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Classification of natural projectors in tensor spaces: (0,3)-tensors

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Abstract This research-expository paper is devoted to application of the natural projector theory to the problem of classification of invariant decompositions of projectors in tensor spaces of covariant valency 3. Using this theory, complete lists of natural projectors, decompositions of natural projectors, and partitions of the tensor space of $(0,3)$-tensors are derived. In order to explain the method of natural projections as clearly as possible, the proofs, based on elementary tensor algebra and projector theory in finite-dimensional real vector spaces, are included.

Keywords Invariant tensor, Natural endomorphism, Natural projector, Partition of vector space.

Mathematics subject classification (2010) 15A72, 20G05, 53A45, 53A55

1 Introduction

This paper is devoted to natural projectors on the space of $(0,3)$-tensors on the real, n-dimensional vector space $\mathbf{R}^n$. The aim is to classify these projectors and to find the partitions of the tensor space $T_3^0\mathbf{R}^n$ induced by them. The second aim is to illustrate the methods of tensor analysis based on natural projection theory in tensor spaces; to this purpose we provide all detailed calculations. It is shown in particular that the natural projectors and the partitions can completely be characterized by elementary means of the theory of (real) quadratic forms and bilinear equations.

The vector space $\mathbf{R}^n$ is considered with the canonical left action of the general linear group $GL_n(\mathbf{R})$, and the tensor space $T_3^0\mathbf{R}^n$ is endowed with the induced tensor action. Since our discussions are $GL_n(\mathbf{R})$-invariant, the decomposition results apply to *any* real, *n*-dimensional vector space *E*, and to the tensor space $T_3^0 E$ over *E*. In the canonical basis e_i of $\mathbf{R}^n$, a tensor $U \in T_3^0\mathbf{R}^n$ is usually denoted in components as $U = U_{ijk}$; an endomorphism $P : T_3^0\mathbf{R}^n \to T_3^0\mathbf{R}^n$ is denoted as $P = P^{ijk}{}_{pqr}$, with standard meaning of the superscripts and the subscripts. For basic theory and notation we follow Krupka [1], [2], where further relevant references can be found.

Sections 2 contain elementary properties of the permutation group S_3; the group S_3 is first used in Section 3, where we derive an explicit formula for the composition of two natural endomorphisms of the tensor space $T_3^0\mathbf{R}^n$. Section 4 contains a criterion for a natural endomorphism to be a natural projector; to this purpose *equations of natural projectors* are derived by means of the composition law for natural endomorphisms. The central topic of this paper is to solve these equations. In Section 5 we introduce some new, *adapted* coordinates to transform these equations to a suitable form. Next in Section 6 we give a complete solution of this system. Namely, a complete list of natural projectors in the tensor space $T_3^0\mathbf{R}^n$ is obtained; another key result is a theorem on the dimensions of image spaces of the natural projectors. Section 7 is devoted to the analysis of decomposition properties of the natural projectors. After recalling the definitions we construct the *decomposability indicatrix* as a simple tool to decide whether a given natural projector is decomposable (in a non-trivial way). The corresponding *indicatrix tables* are obtained from the dimension theorem by a based on a straightforward way. The section concludes by determining the *primitive* (non-decomposable) projectors, and the *partitions* of the tensor space $T_3^0\mathbf{R}^n$.

2 The permutation group S_3

For the reader's convenience and references, we include in this section complete (standard) calculations of basic characteristics of the permutation group S_3. Write elements of S_3 as

$$\nu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \nu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$
$$\nu_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \nu_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \nu_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}. \tag{1}$$

The multiplication table

(2)

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_2	v_3	v_4	v_5	v_4
v_2	v_2	v_2v_2	v_2v_3	v_2v_4	v_2v_5	v_2v_6
v_3	v_3	v_3v_2	v_3v_3	v_3v_4	v_3v_5	v_3v_6
v_4	v_4	v_4v_2	v_4v_3	v_4v_4	v_4v_5	v_4v_6
v_5	v_5	v_5v_2	v_5v_3	v_5v_4	v_5v_5	v_5v_6
v_6	v_6	v_6v_2	v_6v_3	v_6v_4	v_6v_5	v_6v_6

gives

(3)

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	123	132	213	321	312	231
v_2	132	123	312	231	213	321
v_3	213	231	123	312	321	132
v_4	321	312	231	123	132	213
v_5	312	321	132	213	231	123
v_6	231	213	321	132	123	312

or, which is the same,

(4)

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_2	v_3	v_4	v_5	v_6
v_2	v_2	v_1	v_5	v_6	v_3	v_4
v_3	v_3	v_6	v_1	v_5	v_4	v_2
v_4	v_4	v_5	v_6	v_1	v_2	v_3
v_5	v_5	v_4	v_2	v_3	v_6	v_1
v_6	v_6	v_3	v_4	v_2	v_1	v_5

This table can also be expressed as

(5)
$$v_1 = v_1v_1, v_2v_2, v_3v_3, v_4v_4, v_5v_6, v_6v_5,$$
$$v_2 = v_1v_2, v_2v_1, v_3v_6, v_4v_5, v_5v_3, v_6v_4,$$
$$v_3 = v_1v_3, v_2v_5, v_3v_1, v_4v_6, v_5v_4, v_6v_2,$$

$$
\begin{aligned}
&v_4 = v_1v_4, v_2v_6, v_3v_5, v_4v_1, v_5v_2, v_6v_3,\\
&v_5 = v_1v_5, v_2v_3, v_3v_4, v_4v_2, v_5v_1, v_6v_6,\\
&v_6 = v_1v_6, v_2v_4, v_3v_2, v_4v_3, v_5v_5, v_6v_1,
\end{aligned} \tag{5}
$$

which, in particular, immediately determines the *inverse elements* and *conjugacy classes* of elements of the permutation group S_3.

The inverse elements are

$$
\begin{aligned}
&v_1^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = v_1, \quad v_2^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = v_2, \quad v_3^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = v_3,\\
&v_4^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = v_4, \quad v_5^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = v_6, \quad v_6^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = v_5.
\end{aligned} \tag{6}
$$

The table of *conjugate elements* can be obtained by immediate multiplication of permutations (5) and (6). We have

(7)

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	$v_1v_1v_1^{-1}$	$v_2v_1v_2^{-1}$	$v_3v_1v_3^{-1}$	$v_4v_1v_4^{-1}$	$v_5v_1v_5^{-1}$	$v_6v_1v_6^{-1}$
v_2	$v_1v_2v_1^{-1}$	$v_2v_2v_2^{-1}$	$v_3v_2v_3^{-1}$	$v_4v_2v_4^{-1}$	$v_5v_2v_5^{-1}$	$v_6v_2v_6^{-1}$
v_3	$v_1v_3v_1^{-1}$	$v_2v_3v_2^{-1}$	$v_3v_3v_3^{-1}$	$v_4v_3v_4^{-1}$	$v_5v_3v_5^{-1}$	$v_6v_3v_6^{-1}$
v_4	$v_1v_4v_1^{-1}$	$v_2v_4v_2^{-1}$	$v_3v_4v_3^{-1}$	$v_4v_4v_4^{-1}$	$v_5v_4v_5^{-1}$	$v_6v_4v_6^{-1}$
v_5	$v_1v_5v_1^{-1}$	$v_2v_5v_2^{-1}$	$v_3v_5v_3^{-1}$	$v_4v_5v_4^{-1}$	$v_5v_5v_5^{-1}$	$v_6v_5v_6^{-1}$
v_6	$v_1v_6v_1^{-1}$	$v_2v_6v_2^{-1}$	$v_3v_6v_3^{-1}$	$v_4v_6v_4^{-1}$	$v_5v_6v_5^{-1}$	$v_6v_6v_6^{-1}$

that is,

(8)

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_1	v_1	v_1	v_1	v_1
v_2	v_2	v_2	$v_3v_2v_3^{-1}$	$v_4v_2v_4^{-1}$	$v_5v_2v_5^{-1}$	$v_6v_2v_6^{-1}$
v_3	v_3	$v_2v_3v_2^{-1}$	v_3	$v_4v_3v_4^{-1}$	$v_5v_3v_5^{-1}$	$v_6v_3v_6^{-1}$
v_4	v_4	$v_2v_4v_2^{-1}$	$v_3v_4v_3^{-1}$	v_4	$v_5v_4v_5^{-1}$	$v_6v_4v_6^{-1}$
v_5	v_5	$v_2v_5v_2^{-1}$	$v_3v_5v_3^{-1}$	$v_4v_5v_4^{-1}$	v_5	$v_6v_5v_6^{-1}$
v_6	v_6	$v_2v_6v_2^{-1}$	$v_3v_6v_3^{-1}$	$v_4v_6v_4^{-1}$	$v_5v_6v_5^{-1}$	v_6

Computing conjugacy classes,

(9)
$$\begin{aligned}
&v_3v_2v_3^{-1}(1,2,3)=v_3v_2v_3(1,2,3)=v_3v_2(2,1,3)=v_3(3,1,2)=(3,2,1)=v_4,\\
&v_6v_2v_6^{-1}(1,2,3)=v_6v_2v_5(1,2,3)=v_6v_2(3,1,2)=v_6(2,1,3)=(3,2,1)=v_4,\\
&v_5v_2v_5^{-1}(1,2,3)=v_5v_2v_6(1,2,3)=v_5v_2(2,3,1)=v_5(3,2,1)=(2,1,3)=v_3,\\
&v_4v_2v_4^{-1}(1,2,3)=v_4v_2v_4(1,2,3)=v_4v_2(3,2,1)=v_4(2,3,1)=(2,1,3)=v_3,
\end{aligned}$$

(10)
$$\begin{aligned}
&v_2v_3v_2^{-1}(1,2,3)=v_2v_3v_2(1,2,3)=v_2v_3(1,3,2)=v_2(2,3,1)=(3,2,1)=v_4,\\
&v_6v_3v_6^{-1}(1,2,3)=v_6v_3v_5(1,2,3)=v_6v_3(3,1,2)=v_6(3,2,1)=(1,3,2)=v_2,\\
&v_5v_3v_5^{-1}(1,2,3)=v_5v_3v_6(1,2,3)=v_5v_3(2,3,1)=v_5(1,3,2)=(3,2,1)=v_4,\\
&v_4v_3v_4^{-1}(1,2,3)=v_4v_3v_4(1,2,3)=v_4v_3(3,2,1)=v_4(3,1,2)=(1,3,2)=v_2,
\end{aligned}$$

(11)
$$\begin{aligned}
&v_2v_6v_2^{-1}(1,2,3)=v_2v_6v_2(1,2,3)=v_2v_6(1,3,2)=v_2(2,1,3)=(3,1,2)=v_5,\\
&v_3v_6v_3^{-1}(1,2,3)=v_3v_6v_3(1,2,3)=v_3v_6(2,1,3)=v_3(3,2,1)=(3,1,2)=v_5,\\
&v_5v_6v_5^{-1}(1,2,3)=v_5v_6v_6(1,2,3)=v_5v_6(2,3,1)=v_5(3,1,2)=(2,3,1)=v_6,\\
&v_4v_6v_4^{-1}(1,2,3)=v_4v_6v_4(1,2,3)=v_4v_6(3,2,1)=v_4(1,3,2)=(3,1,2)=v_5,
\end{aligned}$$

(12)
$$\begin{aligned}
&v_2v_5v_2^{-1}(1,2,3)=v_2v_5v_2(1,2,3)=v_2v_5(1,3,2)=v_2(3,2,1)=(2,3,1)=v_6,\\
&v_3v_5v_3^{-1}(1,2,3)=v_3v_5v_3(1,2,3)=v_3v_5(2,1,3)=v_3(1,3,2)=(2,3,1)=v_6,\\
&v_6v_5v_6^{-1}(1,2,3)=v_6v_5v_5(1,2,3)=v_6v_5(3,1,2)=v_6(2,3,1)=(3,1,2)=v_5,\\
&v_4v_5v_4^{-1}(1,2,3)=v_4v_5v_4(1,2,3)=v_4v_5(3,2,1)=v_4(2,1,3)=(2,3,1)=v_6,
\end{aligned}$$

(13)
$$\begin{aligned}
&v_2v_4v_2^{-1}(1,2,3)=v_2v_4v_2(1,2,3)=v_2v_4(1,3,2)=v_2(3,1,2)=(2,1,3)=v_3,\\
&v_3v_4v_3^{-1}(1,2,3)=v_3v_4v_3(1,2,3)=v_3v_4(2,1,3)=v_3(2,3,1)=(1,3,2)=v_2,\\
&v_6v_4v_6^{-1}(1,2,3)=v_6v_4v_5(1,2,3)=v_6v_4(3,1,2)=v_6(1,3,2)=(2,1,3)=v_3,\\
&v_5v_4v_5^{-1}(1,2,3)=v_5v_4v_6(1,2,3)=v_5v_4(2,3,1)=v_5(2,1,3)=(1,3,2)=v_2.
\end{aligned}$$

Summarizing these calculations, we obtain the following table of conjugate elements

(14)

	v_1	v_2	v_3	v_6	v_5	v_4
v_1	v_1	v_1	v_1	v_1	v_1	v_1
v_2	v_2	v_2	v_4	v_4	v_3	v_3
v_3	v_3	v_4	v_3	v_2	v_4	v_2
v_6	v_6	v_5	v_5	v_6	v_6	v_5
v_5	v_5	v_6	v_6	v_5	v_5	v_6
v_4	v_4	v_3	v_2	v_3	v_2	v_4

or, in explicit notation,

(15)

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	123	123	123	123	123	123
v_2	132	132	321	213	213	321
v_3	213	321	213	132	321	132
v_4	321	213	132	321	132	213
v_5	312	231	231	231	312	312
v_6	231	312	312	312	231	231

Thus, we have three classes of conjugate elements $\{v_1\}$, $\{v_2,v_3,v_4\}$ and $\{v_5,v_6\}$. Using explicit notation, the classes are $\{123\}$, $\{132,213,321\}$, and $\{312,231\}$. The corresponding *cycle structure* is $(1,1,1)$, $(2,1)$, and 3.

3 Composition of natural endomorphisms in $T_3^0\mathbf{R}^n$

Recall that a *natural endomorphism P* of the tensor space $T_3^0\mathbf{R}^n$ has an expression

(1) $$P^{i_1i_2i_3}{}_{j_1j_2j_3} = \sum_{\kappa\in S_3} a_\kappa \delta^{i_1}_{j_{\kappa(1)}} \delta^{i_2}_{j_{\kappa(2)}} \delta^{i_3}_{j_{\kappa(3)}},$$

where the components a_κ are real numbers. If Q is another natural endomorphism, expressed as

(2) $$Q^{i_1i_2i_3}{}_{j_1j_2j_3} = \sum_{\lambda\in S_3} b_\lambda \delta^{i_1}_{j_{\lambda(1)}} \delta^{i_2}_{j_{\lambda(2)}} \delta^{i_3}_{j_{\lambda(3)}},$$

then the composite $R = QP$ has an expression

(3) $$R^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \sum_{\nu \in S_3} c_\nu \delta^{i_1}_{j_{\nu(1)}} \delta^{i_2}_{j_{\nu(2)}} \delta^{i_3}_{j_{\nu(3)}},$$

where the components satisfy

(4) $$c_\nu = \sum_{\nu = \tau\sigma, \tau, \sigma \in S_3} a_\sigma b_\tau$$

(Krupka [1]).

In different notation, if the natural projectors P and Q are expressed by

(5) $$\begin{aligned} P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} &= a_1 \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + a_2 \delta^{i_1}_{j_1} \delta^{i_2}_{j_3} \delta^{i_3}_{j_2} + a_3 \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \delta^{i_3}_{j_3} \\ &+ a_4 \delta^{i_1}_{j_3} \delta^{i_2}_{j_2} \delta^{i_3}_{j_1} + a_5 \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} + a_6 \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1}, \end{aligned}$$

and

(6) $$\begin{aligned} Q^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} &= b_1 \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + b_2 \delta^{i_1}_{j_1} \delta^{i_2}_{j_3} \delta^{i_3}_{j_2} + b_3 \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \delta^{i_3}_{j_3} \\ &+ b_4 \delta^{i_1}_{j_3} \delta^{i_2}_{j_2} \delta^{i_3}_{j_1} + b_5 \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} + b_6 \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1}, \end{aligned}$$

where $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbf{R}$, $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbf{R}$, then the natural projector R is given by $R^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = Q^{k_1 k_2 k_3}{}_{j_1 j_2 j_3} P^{i_1 i_2 i_3}{}_{k_1 k_2 k_3}$, where

(7) $$\begin{aligned} &R^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} \\ &= (a_1 \delta^{i_1}_{k_1} \delta^{i_2}_{k_2} \delta^{i_3}_{k_3} + a_2 \delta^{i_1}_{k_1} \delta^{i_2}_{k_3} \delta^{i_3}_{k_2} + a_3 \delta^{i_1}_{k_2} \delta^{i_2}_{k_1} \delta^{i_3}_{k_3} \\ &+ a_4 \delta^{i_1}_{k_3} \delta^{i_2}_{k_2} \delta^{i_3}_{k_1} + a_5 \delta^{i_1}_{k_3} \delta^{i_2}_{k_1} \delta^{i_3}_{k_2} + a_6 \delta^{i_1}_{k_2} \delta^{i_2}_{k_3} \delta^{i_3}_{k_1}) \\ &\cdot (b_1 \delta^{k_1}_{j_1} \delta^{k_2}_{j_2} \delta^{k_3}_{j_3} + b_2 \delta^{k_1}_{j_1} \delta^{k_2}_{j_3} \delta^{k_3}_{j_2} + b_3 \delta^{k_1}_{j_2} \delta^{k_2}_{j_1} \delta^{k_3}_{j_3} \\ &+ b_4 \delta^{k_1}_{j_3} \delta^{k_2}_{j_2} \delta^{k_3}_{j_1} + b_5 \delta^{k_1}_{j_3} \delta^{k_2}_{j_1} \delta^{k_3}_{j_2} + b_6 \delta^{k_1}_{j_2} \delta^{k_2}_{j_3} \delta^{k_3}_{j_1}). \end{aligned}$$

One can easily check thatexpressions (3) and (7) for the composite R agree. Consider for example, the term $a_5 b_3 \delta^{k_1}_{j_2} \delta^{k_2}_{j_1} \delta^{k_3}_{j_3} \delta^{i_1}_{k_3} \delta^{i_2}_{k_1} \delta^{i_3}_{k_2} = a_5 b_3 \delta^{i_1}_{j_3} \delta^{i_2}_{j_2} \delta^{i_3}_{j_1}$ in (7), referring to the product $\nu_3 \nu_5$ of the permutations ν_5 and ν_3; since $\nu_3 \nu_5 = \nu_4$, the corresponding summand in equation (3) is labelled by $\nu = \nu_3 \nu_5 = \nu_4$, etc..

Explicit expression for the composition $R = QP$ of natural endomorphisms P and Q can now be obtained from formula (4), with the help of the multiplication table of the permutation group S_3 (Sec. 2, (5)); indeed, one can equivalently use the composition formula (7). We have

$$
\begin{aligned}
c_1 &= b_1a_1 + b_2a_2 + b_3a_3 + b_4a_4 + b_5a_6 + b_6a_5\,,\\
c_2 &= b_1a_2 + b_2a_1 + b_3a_6 + b_4a_5 + b_5a_3 + b_6a_4\,,\\
c_3 &= b_1a_3 + b_2a_5 + b_3a_1 + b_4a_6 + b_5a_4 + b_6a_2\,,\\
c_4 &= b_1a_4 + b_2a_6 + b_3a_5 + b_4a_1 + b_5a_2 + b_6a_3\,,\\
c_5 &= b_1a_5 + b_2a_3 + b_3a_4 + b_4a_2 + b_5a_1 + b_6a_6\,,\\
c_6 &= b_1a_6 + b_2a_4 + b_3a_2 + b_4a_3 + b_5a_5 + b_6a_1\,.
\end{aligned} \tag{8}
$$

4 Natural projector equations

Formulas (8), Sec. 3 can be applied to natural projectors. Suppose that we have a natural endomorphism

$$
\begin{aligned}
P^{i_1i_2i_3}{}_{j_1j_2j_3} &= a_1\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + a_2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + a_3\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}\\
&\quad + a_4\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1} + a_5\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + a_6\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}\,.
\end{aligned} \tag{1}
$$

Then the following two conditions are equivalent:

(1) P is a projector, that is,

$$P^2 = P. \tag{2}$$

(2) The coefficients $a_1, a_2, a_3, a_4, a_5, a_6$ satisfy the system

$$
\begin{aligned}
a_1 &= a_1a_1 + a_2a_2 + a_3a_3 + a_4a_4 + a_5a_6 + a_6a_5\,,\\
a_2 &= a_1a_2 + a_2a_1 + a_3a_6 + a_4a_5 + a_5a_3 + a_6a_4\,,\\
a_3 &= a_1a_3 + a_2a_5 + a_3a_1 + a_4a_6 + a_5a_4 + a_6a_2\,,\\
a_4 &= a_1a_4 + a_2a_6 + a_3a_5 + a_4a_1 + a_5a_2 + a_6a_3\,,\\
a_5 &= a_1a_5 + a_2a_3 + a_3a_4 + a_4a_2 + a_5a_1 + a_6a_6\,,\\
a_6 &= a_1a_6 + a_2a_4 + a_3a_2 + a_4a_3 + a_5a_5 + a_6a_1\,.
\end{aligned} \tag{3}
$$

If a 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a solution of these equations, representing a projector P, then the 6-tuple $(1-a_1, -a_2, -a_3, -a_4, -a_5, -a_6)$ is also a solution; this solution represents the complementary projector $\mathrm{Id} - P$.

Lemma 1 *A natural endomorphism* (1) *is a projector if and only if its coefficients satisfy*

$$a_1(1-a_1) = a_2^2 + a_3^2 + a_4^2 + 2a_5a_6\,, \tag{4}$$

$$a_2(1-2a_1)=(a_3+a_4)(a_5+a_6),$$
(5) $$a_3(1-2a_1)=(a_2+a_4)(a_5+a_6),$$
$$a_4(1-2a_1)=(a_2+a_3)(a_5+a_6),$$

(6) $$(a_5+a_6)(1-2a_1)=2(a_2a_3+a_3a_4+a_4a_2)+a_5^2+a_6^2, \quad (a_5-a_6)(1-2a_1+a_5+a_6)=0.$$

Proof The system (3) can equivalently be expressed as

(7) $$a_1=a_1^2+a_2^2+a_3^2+a_4^2+2a_5a_6,$$

$$a_2=2a_1a_2+a_3a_6+a_4a_5+a_5a_3+a_6a_4,$$
(8) $$a_3=2a_1a_3+a_2a_5+a_4a_6+a_5a_4+a_6a_2,$$
$$a_4=2a_1a_4+a_2a_6+a_3a_5+a_5a_2+a_6a_3,$$

(9) $$a_5=2a_1a_5+a_2a_3+a_3a_4+a_4a_2+a_6^2, \quad a_6=2a_1a_6+a_2a_4+a_3a_2+a_4a_3+a_5^2,$$

which is obviously equivalent to the system (4)–(6).

We call equations (4)–(6) the *natural projector equations*. Note that the mapping $(a_1,a_2,a_3,a_4,a_5,a_6)\to(1-a_1,-a_2,-a_3,-a_4,-a_5,-a_6)$ transforms solutions of the natural projector equations to solutions.

5 Natural projector equations: Adapted coordinates

Recall that the *trace* of a natural projector P,

(1) $$P^{i_1i_2i_3}{}_{i_1i_2i_3}=a_1n^3+(a_2+a_3+a_4)n^2+(a_5+a_6)n,$$

is a non-negative integer, equal to the *dimension* of the image of P. In this paragraph we transform the natural projector equations to new variables, closely related to the trace formula (1).

Lemma 2 *Equations*

(2) $$\alpha=a_1, \quad \beta=a_2+a_3+a_4, \quad \gamma=a_5+a_6,$$

(3) $$\Lambda=a_5-a_6,$$

(4) $\Phi = a_2 - a_3, \quad \Psi = a_3 - a_4$

define a linear transformation of the tensor space $T_3^0\mathbf{R}^n$ *. The inverse linear transformation is given by*

(5) $a_1 = \alpha,$

(6)
$$a_2 = \frac{1}{3}\beta + \frac{1}{3}(\Psi + 2\Phi),$$
$$a_3 = \frac{1}{3}\beta + \frac{1}{3}(\Psi - \Phi),$$
$$a_4 = \frac{1}{3}\beta - \frac{1}{3}(\Phi + 2\Psi),$$

(7) $a_5 = \frac{1}{2}(\gamma + \Lambda), \quad a_6 = \frac{1}{2}(\gamma - \Lambda).$

Proof It is immediately seen that formulas (2), (3), (4) can be solved with respect to $a_1, a_2, a_3, a_4, a_5, a_6$. We derive equations (6). Since

(8)
$$\Phi - \Psi = a_2 - a_3 - a_3 + a_4 = a_2 + a_3 + a_4 - 3a_3$$
$$= \beta - 3a_3,$$

hence

(9)
$$a_3 = \frac{\beta + \Psi - \Phi}{3},$$
$$a_2 = \Phi + a_3 = \Phi + \frac{\beta + \Psi - \Phi}{3} = \frac{\beta + \Psi + 2\Phi}{3},$$
$$a_4 = a_3 - \Psi = \frac{\beta + \Psi - \Phi}{3} - \Psi = \frac{\beta - 2\Psi - \Phi}{3}.$$

The functions $\alpha, \beta, \gamma, \Phi, \Psi, \Lambda$, defined by Lemma 2, are called the *adapted coordinates* on the tensor space $T_3^0\mathbf{R}^n$.

We apply linear transformation (2)–(4) to natural endomorphisms.

Lemma 3 *The components* $P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}$ *of a natural endomorphism P can be uniquely expressed as*

(10) $P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \underset{0}{P}{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} + Q^{i_1 i_2 i_3}{}_{j_1 j_2 j_3},$

where

$$P_0{}^{i_1i_2i_3}{}_{j_1j_2j_3} = \alpha\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \frac{1}{3}\beta(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) + \frac{1}{2}\gamma(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \tag{11}$$

and

$$\begin{aligned} Q^{i_1i_2i_3}{}_{j_1j_2j_3} &= \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \end{aligned} \tag{12}$$

for some constants $\alpha,\beta,\gamma,\Phi,\Psi,\Lambda \in \mathbf{R}$. *The tensors* $P_0 = P_0{}^{i_1i_2i_3}{}_{j_1j_2j_3}$ *and* $Q = Q^{i_1i_2i_3}{}_{j_1j_2j_3}$ *satisfy*

$$P^{i_1i_2i_3}{}_{i_1i_2i_3} = P_0{}^{i_1i_2i_3}{}_{i_1i_2i_3}, \tag{13}$$

and

$$Q^{i_1i_2i_3}{}_{i_1i_2i_3} = 0. \tag{14}$$

Proof Immediate: Substituting from formulas (5)–(7) into expression (1), Sec. 4, for the components $a_1, a_2, a_3, a_4, a_5, a_6$ of P, we get

$$\begin{aligned} P^{i_1i_2i_3}{}_{j_1j_2j_3} &= \alpha\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \frac{1}{3}\beta(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{2}\gamma(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) + \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}). \end{aligned} \tag{15}$$

Verification of the trace formulas (13), (14) is also straightforward, because all tensors, standing at the coefficients Φ, Ψ, and Λ in formula (13) are traceless.

The following lemma describes the *natural projector equations* in the adapted coordinates.

Lemma 4 *A natural endomorphism* $P = P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}$ (10) *is a natural projector if and only if*

$$\alpha(1-\alpha) = \frac{1}{3}\beta^2 + \frac{1}{9}((2\Phi+\Psi)^2 + (\Psi-\Phi)^2 + (\Phi+2\Psi)^2) + \frac{1}{2}(\gamma^2 - \Lambda^2), \tag{16}$$

$$\begin{aligned} &\beta(1-2\alpha-2\gamma) = 0,\\ &\Phi(1-2\alpha+\gamma) = 0,\\ &\Psi(1-2\alpha+\gamma) = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &\gamma(1-2\alpha) = \frac{2}{3}\beta^2 - \frac{1}{9}((2\Phi+\Psi)^2 + (\Psi-\Phi)^2 + (\Phi+2\Psi)^2) + \frac{1}{2}(\gamma^2 + \Lambda^2),\\ &\Lambda(1-2\alpha+\gamma) = 0. \end{aligned} \tag{18}$$

Proof 1. Applying Lemma 2 to equation (4), Sec. 4, we have

$$\begin{aligned} \alpha(1-\alpha) &= a_2^2 + a_3^2 + a_4^2 + 2a_5 a_6\\ &= \frac{1}{9}\beta^2 + \frac{2}{9}\beta(\Psi + 2\Phi) + \frac{1}{9}(\Psi + 2\Phi)^2\\ &+ \frac{1}{9}\beta^2 + \frac{2}{9}\beta(\Psi - \Phi) + \frac{1}{9}(\Psi - \Phi)^2\\ &+ \frac{1}{9}\beta^2 - \frac{2}{9}\beta(\Phi + 2\Psi) + \frac{1}{9}(\Phi + 2\Psi)^2\\ &+ \frac{1}{2}(\gamma^2 - \Lambda^2), \end{aligned} \tag{19}$$

proving (16).

2. Consider equations (5), Sec. 4. Simple algebraic operations imply

$$\begin{aligned} &(a_2 + a_3 + a_4)(1 - 2a_1 - 2(a_5 + a_6)) = 0,\\ &(a_2 - a_3)(1 - 2a_1 + a_5 + a_6) = 0,\\ &(a_3 - a_4)(1 - 2a_1 + a_5 + a_6) = 0,\\ &(a_4 - a_2)(1 - 2a_1 + a_5 + a_6) = 0. \end{aligned} \tag{20}$$

Note that the last equation in this system is dependent.

We have derived this system from equations (5), Sec. 4. However, it is immediately verified that this system is *equivalent* to (5), Sec. 4. Indeed, if (20) holds, then

$$\begin{aligned}(21)\qquad &(a_2+a_3+a_4)(1-2a_1-2(a_5+a_6))\\ &\quad -(a_2-a_3)(1-2a_1)+a_5+a_6)\\ &\quad +(a_3-a_4)(1-2a_1)+a_5+a_6)\\ &\quad =(a_2+a_3+a_4-a_2+a_3+a_3-a_4)(1-2a_1)\\ &\quad -2(a_2+a_3+a_4)(a_5+a_6)-(a_2-a_3)(a_5+a_6)\\ &\quad +(a_3-a_4)(a_5+a_6)=0,\end{aligned}$$

thus

$$\begin{aligned}(22)\qquad &3a_3(1-2a_1)-(2(a_2+a_3+a_4)+a_2-a_3-a_3+a_4)(a_5+a_6)\\ &\quad =3a_3(1-2a_1)-3(a_2+a_4)(a_5+a_6)=0.\end{aligned}$$

This is, however, the second one of equations (5), Sec. 4. The remaining two equations can be verified in the same way.

Now equations (20) can immediately be expressed in the variables α, β, γ, Φ, Ψ, Λ; the arising system is (17).

3. It remains to transform subsystem (6), Sec. 4 to the variables α, β, γ, Φ, Ψ, Λ. Substitutions yield

$$\begin{aligned}(23)\qquad &\gamma(1-2\alpha)=\frac{2}{9}((\beta+\Psi+2\Phi)(\beta+\Psi-\Phi)\\ &\quad +(\beta+\Psi-\Phi)(\beta-\Phi-2\Psi)+(\beta-\Phi-2\Psi)(\beta+\Psi+2\Phi))\\ &\quad +\frac{1}{4}(\gamma^2+2\gamma\Lambda+\Lambda^2+\gamma^2-2\gamma\Lambda+\Lambda^2)\\ &\quad =\frac{2}{3}\beta^2\\ &\quad +\frac{2}{9}\beta(\Psi-\Phi+\Psi+2\Phi-\Phi-2\Psi+\Psi-\Phi+\Psi+2\Phi-\Phi-2\Psi)\\ &\quad +\frac{2}{9}((\Psi+2\Phi)(\Psi-\Phi)-(\Psi-\Phi)(\Phi+2\Psi)-(\Phi+2\Psi)(\Psi+2\Phi))\\ &\quad +\frac{1}{2}(\gamma^2+\Lambda^2)\end{aligned}$$

hence

(24)
$$\begin{aligned}&\gamma(1-2\alpha)\\&=\frac{2}{3}\beta^2+\frac{2}{9}((\Phi-\Psi)(\Psi-\Phi)-(\Phi+2\Psi)(\Psi+2\Phi)+\frac{1}{2}(\gamma^2+\Lambda^2)\\&=\frac{2}{3}\beta^2-\frac{2}{3}(\Psi^2+\Phi^2+\Phi\Psi)+\frac{1}{2}(\gamma^2+\Lambda^2).\end{aligned}$$

Now, however, applying the identity

(25)
$$\begin{aligned}&(2\Phi+\Psi)^2+(\Phi-\Psi)^2+(\Phi+2\Psi)^2\\&=6(\Phi^2+\Psi^2+\Psi\Phi),\end{aligned}$$

we get (18). The second equation (6), Sec. 4, is obviously of the form (18).
This completes the proof.

Remark 1 Every solution $(\alpha,\beta,\gamma,\Phi,\Psi,\Lambda)$ of equations of natural projectors (16) - (18) such that $\alpha=0$ or $\alpha=1$, is the trivial solution

(26) $$\alpha=0,\quad \beta=0,\quad \gamma=0,\quad \Phi=0,\quad \Psi=0,\quad \Lambda=0$$

or

(27) $$\alpha=1,\quad \beta=0,\quad \gamma=0,\quad \Phi=0,\quad \Psi=0,\quad \Lambda=0.$$

Clearly, only formulas (26) need verification, because equations (27) characterize the complementary projector. If $\alpha=0$, then system (16)–(18) reduces to the equations

(28) $$\frac{1}{3}\beta^2+\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)+\frac{1}{2}(\gamma^2-\Lambda^2)=0,$$

(29) $$\beta(1-2\gamma)=0,\quad \Phi(1+\gamma)=0,\quad \Psi(1+\gamma)=0,$$

(30)
$$\begin{aligned}&\gamma=\frac{2}{3}\beta^2-\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)+\frac{1}{2}(\gamma^2+\Lambda^2),\\&\Lambda(1+\gamma)=0.\end{aligned}$$

If $\Phi\neq 0$, then $\gamma=-1$ hence $\beta=0$, and

(31) $$\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)+\frac{1}{2}(1-\Lambda^2)=0,$$

(32) $$-1=-\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)+\frac{1}{2}(1+\Lambda^2).$$

But this is a contradiction, showing that $\Phi = 0$. The same procedure shows that $\Psi = 0$. Consequently, equations (16)–(18) reduce to

$$\frac{1}{3}\beta^2 + \frac{1}{2}(\gamma^2 - \Lambda^2) = 0, \tag{33}$$

$$\beta(1 - 2\gamma) = 0, \tag{34}$$

$$\begin{aligned} &\gamma = \frac{2}{3}\beta^2 + \frac{1}{2}(\gamma^2 + \Lambda^2), \\ &\Lambda(1 + \gamma) = 0. \end{aligned} \tag{35}$$

If $\Lambda \neq 0$, then $\gamma = -1$ hence $\beta = 0$ and we have again a contradiction

$$-1 = \frac{1}{2}(\gamma^2 + \Lambda^2), \tag{36}$$

showing that $\Lambda = 0$. The remaining equations

$$\frac{1}{3}\beta^2 + \frac{1}{2}\gamma^2 = 0, \quad \beta(1 - 2\gamma) = 0, \quad \gamma = \frac{2}{3}\beta^2 + \frac{1}{2}\gamma^2 \tag{37}$$

do not admit non-trivial solutions. Thus, if $\alpha = 0$, then, $\beta = 0$, $\gamma = 0$, $\Phi = 0$, $\Psi = 0$, and $\Lambda = 0$ as required.

6 Classification of natural projectors

Our aim in this section is togive a complete list of natural projectors in the tensor space $T_3^0\mathbf{R}^n$. According to Lemma 3, Sec. 5, a natural endomorphism $P = P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}$ of the tensor space $T_3^0\mathbf{R}^n$ can be expressed as

$$P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = P_0^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} + Q^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}, \tag{1}$$

where

$$\begin{aligned} P_0^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} &= \alpha \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \frac{1}{3}\beta(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{2}\gamma(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \end{aligned} \tag{2}$$

is the *principal part* of P, and

$$Q^{i_1i_2i_3}{}_{j_1j_2j_3} = \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$

$$(3) \qquad + \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$

$$+ \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$

is the *traceless part* of P. Our objective now will be to specify the components $\alpha,\beta,\gamma,\Phi,\Psi,\Lambda \in \mathbf{R}$ in such a way that P be a natural projector.

Lemma 5 *Let* $P = P^{i_1i_2i_3}{}_{j_1j_2j_3}$ *be a natural endomorphism. The following two conditions are equivalent:*

(a) *P is a natural projector.*

(b) *The constants* $\alpha,\beta,\gamma,\Phi,\Psi,\Lambda$ *satisfy one of the conditions* (A11)–(A14), (A21)–(A24),

(A11) $\alpha = 0$, $\beta = 0$, $\gamma = 0$,
(A12) $\alpha = 1$, $\beta = 0$, $\gamma = 0$,
(A13) $\alpha = 2/3$, $\beta = 0$, $\gamma = -2/3$,
(A14) $\alpha = 1/3$, $\beta = 0$, $\gamma = 2/3$,
(A21) $\alpha = 5/6$, $\beta = -1/2$, $\gamma = -1/3$,
(A22) $\alpha = 5/6$, $\beta = 1/2$, $\gamma = -1/3$,
(A23) $\alpha = 1/6$, $\beta = 1/2$, $\gamma = 1/3$,
(A24) $\alpha = 1/6$, $\beta = -1/2$, $\gamma = 1/3$,

and the conditions

$$(4) \qquad \Phi = 0, \quad \Psi = 0, \quad \Lambda = 0,$$

or one of the conditions (B1)–(B4),

(B1) $\alpha = 1/2$, $\beta = 1/2$, $\gamma = 0$,
(B2) $\alpha = 1/2$, $\beta = -1/2$, $\gamma = 0$,
(B3) $\alpha = 2/3$, $\beta = 0$, $\gamma = 1/3$,
(B4) $\alpha = 1/3$, $\beta = 0$, $\gamma = -1/3$,

and the condition

$$(5) \qquad \frac{2}{9}((2\Phi+\Psi)^2 + (\Psi-\Phi)^2 + (\Phi+2\Psi)^2) = \Lambda^2 + \frac{1}{3}.$$

Proof To prove Lemma 5 we use equations (16)–(18), Sec. 5. The proof is divided in two parts; first we suppose that the coefficients satisfy condition (A) $(2\Phi+\Psi)^2 + (\Psi-\Phi)^2 + (\Phi+2\Psi)^2 = 0$, and then we study the complementary possibility (B) $(2\Phi+\Psi)^2 + (\Psi-\Phi)^2 + (\Phi+2\Psi)^2 \neq 0$.

(A) Suppose that

(6) $$(2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2=0.$$

Clearly, this condition is equivalent to

(7) $$\Psi=\Phi=0.$$

Then equations (16)–(18), Sec. 5, reduce to

(8) $$\alpha(1-\alpha)=\frac{1}{3}\beta^2+\frac{1}{2}(\gamma^2-\Lambda^2),$$

(9) $$\beta(1-2\alpha-2\gamma)=0,$$

(10) $$\gamma(1-2\alpha)=\frac{2}{3}\beta^2+\frac{1}{2}(\gamma^2+\Lambda^2),\quad \Lambda(1-2\alpha+\gamma)=0.$$

Since equations (10) do not admit solutions such that $\Lambda\neq 0$, we also have

(11) $$\Lambda=0,$$

and the system to be considered reduces to

(12) $$\alpha(1-\alpha)=\frac{1}{3}\beta^2+\frac{1}{2}\gamma^2,$$

(13) $$\beta(1-2\alpha-2\gamma)=0,$$

(14) $$\gamma(1-2\alpha)=\frac{2}{3}\beta^2+\frac{1}{2}\gamma^2.$$

Now two possibilities should be studied separately, (A1) $\beta=0$, and (A2) $\beta\neq 0$.

(A1) $\beta=0$. In this case system (12)–(14) leads to the following equations for α and γ:

(15) $$\alpha(1-\alpha)=\frac{1}{2}\gamma^2,\quad \gamma(1-2\alpha)=\frac{1}{2}\gamma^2.$$

If $\gamma=0$, then $\alpha=0,1$ (trivial solutions). If $\gamma\neq 0$, then the second equation yields $2(1-2\alpha)=\gamma$ hence $\alpha(1-\alpha)=2(1-4\alpha+4\alpha^2)$ and we get

(16) $$9\alpha^2-9\alpha+2=0.$$

This quadratic equation has two solutions

(17) $$\alpha=\frac{9\pm 3}{18}=\frac{2}{3},\frac{1}{3}.$$

The corresponding values of γ are

(18) $$\gamma=2(1-2\alpha)=-\frac{2}{3},\frac{2}{3}.$$

Summarizing, for $\beta=0$ we have four solutions

(A11) $\beta=0$, $\gamma=0$, $\alpha=0$.
(A12) $\beta=0$, $\gamma=0$, $\alpha=1$.
(A13) $\beta=0$, $\gamma=-2/3$, $\alpha=2/3$.
(A14) $\beta=0$, $\gamma=2/3$, $\alpha=1/3$.

(A2) $\beta\neq 0$. In this case system (12)–(14) leads to the equations

(19) $$\alpha(1-\alpha)=\frac{1}{3}\beta^2+\frac{1}{2}\gamma^2,$$

(20) $$1-2\alpha-2\gamma=0,$$

(21) $$\gamma(1-2\alpha)=\frac{2}{3}\beta^2+\frac{1}{2}\gamma^2.$$

Substituting from (20) to (21),

(22) $$\gamma(1-2\alpha-2\gamma+2\gamma)=2\gamma^2=\frac{2}{3}\beta^2+\frac{1}{2}\gamma^2,$$

hence

(23) $$\frac{9}{4}\gamma^2=\beta^2.$$

Eliminating β from (19) we get two equations for α and γ

(24) $$\alpha(1-\alpha)=\frac{5}{4}\gamma^2,\quad 1-2\alpha-2\gamma=0.$$

Substitution $\gamma=(1-2\alpha)/2$ yields

(25) $$\alpha(1-\alpha)=\frac{5}{16}(1-4\alpha+4\alpha^2),$$

which results in a quadratic equation

(26) $36\alpha^2 - 36\alpha + 5 = 0.$

Since the discriminant is $36^2 - 4\cdot 36\cdot 5 = 24^2$, we have two solutions

(27) $\alpha = \frac{36 \pm 24}{72} = \frac{60}{72}, \frac{12}{72} = \frac{5}{6}, \frac{1}{6}.$

The corrsponding coefficient γ is

(28) $\gamma = \frac{1-2\alpha}{2} = -\frac{1}{3}, \frac{1}{3}.$

Each value of γ then determines β by formula (23). Now the following results end part (A) of the proof:

(A21) $\alpha = 5/6$, $\gamma = -1/3$, $\beta = 3\gamma/2 = -1/2$,
(A22) $\alpha = 5/6$, $\gamma = -1/3$, $\beta = -3\gamma/2 = 1/2$,
(A23) $\alpha = 1/6$, $\gamma = 1/3$, $\beta = 3\gamma/2 = 1/2$,
(A24) $\alpha = 1/6$, $\gamma = 1/3$, $\beta = -3\gamma/2 = -1/2$.

(B) Now we solve equations (16)–(18), Sec. 5, under assumption

(29) $(2\Phi + \Psi)^2 + (\Psi - \Phi)^2 + (\Phi + 2\Psi)^2 \neq 0.$

This condition implies, in particular, that at least one of the constants Ψ , Φ must be different from 0. Note that $\beta(1-2\alpha-2\gamma) = \beta(1-2\alpha+\gamma) - 3\beta\gamma$; consequently, the system to be solved is of the form

(30)
$$\alpha(1-\alpha) = \frac{1}{3}\beta^2 + \frac{1}{9}((2\Phi + \Psi)^2 + (\Psi - \Phi)^2 + (\Phi + 2\Psi)^2) + \frac{1}{2}(\gamma^2 - \Lambda^2),$$

(31) $\beta\gamma = 0, \quad 1 - 2\alpha + \gamma = 0,$

(32)
$$\gamma(1-2\alpha) = \frac{2}{3}\beta^2 - \frac{1}{9}((2\Phi + \Psi)^2 + (\Psi - \Phi)^2 + (\Phi + 2\Psi)^2) + \frac{1}{2}(\gamma^2 + \Lambda^2).$$

If $\gamma = 0$, then $\alpha = 1/2$ and the system reduces to two equations

(33) $\frac{1}{3}\beta^2 + \frac{1}{9}((2\Phi + \Psi)^2 + (\Psi - \Phi)^2 + (\Phi + 2\Psi)^2) - \frac{1}{2}\Lambda^2 = \frac{1}{4},$

(34) $$\frac{2}{3}\beta^2-\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)+\frac{1}{2}\Lambda^2=0.$$

Thus

(35) $$\beta^2=\frac{1}{4},$$

and we have two solutions:

(B1) $\alpha=1/2$, $\beta=1/2$, $\gamma=0$.
(B2) $\alpha=1/2$, $\beta=-1/2$, $\gamma=0$.

In both cases the constants Φ,Ψ,Λ satisfy

(36) $$\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)-\frac{1}{2}\Lambda^2=\frac{1}{6}.$$

If $\gamma\neq 0$, then $\beta=0$ and the system (30)–(32) reduces to

(37) $$\alpha(1-\alpha)=\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)+\frac{1}{2}(\gamma^2-\Lambda^2),$$

(38) $$1-2\alpha+\gamma=0,$$

(39) $$\gamma(1-2\alpha)=-\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)+\frac{1}{2}(\gamma^2+\Lambda^2).$$

Combining (30) and (32),

(40) $$\alpha(1-\alpha)+\gamma(1-2\alpha)=\gamma^2.$$

Substitution from (31) yields $1-\gamma^2=8\gamma^2$ hence

(41) $$9\gamma^2=1.$$

Therefore, we have two solutions:

(B3) $\alpha=2/3$, $\beta=0$, $\gamma=1/3$,
(B4) $\alpha=1/3$, $\beta=0$, $\gamma=-1/3$.

Again in both cases the constants Φ,Ψ,Λ should satisfy

(42) $$\frac{1}{9}((2\Phi+\Psi)^2+(\Psi-\Phi)^2+(\Phi+2\Psi)^2)-\frac{1}{2}\Lambda^2=\frac{1}{6}.$$

This completes part (B) of the proof.

All possibilities for the principal parts of natural projectors, expressed in an explicit form, are summarized in the following lemma.

Lemma 6 (Principal parts of natural projectors) *The following list includes explicit formulas for principal parts of natural projectors:*

(A11) $\alpha = 0$, $\beta = 0$ $\gamma = 0$,

$$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = 0.$$

(A12) $\alpha = 1$, $\beta = 0$, $\gamma = 0$,

$$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3}.$$

(A13) $\alpha = 2/3$, $\beta = 0$, $\gamma = -2/3$,

$$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} - \frac{1}{3}(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1}).$$

(A14) $\alpha = 1/3$, $\beta = 0$, $\gamma = 2/3$,

$$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{3}(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1}).$$

(A21) $\alpha = 5/6$, $\beta = -1/2$, $\gamma = -1/3$,

$$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_1} \delta^{i_2}_{j_3} \delta^{i_3}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \delta^{i_3}_{j_3} + \delta^{i_1}_{j_3} \delta^{i_2}_{j_2} \delta^{i_3}_{j_1}$$
$$+ \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1}).$$

(A22) $\alpha = 5/6$, $\beta = 1/2$, $\gamma = -1/3$,

$$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + \frac{1}{6}(\delta^{i_1}_{j_1} \delta^{i_2}_{j_3} \delta^{i_3}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \delta^{i_3}_{j_3} + \delta^{i_1}_{j_3} \delta^{i_2}_{j_2} \delta^{i_3}_{j_1})$$
$$- \frac{1}{6}(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1}).$$

(A23) $\alpha = 1/6$, $\beta = 1/2$, $\gamma = 1/3$,

$$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{6}(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1}$$
$$+ \delta^{i_1}_{j_1} \delta^{i_2}_{j_3} \delta^{i_3}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \delta^{i_3}_{j_3} + \delta^{i_1}_{j_3} \delta^{i_2}_{j_2} \delta^{i_3}_{j_1}).$$

(A24) $\alpha = 1/6$, $\beta = -1/2$, $\gamma = 1/3$,

$$P_0{}^{i_1i_2i_3}{}_{j_1j_2j_3} = \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$-\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}).$$

(B1) $\alpha = 1/2$, $\beta = 1/2$, $\gamma = 0$,

$$P_0{}^{i_1i_2i_3}{}_{j_1j_2j_3} = \frac{1}{6}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_2}_{j_3}\delta^{i_3}_{j_2}) + \frac{1}{6}\delta^{i_2}_{j_2}(\delta^{i_1}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_3}_{j_1})$$
$$+\frac{1}{6}\delta^{i_3}_{j_3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}).$$

(B2) $\alpha = 1/2$, $\beta = -1/2$, $\gamma = 0$,

$$P_0{}^{i_1i_2i_3}{}_{j_1j_2j_3} = \frac{1}{6}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \delta^{i_2}_{j_3}\delta^{i_3}_{j_2}) + \frac{1}{6}\delta^{i_2}_{j_2}(\delta^{i_1}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_3}_{j_1})$$
$$+\frac{1}{6}\delta^{i_3}_{j_3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}).$$

(B3) $\alpha = 2/3$, $\beta = 0$, $\gamma = 1/3$,

$$P_0{}^{i_1i_2i_3}{}_{j_1j_2j_3} = \frac{2}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

(B4) $\alpha = 1/3$, $\beta = 0$, $\gamma = -1/3$,

$$P_0{}^{i_1i_2i_3}{}_{j_1j_2j_3} = \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

Proof Explicit formulas follow from Lemma 5 and formula (2).

Natural projector (A11) is the *trivial* (*zero*) projector, (A12) is the *identity* projector, (A14) is the *cycle projector*. (A23) and (A24) are the *symmetrization* and *alternation* projectors.

Now we wish to determine traceless parts of natural projectors. Note that the identity

(43) $$(2\Phi + \Psi)^2 + (\Phi - \Psi)^2 + (\Phi + 2\Psi)^2 = 6(\Phi^2 + \Phi\Psi + \Psi^2)$$

allows us to express condition (5) in an equivalent way as the *indicatrix* of the bilinear form $4(\Phi^2 + \Phi\Psi + \Psi^2) - 3\Lambda^2$,

(44) $$4(\Phi^2 + \Phi\Psi + \Psi^2) - 3\Lambda^2 = 1.$$

Lemma 7 *Let* $\Lambda \in \mathbf{R}$ *be given, and let* $\Phi_0, \Psi_0 \in \mathbf{R}$ *be two real numbers. The following two conditions are equivalent:*

(a) *The pair* (Φ_0, Ψ_0) *is a solution of equation* (44).

(b) Φ_0 *and* Ψ_0 *satisfy inequalities*

$$-\sqrt{\frac{1+3\Lambda^2}{3}} \le \Phi_0, \Psi_0 \le \sqrt{\frac{1+3\Lambda^2}{3}}, \tag{45}$$

and one of the following four conditions:

$$\begin{aligned} \Psi_0 &= \frac{-\Phi_0 + \sqrt{-3\Phi_0^2 + 4}}{2}, \\ \Psi_0 &= \frac{-\Phi_0 - \sqrt{-3\Phi_0^2 + 4}}{2}, \\ \Phi_0 &= \frac{-\Psi_0 + \sqrt{-3\Psi_0^2 + 4}}{2}, \\ \Phi_0 &= \frac{-\Psi_0 - \sqrt{-3\Psi_0^2 + 4}}{2}. \end{aligned} \tag{46}$$

Proof 1. First consider the bilinear equation in two unknowns

$$x^2 + xy + y^2 = 1. \tag{47}$$

It is easily seen that a pair (x_0, y_0) is a solution of this equation if and only if

$$-\frac{2}{\sqrt{3}} \le x_0, y_0 \le \frac{2}{\sqrt{3}}, \tag{48}$$

and one of the following four conditions is satisfied:

$$\begin{aligned} x_0 &= \frac{-y_0 + \sqrt{-3y_0^2 + 4}}{2}, \\ x_0 &= \frac{-y_0 - \sqrt{-3y_0^2 + 4}}{2}, \\ y_0 &= \frac{-x_0 + \sqrt{-3x_0^2 + 4}}{2}, \\ y_0 &= \frac{-x_0 - \sqrt{-3x_0^2 + 4}}{2}. \end{aligned} \tag{49}$$

Indeed, suppose (x_0, y_0) is a solution. Then x_0 is a solution of the quadratic equation $x^2 + xy_0 + y_0^2 = 1$; in particular, the discriminant $D = -3y_0^2 + 4$ must be non-negative, thus, y_0 satisfies (48). Then the solutions x are

(50) $$x = \frac{-y_0 \pm \sqrt{-3y_0^2 + 4}}{2},$$

so x_0 must be equal to one of them. It remains to verify that the solution x_0 satisfies inequality (48). From (50) we have $2x_0 + y_0 = \pm\sqrt{-3y_0^2 + 4}$ hence $x_0^2 + x_0 y_0 + y_0^2 = 1$; but y_0 as a solution of this quadratic equation satisfies the discriminant condition $-3x_0^2 + 4 \geq 0$, proving (46).

The converse is obvious.

2. To apply these results to equation (44), we set

(51) $$x = \frac{2}{\sqrt{1 + 3\Lambda^2}}\Phi, \quad y = \frac{2}{\sqrt{1 + 3\Lambda^2}}\Psi,$$

and use equation (47).

Lemma 7 completes the description of all possible *traceless parts* of natural projectors.

We are now in a position to restate results of Lemma 5 and Lemma 7 in a complete, closed classification form.

Theorem 1 (Classification) *Let* $n \geq 3$. *Then every natural projector* $P : T_3^0\mathbf{R}^n \to T_3^0\mathbf{R}^n$, $P = P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}$, *is of the form*

(52) $$P = P_0 + Q,$$

where

(53) $$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \alpha \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \frac{1}{3}\beta(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) + \frac{1}{2}\gamma(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$

(54) $$Q^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) + \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) + \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$

and the coefficients α , β , γ , Λ , Φ , Ψ *are given by one row in the following tables* (A) *and* (B)*:*

(A)

P	P_0			Q		
	α	β	γ	Λ	Φ	Ψ
(A11)	0	0	0	0	0	0
(A12)	1	0	0			
(A13)	$\frac{2}{3}$	0	$-\frac{2}{3}$			
(A14)	$\frac{1}{3}$	0	$\frac{2}{3}$			
(A21)	$\frac{5}{6}$	$-\frac{1}{2}$	$-\frac{1}{3}$			
(A22)	$\frac{5}{6}$	$\frac{1}{2}$	$-\frac{1}{3}$			
(A23)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$			
(A24)	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{3}$			

(B)

P	P_0			Q		
	α	β	γ	Λ	Φ	Ψ
(B1)	$\frac{1}{2}$	$\frac{1}{2}$	0	*)	**)	**)
(B2)	$\frac{1}{2}$	$-\frac{1}{2}$	0			
(B3)	$\frac{2}{3}$	0	$\frac{1}{3}$			
(B4)	$\frac{1}{3}$	0	$-\frac{1}{3}$			

*) Λ arbitrary

**) Φ, Ψ arbitrary solutions of equation

$$4(\Phi^2 + \Phi\Psi + \Psi^2) - 3\Lambda^2 = 1$$

Proof The tables (A) and (B) summarize assertions of Lemma 5 and Lemma 7, and include *all* solutions of the natural projector equations (16)–(18), Sec. 5.

Now we discuss several examples of natural projectors.

Example 1 (Self-adjoint natural projectors) *Self-adjoint* natural projectors $P = P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}$ as considered in this example and further on, are defined to be self-adjoint with respect to the canonical (Euclidean) scalar product on the tensor space $T_3^0\mathbf{R}^n$. Equivalently, they can be defined by the condition $P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = P^{j_1 j_2 j_3}{}_{i_1 i_2 i_3}$, stating that the matrix $P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}$ is *symmetric* or, which is the same, by $\Lambda = 0$. All natural projectors in table (A) are self-adjoint. Self-adjoint projectors in table (B) correspond to the choice $\Lambda = 0$ and are determined as solutions of equation (44), $4(\Phi^2 + \Phi\Psi + \Psi^2) = 1$. According to Lemma 7, we get one-parameter families of self-adjoint projectors, with parameter (Φ of Ψ) in the closed interval $[-1/\sqrt{3}, 1/\sqrt{3}]$.

Example 2 (Symmetrization in two indices) We introduce the symmetrization in a standard way and show the the corresponding natural projector is self-adjoint. Consider a natural endomorphism $P = P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}$, expressed as

$$\begin{aligned} P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} &= a_1 \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + a_2 \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + a_3 \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} \\ &\quad + a_4 \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1} + a_5 \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + a_6 \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}. \end{aligned}$$

Choose

$$a_1 = a_2 = \frac{1}{2}, \quad a_3 = a_4 = a_5 = a_6 = 0.$$

Then

$$P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{2}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_2}_{j_3}\delta^{i_3}_{j_2}).$$

This natural endomorphism is obviously a projector:

$$\begin{aligned} P^{j_1 j_2 j_3}{}_{k_1 k_2 k_3} P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} &= \frac{1}{2} P^{j_1 j_2 j_3}{}_{k_1 k_2 k_3} \delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_2}_{j_3}\delta^{i_3}_{j_2}) \\ &= \frac{1}{2}\left(\frac{1}{2}\delta^{i_1}_{k_1}(\delta^{i_2}_{k_2}\delta^{i_3}_{k_3} + \delta^{i_2}_{k_3}\delta^{i_3}_{k_2}) + \frac{1}{2}\delta^{i_1}_{k_1}(\delta^{i_3}_{k_2}\delta^{i_2}_{k_3} + \delta^{i_3}_{k_3}\delta^{i_2}_{k_2})\right) \\ &= \frac{1}{2}\delta^{i_1}_{k_1}(\delta^{i_2}_{k_2}\delta^{i_3}_{k_3} + \delta^{i_2}_{k_3}\delta^{i_3}_{k_2}). \end{aligned}$$

Passing to the adapted coordinates, we have

$$\alpha = a_1 = \frac{1}{2}, \quad \beta = a_2 + a_3 + a_4 = \frac{1}{2}, \quad \gamma = a_5 + a_6 = 0,$$

and

$$\Lambda = a_5 - a_6 = 0, \quad \Phi = a_2 - a_3 = \frac{1}{2}, \quad \Psi = a_3 - a_4 = 0.$$

Consequently, P belongs to the family (B1) and is self-adjoint. Its explicit expression in adapted coordinates is

$$P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{2}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+ \frac{1}{6}(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}).$$

The dimension of the image space of P is

$$P^{i_1 i_2 i_3}{}_{i_1 i_2 i_3} = \frac{1}{2}n^2(n+1).$$

Example 3 (Alternation in two indices) We show the the natural projector, defined by alternation in two indices, is self-adjoint.

Consider a natural endomorphism $P = P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3}$, expressed as

$$P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = a_1\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + a_2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + a_3\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}$$
$$+ a_4\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1} + a_5\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + a_6\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$

(cf. (1), Sec. 4). Choosing

$$a_1 = \frac{1}{2}, \quad a_2 = -\frac{1}{2}, \quad a_3 = a_4 = a_5 = a_6 = 0,$$

we have

$$P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{2}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \delta^{i_2}_{j_3}\delta^{i_3}_{j_2}).$$

This natural endomorphism is obviously a projector:

$$P^{j_1 j_2 j_3}{}_{k_1 k_2 k_3} P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{2}\delta^{i_1}_{k_1}(\delta^{i_2}_{k_2}\delta^{i_3}_{k_3} - \delta^{i_2}_{k_3}\delta^{i_3}_{k_2}).$$

In the adapted coordinates,

$$\alpha = a_1 = \frac{1}{2}, \quad \beta = a_2 + a_3 + a_4 = -\frac{1}{2}, \quad \gamma = a_5 + a_6 = 0,$$

and

$$\Lambda = a_5 - a_6 = 0, \quad \Phi = a_2 - a_3 = -\frac{1}{2}, \quad \Psi = a_3 - a_4 = 0.$$

Consequently, P belongs to the family (B2) and is self-adjoint. Its explicit expression in adapted coordinates is

$$P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{2}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$- \frac{1}{6}(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}).$$

The dimension of the image space of P is

$$P^{i_1 i_2 i_3}{}_{i_1 i_2 i_3} = \frac{1}{2}n^2(n-1).$$

Example 4 (The Young symmetrizers) By the *Young symmetrizer* in the tensor space $T_3^0\mathbf{R}^n$ we mean any of the projectors S, P_1, P_2, P_3, P_4, P_5, P_6, and A, defined as follows:

$$SU = \frac{1}{6}(U_{ijk} + U_{kij} + U_{jki} + U_{ikj} + U_{kji} + U_{jik}),$$

$$P_1U = \frac{1}{3}(U_{ijk} + U_{jik} - U_{kji} - U_{jki}),$$
$$P_2U = \frac{1}{3}(U_{ijk} + U_{kji} - U_{jik} - U_{kij}),$$
$$P_3U = \frac{1}{3}(U_{ijk} + U_{ikj} - U_{jik} - U_{jki}),$$

$$P_4U = \frac{1}{3}(U_{ijk} + U_{jik} - U_{ikj} - U_{kij}),$$
$$P_5U = \frac{1}{3}(U_{ijk} + U_{kji} - U_{ikj} - U_{jki}),$$
$$P_6U = \frac{1}{3}(U_{ijk} + U_{ikj} - U_{kji} + U_{kij}),$$

$$AU = \frac{1}{6}(U_{ijk} + U_{kij} + U_{jki} - U_{ikj} - U_{kji} - U_{jik})$$

These formulas define the well-known natural projectors, used in the Young decomposition theory.

Clearly, the Young symmetrizers can be expressed in the form of natural endomorphisms; S is the *symmetrization*, and A is the *alternation projector*. We shall discuss in more detail, from the point of view of the naturaal projector theory, the Young symmetrizers P_1 (symmetrization in i,j followed by alternation in i,k), and P_3 (symmetrization in j,k followed by alternation in i,j).

P_1 has an expression

$$P_1{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

In the canonical coordinates $a_1 = a_3 = 1/3$, $a_4 = a_6 = -1/3$, $a_2 = a_5 = 0$, and in the adapted coordinates,

$$\alpha = a_1 = \frac{1}{3}, \quad \beta = a_2 + a_3 + a_4 = 0, \quad \gamma = a_5 + a_6 = -\frac{1}{3},$$
$$\Lambda = a_5 - a_6 = \frac{1}{3}, \quad \Phi = a_2 - a_3 = -\frac{1}{3}, \quad \Psi = a_3 - a_4 = \frac{2}{3}.$$

Parameters Λ, Φ, and Ψ satisfy equation $4(\Phi^2 + \Phi\Psi + \Psi^2) - 3\Lambda^2 = 1$, consequently, P_1 belongs to the family (B4). The principal and traceless parts of P_1 are

$$P_0{}^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$

and

$$\begin{aligned} Q^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} &= -\frac{1}{9}(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{2}{9}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \qquad . \\ &+ \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}). \end{aligned}$$

Note that the Young symmetrizer P_1 is defined by the condition $\Psi + 2\Phi = 0$ in formula (5) of Lemma 5. P_1 is *not* self-adjoint. The dimension of its image vector space is equal to its trace, $\dim \operatorname{Im} P_1 = P_1{}^{i_1 i_2 i_3}{}_{i_1 i_2 i_3} = (n/3)(n^2 - 1)$.

The Young symmetrizer P_3 is given by

$$P^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \frac{1}{3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

Its canonical and adapted coordinates are $a_1 = a_2 = 1/3$, $a_3 = a_6 = -1/3$, $a_4 = a_5 = 0$, and

$$\alpha = a_1 = \frac{1}{3}, \quad \beta = a_2 + a_3 + a_4 = 0, \quad \gamma = a_5 + a_6 = -\frac{1}{3},$$
$$\Lambda = a_5 - a_6 = \frac{1}{3}, \quad \Phi = a_2 - a_3 = \frac{2}{3}, \quad \Psi = a_3 - a_4 = -\frac{1}{3}.$$

Consequently, the Young symmetrizer P_3 also belongs to the family (B4). Similarly as for P_1, the choice of coefficients yields $\Phi + 2\Psi = 0$ and again simplifies formula (5) of Lemma 5. P_3 is *not* self-adjoint, and the dimension of its image vector space $\operatorname{Im} P_3$ is $\dim \operatorname{Im} P_2 = (n/3)(n^2 - 1)$.

Now we can determine the dimensions of the image spaces $\operatorname{Im} P$.

Theorem 2 (Dimensions) *The dimensions of the image spaces of the natural projectors P* (A11)–(A14), A(21)–(A24), *and* (B1)–(B4) *in the tensor space $T^0_3\mathbf{R}^n$ are given by the following tables:*

(A)

P	$\dim \operatorname{Im} P$
(A11)	0
(A12)	n^3
(A13)	$\frac{2}{3}n(n^2-1)$
(A14)	$\frac{1}{3}n(n^2+2)$
(A21)	$\frac{1}{6}n(5n^2-3n-2)$
(A22)	$\frac{1}{6}n(5n^2+3n-2)$
(A23)	$\frac{1}{6}n(n^2+3n+2)$
(A24)	$\frac{1}{6}n(n^2-3n+2)$

(B)

	$\dim \operatorname{Im} P$
(B1)	$\frac{1}{2}n^2(n+1)$
(B2)	$\frac{1}{2}n^2(n-1)$
(B3)	$\frac{1}{3}n(2n^2+1)$
(B4)	$\frac{1}{3}n(n^2-1)$

Proof According to the rank formula, $\dim \operatorname{Im} P = P^{i_1 i_2 i_3}{}_{i_1 i_2 i_3}$. But by (52), a natural projector P has a decomposition $P = P_0 + Q$, where P_0 is the principal, and Q is the traceless part of P. Thus, $\dim \operatorname{Im} P = \dim \operatorname{Im} P_0$, and the dimensions can be completely determined by explicit expressions for the principal parts of natural projectors (Lemma 6).

7 Decomposability

In this paragraph we study the decomposability problem for natural projectors $P : T^0_3\mathbf{R}^n \to T^0_3\mathbf{R}^n$. The family of natural projectors is denoted by $\mathcal{P} = \{P_r\}$, where P_r runs through the set of eight natural projectors (A11) – (A14), (A21) – (A24), and four families (B1) – (B4). For further use, we write explicitly

(1)
$$\begin{aligned} &P_1 = (\text{A}11), \quad P_2 = (\text{A}12), \quad P_3 = (\text{A}13), \quad P_4 = (\text{A}14), \\ &P_5 = (\text{A}21), \quad P_6 = (\text{A}22), \quad P_7 = (\text{A}23), \quad P_8 = (\text{A}24), \\ &P_9 = (\text{B}1), \quad P_{10} = (\text{B}2), \quad P_{11} = (\text{B}3), \quad P_{12} = (\text{B}4). \end{aligned}$$

Recall that a natural projector P is *decomposable*, if there exist two natural projectors Q_1 and Q_2, different from P, such that

(2)
$$\operatorname{Im} P = \operatorname{Im} Q_1 \oplus \operatorname{Im} Q_2.$$

The natural projector Q_1 is said to *decompose* P, if there exists Q_2 such that $\operatorname{Im} P = \operatorname{Im} Q_1 \oplus \operatorname{Im} Q_2$. A natural projector, which is *not* decomposable, is called *primitive*.

The zero projector is primitive. The identity projector Id is decomposable: each natural projector together with its complementary projector defines a decomposition (1) of the identity Id .

Given a natural projector P, a *necessary* condition for P to be decomposable, the *dimension decomposability condition*, is the existence of natural projectors Q_1 and Q_2 , different from P, such that

$$\dim \mathrm{Im}\, P = \dim \mathrm{Im}\, Q_1 + \dim \mathrm{Im}\, Q_2 . \tag{3}$$

Since we already have a complete classification of natural projectors and their dimensions (Sec. 6, Theorem 1 and Theorem 2), we can immediately apply these dimension arguments to the decomposability problem. To this purpose we determine the *decomposability indicatrix* $\mathcal{I} = \{I_{rs}\}$ for the family $\mathcal{P} = \{P_r\}$ (Sec. 6.1). By definition, I_{rs} are positive integers

$$I_{rs} = \dim \mathrm{Im}\, P_r + \dim \mathrm{Im}\, P_s , \tag{4}$$

such that $s > r$ for $r = 1,2,\dots,8$ and $s \geq r$ for $r = 9,10,11,12$.

For convenience, the decomposability indicatrix can be expressed as a collection of tables with entries I_{rs} :

$r = 1, \quad \dim \mathrm{Im}\, P_2 = n^3$

P_s	$\dim \mathrm{Im}\, P_s$	$\dim \mathrm{Im}\, P_1 + \dim \mathrm{Im}\, P_s$	
	$\dim \mathrm{Im}\, P_s$	$\dim \mathrm{Im}\, P_s$	

$r = 2, \quad \dim \mathrm{Im}\, P_2 = n^3$

P_s	$\dim \mathrm{Im}\, P_s$	$\dim \mathrm{Im}\, P_2 + \dim \mathrm{Im}\, P_s$	
	$\dim \mathrm{Im}\, P_s$	$n^3 + \dim \mathrm{Im}\, P_s$	–

$r = 3, \quad \dim \mathrm{Im}\, P_3 = \frac{2}{3} n(n^2 - 1)$

P_s	$\dim \mathrm{Im}\, P_s$	$\dim \mathrm{Im}\, P_3 + \dim \mathrm{Im}\, P_s$	
P_4	$\frac{1}{3} n(n^2 + 2)$	n^3	P_2
P_5	$\frac{1}{6} n(5n^2 - 3n - 2)$	$\frac{1}{2} n(3n^2 - n - 2)$	–

P_6	$\frac{1}{6}n(5n^2+3n-2)$	$\frac{1}{2}n(3n^2+n-2)$	–
P_7	$\frac{1}{6}n(n^2+3n+2)$	$\frac{1}{6}n(5n^2+3n-2)$	P_6
P_8	$\frac{1}{6}n(n^2-3n+2)$	$\frac{1}{6}n(5n^2-3n-2)$	P_5
P_9	$\frac{1}{2}n^2(n+1)$	$\frac{1}{6}n(7n^2+3n-4)$	–
P_{10}	$\frac{1}{2}n^2(n-1)$	$\frac{1}{6}n(7n^2-3n-4)$	–
P_{11}	$\frac{1}{3}n(2n^2+1)$	$\frac{1}{3}n(4n^2-1)$	–
P_{12}	$\frac{1}{3}n(n^2-1)$	$n(n^2-1)$	–

$r=4,\quad \dim \operatorname{Im} P_4 = \frac{1}{3}n(n^2+2)$

P_s	$\dim \operatorname{Im} P_2$	$\dim \operatorname{Im} P_1 + \dim \operatorname{Im} P_2$	
P_5	$\frac{1}{6}n(5n^2-3n-2)$	$\frac{1}{6}n(7n^2-3n+2)$	–
P_6	$\frac{1}{6}n(5n^2+3n-2)$	$\frac{1}{6}n(7n^2+3n+2)$	–
P_7	$\frac{1}{6}n(n^2+3n+2)$	$\frac{1}{2}n(n^2+n+2)$	–
P_8	$\frac{1}{6}n(n^2-3n+2)$	$\frac{1}{2}n(n^2-n+2)$	–
P_9	$\frac{1}{2}n^2(n+1)$	$\frac{1}{6}n(5n^2+3n+4)$	–
P_{10}	$\frac{1}{2}n^2(n-1)$	$\frac{1}{6}n(5n^2-3n+4)$	–
P_{11}	$\frac{1}{3}n(2n^2+1)$	$n(n^2+1)$	–
P_{12}	$\frac{1}{3}n(n^2-1)$	$\frac{1}{3}n(2n^2+1)$	P_{11}

$$r=5,\quad \dim \operatorname{Im} P_5 = \frac{1}{6}n(5n^2-3n-2)$$

P_s	$\dim \operatorname{Im} P_s$	$\dim \operatorname{Im} P_5 + \dim \operatorname{Im} P_s$	
P_6	$\frac{1}{6}n(5n^2+3n-2)$	$\frac{1}{3}n(5n^2-2)$	$-$
P_7	$\frac{1}{6}n(n^2+3n+2)$	n^3	P_2
P_8	$\frac{1}{6}n(n^2-3n+2)$	$n^2(n-1)$	$-$
P_9	$\frac{1}{2}n^2(n+1)$	$\frac{1}{3}n(4n^2-1)$	$-$
P_{10}	$\frac{1}{2}n(n^2-1)$	$\frac{1}{6}n(8n^2-3n-5)$	$-$
P_{11}	$\frac{1}{3}n(2n^2+1)$	$\frac{1}{2}n^2(3n-1)$	$-$
P_{12}	$\frac{1}{3}n(n^2-1)$	$\frac{1}{6}n(7n^2-3n-4)$	$-$

$$r=6,\quad \dim \operatorname{Im} P_6 = \frac{1}{6}n(5n^2+3n-2)$$

$$r=6,\quad \dim \operatorname{Im} P_6 = \frac{1}{6}n(5n^2+3n-2)$$

P_s	$\dim \operatorname{Im} P_s$	$\dim \operatorname{Im} P_6 + \dim \operatorname{Im} P_s$	
P_7	$\frac{1}{6}n(n^2+3n+2)$	$n^2(n+1)$	$-$
P_7	$\frac{1}{6}n(n^2-3n+2)$	n^3	P_2
P_9	$\frac{1}{2}n^2(n+1)$	$\frac{1}{3}n(4n^2+3n-1)$	$-$
P_{10}	$\frac{1}{2}n^2(n-1)$	$\frac{1}{3}n(4n^2-1)$	$-$
P_{11}	$\frac{1}{3}n(2n^2+1)$	$\frac{1}{2}n^2(3n+1)$	$-$
P_{12}	$\frac{1}{3}n(n^2-1)$	$\frac{1}{6}n(7n^2+3n-4)$	$-$

$r = 7,\quad \dim \operatorname{Im} P_7 = \frac{1}{6}n(n^2 + 3n + 2)$

P_s	$\dim \operatorname{Im} P_s$	$\dim \operatorname{Im} P_7 + \dim \operatorname{Im} P_s$	
(A24)	$\frac{1}{6}n(n^2 - 3n + 2)$	$\frac{1}{3}n(n^2 + 2)$	P_4
(B1)	$\frac{1}{2}n^2(n+1)$	$\frac{1}{3}n(2n^2 + 3n + 1)$	–
(B2)	$\frac{1}{2}n^2(n-1)$	$\frac{1}{3}n(2n^2 + 1)$	P_{11}
(B3)	$\frac{1}{3}n(2n^2 + 1)$	$\frac{1}{6}n(5n^2 + 3n + 4)$	–
(B4)	$\frac{1}{3}n(n^2 - 1)$	$\frac{1}{2}n^2(n+1)$	P_9

$r = 8,\quad \dim \operatorname{Im} P_8 = \frac{1}{6}n(n^2 - 3n + 2)$

P_s	$\dim \operatorname{Im} P_s$	$\dim \operatorname{Im} P_8 + \dim \operatorname{Im} P_s$	
P_9	$\frac{1}{2}n^2(n+1)$	$\frac{1}{3}n(2n^2 + 1)$	P_{11}
P_{10}	$\frac{1}{2}n^2(n-1)$	$\frac{1}{3}n(2n^2 - 3n + 1)$	–
P_{11}	$\frac{1}{3}n(2n^2 + 1)$	$\frac{1}{6}n(5n^2 - 3n + 4)$	–
P_{12}	$\frac{1}{3}n(n^2 - 1)$	$\frac{1}{2}n^2(n-1)$	P_{10}

$r = 9,\quad \dim \operatorname{Im} P_9 = \frac{1}{2}n^2(n+1)$

P_s	$\dim \operatorname{Im} P_s$	$\dim \operatorname{Im} P_9 + \dim \operatorname{Im} P_s$	
P_9	$\frac{1}{2}n^2(n+1)$	$n^2(n+1)$	–
P_{10}	$\frac{1}{2}n^2(n-1)$	n^3	P_2

P_{11}	$\frac{1}{3}n(2n^2+1)$	$\frac{1}{6}n(7n^2+3n+2)$	–
P_{12}	$\frac{1}{3}n(n^2-1)$	$\frac{1}{6}n(5n^2+3n-2)$	P_6

$r = 10, \quad \dim \operatorname{Im} P_{10} = \frac{1}{2}n^2(n-1)$

P_s	$\dim \operatorname{Im} P_s$	$\dim \operatorname{Im} P_{10} + \dim \operatorname{Im} P_s$	
P_{10}	$\frac{1}{2}n^2(n-1)$	$n^2(n-1)$	–
P_{11}	$\frac{1}{3}n(2n^2+1)$	$\frac{1}{6}n(7n^2-3n+2)$	–
P_{12}	$\frac{1}{3}n(n^2-1)$	$\frac{1}{6}n(5n^2-3n-2)$	P_5

$r = 11, \quad \dim \operatorname{Im} P_{11} = \frac{1}{3}n(2n^2+1)$

P_s	$\dim \operatorname{Im} P_s$	$\dim \operatorname{Im} P_{11} + \dim \operatorname{Im} P_s$	
P_{11}	$\frac{1}{3}n(2n^2+1)$	$\frac{2}{3}n(2n^2+1)$	–
P_{12}	$\frac{1}{3}n(n^2-1)$	n^3	P_2

$r = 12, \quad \dim \operatorname{Im} P_{12} = \frac{1}{3}n(n^2-1)$

P_s	$\dim \operatorname{Im} P_s$	$\dim \operatorname{Im} P_{12} + \dim \operatorname{Im} P_s$	
P_{12}	$\frac{1}{3}n(n^2-1)$	$\frac{2}{3}n(n^2-1)$	P_3

We are now in a position to prove main results of this paper – classification of decomposable natural projectors and primitive natural projectors, and classification of natural partitions of the tensor space $T_3^0\mathbf{R}^n$. The proofs are immediate consequences of the decomposability indicatrix $\mathcal{I} = \{I_{rs}\}$ for the family of natural projectors $\mathcal{P} = \{P_r\}$, and of explicit expressions derived in Theorem 1, Sec. 6.

Theorem 3 (Decomposability) *Let P be a natural projector. The following two conditions are equivalent:*

(a) *P is decomposable.*

(b) *P is equal to one of the natural projectors P_2, P_3, P_4, P_5, P_6 or to one element of the families P_9, P_{10}, P_{11}.*

Proof 1. Suppose that P is decomposable. Then P must belong to the fourth column of the decomposability indicatrix tables, proving (b).

2. To prove that (b) implies (a), we study decomposability of the projectors P_2, P_4, P_5, P_6 and P_9, P_{10}, P_{11} separately.

P_2 is the identity projector, hence it is decomposable.

We show that there exist two natural projectors R_{12} and S_{12} in the family P_{12} such that $P_3 = R_{12} + S_{12}$. Since $P_{12} = (\mathrm{B}4)$, we have

$$\begin{aligned}R_{12} = {} & \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \\ & + \frac{1}{3}\Phi_1(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ & + \frac{1}{3}\Psi_1(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ & + \frac{1}{2}\Lambda_1(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),\end{aligned}$$

and

$$\begin{aligned}S_{12} = {} & \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \\ & + \frac{1}{3}\Phi_2(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ & + \frac{1}{3}\Psi_2(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ & + \frac{1}{2}\Lambda_2(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).\end{aligned}$$

The sum of principal parts is

$$\begin{aligned}& \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \\ & + \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \\ & = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}\end{aligned}$$

$$= \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$+ \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$
$$= \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

But this expression defines the natural projector $P_3 = (\text{A}13)$. To prove formula $P_3 = R_{12} + S_{12}$, note that the traceless parts of R_{12} and S_{12} can be chosen in such a way that

$$\Phi_2 = -\Phi_1, \quad \Psi_2 = -\Psi_1, \quad \Lambda_2 = -\Lambda_1.$$

This choice proves existence of the desired decomposition.

We show that $P_4 = P_7 + P_8$. Since $P_7 = (\text{A}23)$ and $P_8 = (\text{A}24)$, then

$$P_7 = \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$+ \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}),$$
$$P_8 = \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$- \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}),$$

hence

$$P_7 + P_8$$
$$= \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1} + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$+ \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1} - \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$= \frac{1}{3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

But this expression is exactly $P_4 = (\text{A}14)$.

We show that $P_5 = P_3 + P_8$. Since $P_3 = (\text{A}13)$ and $P_8 = (\text{A}24)$, then

$$P_3 = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$
$$P_8 = \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$- \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}),$$

hence

$$P_3+P_8=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$
$$+\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$-\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}-\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$=\frac{5}{6}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{2}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})+\frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$-\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}-\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$+\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}),$$

which is equal to $P_5=(\text{A}21)$.

We show that $P_6=P_3+P_7$. Since $P_3=(\text{A}13)$ and $P_7=(\text{A}23)$, then

$$P_3=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$
$$P_7=\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$+\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}),$$

hence

$$P_3+P_7=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$
$$+\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$+\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$
$$+\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}),$$

but this is equal to $P_6=(\text{A}22)$.

Now consider a natural projector, belonging to the family P_9. We show that there is a natural projector belonging to the family P_{12}, such that $P_9=P_{12}+P_7$. Since $P_{12}=(\text{B}4)$ and $P_7=(\text{A}23)$, we have

$$P_{12} = \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$

and

$$P_7 = \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$+\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$

hence

$$P_{12} + P_7 = \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$
$$+\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$
$$+\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$
$$= \frac{3}{6}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

However, this expression is equal to $P_9 = \text{(B1)}$.

Analogously, consider a natural projector, belonging to the family P_{10}.

We show that there is a natural projector belonging to the family P_{12}, such that $P_{10} = P_{12} + P_8$. Since $P_{12} = (\mathrm{B}4)$ and $P_8 = (\mathrm{A}24)$, we have

$$\begin{aligned}
P_{12} &= \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),
\end{aligned}$$

and

$$\begin{aligned}
P_8 &= \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1} \\
&\quad - \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}),
\end{aligned}$$

Thus

$$\begin{aligned}
P_{12} + P_8 &= \frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1} \\
&\quad - \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \\
&= \frac{3}{6}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} - \frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\
&\quad + \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).
\end{aligned}$$

But this expression is equal to $P_{10} = (\mathrm{B}2)$.

Finally, we show that $P_{11} = P_8 + P_9$. Since $P_8 = (\mathrm{A}24)$ and $P_9 = (\mathrm{B}1)$, we have

$$\begin{aligned} P_8 = \frac{1}{6}(&\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1} \\ &- \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}), \end{aligned}$$

and

$$\begin{aligned} P_9 = &\frac{1}{6}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_2}_{j_3}\delta^{i_3}_{j_2}) + \frac{1}{6}\delta^{i_2}_{j_2}(\delta^{i_1}_{j_1}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{6}\delta^{i_3}_{j_3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}) \\ &+ \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}), \end{aligned}$$

then

$$\begin{aligned} P_8 + P_9 = &\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1} \\ &- \cancel{\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}} - \cancel{\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}} - \cancel{\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}}) \\ &+ \frac{1}{6}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \cancel{\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}}) + \frac{1}{6}\delta^{i_2}_{j_2}(\delta^{i_1}_{j_1}\delta^{i_3}_{j_3} + \cancel{\delta^{i_1}_{j_3}\delta^{i_3}_{j_1}}) \\ &+ \frac{1}{6}\delta^{i_3}_{j_3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \cancel{\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}}) \\ &+ \frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3} - 2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}) \\ &+ \frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \\ = &\frac{2}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}) \end{aligned}$$

$$+\frac{1}{3}\Phi(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}-\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Psi(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}-2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{2}\Lambda(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

But this expression is equal to $P_{11}=(\mathrm{B}3)$.

Theorem 4 (Decomposition formulas) *Decomposable natural projectors* $P_2,P_3,P_4,P_5,P_6,P_9,P_{10},P_{11}$ *admit the following decompositions:*

$$P_3=R_{12}+S_{12},\quad P_4=P_7+P_8,$$
$$P_5=P_3+P_8=P_{10}+P_{12},$$
$$P_6=P_3+P_7=P_9+P_{12},$$
$$P_9=P_{12}+P_7,\quad P_{10}=P_{12}+P_8,\quad P_{11}=P_8+P_9.$$

Proof Some of these formulas have already been proved (cf. the proof of Theorem 3). It remains to show that $P_5=P_{10}+P_{12}$ and $P_6=P_9+P_{12}$. Consider the natural projectors $P_{10}=(\mathrm{B}2)$ and $P_{12}=(\mathrm{B}4)$, with principal parts

$$\frac{1}{6}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\delta^{i_2}_{j_3}\delta^{i_3}_{j_2})+\frac{1}{6}\delta^{i_2}_{j_2}(\delta^{i_1}_{j_1}\delta^{i_3}_{j_3}-\delta^{i_1}_{j_3}\delta^{i_3}_{j_1})+\frac{1}{6}\delta^{i_3}_{j_3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}),$$
$$\frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$

and traceless parts

$$\frac{1}{3}\Phi_1(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}-\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Psi_1(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}-2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{2}\Lambda_1(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$
$$\frac{1}{3}\Phi_2(2\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}-\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{3}\Psi_2(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}-2\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1})$$
$$+\frac{1}{2}\Lambda_2(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

The sum of the principal parts is

$$\frac{1}{6}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\delta^{i_2}_{j_3}\delta^{i_3}_{j_2})+\frac{1}{6}\delta^{i_2}_{j_2}(\delta^{i_1}_{j_1}\delta^{i_3}_{j_3}-\delta^{i_1}_{j_3}\delta^{i_3}_{j_1})+\frac{1}{6}\delta^{i_3}_{j_3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_1})$$

$$+\frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$

$$=\frac{5}{6}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{6}\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}-\frac{1}{6}\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}-\frac{1}{6}\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}$$

$$-\frac{1}{6}\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}-\frac{1}{6}\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}$$

$$=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}$$

$$+\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$

and coincides with $P_5=(\mathrm{A}21)$. Consequently, the choice

$$\Phi_1=-\Phi_2,\quad \Psi_1=-\Psi_2,\quad \Lambda_1=-\Lambda_2$$

gives $P_5=P_{10}+P_{12}$.

Similarly, consider the natural projectors $P_9=(\mathrm{B}1)$ and $P_{12}=(\mathrm{B}4)$, with principal parts

$$\frac{1}{6}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_2}_{j_3}\delta^{i_3}_{j_2})+\frac{1}{6}\delta^{i_2}_{j_2}(\delta^{i_1}_{j_1}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_3}_{j_1})$$

$$+\frac{1}{6}\delta^{i_3}_{j_3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}),$$

$$\frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}),$$

and traceless parts expressed as above. The sum of the principal parts is

$$\frac{1}{6}\delta^{i_1}_{j_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\delta^{i_2}_{j_3}\delta^{i_3}_{j_2})+\frac{1}{6}\delta^{i_2}_{j_2}(\delta^{i_1}_{j_1}\delta^{i_3}_{j_3}+\delta^{i_1}_{j_3}\delta^{i_3}_{j_1})$$

$$+\frac{1}{6}\delta^{i_3}_{j_3}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1})$$

$$+\frac{1}{3}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\frac{1}{6}(\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1})$$

$$=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}+\frac{1}{6}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\delta^{i_3}_{j_2}+\delta^{i_1}_{j_3}\delta^{i_2}_{j_2}\delta^{i_3}_{j_1}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}$$

$$-\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}-\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}-\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}).$$

and is equal to $P_6 = (\mathrm{A}22)$. Thus, the choice

$$\Phi_1 = -\Phi_2, \quad \Psi_1 = -\Psi_2, \quad \Lambda_1 = -\Lambda_2$$

gives $P_6 = P_9 + P_{12}$.

Theorem 5 (Primitive natural projectors) (a) *The natural projectors* P_1, P_7, P_8, *and every natural projector belonging to the family* P_{12}, *are primitive. There are no other primitive natural projectors in* $T_3^0\mathbf{R}^n$.

(b) *The decomposible natural projectors can be uniquely expressed by the formulas*

$$P_3 = R_{12} + S_{12}, \quad P_4 = P_7 + P_8,$$
$$P_5 = R_{12} + S_{12} + P_8, \quad P_6 = R_{12} + P_7 + S_{12},$$
$$P_9 = P_{12} + P_7, \quad P_{10} = P_{12} + P_8, \quad P_{11} = P_8 + P_{12} + P_7.$$

Proof The natural projectors P_1, P_7, P_8 and P_{12} do not appear in the fourth column of the decomposability indicatrix tables, thus, dimension argument show that none of them is decomposable.

Now we can describe all *partitions* of the tensor space $T_3^0\mathbf{R}^n$, formed by natural projectors. Canonical partitions arise from the pairs of complementary natural projectors; we have the trivial partition $\{P_1, P_2\}$, and the partitions $\{P_3, P_4\}$, $\{P_5, P_7\}$, $\{P_6, P_8\}$, $\{P_9, P_{10}\}$, $\{P_{11}, P_{12}\}$. Of specific interest are partitions formed by *primitive* natural projectors, the *primitive natural partitions*.

Theorem 6 (Partitions) *The tensor space* $T_3^0\mathbf{R}^n$ *admits partitions, whose elements are primitive natural projectors. Every partition with these properties is of the form* $\{P_7, P_8, R_{12}, S_{12}\}$, *where* R_{12} *and* S_{12} *are different natural projectors, belonging to the family* P_{12} .

Proof Starting with the canonical partitions, we get with the help of Theorem 5 five decompositions of the identity by means of complementary natural projectors,

$$\mathrm{Id} = P_3 + P_4 = R_{12} + S_{12} + P_7 + P_8,$$

$$\mathrm{Id} = P_5 + P_7 = R_{12} + S_{12} + P_8 + P_7,$$

$$\mathrm{Id} = P_6 + P_8 = R_{12} + S_{12} + P_7 + P_8,$$

$$\mathrm{Id} = P_9 + P_{10} = R_{12} + P_7 + S_{12} + P_8,$$

$$\mathrm{Id} = P_6 + P_8 = R_{12} + S_{12} + P_7 + P_8 .$$

Since this list includes *all* nontrivial natural partitions, and all of them are identical, Theorem 6 is proved.

Remark 2 If we search for partitions of the tensor space $T_3^0\mathbf{R}^n$, which do not contain elements of the family P_{12}, then

$$\mathrm{Id} = P_3 + P_7 + P_8 .$$

Thus, the identity of the tensor space $T_3^0\mathbf{R}^n$ can be decomposed by the complementary to the cycle projector, symmetrization and alternation.

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