$

Indeed, in this case (6) implies

(9) $$\begin{aligned} a_\tau-\sum_{\sigma\nu=\tau} a_\sigma a_\nu &= a_\tau-\sum_{\sigma\varepsilon=\tau} a_\sigma a_\varepsilon-\sum_{\sigma\nu=\tau,\nu\neq\varepsilon} a_\sigma a_\nu \\ &= a_\tau-a_\tau a_\varepsilon-\sum_{\varepsilon\nu=\tau,\nu\neq\varepsilon} a_\varepsilon a_\nu-\sum_{\sigma\nu=\tau,\sigma,\nu\neq\varepsilon} a_\sigma a_\nu \\ &= a_\tau-2a_\varepsilon a_\tau-\sum_{\sigma\nu=\tau,\sigma,\nu\neq\varepsilon} a_\sigma a_\nu. \end{aligned}$$

The following are immediate consequences of Theorem 5.

Lemma 7 *Suppose that* $r\leq n$.

(a) *If the system* (6) *has a solution* $a_\tau=b_\tau$, *then it also has a solution*

(10) $$a_\tau=\begin{cases} 1-b_\varepsilon, & \tau=\varepsilon, \\ -b_\tau, & \tau\neq\varepsilon. \end{cases}$$

(b) *The adjoint of a natural projector is a natural projector. If P is expressed by* (1)*, then*

(11) $$^tP=\sum_{\tau\in S_r} a_{\tau^{-1}}\Delta_\tau.$$

(c) *A natural projector* (1) *is self-adjoint if and only if*

(12) $$a_\tau=a_{\tau^{-1}}.$$

Proof (a) Suppose that b_τ is a solution,

(13) $$b_\tau-\sum_{\sigma\nu=\tau} b_\sigma b_\nu=0.$$

Then if $\tau = \varepsilon$,

$$(14)\qquad \begin{aligned} a_{\varepsilon} - \sum_{\sigma} a_{\sigma} a_{\sigma^{-1}} &= a_{\varepsilon} - a_{\varepsilon}^{2} - \sum_{\sigma \neq \varepsilon} a_{\sigma} a_{\sigma^{-1}} = 1 - b_{\varepsilon} - (1 - b_{\varepsilon})^{2} - \sum_{\sigma \neq \varepsilon} b_{\sigma} b_{\sigma^{-1}} \\ &= 1 - b_{\varepsilon} - 1 + 2 b_{\varepsilon} - b_{\varepsilon}^{2} - \sum_{\sigma \neq \varepsilon} b_{\sigma} b_{\sigma^{-1}} = b_{\varepsilon} - b_{\varepsilon}^{2} - \sum_{\sigma \neq \varepsilon} b_{\sigma} b_{\sigma^{-1}} \equiv 0, \end{aligned}$$

and if $\tau \neq \varepsilon$,

$$(15)\qquad \begin{aligned} a_{\tau} - \sum_{\sigma\nu = \tau} a_{\sigma} a_{\nu} &= a_{\tau} - a_{\tau} a_{\varepsilon} - a_{\varepsilon} a_{\tau} - \sum_{\sigma\nu = \tau, \sigma, \nu \neq \varepsilon} a_{\sigma} a_{\nu} \\ &= -b_{\tau} + b_{\tau}(1 - b_{\varepsilon}) + (1 - b_{\varepsilon}) b_{\tau} - \sum_{\sigma\nu = \tau, \sigma, \nu \neq \varepsilon} b_{\sigma} b_{\nu} \\ &= b_{\tau} - b_{\tau} b_{\varepsilon} - b_{\varepsilon} b_{\tau} - \sum_{\sigma\nu = \tau, \sigma, \nu \neq \varepsilon} b_{\sigma} b_{\nu} \\ &= b_{\tau} - \sum_{\sigma\nu = \tau} b_{\sigma} b_{\nu} = 0, \end{aligned}$$

proving that a_{τ} (10) is also a solution.

(b) The adjoint of the natural projector (1) is the endomorphism

$$(16)\qquad {}^{t}P = \sum_{\tau \in S_{r}} a_{\tau} {}^{t}\Delta_{\tau},$$

where the transposed matrix is ${}^{t}\Delta_{\tau} = \Delta_{\tau^{-1}}$. Hence

$$(17)\qquad {}^{t}P = \sum_{\tau \in S_{r}} a_{\tau} \Delta_{\tau^{-1}} = \sum_{\tau \in S_{r}} a_{\tau^{-1}} \Delta_{\tau} = \sum_{\tau \in S_{r}} b_{\tau} \Delta_{\tau},$$

where $b_{\tau} = a_{\tau^{-1}}$. But this is obviously a natural projector.

(c) The condition for a projector (1) to be self-adjoint reads

$$(18)\qquad \sum_{\tau \in S_{r}} (a_{\tau} - a_{\tau^{-1}}) \Delta_{\tau} = 0.$$

Since the endomorphisms $\Delta_{(\tau)}$ are linearly independent, this is equivalent with (12).

Remark 6 Theorem 5 reduces the problem of finding natural projectors in a covariant tensor space to a system of bilinear equations for the components of natural projectors. The system can be expressed explicitly for any concrete dimension. Now we have a general remark. Quadratic equation (7)

$$a_\varepsilon^2 - a_\varepsilon + \sum_{\sigma \neq \varepsilon} a_\sigma a_{\sigma^{-1}} = 0$$

gives a necessary and sufficient discriminant condition for existence of a solution a_ε ,

$$4\sum_{\sigma \neq \varepsilon} a_\sigma a_{\sigma^{-1}} \leq 1.$$

Then

$$a_\varepsilon = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\sum_{\sigma \neq \varepsilon} a_\sigma a_{\sigma^{-1}}}\right).$$

Equation (8) becomes

$$a_\tau(1 - 2a_\varepsilon) - \sum_{\sigma\nu^{-1} = \tau, \sigma, \nu \neq \varepsilon} a_\sigma a_{\nu^{-1}} = 0,$$

that is,

$$\pm a_\tau \sqrt{1 - 4\sum_{\sigma \neq \varepsilon} a_\sigma a_{\sigma^{-1}}} + \sum_{\sigma\nu^{-1} = \tau, \sigma, \nu \neq \varepsilon} a_\sigma a_{\nu^{-1}} = 0.$$

A solution is $a_\tau = 0$. Solutions such that $a_\tau \neq 0$ for some τ satisfy

$$\sqrt{1 - 4\sum_{\sigma \neq \varepsilon} a_\sigma a_{\sigma^{-1}}} = \mp \frac{1}{a_\tau} \sum_{\sigma\nu^{-1} = \tau, \sigma, \nu \neq \varepsilon} a_\sigma a_{\nu^{-1}} .$$

In particular, for any two $a_\tau, a_\kappa \neq 0$ we get a notable identity

$$\frac{1}{a_\tau} \sum_{\sigma\nu^{-1} = \tau, \sigma, \nu \neq \varepsilon} a_\sigma a_{\nu^{-1}} = \frac{1}{a_\kappa} \sum_{\sigma\nu^{-1} = \kappa, \sigma, \nu \neq \varepsilon} a_\sigma a_{\nu^{-1}} .$$

6 Decomposability of (0,*r*)-tensors

Decomposability means in this section $\mathcal{P}$ *-decomposability*, where $\mathcal{P}$ is the set of natural projectors $P : T_r^0\mathbf{R}^n \to T_r^0\mathbf{R}^n$; $\mathcal{P}$ can equivalently be defined as the set of solutions of natural projectors equations (Theorem 5).

Let $P \in \mathcal{P}$ be a natural projector. Recall that the *decomposability problem* for *P* is the problem of existence of two natural projectors *Q* and *R*, such that *decomposability equation*

(1) $$P = Q + R$$

holds. We shall discuss properties of this equation.

Thus, expressing the natural tensors *P*, *Q*, and *R* in components as in formula (1), Sec. 5,

(2) $$P = \sum_{\tau\in S_r} c_\tau \Delta_\tau, \quad P_\iota = \sum_{\tau\in S_r} a_\tau \Delta_\tau, \quad P_\kappa = \sum_{\tau\in S_r} b_\tau \Delta_\tau,$$

the decomposability equation reads

(3) $$\sum_{\tau\in S_r} (c_\tau - a_\tau - b_\tau)\Delta_\tau = 0.$$

If the valency *r* satisfies $r \le n$, then the endomorphisms Δ_τ are linearly independent of Δ_τ so we get the system

(4) $$c_\tau = a_\tau + b_\tau$$

for the unknowns a_τ and b_τ.

It should be pointed out, however, that the coefficients c_τ, a_τ and b_τ represent natural projectors, hence satisfy the natural projector equations (Theorem 5).

Given *P*, some natural projectors *Q* and *R* cannot *a priori* satisfy decomposability equation (1). To exclude these natural projectors, consider the set $\mathcal{P}$, expressed as an indexed family $\mathcal{P} = \{P_\iota\}_{\iota\in J}$, and the decomposability indicatrix $\mathcal{I} = \{I_{\iota\kappa}\}_{\iota,\kappa\in J}$ of the set $\mathcal{P}$.

Lemma 8 (Decomposability indicatrix) *A necessary condition for a natural projector $P \in \mathcal{P}$ to be decomposable is that there exist $\iota,\kappa \in J$ such that*

(5) $$\dim \operatorname{Im} P = I_{\iota\kappa}.$$

Proof This follows from the construction of the decomposability indicatrix $\mathcal{I}$: $I_{\iota\kappa}$ is by definition equal to $\dim \operatorname{Im} P_\iota + \dim \operatorname{Im} P_\kappa$.

The following statement characterizes a key property of every natural projector *P*, namely, the dimension of the image space $\dim \operatorname{Im} P \subset T_r^0 \mathbf{R}^n$. In particular, the statement completely determines the decomposability indicatrix $\mathcal{I}$ provided all solutions of the natural projector equations are given.

In the following theorem, $\chi(\tau)$ denotes the number of cycles of a permutation $\tau \in S_r$.

Theorem 6 *Let* $P : T_r^0\mathbf{R}^n \to T_r^0\mathbf{R}^n$ *be a natural projector, expressed as*

(6) $$P = \sum_{\tau\in S_r} a_\tau \Delta_\tau ,$$

where $a_\tau \in \mathbf{R}$ *. Then*

(7) $$\dim \operatorname{Im} P = \sum_{\tau\in S_r} a_\tau n^{\chi(\tau)} .$$

Proof We know that the dimension of the image space $\operatorname{Im} P$ is equal to the rank of *P* (Lemma 1). Since $\operatorname{tr}\Delta_\tau = \Delta_\tau{}^{i_1 i_2 \dots i_r}{}_{i_1 j_2 \dots j_r} = \delta^{i_1}_{i_{\tau(1)}} \delta^{i_2}_{i_{\tau(2)}} \dots \delta^{i_r}_{i_{\tau(r)}}$, we get

(8) $$\dim \operatorname{Im} P = \sum_{\tau\in S_r} a_\tau \operatorname{tr}\Delta_\tau = \sum_{\tau\in S_r} a_\tau n^{\chi(\tau)} .$$

Theorem 6 together with Theorem 5 includes all information needed for the study of decomposability of natural projectors and natural partitions of the tensor space $T_r^0\mathbf{R}^n$, for every fixed *r*. Summarizing, we get the following method of finding all decomposable natural projectors, all decomposition formulas, and all natural partitions of the tensor space $T_r^0\mathbf{R}^n$:

(a) find all solutions of natural projector equations, that is, the set $\mathcal{P} = \{P_\iota\}_{\iota\in J}$;

(b) determine dimensions of the image spaces of these natural projectors, using Lemma 1, formula (6); clearly, the set of positive integers *N* expressible as $N = \dim \operatorname{Im} P_\iota$, where P_ι runs through the family $\mathcal{P}$, is always *finite* (cf. Remark 4),

(c) construct the decomposability indicatrix: use the finite set of pairs of positive integers $(\dim \operatorname{Im} P_\iota, \dim \operatorname{Im} P_\kappa)$, where $P_\iota, P_\kappa \in \mathcal{P}$, and get a finite set of positive integers $I_{\iota\kappa} = \dim \operatorname{Im} P_\iota + \dim \operatorname{Im} P_\kappa$, defining the decomposability indicatrix $\mathcal{I} = \{I_{\iota\kappa}\}_{\iota,\kappa\in J}$;

(d) using the decomposability indicatrix, consider the decomposibility equations, satisfying necessary decomposability condition, given by Lemma 8, and solve these equations;

(e) find partitions of $T_r^0\mathbf{R}^n$, consider for any natural projector *P* the canonical partition $\{P, \mathrm{Id} - P\}$, and apply decomposability criteria to the natural projectors *P* and $\mathrm{Id} - P$.

This method, consisting is solving a system of bilinear equations, applies to tensor spaces of arbitrary covariant valency, is applied to the tensor space $T_3^0\mathbf{R}^n$ in Krupka [5]. This paper provides a complete analysis of the system comprehensive discussion of decomposability, leading to a complete list of decompositions of the natural projectors. Clearly, the method extends to mixed tensor spaces. Complete description of the method and complete classification of natural projectors and their decompositions is given in Krupka [6]. In these two particular cases, the reader can easily compare differences between the methods and results of natural projector decomposition theory and the group representation theory.

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