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Classification of natural projectors in tensor spaces: (1,2)-tensors

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Abstract In this research-expository article an example of an invariant decomposition of a tensor space, extending the natural projector decomposition method to mixed tensor spaces, is studied. Complete list of the natural projectors, decompositions of natural projectors, and partitions of the tensor space of $(1,2)$-tensors are given. All proofs, based on elementary tensor algebra and projector theory in finite-dimensional real vector spaces, are included.

Keywords Invariant tensor, Natural endomorphism, Projector, Partition of vector space, Torsion

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1 Introduction

In this note we apply the natural projector method to the decomposition problem of tensors of type $(1,2)$ on the *n*-dimensional real vector space $\mathbf{R}^n$ (Krupka, [2], [3], [4]). Our aim is twofold: (1) to test the possibility of extending the theory of natural projectors for tensors of type $(0,s)$ to tensors of type (r,s) with $r>0$, and (2) to study in detail natural projectors in the tensor space $T_2^1\mathbf{R}^n = \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ of $(1,2)$-tensors (in differential geometry the *torsion tensors*).

In this paper the vector space $\mathbf{R}^n$ is considered with the canonical left action of the general linear group $GL_n(\mathbf{R})$, and the tensor space $T_2^1\mathbf{R}^n$ is endowed with the induced tensor action. Since our discussions are $GL_n(\mathbf{R})$-invariant, the results apply, in the well-known sense, to *any* real, *n*-dimensional vector space *E*, and to the tensor space $T^1_2 E$. In the canonical basis e_i of $\mathbf{R}^n$, a tensor $U \in T^1_2 \mathbf{R}^n$ is usually denoted in components as $U = U^i_{jk}$; an endomorphism $P: T^1_2\mathbf{R}^n \to T^1_2\mathbf{R}^n$ is denoted as $P = P^{i\ \ pq}_{jk\ r}$, with standard meaning of the superscripts and the subscripts.

In Section 2 basic composition law for natural linear operators in the vector space $T^1_2\mathbf{R}^n$ is recalled, serving later in Section 3 for derivation of the natural projector equations. Section 4 contains solutions of these equations, expressed as a complete list of all natural projectors. It should be pointed out that the list includes, beside the natural projectors with constant coefficients, also *parameterized families* of natural pojectors, depending on several real parameters. Section 5 is devoted to the decomposition theory of natural projectors; all invariant decompositions and the dimensions of the corresponding image subspaces of the tensor space $T^1_2\mathbf{R}^n$ are found. Since the set of dimensions turns out to be *finite*, as the main investigation tool for the decomposition problem we propose the *decomposability indicatrix*, a finite subset of integers, constructed from the dimensions of subspaces of the tensor space $T^1_2\mathbf{R}^n$ and the sums of these dimensions. Finally, in Section 6 we prove a theorem on natural decompositions of the identity projector in the tensor space $T^1_2\mathbf{R}^n$; we find the natural projectors, which define *primitive natural partitions* of the tensor space $T^1_2\mathbf{R}^n$.

Many sources dealing with decompositions of tensor spaces of type $(0,s)$ (or $(r,0)$) as a part of the group representation theory can be found in the literature; exposition is usually given in a simplified setting for vector spaces over the field of complex numbers. It seems, however, that the case of mixed (r,s)-tensors with $r,s \neq 0$, or even examples of decompositions of mixed tensors for small *r*, and *s*, has not been discussed. Our results, esp. appearance of *parameterized families* of natural projectors, show that such extension of the representation theory to mixed real tensor spaces would not be straightforward.

2 Natural endomorphisms

Let $P: T^1_2\mathbf{R}^n \to T^1_2\mathbf{R}^n$, $P = P^{i\ \ qr}_{jk\ p}$, be an endomorphisms. According to the *Gurevich theorem* on the structure of natural tensors (Gurevich [1], Krupka [5]) *P* is *natural* if and only if

(1)
$$P^{i\ qr}_{jk p} = a_1\delta^i_j\delta^q_k\delta^r_p + a_2\delta^i_j\delta^q_p\delta^r_k + a_3\delta^i_k\delta^q_p\delta^r_j + a_4\delta^i_k\delta^q_j\delta^r_p + a_5\delta^i_p\delta^q_j\delta^r_k + a_6\delta^i_p\delta^q_k\delta^r_j,$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ are some real numbers. The vector space of natural endomorphisms of $T^1_2\mathbf{R}^n$ is denoted by $\mathcal{N}(T^1_2\mathbf{R}^n)$. According to formula (1), $\dim \mathcal{N}(T^1_2\mathbf{R}^n) = 6$.

We need the composition law for natural endomorphisms. Its derivation is straightforward. Consider a natural endomorphism (1), and another natural endomorphism $Q = Q^{a}_{bc}{}^{qr}_{p}$, where

$$Q^{a}_{bc}{}^{qr}_{p} = b_1\delta^a_b\delta^q_c\delta^r_p + b_2\delta^a_b\delta^q_p\delta^r_c + b_3\delta^a_c\delta^q_p\delta^r_b + b_4\delta^a_c\delta^q_b\delta^r_p + b_5\delta^a_p\delta^q_b\delta^r_c + b_6\delta^a_p\delta^q_c\delta^r_b .$$

Lemma 1 *The composed endomorphism* $R = PQ = R^{i}_{jk}{}^{qr}_{p}$ *is a natural endomorphism expressed by*

$$R^{i}_{jk}{}^{qr}_{p} = c_1\delta^i_j\delta^q_k\delta^r_p + c_2\delta^i_j\delta^q_p\delta^r_k + c_3\delta^i_k\delta^q_p\delta^r_j + c_4\delta^i_k\delta^q_j\delta^r_p + c_5\delta^i_p\delta^q_j\delta^r_k + c_6\delta^i_p\delta^q_k\delta^r_j ,$$

where

$$\begin{aligned}
c_1 &= a_1b_1 + na_1b_4 + a_1b_5 + na_2b_1 + a_2b_4 + a_2b_6 + a_5b_1 + a_6b_4 ,\\
c_2 &= a_1b_2 + na_1b_3 + a_1b_6 + na_2b_2 + a_2b_3 + a_2b_5 + a_5b_2 + a_6b_3 ,\\
c_3 &= na_3b_2 + a_3b_3 + a_3b_5 + a_4b_2 + na_4b_3 + a_4b_6 + a_5b_3 + a_6b_2 ,\\
c_4 &= na_3b_1 + a_3b_4 + a_3b_6 + a_4b_1 + na_4b_4 + a_4b_5 + a_5b_4 + a_6b_1 ,\\
c_5 &= a_5b_5 + a_6b_6 ,\\
c_6 &= a_5b_6 + a_6b_5 .
\end{aligned} \tag{2}$$

Proof Formula (2) is obtained by a direct substitution. Indeed, since for any $U \in T^1_2\mathbf{R}^n$, $U = U^p_{qr}$, $RU = \bar{\bar{U}}^i_{jk} = P^{i}_{jk}{}^{bc}_{a}\bar{U}^a_{bc} = P^{i}_{jk}{}^{bc}_{a}Q^{a}_{bc}{}^{qr}_{p}U^p_{qr} = R^{i}_{jk}{}^{qr}_{p}U^p_{qr}$, the coefficients $R^{i}_{jk}{}^{qr}_{p}$ are given by the formula

$$R^{i}_{jk}{}^{qr}_{p} = P^{i}_{jk}{}^{bc}_{a}Q^{a}_{bc}{}^{qr}_{p} . \tag{3}$$

Then

$$\begin{aligned}
R^{i}_{jk}{}^{qr}_{p} = {} & a_1\delta^i_j\delta^b_k\delta^c_a(b_1\delta^a_b\delta^q_c\delta^r_p + b_2\delta^a_b\delta^q_p\delta^r_c + b_3\delta^a_c\delta^q_p\delta^r_b\\
& + b_4\delta^a_c\delta^q_b\delta^r_p + b_5\delta^a_p\delta^q_b\delta^r_c + b_6\delta^a_p\delta^q_c\delta^r_b)\\
& + a_2\delta^i_j\delta^b_a\delta^c_k(b_1\delta^a_b\delta^q_c\delta^r_p + b_2\delta^a_b\delta^q_p\delta^r_c + b_3\delta^a_c\delta^q_p\delta^r_b
\end{aligned}$$

$$+b_4\delta^a_c\delta^q_b\delta^r_p+b_5\delta^a_p\delta^q_b\delta^r_c+b_6\delta^a_p\delta^q_c\delta^r_b)$$
$$+a_3\delta^i_k\delta^b_a\delta^c_j(b_1\delta^a_b\delta^q_c\delta^r_p+b_2\delta^a_b\delta^q_p\delta^r_c+b_3\delta^a_c\delta^q_p\delta^r_b$$
$$+b_4\delta^a_c\delta^q_b\delta^r_p+b_5\delta^a_p\delta^q_b\delta^r_c+b_6\delta^a_p\delta^q_c\delta^r_b)$$
$$+a_4\delta^i_k\delta^b_j\delta^c_a(b_1\delta^a_b\delta^q_c\delta^r_p+b_2\delta^a_b\delta^q_p\delta^r_c+b_3\delta^a_c\delta^q_p\delta^r_b$$
$$+b_4\delta^a_c\delta^q_b\delta^r_p+b_5\delta^a_p\delta^q_b\delta^r_c+b_6\delta^a_p\delta^q_c\delta^r_b)$$
$$+a_5\delta^i_a\delta^b_j\delta^c_k(b_1\delta^a_b\delta^q_c\delta^r_p+b_2\delta^a_b\delta^q_p\delta^r_c+b_3\delta^a_c\delta^q_p\delta^r_b$$
$$+b_4\delta^a_c\delta^q_b\delta^r_p+b_5\delta^a_p\delta^q_b\delta^r_c+b_6\delta^a_p\delta^q_c\delta^r_b)$$
$$+a_6\delta^i_a\delta^b_k\delta^c_j(b_1\delta^a_b\delta^q_c\delta^r_p+b_2\delta^a_b\delta^q_p\delta^r_c+b_3\delta^a_c\delta^q_p\delta^r_b$$
$$+b_4\delta^a_c\delta^q_b\delta^r_p+b_5\delta^a_p\delta^q_b\delta^r_c+b_6\delta^a_p\delta^q_c\delta^r_b),$$

hence

$$R^{i\ \ qr}_{jk\ p}=a_1(b_1\delta^i_j\delta^q_k\delta^r_p+b_2\delta^i_j\delta^q_p\delta^r_k+nb_3\delta^i_j\delta^q_p\delta^r_k$$
$$+nb_4\delta^i_j\delta^q_k\delta^r_p+b_5\delta^i_j\delta^q_k\delta^r_p+b_6\delta^i_j\delta^q_p\delta^r_k)$$
$$+a_2(nb_1\delta^i_j\delta^q_k\delta^r_p+nb_2\delta^i_j\delta^q_p\delta^r_k+b_3\delta^i_j\delta^q_p\delta^r_k$$
$$+b_4\delta^i_j\delta^q_k\delta^r_p+b_5\delta^i_j\delta^q_p\delta^r_k+b_6\delta^i_j\delta^q_k\delta^r_p)$$
$$+a_3(nb_1\delta^i_k\delta^q_j\delta^r_p+nb_2\delta^i_k\delta^q_p\delta^r_j+b_3\delta^i_k\delta^q_p\delta^r_j$$
$$+b_4\delta^i_k\delta^q_j\delta^r_p+b_5\delta^i_k\delta^q_p\delta^r_j+b_6\delta^i_k\delta^q_j\delta^r_p)$$
$$+a_4(b_1\delta^i_k\delta^q_j\delta^r_p+b_2\delta^i_k\delta^q_p\delta^r_j+nb_3\delta^i_k\delta^q_p\delta^r_j$$
$$+nb_4\delta^i_k\delta^q_j\delta^r_p+b_5\delta^i_k\delta^q_j\delta^r_p+b_6\delta^i_k\delta^q_p\delta^r_j)$$
$$+a_5(b_1\delta^i_j\delta^q_k\delta^r_p+b_2\delta^i_j\delta^q_p\delta^r_k+b_3\delta^i_k\delta^q_p\delta^r_j$$
$$+b_4\delta^i_k\delta^q_j\delta^r_p+b_5\delta^i_p\delta^q_j\delta^r_k+b_6\delta^i_p\delta^q_k\delta^r_j)$$
$$+a_6(b_1\delta^i_k\delta^q_j\delta^r_p+b_2\delta^i_k\delta^q_p\delta^r_j+b_3\delta^i_j\delta^q_p\delta^r_k$$
$$+b_4\delta^i_j\delta^q_k\delta^r_p+b_5\delta^i_p\delta^q_k\delta^r_j+b_6\delta^i_p\delta^q_j\delta^r_k)$$

and

$$R^{i\ \ qr}_{jk\ p}=a_1b_1\delta^i_j\delta^q_k\delta^r_p+a_1b_2\delta^i_j\delta^q_p\delta^r_k+na_1b_3\delta^i_j\delta^q_p\delta^r_k$$
$$+na_1b_4\delta^i_j\delta^q_k\delta^r_p+a_1b_5\delta^i_j\delta^q_k\delta^r_p+a_1b_6\delta^i_j\delta^q_p\delta^r_k$$

$$
\begin{aligned}
&+na_2b_1\delta^i_j\delta^q_k\delta^r_p + na_2b_2\delta^i_j\delta^q_p\delta^r_k + a_2b_3\delta^i_j\delta^q_p\delta^r_k\\
&+a_2b_4\delta^i_j\delta^q_k\delta^r_p + a_2b_5\delta^i_j\delta^q_p\delta^r_k + a_2b_6\delta^i_j\delta^q_k\delta^r_p\\
&+na_3b_1\delta^i_k\delta^q_j\delta^r_p + na_3b_2\delta^i_k\delta^q_p\delta^r_j + a_3b_3\delta^i_k\delta^q_p\delta^r_j\\
&+a_3b_4\delta^i_k\delta^q_j\delta^r_p + a_3b_5\delta^i_k\delta^q_p\delta^r_j + a_3b_6\delta^i_k\delta^q_j\delta^r_p\\
&+a_4b_1\delta^i_k\delta^q_j\delta^r_p + a_4b_2\delta^i_k\delta^q_p\delta^r_j + na_4b_3\delta^i_k\delta^q_p\delta^r_j\\
&+na_4b_4\delta^i_k\delta^q_j\delta^r_p + a_4b_5\delta^i_k\delta^q_j\delta^r_p + a_4b_6\delta^i_k\delta^q_p\delta^r_j\\
&+a_5b_1\delta^i_j\delta^q_k\delta^r_p + a_5b_2\delta^i_j\delta^q_p\delta^r_k + a_5b_3\delta^i_k\delta^q_p\delta^r_j\\
&+a_5b_4\delta^i_k\delta^q_j\delta^r_p + a_5b_5\delta^i_p\delta^q_j\delta^r_k + a_5b_6\delta^i_p\delta^q_k\delta^r_j\\
&+a_6b_1\delta^i_k\delta^q_j\delta^r_p + a_6b_2\delta^i_k\delta^q_p\delta^r_j + a_6b_3\delta^i_j\delta^q_p\delta^r_k\\
&+a_6b_4\delta^i_j\delta^q_k\delta^r_p + a_6b_5\delta^i_p\delta^q_k\delta^r_j + a_6b_6\delta^i_p\delta^q_j\delta^r_k\\
&=(a_1b_1 + na_1b_4 + a_1b_5 + na_2b_1 + a_2b_4 + a_2b_6 + a_5b_1 + a_6b_4)\delta^i_j\delta^q_k\delta^r_p\\
&+(a_1b_2 + na_1b_3 + a_1b_6 + na_2b_2 + a_2b_3 + a_2b_5 + a_5b_2 + a_6b_3)\delta^i_j\delta^q_p\delta^r_k\\
&+(na_3b_1 + a_3b_4 + a_3b_6 + a_4b_1 + na_4b_4 + a_4b_5 + a_5b_4 + a_6b_1)\delta^i_k\delta^q_j\delta^r_p\\
&+(na_3b_2 + a_3b_3 + a_3b_5 + a_4b_2 + na_4b_3 + a_4b_6 + a_5b_3 + a_6b_2)\delta^i_k\delta^q_p\delta^r_j\\
&+(a_5b_5 + a_6b_6)\delta^i_p\delta^q_j\delta^r_k + (a_5b_6 + a_6b_5)\delta^i_p\delta^q_k\delta^r_j,
\end{aligned}
$$

proving (2).

3 Natural projector equations

Consider a natural endomorphism $P : T^1_2\mathbf{R}^n \to T^1_2\mathbf{R}^n$, expressed as

(1)
$$
\begin{aligned}
P^{i\ qr}_{jk\,p} &= a_1\delta^i_j\delta^q_k\delta^r_p + a_2\delta^i_j\delta^q_p\delta^r_k + a_3\delta^i_k\delta^q_p\delta^r_j + a_4\delta^i_k\delta^q_j\delta^r_p\\
&+ a_5\delta^i_p\delta^q_j\delta^r_k + a_6\delta^i_p\delta^q_k\delta^r_j.
\end{aligned}
$$

Lemma 2 *P is a natural projector if and only if its components satisfy the system*

(2)
$$
\begin{aligned}
&(a_1 - a_3)(a_1 + a_3 + n(a_2 + a_4) + 2a_5 - 1) = 0,\\
&(a_2 - a_4)(a_1 + a_3 + n(a_2 + a_4) + 2a_5 - 1) = 0,
\end{aligned}
$$

(3)
$$
\begin{aligned}
&(a_1+a_3)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\
&\quad +2((a_2+a_4)a_6+a_2a_4-a_1a_3)=0,\\
&(a_2+a_4)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\
&\quad +2((a_1+a_3)a_6+n(a_1a_3-a_2a_4))=0,
\end{aligned}
$$

(4)
$$
\begin{aligned}
&a_5^2+a_6^2=a_5,\\
&2a_5a_6=a_6.
\end{aligned}
$$

Proof Endomorphism (1) is a projector if and only if it satisfies the projector equation $P^{i\ \ vw}_{jk\ u}P^{u\ \ qr}_{vw\ p}=P^{i\ \ qr}_{jk\ p}$. Substituting $Q=P$ and $R=P$ into formula (3), Sec. 2, or, which is the same, setting $b_i=a_i$ and $c_i=a_i$ in equation (2), Sec. 2, we obtain

$$
\begin{aligned}
&a_1^2+na_1a_2+na_1a_4+2a_1a_5+a_2a_4+a_2a_6+a_4a_6=a_1,\\
&a_1a_2+na_1a_3+a_1a_6+na_2^2+a_2a_3+2a_2a_5+a_3a_6=a_2,\\
&na_2a_3+a_2a_4+a_2a_6+a_3^2+na_3a_4+2a_3a_5+a_4a_6=a_3,\\
&na_1a_3+a_1a_4+a_1a_6+a_3a_4+a_3a_6+na_4^2+2a_4a_5=a_4,\\
&a_5^2+a_6^2=a_5,\\
&2a_5a_6=a_6.
\end{aligned}
$$

Equivalence of this system and the system (2), (3), and (4) is immediate:

$$
\begin{aligned}
a_1-a_3&=a_1^2+na_1a_2+na_1a_4+2a_1a_5+a_2a_4+a_2a_6+a_4a_6\\
&\quad -na_2a_3-a_2a_4-a_2a_6-a_3^2-na_3a_4-2a_3a_5-a_4a_6\\
&=(a_1-a_3)(a_1+a_3)+na_2(a_1-a_3)+na_4(a_1-a_3)+2a_5(a_1-a_3),\\
&=(a_1-a_3)(a_1+a_3+n(a_2+a_4)+2a_5),
\end{aligned}
$$

$$
\begin{aligned}
a_1+a_3&=a_1^2+na_1a_2+na_1a_4+2a_1a_5+a_2a_4+a_2a_6+a_4a_6\\
&\quad +na_2a_3+a_2a_4+a_2a_6+a_3^2+na_3a_4+2a_3a_5+a_4a_6\\
&=a_1^2+a_3^2+na_2(a_1+a_3)+na_4(a_1+a_3)+2a_5(a_1+a_3)\\
&\quad +2a_2a_4+2a_2a_6+2a_4a_6\\
&=a_1^2+a_3^2+(a_1+a_3)(n(a_2+a_4)+2a_5)\\
&\quad +2a_2a_4+2a_2a_6+2a_4a_6
\end{aligned}
$$

$$= a_1^2 + a_3^2 + 2a_1a_3 + (a_1 + a_3)(n(a_2 + a_4) + 2a_5)$$
$$+ 2(a_2a_4 + a_2a_6 + a_4a_6 - a_1a_3)$$
$$= (a_1 + a_3)(a_1 + a_3 + n(a_2 + a_4) + 2a_5)$$
$$+ 2(a_2a_4 + a_2a_6 + a_4a_6 - a_1a_3),$$

and

$$a_2 - a_4 = a_1a_2 + na_1a_3 + a_1a_6 + na_2^2 + a_2a_3 + 2a_2a_5 + a_3a_6$$
$$- na_1a_3 - a_1a_4 - a_1a_6 - a_3a_4 - a_3a_6 - na_4^2 - 2a_4a_5$$
$$= n(a_2^2 - a_4^2) + a_1(a_2 - a_4) + a_3(a_2 - a_4) + 2a_5(a_2 - a_4)$$
$$= (a_2 - a_4)(n(a_2 + a_4) + a_1 + a_3 + 2a_5),$$

$$a_2 + a_4 = a_1(a_2 + a_4) + 2na_1a_3 + 2a_1a_6 + na_2^2 + a_3(a_2 + a_4)$$
$$+ 2a_5(a_2 + a_4) + 2a_3a_6 + na_4^2$$
$$= na_2^2 + na_4^2 + (a_2 + a_4)(a_1 + a_3 + 2a_5)$$
$$+ 2na_1a_3 + 2a_1a_6 + 2a_3a_6$$
$$= n((a_2 + a_4)^2 - 2a_2a_4) + (a_2 + a_4)(a_1 + a_3 + 2a_5)$$
$$+ 2na_1a_3 + 2a_1a_6 + 2a_3a_6$$
$$= (a_2 + a_4)(n(a_2 + a_4) + a_1 + a_3 + 2a_5)$$
$$+ 2(na_1a_3 + (a_1 + a_3)a_6 - na_2a_4).$$

Remark 1 The *identity endomorphism* Id, and the *zero endomorphism*, represented by the 6-tuples $(0,0,0,0,1,0)$ and $(0,0,0,0,0,0)$, are both natural projectors. If $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a solution of equations (2), (3), (4), representing a natural projector P, then $(-a_1, -a_2, -a_3, -a_4, 1 - a_5, -a_6)$ is also a solution; this solution represents the *complementary projector* $\mathrm{Id} - P$.

Equations (2), (3), and (4) can be further simplified. Set

$$\alpha = a_1 + a_3, \quad \beta = a_2 + a_4.$$

Lemma 3 *P is a natural projector if and only if*

$$(a_1 - a_3)(\alpha + n\beta + 2a_5 - 1) = 0, \tag{5}$$
$$(a_2 - a_4)(\alpha + n\beta + 2a_5 - 1) = 0,$$

(6)
$$\begin{aligned}&\beta(\alpha+n\beta+2a_5-1)+2a_6\alpha=0,\\&\alpha(\alpha+n\beta+2a_5-1)+2a_6\beta=0,\end{aligned}$$

(7)
$$\begin{aligned}&a_5^2+a_6^2=a_5,\\&2a_5a_6=a_6.\end{aligned}$$

Proof Subsystem (2) yields formula (5). Consider subsystem (3). Subtracting and adding these equations we find

$$\begin{aligned}&n(a_1+a_3)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\&\quad+2n(a_2a_4+a_2a_6+a_4a_6-a_1a_3)\\&\quad-(a_2+a_4)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\&\quad-2(na_1a_3+a_1a_6+a_3a_6-na_2a_4)=0,\end{aligned}$$

that is

$$\begin{aligned}&(n(a_1+a_3)-a_2-a_4)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\&\quad+2a_6(n(a_2+a_4)-a_1-a_3)=0,\end{aligned}$$

and

$$\begin{aligned}&n(a_1+a_3)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\&\quad+2n(a_2a_4+a_2a_6+a_4a_6-a_1a_3)\\&\quad+(a_2+a_4)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\&\quad+2(na_1a_3+a_1a_6+a_3a_6-na_2a_4)\\&\quad=(n(a_1+a_3)+a_2+a_4)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\&\quad+2(na_2a_4+na_2a_6+na_4a_6-na_1a_3+na_1a_3\\&\quad+a_1a_6+a_3a_6-na_2a_4)\\&\quad=0,\end{aligned}$$

that is

$$\begin{aligned}&(n(a_1+a_3)+a_2+a_4)(a_1+a_3+n(a_2+a_4)+2a_5-1)\\&\quad+2a_6(n(a_2+a_4)+a_1+a_3)=0.\end{aligned}$$

Consequently, in new variables $\alpha=a_1+a_3$ and $\beta=a_2+a_4$ subsystem (3) is of the form

$$(n\alpha - \beta)(\alpha + n\beta + 2a_5 - 1) + 2a_6(n\beta - \alpha) = 0,$$
$$(n\alpha + \beta)(\alpha + n\beta + 2a_5 - 1) + 2a_6(n\beta + \alpha) = 0.$$

Subtracting and adding expressions on the left hand side,

$$\begin{aligned}
&(n\alpha - \beta)(\alpha + n\beta + 2a_5 - 1) + 2a_6(n\beta - \alpha) \\
&\quad -(n\alpha + \beta)(\alpha + n\beta + 2a_5 - 1) - 2a_6(n\beta + \alpha) \\
&\quad = -2\beta(\alpha + n\beta + 2a_5 - 1) - 4a_6\alpha, \\
&(n\alpha - \beta)(\alpha + n\beta + 2a_5 - 1) + 2a_6(n\beta - \alpha) \\
&\quad +(n\alpha + \beta)(\alpha + n\beta + 2a_5 - 1) + 2a_6(n\beta + \alpha) \\
&\quad = 2n\alpha(\alpha + n\beta + 2a_5 - 1) + 4na_6\beta = 0,
\end{aligned}$$

proving formula (6).

4 Natural projector equations: Solutions

Equations of natural projectors (Sec. 3, (5), (6), (7)) will be solved by elimination of variables. Recall for convenience that subsystems (5) and (6) are of the form

$$(1) \qquad \begin{aligned} &(a_1 - a_3)(\alpha + n\beta + 2a_5 - 1) = 0, \\ &(a_2 - a_4)(\alpha + n\beta + 2a_5 - 1) = 0, \end{aligned}$$

$$(2) \qquad \begin{aligned} &\beta(\alpha + n\beta + 2a_5 - 1) + 2a_6\alpha = 0, \\ &\alpha(\alpha + n\beta + 2a_5 - 1) + 2a_6\beta = 0. \end{aligned}$$

Subsystem (7) has four solutions

$$(a_5, a_6) = (0,0), (1,0), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right).$$

These solutions (a_5, a_6) split the system (1), (2) in four subsystems (A), (B), (C), and (D), which will be studied separately.

(A) $(a_5, a_6) = (0,0)$.
From equations (1) and (2) we get the system of bilinear equations

$$(3) \qquad (a_1 - a_3)(\alpha + n\beta - 1) = 0, \quad (a_2 - a_4)(\alpha + n\beta - 1) = 0,$$

(4) $\beta(\alpha+n\beta-1)=0, \quad \alpha(\alpha+n\beta-1)=0.$

(A1) $(a_5,a_6)=(0,0)$, $\alpha+n\beta-1=0$.
Then since $\alpha=a_1+a_3$ and $\beta=a_2+a_4$,

$$a_1+a_3+n(a_2+a_4)-1=0,$$

and equations (3) and (4) do not give a new condition. Thus, case (A1) represents a 3-parameter family of solutions

(I) $$(a_1,a_2,a_3,a_4,a_5,a_6)=(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0), \quad \nu,\mu,\kappa\in\mathbf{R}.$$

(A2) $(a_5,a_6)=(0,0)$, $\alpha+n\beta-1\neq 0$.
In this case

(II) $$(a_1,a_2,a_3,a_4,a_5,a_6)=(0,0,0,0,0,0).$$

(B) $(a_5,a_6)=(1,0)$.
In this case equations (1) and (2) yield

$$(a_1-a_3)(\alpha+n\beta+1)=0, \quad (a_2-a_4)(\alpha+n\beta+1)=0,$$

$$\beta(\alpha+n\beta+1)=0, \quad \alpha(\alpha+n\beta+1)=0.$$

(B1) $(a_5,a_6)=(1,0)$, $\alpha+n\beta+1=0$.
In this case equations do not provide a new condition. Consequently,

$$a_1+a_3+n(a_2+a_4)+1=0,$$

and we have a 3-parameter family of solutions

(III) $$(a_1,a_2,a_3,a_4,a_5,a_6)=(-1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,1,0), \quad \nu,\mu,\kappa\in\mathbf{R}.$$

(B2) $(a_5,a_6)=(1,0)$, $\alpha+n\beta\neq 0$.
In this case $(a_1,a_2,a_3,a_4,a_5,a_6)=(0,0,0,0,1,0)$

(IV) $$(a_1,a_2,a_3,a_4,a_5,a_6)=(0,0,0,0,1,0).$$

(C) $(a_5,a_6)=(1/2,1/2)$.

In this case, equations (1), (2) transform to the system

$$(a_1 - a_3)(\alpha + n\beta) = 0, \quad (a_2 - a_4)(\alpha + n\beta) = 0,$$

$$\beta(\alpha + n\beta) + \alpha = 0, \quad \alpha(\alpha + n\beta) + \beta = 0.$$

(C1) $(a_5, a_6) = (1/2, 1/2)$, $\alpha + n\beta = 0$.

This assumption leads to equations $\alpha = 0$ and $\beta = 0$ hence $a_1 = -a_3$ and $a_2 = -a_4$. The corresponding solution is

(V) $$(a_1, a_2, a_3, a_4, a_5, a_6) = \left(-\mu, -\nu, \mu, \nu, \frac{1}{2}, \frac{1}{2}\right), \quad \mu, \nu \in \mathbf{R}.$$

(C2) $(a_5, a_6) = (1/2, 1/2)$, $\alpha + n\beta \neq 0$.

In this case $a_1 = a_3$ and $a_2 = a_4$, and $\beta^2 = \alpha^2$ that is, $a_1^2 = a_2^2$ hence $a_1 = \pm a_2$.

(C21) $(a_5, a_6) = (1/2, 1/2)$, $a_1 = a_3$, $a_2 = a_4$, $a_1 = a_2$.

In this case $\alpha = 2a_1 = 2a_2 = \beta$ hence $\alpha \neq 0$ and $\alpha = -1/(1+n)$. Thus,

(VI) $$(a_1, a_2, a_3, a_4, a_5, a_6) = \left(-\frac{1}{2(1+n)}, -\frac{1}{2(1+n)}, -\frac{1}{2(1+n)}, -\frac{1}{2(1+n)}, \frac{1}{2}, \frac{1}{2}\right).$$

(C22) $(a_5, a_6) = (1/2, 1/2)$, $a_1 = a_3$, $a_2 = a_4$, $a_1 = -a_2$.

In this case $\alpha = 2a_1 = -2a_2 = -\beta$ hence $\alpha \neq 0$ and equation $-\alpha(\alpha + n\alpha) + \alpha = 0$ yields $\alpha = 1/(1+n)$. Thus,

(VII) $$(a_1, a_2, a_3, a_4, a_5, a_6) = \left(\frac{1}{2(1+n)}, -\frac{1}{2(1+n)}, \frac{1}{2(1+n)}, -\frac{1}{2(1+n)}, \frac{1}{2}, \frac{1}{2}\right).$$

(D) $(a_5, a_6) = (1/2, -1/2)$.

In this case system (1), (2) transforms to

(5) $$(a_1 - a_3)(\alpha + n\beta) = 0, \quad (a_2 - a_4)(\alpha + n\beta) = 0,$$

(6) $$\beta(\alpha + n\beta) - \alpha = 0, \quad \alpha(\alpha + n\beta) - \beta = 0.$$

(D1) $(a_5, a_6) = (1/2, -1/2)$, $\alpha + n\beta = 0$.

This assumption gives $\alpha = 0$ and $\beta = 0$ hence $a_1 = -a_3$ and $a_2 = -a_4$. The corresponding solution is

(VIII) $(a_1,a_2,a_3,a_4,a_5,a_6)=\left(-\mu,-\nu,\mu,\nu,\frac{1}{2},-\frac{1}{2}\right),\quad \mu,\nu\in\mathbf{R}.$

(D2) $(a_5,a_6)=(1/2,-1/2)$, $\alpha+n\beta\neq 0$.

In this case condition $\alpha+n\beta\neq 0$ implies that at least one of the numbers α, β must be different from 0. Since from (6), $\alpha^2=\beta^2$ hence

$$\alpha=\pm\beta\neq 0.$$

(D21) $(a_5,a_6)=(1/2,-1/2)$, $\alpha+n\beta\neq 0$, $\alpha=\beta\neq 0$.

Equations (5), (6) imply $a_1-a_3=0$, $a_2-a_4=0$ and $\alpha+n\alpha-1=0$. Thus, $\alpha=1/(1+n)$ and since

$$a_1+a_3=a_2+a_4=2a_1=2a_2=\frac{1}{1+n},$$

we have

$$a_1=a_3=a_2=a_4=\frac{1}{2(1+n)}.$$

Summarizing,

(IX) $(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right).$

(D22) $(a_5,a_6)=(1/2,-1/2)$, $\alpha+n\beta\neq 0\neq 0$, $\alpha=-\beta\neq 0$.

Equations (5), (6) imply $a_1-a_3=0$, $a_2-a_4=0$ and $\alpha-n\alpha+1=0$. Consequently, $\alpha=-1/(1-n)$. But

$$a_1+a_3=-a_2-a_4=2a_1=-2a_2=-\frac{1}{1+n},$$

so we have

$$a_1=a_3=-a_2=-a_4=-\frac{1}{2(1+n)}.$$

The corresponding solution is

(X) $(a_1,a_2,a_3,a_4,a_5,a_6)=\left(-\frac{1}{2(1-n)},\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2(1-n)},\frac{1}{2},-\frac{1}{2}\right).$

Remark 2 (Complementary natural projectors) Formulas (I)–(X) show, that the pairs $({}^{\mathrm{I}}P,{}^{\mathrm{III}}P)$, $({}^{\mathrm{II}}P,{}^{\mathrm{IV}}P)$, $({}^{\mathrm{V}}P,{}^{\mathrm{VIII}}P)$, $({}^{\mathrm{VI}}P,{}^{\mathrm{IX}}P)$, and $({}^{\mathrm{VII}}P,{}^{\mathrm{X}}P)$ include *complementary* projectors. To express this fact explicitly it is sometimes convenient to use for the pairs $({}^{\mathrm{I}}P,{}^{\mathrm{III}}P)$ and $({}^{\mathrm{V}}P,{}^{\mathrm{VIII}}P)$ *complementary parameterizations*, that is, to write the families ${}^{\mathrm{III}}P$ and ${}^{\mathrm{VIII}}P$ as

(III) $${}^{\mathrm{III}}P$$
$$(a_1,a_2,a_3,a_4,a_5,a_6)=(-1+\mu+n(\nu+\kappa),-\nu,-\mu,-\kappa,1,0),$$
$$\nu,\mu,\kappa\in\mathbf{R},$$

and

(VIII) $${}^{\mathrm{VIII}}P$$
$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\mu,\nu,-\mu,-\nu,\frac{1}{2},-\frac{1}{2}\right),\quad \mu,\nu\in\mathbf{R}.$$

We can now summarize our results in a complete list of natural projectors. In the following theorem M is an index running through the index set $\{\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV},\mathrm{V},\mathrm{VI},\mathrm{VII},\mathrm{VIII},\mathrm{IX},\mathrm{X}\}$ (Greek numbers); in this notation, $P={}^{\mathrm{M}}P$ is an element of the family of natural projectors $\mathscr{P}=\{{}^{\mathrm{I}}P,{}^{\mathrm{II}}P,\ldots,{}^{\mathrm{X}}P\}$.

Theorem 1 *Let* $P:T^1_2\mathbf{R}^n\to T^1_2\mathbf{R}^n$ *be an endomorphism. The following two conditions are equivalent:*

(a) *P is a natural projector.*

(b) $P={}^{\mathrm{M}}P$ *for some* M *, where in components* ${}^{\mathrm{M}}P={}^{\mathrm{M}}P^{i\,qr}_{jk\,p}$,

$$\begin{aligned}{}^{\mathrm{M}}P^{i\,qr}_{jk\,p}&=a_1\delta^i_j\delta^q_k\delta^r_p+a_2\delta^i_j\delta^q_p\delta^r_k+a_3\delta^i_k\delta^q_p\delta^r_j+a_4\delta^i_k\delta^q_j\delta^r_p\\&\quad+a_5\delta^i_p\delta^q_j\delta^r_k+a_6\delta^i_p\delta^q_k\delta^r_j,\end{aligned}$$

and the numbers $(a_1,a_2,a_3,a_4,a_5,a_6)$ *are determined by the following formulas:*

(I) $${}^{\mathrm{I}}P$$
$$(a_1,a_2,a_3,a_4,a_5,a_6)=(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0),$$
$$\nu,\mu,\kappa\in\mathbf{R},$$

(II) $${}^{\mathrm{II}}P$$
$$(a_1,a_2,a_3,a_4,a_5,a_6)=(0,0,0,0,0,0),$$

(III) ${}^{\mathrm{III}}P$

$$(a_1,a_2,a_3,a_4,a_5,a_6)=(-1+\mu+n(\nu+\kappa),-\nu,-\mu,-\kappa,1,0),$$
$$\nu,\mu,\kappa\in\mathbf{R},$$

(IV) ${}^{\mathrm{IV}}P$

$$(a_1,a_2,a_3,a_4,a_5,a_6)=(0,0,0,0,1,0),$$

(V) ${}^{\mathrm{V}}P$

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(-\mu,-\nu,\mu,\nu,\frac{1}{2},\frac{1}{2}\right),\quad \mu,\nu\in\mathbf{R},$$

(VI) ${}^{\mathrm{VI}}P$

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},\frac{1}{2},\frac{1}{2}\right),$$

(VII) ${}^{\mathrm{VII}}P$

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2},\frac{1}{2}\right),$$

(VIII) ${}^{\mathrm{VIII}}P$

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\mu,\nu,-\mu,-\nu,\frac{1}{2},-\frac{1}{2}\right),\quad \mu,\nu\in\mathbf{R},$$

(IX) ${}^{\mathrm{IX}}P$

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right),$$

(X) ${}^{\mathrm{X}}P$

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(-\frac{1}{2(1-n)},\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2(1-n)},\frac{1}{2},-\frac{1}{2}\right).$$

Proof See formulas (I), (II), (III), …, (X).

Theorem 1 can be expressed in terms of *equations* of natural projectors

$$V^i_{jk}={}^{\mathrm{M}}P^{i\ qr}_{jk\ p}U^p_{qr},$$

where $U \in T_2^1\mathbf{R}^n$, $U = U_{qr}^p$, and ${}^{\mathrm{M}}P = {}^{\mathrm{M}}P_{jk\,p}^{i\,qr}$.

Theorem 2 (Equations of natural projectors) *Natural projectors* ${}^{\mathrm{M}}P$ *,* $\mathrm{M} = \mathrm{I},\mathrm{II},\ldots,\mathrm{X}$ *, are expressed by the following equations:*

(I) ${}^{\mathrm{I}}P$

$$V_{jk}^i = (1-\mu-n(\nu+\kappa))\delta_j^i U_{ks}^s + \nu\delta_j^i U_{sk}^s + \mu\delta_k^i U_{sj}^s + \kappa\delta_k^i U_{js}^s,$$

(II) ${}^{\mathrm{II}}P$

$$V_{jk}^i = 0,$$

(III) ${}^{\mathrm{III}}P$

$$V_{jk}^i = (-1+\mu+n(\nu+\kappa))\delta_j^i U_{ks}^s - \nu\delta_j^i U_{sk}^s - \mu\delta_k^i U_{sj}^s - \kappa\delta_k^i U_{js}^s + U_{jk}^i,$$

(IV) ${}^{\mathrm{IV}}P$

$$V_{jk}^i = U_{jk}^i,$$

(V) ${}^{\mathrm{V}}P$

$$V_{jk}^i = \mu(\delta_k^i U_{sj}^s - \delta_j^i U_{ks}^s) + \nu(\delta_k^i U_{js}^s - \delta_j^i U_{sk}^s) + \frac{1}{2}(U_{jk}^i + U_{kj}^i),$$

(VI) ${}^{\mathrm{VI}}P$

$$V_{jk}^i = -\frac{1}{2(1+n)}(\delta_j^i(U_{ks}^s + U_{sk}^s) + \delta_k^i(U_{sj}^s + U_{js}^s)) + \frac{1}{2}(U_{jk}^i + U_{kj}^i),$$

(VII) ${}^{\mathrm{VII}}P$

$$V_{jk}^i = \frac{1}{2(1-n)}(\delta_j^i(U_{ks}^s - U_{sk}^s) + \delta_k^i(U_{sj}^s - \delta_k^i U_{js}^s)) + \frac{1}{2}(U_{jk}^i + U_{kj}^i),$$

(VIII) ${}^{\mathrm{VIII}}P$

$$V_{jk}^i = \mu(\delta_j^i U_{ks}^s - \delta_k^i U_{sj}^s) + \nu(\delta_j^i U_{sk}^s - \delta_k^i U_{js}^s) + \frac{1}{2}(U_{jk}^i - U_{kj}^i),$$

(IX) ${}^{\mathrm{IX}}P$

$$V_{jk}^i = \frac{1}{2(1+n)}(\delta_j^i(U_{ks}^s + U_{sk}^s) + \delta_k^i(U_{sj}^s + U_{js}^s)) + \frac{1}{2}(U_{jk}^i - U_{kj}^i),$$

$^{\mathrm{X}}P$

(X) $$V^i_{jk} = \frac{1}{2(1-n)}(\delta^i_j(U^s_{sk} - U^s_{ks}) + \delta^i_k(U^s_{js} - U^s_{sj})) + \frac{1}{2}(U^i_{jk} - U^i_{kj}).$$

Proof These formulas result from Theorem 1 by straightforward substitution for $U = U^p_{qr}$ into expressions $V^i_{jk} = {}^{\mathrm{M}}P^{i\,qr}_{jk\,p}U^p_{qr}$.

Remark 3 (Families of natural projectors) Natural projectors $^{\mathrm{I}}P$, $^{\mathrm{III}}P$, $^{\mathrm{V}}P$, $^{\mathrm{VIII}}P$ depend on real parameters. If for instance $\mu = 1$, $\nu = \kappa = 0$, then Theorem 2, formula (I), yields

$$V^i_{jk} = \delta^i_k U^s_{sj}.$$

and if $\mu = 0$, $\nu = \kappa = 0$, then

$$V^i_{jk} = \delta^i_j U^s_{ks}.$$

If $\mu = 0$, $\nu = 0$ in Theorem 2, (V) and (VIII) then

$$V^i_{jk} = \frac{1}{2}(U^i_{jk} + U^i_{kj}),$$

and

$$V^i_{jk} = \frac{1}{2}(U^i_{jk} - U^i_{kj}).$$

Theorem 2 also shows that every nontrivial natural projector can be obtained as a linear combination of these *elementary natural projectors*. In terms of components, these four elementary projectors are characterized by

$$(a_1, a_2, a_3, a_4, a_5, a_6) = (0,0,1,0,0,0),$$

$$(a_1, a_2, a_3, a_4, a_5, a_6) = (1,0,0,0,0,0),$$

$$(a_1, a_2, a_3, a_4, a_5, a_6) = \left(0,0,0,0,\frac{1}{2},\frac{1}{2}\right),$$

and

$$(a_1, a_2, a_3, a_4, a_5, a_6) = \left(0,0,0,0,\frac{1}{2},-\frac{1}{2}\right).$$

Similarly, the choice $\mu = 1/2$, $\nu = \kappa = 0$ in the family (I) yields

$$V^i_{jk} = \frac{1}{2}(\delta^i_j U^s_{ks} + \delta^i_k U^s_{sj}).$$

5 Decomposability of (1,2)-tensors

In this section we study the problem of decomposing of a natural projector $P : T^1_2\mathbf{R}^n \to T^1_2\mathbf{R}^n$ into the sum of two natural projectors. We shall say that a natural projector P is *decomposable*, if there exist two natural projectors Q and R such that

$$P = Q + R. \tag{1}$$

Equation (1) for the unknowns Q and R is the *decomposability equation* for the natural projector P. The problem of solving this equation includes the *decomposability conditions*, to be satisfied by P, and then, provided the decomposability conditions are satisfied, determining Q and R.

A natural projector, which is *not* decomposable, is called *primitive*.

Clearly, a necessary condition of existence of a solution is the existence of a partition of the number $\dim \operatorname{Im} P$ in two positive integers, satisfying the *dimension condition*

$$\dim \operatorname{Im} P = \dim \operatorname{Im} Q + \dim \operatorname{Im} R \tag{2}$$

for some Q and R from the set of natural projectors, given by Theorem 1.

To apply this condition, we first determine the dimensions of the image spaces $\operatorname{Im} {}^{\mathrm{M}}P \subset T^1_2\mathbf{R}^n$, where ${}^{\mathrm{M}}P$ runs through the family of natural projectors $\mathcal{P} = \{{}^{\mathrm{I}}P, {}^{\mathrm{II}}P, \ldots, {}^{\mathrm{X}}P\}$. If in components ${}^{\mathrm{M}}P = {}^{\mathrm{M}}P^{i\ qr}_{jk\ p}$, where

$$\begin{aligned}{}^{\mathrm{M}}P^{i\ qr}_{jk\ p} &= a_1\delta^i_j\delta^q_k\delta^r_p + a_2\delta^i_j\delta^q_p\delta^r_k + a_3\delta^i_k\delta^q_p\delta^r_j + a_4\delta^i_k\delta^q_j\delta^r_p \\ &\quad + a_5\delta^i_p\delta^q_j\delta^r_k + a_6\delta^i_p\delta^q_k\delta^r_j\end{aligned}$$

(Theorem 1), then the *dimension formula* is

$$\begin{aligned}\dim \operatorname{Im} {}^{\mathrm{M}}P &= {}^{\mathrm{M}}P^{i\ jk}_{jk\ i} = a_1 n + a_2 n^2 + a_3 n + a_4 n^2 + a_5 n^3 + a_6 n^2 \\ &= a_5 n^3 + (a_2 + a_4 + a_6)n^2 + (a_1 + a_3)n.\end{aligned} \tag{3}$$

Second, we determine the *decomposability indicatrix* $\mathcal{I} = \{I_{\Phi\Psi}\}$ for the family $\mathcal{P}$. Recall that by definition, elements $I_{\Phi\Psi}$ of the indicatrix $\mathcal{I}$ are positive integers of the form $I_{\Phi\Psi} = \dim \operatorname{Im} {}^{\Phi}P + \dim \operatorname{Im} {}^{\Psi}P$ such that $\Psi \geq \Phi$ for

$\Phi = \mathrm{I},\mathrm{II},\mathrm{III},\dots,\mathrm{X}$. Third, using the structure of ${}^{\mathrm{M}}P$ and comparing the dimension $\dim \operatorname{Im} {}^{\mathrm{M}}P$ to the elements $I_{\Phi\Psi}$ of the indicatrix $\mathcal{I}$, we get dimension conditions (2) and decomposability equations (1) for ${}^{\mathrm{M}}P$.

Remark 4 The decomposition method for natural projectors in the tensor space $T_2^1\mathbf{R}^n$, described above, follows basic steps of the method for decomposition of $(0,s)$-tensors (see Krupka [2]), and is obviously applicable to general mixed (r,s)-tensors.

Theorem 3 (Dimensions) *The dimensions of the image vector spaces* $\operatorname{Im} {}^{\mathrm{M}}P$*, where* $\mathrm{M} = \mathrm{I},\mathrm{II},\mathrm{III},\dots,\mathrm{X}$*, are given by the following list:*

(I) $\dim \operatorname{Im} {}^{\mathrm{I}}P = n,$

(II) $\dim \operatorname{Im} {}^{\mathrm{II}}P = 0,$

(III) $\dim \operatorname{Im} {}^{\mathrm{III}}P = n(n^2 - 1),$

(IV) $\dim \operatorname{Im} {}^{\mathrm{IV}}P = n^3,$

(V) $\dim \operatorname{Im} {}^{\mathrm{V}}P = \frac{1}{2}n^2(n+1),$

(VI) $\dim \operatorname{Im} {}^{\mathrm{VI}}P = \frac{n^2+n-2}{2}n,$

(VII) $\dim \operatorname{Im} {}^{\mathrm{VII}}P = \frac{n^2+n+2}{2}n,$

(VIII) $\dim \operatorname{Im} {}^{\mathrm{VIII}}P = \frac{1}{2}n^2(n-1),$

(IX) $\dim \operatorname{Im} {}^{\mathrm{IX}}P = \frac{n^2-n+2}{2}n,$

(X) $\dim \operatorname{Im} {}^{\mathrm{X}}P = \frac{n^2-n-2}{2}n.$

Proof We use Theorem 1 and substitute for $a_1, a_2, a_3, a_4, a_5, a_6$ into dimension formula (3):

(I) $\dim \operatorname{Im} {}^{\mathrm{I}}P = (\nu + \kappa)n^2 + (1 - \mu - n(\nu + \kappa) + \mu)n = n,$

(II) $$\dim \operatorname{Im} {}^{\mathrm{II}}P = 0,$$

(III) $$\begin{aligned}\dim \operatorname{Im} {}^{\mathrm{III}}P &= n^3 + (\kappa + \nu)n^2 + (-1 - \mu - n(\nu + \kappa) + \mu)n \\ &= n^3 + (\kappa + \nu)n^2 + (-1 - n(\nu + \kappa))n = n(n^2 - 1),\end{aligned}$$

(IV) $$\dim \operatorname{Im} {}^{\mathrm{IV}}P = n^3,$$

(V) $$\dim \operatorname{Im} {}^{\mathrm{V}}P = \frac{1}{2}n^3 + \left(\nu - \nu + \frac{1}{2}\right)n^2 + (-\mu + \mu)n = \frac{1}{2}n^2(n+1),$$

(VI) $$\begin{aligned}\dim \operatorname{Im} {}^{\mathrm{VI}}P &= \frac{1}{2}n^3 + \left(-\frac{1}{2(1+n)} - \frac{1}{2(1+n)} + \frac{1}{2}\right)n^2 + \left(-\frac{1}{2(1+n)} - \frac{1}{2(1+n)}\right)n \\ &= \frac{n^3+n^4}{2(1+n)} + \left(-\frac{1}{2(1+n)} - \frac{1}{2(1+n)} + \frac{1+n}{2(1+n)}\right)n^2 - \frac{2}{2(1+n)}n \\ &= \frac{n^3+n^4}{2(1+n)} + \frac{n^3-n^2}{2(1+n)} - \frac{2n}{2(1+n)} = \frac{n^3+2n^2-n-2}{2(1+n)}n = \frac{(1+n)(-2+n+n^2)}{2(1+n)}n \\ &= \frac{n^2+n-2}{2}n = \frac{(n-1)(n+2)}{2}n,\end{aligned}$$

(VII) $$\begin{aligned}&\dim \operatorname{Im} {}^{\mathrm{VII}}P \\ &= \frac{1}{2}n^3 + \left(-\frac{1}{2(1-n)} - \frac{1}{2(1-n)} + \frac{1}{2}\right)n^2 + \left(\frac{1}{2(1-n)} + \frac{1}{2(1-n)}\right)n \\ &= \frac{n^3-n^4}{2(1-n)} + \frac{1-n-2}{2(1-n)}n^2 + \frac{2}{2(1-n)}n \\ &= \frac{n^3-n^4-n^3-n^2+2n}{2(1-n)} = \frac{-n^3-n+2}{2(1-n)}n = \frac{(1-n)(n^2+n+2)}{2(1-n)}n \\ &= \frac{n^2+n+2}{2}n,\end{aligned}$$

(VIII) $$\dim \operatorname{Im} {}^{\mathrm{VIII}}P = \frac{1}{2}n^3 + \left(\nu - \nu - \frac{1}{2}\right)n^2 + (-\mu + \mu)n = \frac{1}{2}n^2(n-1),$$

(IX) $$\begin{aligned}&\dim \operatorname{Im} {}^{\mathrm{IX}}P \\ &= \frac{1}{2}n^3 + \left(\frac{1}{2(1+n)} + \frac{1}{2(1+n)} - \frac{1}{2}\right)n^2 + \left(\frac{1}{2(1+n)} + \frac{1}{2(1+n)}\right)n\end{aligned}$$

$$= \frac{1+n}{2(1+n)}n^3 + \frac{2-1-n}{2(1+n)}n^2 + \frac{2}{2(1+n)}n$$
$$= \frac{n^3+n^4+n^2-n^3+2n}{2(1+n)} = \frac{n^3+n+2}{2(1+n)}n = \frac{(n+1)(n^2-n+2)}{2(1+n)}n$$
$$= \frac{n^2-n+2}{2}n,$$

$\dim \operatorname{Im} {}^{\mathrm{X}}P$

(X)
$$= \frac{1}{2}n^3 + \left(\frac{1}{2(1-n)} + \frac{1}{2(1-n)} - \frac{1}{2}\right)n^2 + \left(-\frac{1}{2(1-n)} - \frac{1}{2(1-n)}\right)n$$
$$= \frac{n^3-n^4}{2(1-n)} + \frac{2-1+n}{2(1-n)}n^2 - \frac{2}{2(1-n)}n$$
$$= \frac{n^3-n^4+n^2+n^3-2n}{2(1-n)} = \frac{-n^3+2n^2+n-2}{2(1-n)}n = \frac{(1-n)(n^2-n-2)}{2(1-n)}n$$
$$= \frac{n^2-n-2}{2(1-n)}n.$$

Remark 5 Theorem 3 shows that the dimensions of the image vector spaces of natural projectors $P: T_2^1\mathbf{R}^n \to T_2^1\mathbf{R}^n$ must be equal to one of the positive integers

$$n,\quad 0,\quad n(n^2-1),\quad n^3,\quad \frac{1}{2}n^2(n+1),\quad \frac{n^2+n-2}{2}n,$$
$$\frac{n^2+n+2}{2}n,\quad \frac{1}{2}n^2(n-1),\quad \frac{n^2-n+2}{2}n,\quad \frac{n^2-n-2}{2}n.$$

Remark 6 The sum of any two projectors R, S, belonging to the family of natural projectors ${}^{\mathrm{I}}P$, is *not* a natural projector. Indeed, the image space $\operatorname{Im}(R+S)$ of the sum would be the direct sum $\operatorname{Im} R \oplus \operatorname{Im} S$, whose dimension would be $2n$; but according to Theorem 3, there does not exist any natural projector whose image space is of dimension $2n$. Similar assertion holds for any two projectors, belonging to any family ${}^{\mathrm{M}}P$, $\mathrm{M} = \mathrm{I},\mathrm{III},\mathrm{V},\mathrm{VIII}$.

Now we determine the *decomposability indicatrix*, restricting possible decompositions of natural projectors by *a priori* dimension arguments. Since the set of natural projectors is finite (and contains ten elements), complete results for the indicatrix can be easily obtained from Theorem 3. Below we get seven *indicatrix tables*; the fourth column characterizing the coincidence of the sums of dimensions of the image spaces of the natural projectors ${}^{\mathrm{I}}P, {}^{\mathrm{II}}P, {}^{\mathrm{III}}P, \ldots, {}^{\mathrm{X}}P$ with the dimensions of these image spaces, includes the corresponding *decomposability equation*. Indeed, next step in solving the decomposability problem consists in investigating of these decomposability equations. Clearly, calculations dimensions for the zero projector ${}^{\mathrm{II}}P$ and the identity projector ${}^{\mathrm{IV}}P$ are not needed, so the corresponding (trivial) tables are omitted.

The indicatrix tables

${}^{\mathrm{I}}P$

$\dim \mathrm{Im}\, {}^{\mathrm{I}}P = n$

${}^{\mathrm{M}}P$	$\dim \mathrm{Im}\, {}^{\mathrm{M}}P$	$\dim \mathrm{Im}\, {}^{\mathrm{I}}P + \dim \mathrm{Im}\, {}^{\mathrm{M}}P$	
${}^{\mathrm{I}}P$	n	$2n$	$-$
${}^{\mathrm{III}}P$	$n(n^2-1)$	n^3	${}^{\mathrm{IV}}P = {}^{\mathrm{I}}P + {}^{\mathrm{III}}P$
${}^{\mathrm{V}}P$	$\frac{1}{2}n^2(n+1)$	$\frac{n^2+n+2}{2}n$	${}^{\mathrm{VII}}P = {}^{\mathrm{I}}P + {}^{\mathrm{V}}P$
${}^{\mathrm{VI}}P$	$\frac{n^2+n-2}{2}n$	$\frac{1}{2}n^2(n+1)$	${}^{\mathrm{V}}P = {}^{\mathrm{I}}P + {}^{\mathrm{VI}}P$
${}^{\mathrm{VII}}P$	$\frac{n^2+n+2}{2}n$	$\frac{n^2+n+4}{2}n$	$-$
${}^{\mathrm{VIII}}P$	$\frac{1}{2}n^2(n-1)$	$\frac{n^2-n+2}{2}n$	${}^{\mathrm{IX}}P = {}^{\mathrm{I}}P + {}^{\mathrm{VIII}}P$
${}^{\mathrm{IX}}P$	$\frac{n^2-n+2}{2}n$	$\frac{n^2-n+4}{2}n$	$-$
${}^{\mathrm{X}}P$	$\frac{n^2-n-2}{2}n$	$\frac{1}{2}n^2(n-1)$	${}^{\mathrm{VIII}}P = {}^{\mathrm{I}}P + {}^{\mathrm{X}}P$

Calculations of the sum $\dim \mathrm{Im}\, {}^{\mathrm{I}}P + \dim \mathrm{Im}\, {}^{\mathrm{M}}P$:

$$n + n = 2n,$$

$$n + n(n^2-1) = n^3,$$

$$n + \frac{1}{2}n^2(n+1) = n + \frac{1}{2}n^3 + \frac{1}{2}n^2 = \frac{1}{2}n(n^2+n+2),$$

$$n+\frac{1}{2}n(n-1)(n+2)=\frac{2n}{2}+\frac{n^3+n^2-2n}{2}=\frac{1}{2}n^2(n+1),$$
$$n+\frac{n^2+n+2}{2}n=\frac{2n+n^3+n^2+2n}{2}=\frac{n^2+n+4}{2}n,$$
$$n+\frac{1}{2}n^2(n-1)=\frac{2n+n^3-n^2}{2}=\frac{n^2-n+2}{2}n,$$
$$n+\frac{n^2-n+2}{2}n=\frac{2n+n^3-n^2+2n}{2}=\frac{n^2-n+4}{2}n,$$
$$n+\frac{n^2-n-2}{2}n=\frac{1}{2}n^2(n-1).$$

${}^{\mathrm{III}}P$

$\dim \mathrm{Im}\, {}^{\mathrm{III}}P = n(n^2-1)$

${}^{\mathrm{M}}P$	$\dim \mathrm{Im}\, {}^{\mathrm{M}}P$	$\dim \mathrm{Im}\, {}^{\mathrm{III}}P + \dim \mathrm{Im}\, {}^{\mathrm{M}}P$	
${}^{\mathrm{III}}P$	$n(n^2-1)$	$2n(n^2-1)$	–
${}^{\mathrm{V}}P$	$\frac{1}{2}n^2(n+1)$	$\frac{3n^2+n-2}{2}n$	–
${}^{\mathrm{VI}}P$	$\frac{n^2+n-2}{2}n$	$\frac{3n^2+n-4}{2}n$	–
${}^{\mathrm{VII}}P$	$\frac{n^2+n+2}{2}n$	$\frac{3n+1}{2}n^2$	–
${}^{\mathrm{VIII}}P$	$\frac{1}{2}n^2(n-1)$	$\frac{3n^2-n-2}{2}n$	–
${}^{\mathrm{IX}}P$	$\frac{n^2-n+2}{2}n$	$\frac{3n-1}{2}n^2$	–
${}^{\mathrm{X}}P$	$\frac{n^2-n-2}{2}n$	$\frac{3n^2-n-4}{2}n$	–

Calculations of $\dim \mathrm{Im}\, {}^{\mathrm{III}}P + \dim \mathrm{Im}\, {}^{\mathrm{M}}P$:

$$n(n^2-1)+n(n^2-1)=2n(n^2-1),$$
$$n(n^2-1)+\frac{1}{2}n^2(n+1)=\frac{2n^3-2n+n^3+n^2}{2}=\frac{3n^2+n-2}{2}n,$$

$$n(n^2-1)+\frac{n^2+n-2}{2}n=\frac{2n^2-2+n^2+n-2}{2}n=\frac{3n^3+n^2-4n}{2}$$
$$=\frac{3n^2+n-4}{2}n,$$
$$n(n^2-1)+\frac{n^2+n+2}{2}n=\frac{2n^2-2+n^2+n+2}{2}n=\frac{3n+1}{2}n^2,$$
$$n(n^2-1)+\frac{1}{2}n^2(n-1)=\frac{2n^2-2+n^2-n}{2}n$$
$$=\frac{3n^2-n-2}{2}n=\frac{(n-1)(3n+2)}{2}n,$$
$$n(n^2-1)+\frac{n^2-n+2}{2}n=\frac{2n^2-2+n^2-n+2}{2}n=\frac{3n-1}{2}n^2,$$
$$n(n^2-1)+\frac{n^2-n-2}{2}n=\frac{2n^2-2+n^2-n-2}{2}n=\frac{3n^2-n-4}{2}n.$$

${}^{\mathrm{V}}P$

$$\dim\operatorname{Im}{}^{\mathrm{V}}P=\frac{1}{2}n^2(n+1)$$

${}^{\mathrm{M}}P$	$\dim\operatorname{Im}{}^{\mathrm{M}}P$	$\dim\operatorname{Im}{}^{\mathrm{V}}P+\dim\operatorname{Im}{}^{\mathrm{M}}P$	
${}^{\mathrm{V}}P$	$\frac{1}{2}n^2(n+1)$	$n^2(n+1)$	–
${}^{\mathrm{VI}}P$	$\frac{n^2+n-2}{2}n$	$n(n^2+n-1)$	–
${}^{\mathrm{VII}}P$	$\frac{n^2+n+2}{2}n$	$n(n^2+n+1)$	–
${}^{\mathrm{VIII}}P$	$\frac{1}{2}n^2(n-1)$	n^3	${}^{\mathrm{IV}}P={}^{\mathrm{V}}P+{}^{\mathrm{VIII}}P$
${}^{\mathrm{IX}}P$	$\frac{n^2-n+2}{2}n$	$n(n^2+1)$	–
${}^{\mathrm{X}}P$	$\frac{n^2-n-2}{2}n$	$n(n^2-1)$	${}^{\mathrm{III}}P={}^{\mathrm{V}}P+{}^{\mathrm{X}}P$

Calculations of $\dim \operatorname{Im} {}^{\mathrm{V}}P + \dim \operatorname{Im} {}^{\mathrm{M}}P$:

$$\frac{1}{2}n^2(n+1)+\frac{1}{2}n^2(n+1)=n^2(n+1),$$

$$\frac{1}{2}n^2(n+1)+\frac{n^2+n-2}{2}n=\frac{1}{2}n(n^2+n+n^2+n-2)$$
$$=\frac{1}{2}n(2n^2+2n-2)=n(n^2+n-1),$$

$$\frac{1}{2}n^2(n+1)+\frac{n^2+n+2}{2}n=\frac{1}{2}n(n^2+n+n^2+n+2)$$
$$=\frac{1}{2}n(2n^2+2n+2)=\frac{1}{2}n(2n^2+2n+2)=n(n^2+n+1),$$

$$\frac{1}{2}n^2(n+1)+\frac{1}{2}n^2(n-1)=n^3,$$

$$\frac{1}{2}n^2(n+1)+\frac{n^2-n+2}{2}n=\frac{1}{2}n(n^2+n+n^2-n+2)$$
$$=n(n^2+1),$$

$$\frac{1}{2}n^2(n+1)+\frac{n^2-n-2}{2}n=\frac{1}{2}n(n^2+n+n^2-n-2)$$
$$=n(n^2-1).$$

${}^{\mathrm{VI}}P$

$$\dim \operatorname{Im} {}^{\mathrm{VI}}P=\frac{1}{2}n(n-1)(n+2)=\frac{n^2+n-2}{2}n$$

${}^{\mathrm{M}}P$	$\dim \operatorname{Im} {}^{\mathrm{M}}P$	$\dim \operatorname{Im} {}^{\mathrm{VI}}P + \dim \operatorname{Im} {}^{\mathrm{M}}P$	
${}^{\mathrm{VII}}P$	$\frac{n^2+n+2}{2}n$	$n^2(n+1)$	–
${}^{\mathrm{VIII}}P$	$\frac{1}{2}n^2(n-1)$	$n(n^2-1)$	${}^{\mathrm{III}}P = {}^{\mathrm{VI}}P + {}^{\mathrm{VIII}}P$
${}^{\mathrm{IX}}P$	$\frac{n^2-n+2}{2}n$	n^3	${}^{\mathrm{IV}}P = {}^{\mathrm{VI}}P + {}^{\mathrm{IX}}P$
${}^{\mathrm{X}}P$	$\frac{n^2-n-2}{2}n$	$n(n^2-2)$	–

Calculations of $\dim \operatorname{Im} {}^{\mathrm{VI}}P + \dim \operatorname{Im} {}^{\mathrm{M}}P$:

$$\frac{1}{2}n(n-1)(n+2)+\frac{n^2+n+2}{2}n=\frac{1}{2}n(n^2+n-2+n^2+n+2)$$
$$=n^2(n+1),$$
$$\frac{1}{2}n(n-1)(n+2)+\frac{1}{2}n^2(n-1)=\frac{1}{2}n(n^2+n-2+n^2-n)$$
$$=n(n^2-1),$$
$$\frac{1}{2}n(n-1)(n+2)+\frac{n^2-n+2}{2}n=\frac{1}{2}n(n^2+n-2+n^2-n+2)=n^3,$$
$$\frac{1}{2}n(n-1)(n+2)+\frac{n^2-n-2}{2}n=\frac{1}{2}n(n^2+n-2+n^2-n-2)$$
$$=n(n^2-2).$$

${}^{\mathrm{VII}}P$

$$\dim \operatorname{Im} {}^{\mathrm{VII}}P=\frac{n^2+n+2}{2}n$$

${}^{\mathrm{M}}P$	$\dim \operatorname{Im} {}^{\mathrm{M}}P$	$\dim \operatorname{Im} {}^{\mathrm{VII}}P + \dim \operatorname{Im} {}^{\mathrm{M}}P$	
${}^{\mathrm{VIII}}P$	$\frac{1}{2}n^2(n-1)$	$n(n^2+1)$	–
${}^{\mathrm{IX}}P$	$\frac{n^2-n+2}{2}n$	$n(n^2+2)$	–
${}^{\mathrm{X}}P$	$\frac{n^2-n-2}{2}n$	n^3	${}^{\mathrm{IV}}P = {}^{\mathrm{VII}}P + {}^{\mathrm{X}}P$

Calculations of $\dim \operatorname{Im} {}^{\mathrm{VII}}P + \dim \operatorname{Im} {}^{\mathrm{M}}P$:

$$\frac{n^2+n+2}{2}n+\frac{1}{2}n^2(n-1)=\frac{1}{2}n(n^2+n+2+n^2-n)=n(n^2+1),$$
$$\frac{n^2+n+2}{2}n+\frac{n^2-n+2}{2}n=\frac{1}{2}n(n^2+n+2+n^2-n+2)=n(n^2+2),$$
$$\frac{n^2+n+2}{2}n+\frac{n^2-n-2}{2}n=\frac{1}{2}n(n^2+n+2+n^2-n-2)=n^3.$$

${}^{\mathrm{VIII}}P$

$\dim\operatorname{Im}{}^{\mathrm{VIII}}P = \frac{1}{2}n^2(n-1)$

${}^{\mathrm{M}}P$	$\dim\operatorname{Im}{}^{\mathrm{M}}P$	$\dim\operatorname{Im}{}^{\mathrm{VIII}}P + \dim\operatorname{Im}{}^{\mathrm{M}}P$	
${}^{\mathrm{VIII}}P$	$\frac{1}{2}n^2(n-1)$	$n^2(n-1)$	–
${}^{\mathrm{IX}}P$	$\frac{n^2-n+2}{2}n$	$n(n^2-n+1)$	–
${}^{\mathrm{X}}P$	$\frac{n^2-n-2}{2}n$	$n(n^2-n-1)$	–

Calculations of $\dim\operatorname{Im}{}^{\mathrm{VIII}}P + \dim\operatorname{Im}{}^{\mathrm{M}}P$:

$$\frac{1}{2}n^2(n-1)+\frac{1}{2}n^2(n-1)=n^2(n-1),$$
$$\frac{1}{2}n^2(n-1)+\frac{n^2-n+2}{2}n=\frac{1}{2}n(n^2-n+n^2-n+2)=n(n^2-n+1),$$
$$\frac{1}{2}n^2(n-1)+\frac{n^2-n-2}{2}n=\frac{1}{2}n(n^2-n+n^2-n-2)=n(n^2-n-1).$$

${}^{\mathrm{IX}}P$

$\dim\operatorname{Im}{}^{\mathrm{IX}}P = \frac{n^2-n+2}{2}n$

${}^{\mathrm{M}}P$	$\dim\operatorname{Im}{}^{\mathrm{M}}P$	$\dim\operatorname{Im}{}^{\mathrm{IX}}P + \dim\operatorname{Im}{}^{\mathrm{M}}P$	
${}^{\mathrm{X}}P$	$\frac{n^2-n-2}{2}n$	$n^2(n-1)$	–

Calculations of $\dim\operatorname{Im}{}^{\mathrm{IX}}P + \dim\operatorname{Im}{}^{\mathrm{M}}P$:

$$\frac{n^2-n+2}{2}n+\frac{1}{2}n(n+1)(n-2)=\frac{1}{2}n(n^2-n+2+n^2-n-2)$$
$$=n^2(n-1).$$

The following are results of the decomposability indicatrix tables. Note that $\dim\operatorname{Im}{}^{\mathrm{IV}}P=\dim\operatorname{Im}\mathrm{Id}=\dim T_2^1\mathbf{R}^n=n^3$.

Lemma 4 (Decomposability indicatrix) *The natural projectors* ${}^{\mathrm{I}}P$, ${}^{\mathrm{II}}P$, ${}^{\mathrm{III}}P$, ..., ${}^{\mathrm{X}}P$ *satisfy the following conditions:*

$$
\begin{aligned}
&\dim \mathrm{Im}\, {}^{\mathrm{III}}P = \dim \mathrm{Im}\, {}^{\mathrm{V}}P + \dim \mathrm{Im}\, {}^{\mathrm{X}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{III}}P = \dim \mathrm{Im}\, {}^{\mathrm{VI}}P + \dim \mathrm{Im}\, {}^{\mathrm{VIII}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{IV}}P = \dim \mathrm{Im}\, {}^{\mathrm{I}}P + \dim \mathrm{Im}\, {}^{\mathrm{III}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{IV}}P = \dim \mathrm{Im}\, {}^{\mathrm{V}}P + \dim \mathrm{Im}\, {}^{\mathrm{VIII}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{IV}}P = \dim \mathrm{Im}\, {}^{\mathrm{VI}}P + \dim \mathrm{Im}\, {}^{\mathrm{IX}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{IV}}P = \dim \mathrm{Im}\, {}^{\mathrm{VII}}P + \dim \mathrm{Im}\, {}^{\mathrm{X}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{V}}P = \dim \mathrm{Im}\, {}^{\mathrm{I}}P + \dim \mathrm{Im}\, {}^{\mathrm{VI}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{VII}}P = \dim \mathrm{Im}\, {}^{\mathrm{I}}P + \dim \mathrm{Im}\, {}^{\mathrm{V}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{VIII}}P = \dim \mathrm{Im}\, {}^{\mathrm{I}}P + \dim \mathrm{Im}\, {}^{\mathrm{X}}P,\\
&\dim \mathrm{Im}\, {}^{\mathrm{IX}}P = \dim \mathrm{Im}\, {}^{\mathrm{I}}P + \dim \mathrm{Im}\, {}^{\mathrm{VIII}}P.
\end{aligned}
\tag{4}
$$

Proof These assertions are consequences of the decomposability indicatrix tables for the projectors ${}^{\mathrm{M}}P$.

Lemma 5 *The natural projectors* ${}^{\mathrm{I}}P, {}^{\mathrm{II}}P, {}^{\mathrm{VI}}P, {}^{\mathrm{X}}P$ *are primitive*.

Proof According to Lemma 4, every decomposable natural projector satisfies one of the formulas (4), with non-trivial summands.

Our aim now will be to investigate the decomposability equations suggested by dimension formulas (4). Clearly, it is sufficient to study decomposability of the natural projectors ${}^{\mathrm{III}}P, {}^{\mathrm{V}}P, {}^{\mathrm{VII}}P, {}^{\mathrm{VIII}}P, {}^{IX}P$, different from primitive natural projectors and the identity projector; the corresponding decomposability equations are

$$
\begin{gathered}
{}^{\mathrm{III}}P = {}^{\mathrm{V}}P + {}^{\mathrm{X}}P, \quad {}^{\mathrm{III}}P = {}^{\mathrm{VI}}P + {}^{\mathrm{VIII}}P, \quad {}^{\mathrm{V}}P = {}^{\mathrm{I}}P + {}^{\mathrm{VI}}P,\\
{}^{\mathrm{VII}}P = {}^{\mathrm{I}}P + {}^{\mathrm{V}}P, \quad {}^{\mathrm{VIII}}P = {}^{\mathrm{I}}P + {}^{\mathrm{X}}P, \quad {}^{\mathrm{IX}}P = {}^{\mathrm{I}}P + {}^{\mathrm{VIII}}P.
\end{gathered}
$$

Using Theorem 1, we find explicit forms of these equations, in which the unknowns are parameters of the families ${}^{\mathrm{M}}P$. Then we give complete solutions of these equations.

Consider the family of natural projectors ${}^{\mathrm{III}}P$, expressed as

$$
\begin{gathered}
(a_1, a_2, a_3, a_4, a_5, a_6) = (-1 + \mu + n(\nu + \kappa), -\nu, -\mu, -\kappa, 1, 0),\\
\nu, \mu, \kappa \in \mathbf{R}.
\end{gathered}
\tag{5}
$$

Theorem 5 (a) ${}^{\mathrm{III}}P$ *is decomposable if and only if one of the following two conditions holds:*

(6) $$\kappa+\nu=\frac{1}{n-1},$$

or

(7) $$\nu+\kappa=\frac{1}{n+1}.$$

(b) *If condition* (6) *is satisfied, then* ${}^{\mathrm{III}}P$ *is given*

(8) $$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\mu+\frac{1}{n-1},-\nu,-\mu,\nu-\frac{1}{n-1},1,0\right),$$

and has a unique decomposition

(9) $${}^{\mathrm{III}}P={}^{\mathrm{V}}P+{}^{\mathrm{X}}P,$$

where ${}^{\mathrm{V}}P$ *is given by*

(10) $$\begin{aligned}&(b_1,b_2,b_3,b_4,b_5,b_6)\\&=\left(\mu-\frac{1}{2(1-n)},-\nu-\frac{1}{2(1-n)},-\mu+\frac{1}{2(1-n)},\nu+\frac{1}{2(1-n)},\frac{1}{2},\frac{1}{2}\right).\end{aligned}$$

(c) *If condition* (7) *is satisfied, then* ${}^{\mathrm{III}}P$ *is given by*

(11) $$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\mu-\frac{1}{n+1},-\nu,-\mu,\nu-\frac{1}{n+1},1,0\right),$$

and has a unique decomposition

(12) $${}^{\mathrm{III}}P={}^{\mathrm{VIII}}P+{}^{\mathrm{VI}}P,$$

where ${}^{\mathrm{VIII}}P$ *is given by*

(13) $$\begin{aligned}&(b_1,b_2,b_3,b_4,b_5,b_6)\\&=\left(\mu-\frac{1}{2(1+n)},-\nu+\frac{1}{2(1+n)},-\mu+\frac{1}{2(1+n)},\nu-\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right).\end{aligned}$$

Proof 1. To prove necessity, suppose that ${}^{\mathrm{III}}P$ is decomposable. Then either ${}^{\mathrm{III}}P={}^{\mathrm{V}}P+{}^{\mathrm{X}}P$ or ${}^{\mathrm{III}}P={}^{\mathrm{VI}}P+{}^{\mathrm{VIII}}P$ (Lemma 4).

But ${}^{\mathrm{III}}P$, ${}^{\mathrm{V}}P$, and ${}^{\mathrm{X}}P$ are given by

$$(14)\quad \begin{aligned}&(-1+\mu+n(\nu+\kappa),-\nu,-\mu,-\kappa,1,0),\\ &\left(-\mu',-\nu',\mu',\nu',\frac{1}{2},\frac{1}{2}\right),\\ &\left(-\frac{1}{2(1-n)},\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2(1-n)},\frac{1}{2},-\frac{1}{2}\right),\end{aligned}$$

thus, equation ${}^{\mathrm{III}}P={}^{\mathrm{V}}P+{}^{\mathrm{X}}P$ reads

$$(15)\quad \begin{aligned}&-1+\mu+n(\nu+\kappa)=-\mu'-\frac{1}{2(1-n)},\\ &-\nu=-\nu'+\frac{1}{2(1-n)},\quad -\mu=\mu'-\frac{1}{2(1-n)},\quad -\kappa=\nu'+\frac{1}{2(1-n)}.\end{aligned}$$

This system already implies (6), because

$$\nu'=\nu+\frac{1}{2(1-n)}=-\kappa-\frac{1}{2(1-n)}.$$

Analogously, ${}^{\mathrm{III}}P$, ${}^{\mathrm{VI}}P$, and ${}^{\mathrm{VIII}}P$ are given by

$$\begin{aligned}&(-1+\mu+n(\nu+\kappa),-\nu,-\mu,-\kappa,1,0),\\ &\left(-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},\frac{1}{2},\frac{1}{2}\right),\\ &\left(\mu',\nu',-\mu',-\nu',\frac{1}{2},-\frac{1}{2}\right),\end{aligned}$$

so equation ${}^{\mathrm{III}}P={}^{\mathrm{VI}}P+{}^{\mathrm{VIII}}P$ is equivalent to the system

$$(16)\quad \begin{aligned}&-1+\mu+n(\nu+\kappa)=-\frac{1}{2(1+n)}+\mu',\\ &-\nu=-\frac{1}{2(1+n)}+\nu',\quad -\mu=-\frac{1}{2(1+n)}-\mu',\quad -\kappa=-\frac{1}{2(1+n)}-\nu'.\end{aligned}$$

Hence

$$\nu'=-\nu+\frac{1}{2(1+n)}=-\frac{1}{2(1+n)}+\kappa.$$

2. We show that if condition (6) is satisfied, then ${}^{\mathrm{III}}P$ is of the form (8) and can be expressed as ${}^{\mathrm{III}}P={}^{\mathrm{V}}P+{}^{\mathrm{X}}P$, with ${}^{\mathrm{V}}P$ given by (10).

Condition (6) determines the choice of parameters for which ${}^{\mathrm{III}}P$ is decomposable; substituting into (III) gives formula (8). Consider decomposability equation (9), expressed in terms of parameters of the natural projectors ${}^{\mathrm{III}}P$ and ${}^{\mathrm{V}}P$ by the system (15) for the unknowns μ',ν',

$$\begin{aligned}&-1+\mu+n(\nu+\kappa)=-\mu'-\frac{1}{2(1-n)},\\ &-\nu=-\nu'+\frac{1}{2(1-n)},\quad -\mu=\mu'-\frac{1}{2(1-n)},\quad -\kappa=\nu'+\frac{1}{2(1-n)}.\end{aligned}\tag{17}$$

Since (6) holds, the third equation is satisfied identically. The first equation is also an identity: indeed, the left-hand side is

$$\begin{aligned}-1+\mu+n(\nu+\kappa)&=-1+\mu-\frac{n}{1-n}=\mu-\frac{n+1-n}{1-n}=\mu-\frac{1}{1-n}\\ &=-\mu'-\frac{1}{2(1-n)}.\end{aligned}$$

Consequently, (17) implies

$$\nu'=\nu+\frac{1}{2(1-n)},\quad \mu'=-\mu+\frac{1}{2(1-n)}.$$

Substitution in (14) yields

$$\begin{aligned}&\left(-\mu',-\nu',\mu',\nu',\frac{1}{2},\frac{1}{2}\right)\\ &=\left(\mu-\frac{1}{2(1-n)},-\nu-\frac{1}{2(1-n)},-\mu+\frac{1}{2(1-n)},\nu+\frac{1}{2(1-n)},\frac{1}{2},\frac{1}{2}\right),\end{aligned}$$

proving (10).

3. We show that if condition (7) is satisfied, then the natural projector ${}^{\mathrm{III}}P$ is of the form (11), and can be expressed as ${}^{\mathrm{III}}P={}^{\mathrm{VIII}}P+{}^{\mathrm{VI}}P$, with ${}^{\mathrm{VIII}}P$ given by (13).

Formula (11) arises by a direct substitution from (7) to expression (III). Consider decomposability equation (12) for the unknowns μ',ν', expressed by (16),

$$\begin{aligned}&-1+\mu+n(\nu+\kappa)=-\frac{1}{2(1+n)}+\mu',\\ &-\nu=-\frac{1}{2(1+n)}+\nu',\quad -\mu=-\frac{1}{2(1+n)}-\mu',\quad -\kappa=-\frac{1}{2(1+n)}-\nu',\end{aligned}\tag{18}$$

Condition (7) implies that the last and the first equations follow from the remaining equations, which can also be written as

$$(19)\qquad \nu' = -\nu + \frac{1}{2(1+n)}, \quad \mu' = \mu - \frac{1}{2(1+n)}.$$

Indeed,

$$-\frac{1}{2(1+n)} - \nu' = -\frac{1}{2(1+n)} - \frac{1}{2(1+n)} + \nu = -\frac{1}{1+n} + \nu \equiv -\kappa,$$

and

$$-1 + \mu + n(\nu + \kappa) = -1 + \mu + \frac{n}{1+n} = -1 + \frac{1}{2(1+n)} + \mu' + \frac{n}{1+n}$$
$$\equiv -\frac{1}{2(1+n)} + \mu'.$$

Thus, equations (19) solve the decomposability equation (18), or (12), and we have

$$\left(\mu', \nu', -\mu', -\nu', \frac{1}{2}, -\frac{1}{2}\right)$$
$$= \left(\mu - \frac{1}{2(1+n)}, -\nu + \frac{1}{2(1+n)}, -\mu + \frac{1}{2(1+n)}, \nu - \frac{1}{2(1+n)}, \frac{1}{2}, -\frac{1}{2}\right).$$

as required.

Consider the natural projector ${}^{\mathrm{V}}P$, expressed by

$$(a_1, a_2, a_3, a_4, a_5, a_6) = \left(-\mu, -\nu, \mu, \nu, \frac{1}{2}, \frac{1}{2}\right), \quad \mu, \nu \in \mathbf{R},$$

Theorem 6 *For any values of the parameters* $\mu, \nu \in \mathbf{R}$, *the natural projector* ${}^{\mathrm{V}}P$ *is decomposable, and admits a unique decomposition*

$$(20)\qquad {}^{\mathrm{V}}P = {}^{\mathrm{I}}P + {}^{\mathrm{VI}}P.$$

In this formula ${}^{\mathrm{I}}P$ *is given by*

$$(21)\qquad (b_1, b_2, b_3, b_4, b_5, b_6) = \left(-\mu + \frac{1}{2(1+n)}, -\nu + \frac{1}{2(1+n)}, \mu + \frac{1}{2(1+n)}, \nu + \frac{1}{2(1+n)}, 0, 0\right).$$

Proof According to Theorem 1, ${}^{\mathrm{I}}P$ and ${}^{\mathrm{VI}}P$ are given by

(22) $$(1-\mu'-n(\nu'+\kappa'),\nu',\mu',\kappa',0,0),$$

and

$$\left(-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},\frac{1}{2},\frac{1}{2}\right).$$

Then the sum ${}^{\mathrm{I}}P+{}^{\mathrm{VI}}P$ is

$$\Big(1-\mu'-n(\nu'+\kappa')-\frac{1}{2(1+n)},\nu'-\frac{1}{2(1+n)},\mu'-\frac{1}{2(1+n)},$$
$$\kappa'-\frac{1}{2(1+n)},\frac{1}{2},\frac{1}{2}\Big).$$

Decomposability equation (20) gives the following system for the unknowns μ',ν',κ'

$$-\mu=1-\mu'-n(\nu'+\kappa')-\frac{1}{2(1+n)},$$
$$-\nu=\nu'-\frac{1}{2(1+n)},\quad \mu=\mu'-\frac{1}{2(1+n)},\quad \nu=\kappa'-\frac{1}{2(1+n)}.$$

These equations imply

$$\kappa'+\nu'=\frac{1}{1+n},$$

and the first equation becomes an identity, because calculation of the righ-hand side yields

$$1-\mu'-n(\nu'+\kappa')-\frac{1}{2(1+n)}=1-\mu'-\frac{n}{1+n}-\frac{1}{2(1+n)}$$
$$=-\mu'+\frac{2+2n-2n-1}{2(1+n)}=-\mu'+\frac{1}{2(1+n)}\equiv-\mu.$$

Consequently, we have a solution

$$\mu'=\mu+\frac{1}{2(1+n)},\quad \nu'=-\nu+\frac{1}{2(1+n)},\quad \kappa'=\nu+\frac{1}{2(1+n)},$$

uniquely determined by μ and ν. These parameters define the natural projector ${}^{\mathrm{I}}P$ (22) as

$$\begin{aligned}&(1-\mu'-n(\nu'+\kappa'),\nu',\mu',\kappa',0,0)\\&=\left(1-\mu-\frac{1}{2(1+n)}-\frac{n}{1+n},-\nu+\frac{1}{2(1+n)},\mu+\frac{1}{2(1+n)},\nu+\frac{1}{2(1+n)},0,0\right)\\&=\left(-\mu+\frac{2+2n-1-2n}{2(1+n)},-\nu+\frac{1}{2(1+n)},\mu+\frac{1}{2(1+n)},\nu+\frac{1}{2(1+n)},0,0\right)\\&=\left(-\mu+\frac{1}{2(1+n)},-\nu+\frac{1}{2(1+n)},\mu+\frac{1}{2(1+n)},\nu+\frac{1}{2(1+n)},0,0\right),\end{aligned}$$

proving formula (21).

Consider the natural projector ${}^{\mathrm{VII}}P$, given by

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2},\frac{1}{2}\right).$$

Theorem 7 *The natural projector* ${}^{\mathrm{VII}}P$ *is decomposable and has a decomposition*

(23) $${}^{\mathrm{VII}}P={}^{\mathrm{I}}P+{}^{\mathrm{V}}P,$$

where for any values of parameters $\mu,\nu\in\mathbf{R}$, ${}^{\mathrm{I}}P$ *and* ${}^{\mathrm{V}}P$, *are given by*

$$(b_1,b_2,b_3,b_4,b_5,b_6)=\left(-\mu+\frac{1}{1-n},\nu,\mu,-\nu-\frac{1}{1-n},0,0\right)$$

and

$$\begin{aligned}&(c_1,c_2,c_3,c_4,c_5,c_6)\\&=\left(-\frac{1}{2(1-n)}+\mu,-\frac{1}{2(1-n)}-\nu,\frac{1}{2(1-n)}-\mu,\frac{1}{2(1-n)}+\nu,\frac{1}{2},\frac{1}{2}\right).\end{aligned}$$

Proof The natural projectors ${}^{\mathrm{I}}P$ and ${}^{\mathrm{V}}P$ are given by

(24) $$(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0)$$

and

$$\left(-\mu',-\nu',\mu',\nu',\frac{1}{2},\frac{1}{2}\right).$$

Then ${}^{\mathrm{I}}P + {}^{\mathrm{V}}P$ has an expression

$$\left(1-\mu-n(\nu+\kappa)-\mu',\nu-\nu',\mu+\mu',\kappa+\nu',\frac{1}{2},\frac{1}{2}\right),$$

and the decomposability equation (23) is equivalent to the system

(25)
$$1-\mu-n(\nu+\kappa)-\mu'=\frac{1}{2(1-n)},$$
$$\nu-\nu'=-\frac{1}{2(1-n)},\quad \mu+\mu'=\frac{1}{2(1-n)},\quad \kappa+\nu'=-\frac{1}{2(1-n)}$$

for the unknowns μ,ν,κ and μ',ν'. Thus,

$$\nu'=-\frac{1}{2(1-n)}-\kappa=\nu+\frac{1}{2(1-n)},$$

hence

(26)
$$-\frac{1}{1-n}=\nu+\kappa,$$

and the first equation in the system (25) is a consequence of the remaining ones:

$$1-\mu-n(\nu+\kappa)-\mu'=1-\not{\mu}+\frac{n}{1-n}-\frac{1}{2(1-n)}+\not{\mu}$$
$$=\frac{2-2n+2n-1}{2(1-n)}\equiv\frac{1}{2(1-n)}.$$

Thus, if ν and κ satisfy (26), then system (25) has a solution

$$\nu'=\frac{1}{2(1-n)}+\nu,\quad \mu'=\frac{1}{2(1-n)}-\mu.$$

Substitution into (24) and (25) yields

$$\begin{aligned}&(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0)\\ &=\left(1-\mu+\frac{n}{1-n},\nu,\mu,-\frac{1}{1-n}-\nu,0,0\right)\\ &=\left(-\mu+\frac{1}{1-n},\nu,\mu,-\frac{1}{1-n}-\nu,0,0\right),\end{aligned}$$

and

$$\left(-\mu',-\nu',\mu',\nu',\frac{1}{2},\frac{1}{2}\right)$$
$$=\left(-\frac{1}{2(1-n)}+\mu,-\frac{1}{2(1-n)}-\nu,\frac{1}{2(1-n)}-\mu,\frac{1}{2(1-n)}+\nu,\frac{1}{2},\frac{1}{2}\right).$$

Now we study the natural projector ${}^{\mathrm{VIII}}P$, given by

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\mu,\nu,-\mu,-\nu,\frac{1}{2},-\frac{1}{2}\right),\quad \mu,\nu\in\mathbf{R}.$$

Theorem 8 *For any values of the parameters* $\mu,\nu\in\mathbf{R}$, *the natural projector* ${}^{\mathrm{VIII}}P$ *is decomposable, and admits a unique decomposition*

(27) $\qquad {}^{\mathrm{VIII}}P={}^{\mathrm{I}}P+{}^{\mathrm{X}}P,$

where ${}^{\mathrm{I}}P$ *is given by*

(28)
$$(b_1,b_2,b_3,b_4,b_5,b_6)$$
$$=\left(\mu+\frac{1}{2(1-n)},\nu-\frac{1}{2(1-n)},-\mu+\frac{1}{2(1-n)},-\nu-\frac{1}{2(1-n)},0,0\right).$$

Proof Natural projectors ${}^{\mathrm{I}}P$ and ${}^{\mathrm{X}}P$ are given by

(29) $\qquad (1-\mu'-n(\nu'+\kappa'),\nu',\mu',\kappa',0,0)$

and

$$\left(-\frac{1}{2(1-n)},\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2(1-n)},\frac{1}{2},-\frac{1}{2}\right),$$

and their sum ${}^{\mathrm{I}}P+{}^{\mathrm{X}}P$ is

$$\left(1-\mu'-n(\nu'+\kappa')-\frac{1}{2(1-n)},\nu'+\frac{1}{2(1-n)},\mu'-\frac{1}{2(1-n)},\right.$$
$$\left.\kappa'+\frac{1}{2(1-n)},\frac{1}{2},-\frac{1}{2}\right).$$

Decomposability equations (27) for the unknowns μ',ν',κ' can be written as the system

$$1-\mu'-n(\nu'+\kappa')-\frac{1}{2(1-n)}=\mu,$$
$$\nu'+\frac{1}{2(1-n)}=\nu,\quad \mu'-\frac{1}{2(1-n)}=-\mu,\quad \kappa'+\frac{1}{2(1-n)}=-\nu.$$

Thus,

$$\nu'+\kappa'=-\frac{1}{1-n},$$

and the first equation becomes the identity:

$$1-\mu'-n(\nu'+\kappa')-\frac{1}{2(1-n)}=1+\mu-\frac{1}{2(1-n)}+\frac{n}{1-n}-\frac{1}{2(1-n)}$$
$$=\mu+\frac{2-2n-1+2n-1}{2(1-n)}\equiv\mu.$$

Consequently, equations

$$\nu'=\nu-\frac{1}{2(1-n)},\quad \mu'=-\mu+\frac{1}{2(1-n)},\quad \kappa'=-\nu-\frac{1}{2(1-n)}$$

give a solution μ',ν',κ'. Then from (29), the natural projector ${}^{\mathrm{I}}P$ is equal to the expression

$$(1-\mu'-n(\nu'+\kappa'),\nu',\mu',\kappa',0,0)$$
$$=\left(1+\mu-\frac{1}{2(1-n)}+\frac{n}{1-n},\nu-\frac{1}{2(1-n)},-\mu+\frac{1}{2(1-n)},-\nu-\frac{1}{2(1-n)},0,0\right)$$
$$=\left(\mu+\frac{2-2n-1+2n}{2(1-n)},\nu-\frac{1}{2(1-n)},-\mu+\frac{1}{2(1-n)},-\nu-\frac{1}{2(1-n)},0,0\right)$$
$$=\left(\mu+\frac{1}{2(1-n)},\nu-\frac{1}{2(1-n)},-\mu+\frac{1}{2(1-n)},-\nu-\frac{1}{2(1-n)},0,0\right),$$

proving (28).

Consider the natural projector ${}^{\mathrm{IX}}P$, given by

$$(a_1,a_2,a_3,a_4,a_5,a_6)=\left(\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right).$$

Theorem 9 *The natural projector* ${}^{\mathrm{IX}}P$ *is decomposable. For any values of parameters* $\mu,\nu\in\mathbf{R}$*, it admits a decomposition*

$$(30)\qquad {}^{\mathrm{IX}}P={}^{\mathrm{I}}P+{}^{\mathrm{VIII}}P,$$

where the natural projectors ${}^{\mathrm{I}}P$ *and* ${}^{\mathrm{VIII}}P$ *are given by*

$$(31)\qquad (b_1,b_2,b_3,b_4,b_5,b_6)=\left(-\mu+\frac{1}{1+n},\nu,\mu,\frac{1}{1+n}-\nu,0,0\right),$$

and

$$(c_1,c_2,c_3,c_4,c_5,c_6)$$
$$=\left(-\frac{1}{2(1+n)}-\mu,\nu+\frac{1}{2(1+n)},\frac{1}{2(1+n)}+\mu,-\nu-\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right).$$

Proof Natural projectors ${}^{\mathrm{I}}P$ and ${}^{\mathrm{VIII}}P$ are given by

$$(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0)$$

and

$$(32)\qquad \left(\mu',\nu',-\mu',-\nu',\frac{1}{2},-\frac{1}{2}\right).$$

Thus, ${}^{\mathrm{I}}P+{}^{\mathrm{VIII}}P$ is given by

$$\left(1-\mu-n(\nu+\kappa)+\mu',\nu+\nu',\mu-\mu',\kappa-\nu',\frac{1}{2},-\frac{1}{2}\right).$$

Decomposition equation (30) provides a system

$$(33)\qquad \begin{gathered} 1-\mu-n(\nu+\kappa)+\mu'=\frac{1}{2(1+n)},\\ \nu+\nu'=\frac{1}{2(1+n)},\quad \mu-\mu'=\frac{1}{2(1+n)},\quad \kappa-\nu'=\frac{1}{2(1+n)} \end{gathered}$$

for the unknowns μ,ν,κ and μ',ν'. This system implies

$$\kappa+\nu=\frac{1}{1+n}.$$

Hence from (67), ${}^{\mathrm{I}}P$ is of the form

$$(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0)$$
$$=\left(1-\mu-\frac{n}{1+n},\nu,\mu,\frac{1}{1+n}-\nu,0,0\right)$$
$$=\left(-\mu+\frac{1}{1+n},\nu,\mu,\frac{1}{1+n}-\nu,0,0\right),$$

proving formula (31). Since the first equation (33) becomes an identity,

$$1-\mu-n(\nu+\kappa)+\mu'=1-\mu-\frac{n}{1+n}+\mu-\frac{1}{2(1+n)}$$
$$=\frac{2+2n-2n-1}{2(1+n)}\equiv\frac{1}{2(1+n)},$$

therefore

$$\nu'=-\nu+\frac{1}{2(1+n)},\quad \mu'=\mu-\frac{1}{2(1+n)}.$$

Then ${}^{\mathrm{VIII}}P$ is, according to (32),

$$\left(\mu',\nu',-\mu',-\nu',\frac{1}{2},-\frac{1}{2}\right)$$
$$=\left(\mu-\frac{1}{2(1+n)},-\nu+\frac{1}{2(1+n)},-\mu+\frac{1}{2(1+n)},\nu-\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right).$$

Results, included in Theorems 5 – 9, characterize all decomposable natural projectors in the tensor space $T_2^1\mathbf{R}^n$, and their decompositions. The following corollary summarizes the classification of *primitive* natural projectors.

Theorem 10 *The following two conditions are equivalent:*
(a) *P is a primitive natural projector.*
(b) *P is equal to one of the natural projectors* ${}^{\mathrm{I}}P,{}^{\mathrm{II}}P,{}^{\mathrm{VI}}P,{}^{\mathrm{X}}P$ *, or to an element of the family* ${}^{\mathrm{III}}P$ *,* (5)*, such that*

$$\kappa+\nu\neq\frac{1}{n-1},\frac{1}{n+1}.$$

Proof This follows from Lemma 5 and Theorems 5–9, classifying decomposable projectors.

6 Partitions of the tensor space $T_2^1\mathbf{R}^n$

By a *partition* of a vector space E we mean a family $\{P_1,P_2,\ldots,P_k\}$ of non-trivial projectors $P_i : E \to E$ such that

$$P_1 + P_2 + \ldots + P_k = \mathrm{Id}_E .$$

$\{P_1,P_2,\ldots,P_k\}$ is said to be *refinable*, if there exists a partition $\{Q_1,Q_2,\ldots,Q_l\}$ of E, such that for every i, $1 \le i \le k$, either $P_i = Q_\alpha$ for some α, $1 \le \alpha \le l$, or there exist α and β such that $1 \le \alpha,\beta \le l$, and $P_i = Q_\alpha + Q_\beta$. Any partition $\{Q_1,Q_2,\ldots,Q_l\}$ with these properties is called a *refinement* of $\{P_1,P_2,\ldots,P_k\}$.

Every projector $P : E \to E$ defines a partition $\{P,Q\}$, where Q is the complementary projector $\mathrm{Id}_E - P$.

A partition of the vector space $T_2^1\mathbf{R}^n$ is said to be *natural*, if its elements are natural projectors. A natural partition is said to be *primitive*, if it has no natural refinement different from the trivial one.

Our objective in this section is to find all primitive natural partitions of the tensor space $T_2^1\mathbf{R}^n$. To this purpose we write Id (the identity of $T_2^1\mathbf{R}^n$) instead of ${}^{\mathrm{IV}}P$, and consider partitions, defined by *complementary* projectors,

(1)
$$\mathrm{Id} = {}^{\mathrm{I}}P + {}^{\mathrm{III}}P, \quad \mathrm{Id} = {}^{\mathrm{V}}P + {}^{\mathrm{VIII}}P, \quad \mathrm{Id} = {}^{\mathrm{VI}}P + {}^{\mathrm{IX}}P,$$
$$\mathrm{Id} = {}^{\mathrm{VII}}P + {}^{\mathrm{X}}P.$$

All other partitions can be constructed by refining of these four partitions.

Theorem 11 (Primitive natural partitions) (a) *The tensor space of* $(1,2)$*-tensors* $T_2^1\mathbf{R}^n$ *admits exactly two primitive natural partitions, namely* $\{{}^{\mathrm{I}}P, {}^{\mathrm{V}}P, {}^{\mathrm{X}}P\}$ *and* $\{{}^{\mathrm{I}}P, {}^{\mathrm{VI}}P, {}^{\mathrm{VIII}}P\}$.

(b) *The natural partition* $\{{}^{\mathrm{I}}P, {}^{\mathrm{V}}P, {}^{\mathrm{X}}P\}$ *is formed by the natural projectors*

(2)
$$\left(-\mu + \frac{1}{1-n}, \nu, \mu, -\nu - \frac{1}{1-n}, 0, 0\right),$$

$$\left(\mu - \frac{1}{2(1-n)}, -\nu - \frac{1}{2(1-n)}, -\mu + \frac{1}{2(1-n)}, \nu + \frac{1}{2(1-n)}, \frac{1}{2}, \frac{1}{2}\right),$$

and

(3) $$\left(-\frac{1}{2(1-n)},\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2(1-n)},\frac{1}{2},-\frac{1}{2}\right),$$

where $\mu,\nu\in\mathbf{R}$ *are arbitrary parameters.*

(c) *The natural partition* $\{{}^{\mathrm{I}}P,{}^{\mathrm{VI}}P,{}^{\mathrm{VIII}}P\}$ *is formed by the natural projectors*

$$\left(-\mu+\frac{1}{1+n},\nu,\mu,-\nu+\frac{1}{1+n},0,0\right)$$

$$\left(-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},\frac{1}{2},\frac{1}{2}\right),$$

and

(4) $$\left(\mu-\frac{1}{2(1+n)},-\nu+\frac{1}{2(1+n)},-\mu+\frac{1}{2(1+n)},\nu-\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right),$$

where $\mu,\nu\in\mathbf{R}$ *are arbitrary parameters.*

Proof 1. Theorems 5 – 9 provide the following complete list of decompositions of natural projectors in two summands

$$
\begin{gathered}
{}^{\mathrm{III}}P={}^{\mathrm{V}}P+{}^{\mathrm{X}}P,\quad {}^{\mathrm{III}}P={}^{\mathrm{VI}}P+{}^{\mathrm{VIII}}P,\quad {}^{\mathrm{V}}P={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P,\\
{}^{\mathrm{VII}}P={}^{\mathrm{I}}P+{}^{\mathrm{V}}P,\quad {}^{\mathrm{VIII}}P={}^{\mathrm{I}}P+{}^{\mathrm{X}}P,\quad {}^{\mathrm{IX}}P={}^{\mathrm{I}}P+{}^{\mathrm{VIII}}P,
\end{gathered}
$$

and characterize together the parameterizations for which the decompositions are valid. Substitutions into (1) lead to the following equations

(5) $$
\begin{aligned}
&\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{V}}P+{}^{\mathrm{X}}P,\quad \mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P+{}^{\mathrm{VIII}}P,\\
&\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P+{}^{\mathrm{VIII}}P,\quad \mathrm{Id}={}^{\mathrm{V}}P+{}^{\mathrm{I}}P+{}^{\mathrm{X}}P,\\
&\mathrm{Id}={}^{\mathrm{VI}}P+{}^{\mathrm{I}}P+{}^{\mathrm{VIII}}P,\\
&\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{V}}P+{}^{\mathrm{X}}P,
\end{aligned}
$$

where the unknowns are the natural projectors, complementary to the natural projectors ${}^{\mathrm{III}}P$, ${}^{\mathrm{III}}P$, ${}^{\mathrm{V}}P$, ${}^{\mathrm{VIII}}P$, ${}^{\mathrm{IX}}P$, ${}^{\mathrm{VII}}P$, respectively. But equations (1) do not restrict parameterizations, thus, equations (5) always admit solutions. However, this list includes only two different equations; this proves that the identity Id admits no more than two partitions,

$$\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{V}}P+{}^{\mathrm{X}}P,\quad \mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P+{}^{\mathrm{VIII}}P.$$

2. We prove assertion (b). Consider the partition

$$\mathrm{Id} = {}^{\mathrm{I}}P + {}^{\mathrm{III}}P.$$

This formula holds for any fixed parameters $\nu,\mu,\kappa \in \mathbf{R}$ of the families ${}^{\mathrm{I}}P$ and ${}^{\mathrm{III}}P$, allowing us to condider ${}^{\mathrm{I}}P$ and ${}^{\mathrm{III}}P$ as complementary projectors. Explicit expressions are

(6) $$(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0)$$

and

$$(-1+\mu+n(\nu+\kappa),-\nu,-\mu,-\kappa,1,0),$$

respectively. But according to Theorem 5, for some values of parameters the natural projector ${}^{\mathrm{III}}P$ is *decomposable*; if

(7) $$\kappa+\nu=\frac{1}{n-1},$$

that is, if ${}^{\mathrm{III}}P$ is equal to

$$(-1+\mu+n(\nu+\kappa),-\nu,-\mu,-\kappa,1,0).$$
$$=\left(\mu+\frac{1}{n-1},-\nu,-\mu,\nu-\frac{1}{n-1},1,0\right),$$

then

(8) $${}^{\mathrm{III}}P = {}^{\mathrm{V}}P + {}^{\mathrm{X}}P,$$

where ${}^{\mathrm{V}}P$ is given by

(9) $$\left(\mu-\frac{1}{2(1-n)},-\nu-\frac{1}{2(1-n)},-\mu+\frac{1}{2(1-n)},\nu+\frac{1}{2(1-n)},\frac{1}{2},\frac{1}{2}\right),$$

and ${}^{\mathrm{X}}P$ is

(10) $$\left(-\frac{1}{2(1-n)},\frac{1}{2(1-n)},-\frac{1}{2(1-n)},\frac{1}{2(1-n)},\frac{1}{2},-\frac{1}{2}\right).$$

The natural projector ${}^{\mathrm{I}}P$ for which ${}^{\mathrm{III}}P = \mathrm{Id} - {}^{\mathrm{I}}P$ has decomposition (8), arises from (6) for parameters satisfying compatibility condition (7); this is the natural projector

(11) $$(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0)$$

$$=\left(1-\mu-\frac{n}{n-1},\nu,\mu,-\nu+\frac{1}{n-1},0,0\right)$$
$$=\left(-\mu-\frac{1}{n-1},\nu,\mu,-\nu+\frac{1}{n-1},0,0\right).$$

Summarizing, we have found parameters in such a way, that the families ${}^{\mathrm{I}}P$, ${}^{\mathrm{III}}P$, and ${}^{\mathrm{V}}P$, ${}^{\mathrm{X}}P$ satisfy equations $\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{III}}P$ and ${}^{\mathrm{III}}P={}^{\mathrm{V}}P+{}^{\mathrm{X}}P$; for these parameters, restricted by condition (7), equation

(12) $\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{V}}P+{}^{\mathrm{X}}P$

holds. The natural projectors on the right-hand side are determined by (9), (10), and (11). This proves statement (b).

3. We prove assertion (c). Consider the partition

$$\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{III}}P,$$

where ${}^{\mathrm{I}}P$ and ${}^{\mathrm{III}}P$ are given by

(13) $(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0),$

and

$$(-1+\mu+n(\nu+\kappa),-\nu,-\mu,-\kappa,1,0).$$

If according to Theorem 5

(14) $\kappa+\nu=\frac{1}{n+1},$

then for some values of parameters the natural projector ${}^{\mathrm{III}}P$ is *decomposable*. In this case ${}^{\mathrm{III}}P$ is equal to

$$\left(\mu-\frac{1}{n+1},-\nu,-\mu,\nu-\frac{1}{n+1},1,0\right),$$

and admits a unique decomposition

$${}^{\mathrm{III}}P={}^{\mathrm{VIII}}P+{}^{\mathrm{VI}}P,$$

where ${}^{\mathrm{VIII}}P$ is given by

(15) $\left(\mu-\frac{1}{2(1+n)},-\nu+\frac{1}{2(1+n)},-\mu+\frac{1}{2(1+n)},\nu-\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right),$

and ${}^{\mathrm{VI}}P$ is

(16) $$\left(-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},-\frac{1}{2(1+n)},\frac{1}{2},\frac{1}{2}\right).$$

The natural projector ${}^{\mathrm{I}}P$ for which ${}^{\mathrm{III}}P=\mathrm{Id}-{}^{\mathrm{I}}P$ has decomposition (4), is given by (13) with parameters satisfying compatibility condition (14); this is the natural projector

(17) $$\begin{aligned}&(1-\mu-n(\nu+\kappa),\nu,\mu,\kappa,0,0)\\&=\left(1-\mu-\frac{n}{n+1},\nu,\mu,\frac{1}{n+1}-\nu,0,0\right)\\&=\left(-\mu+\frac{1}{n+1},\nu,\mu,\frac{1}{n+1}-\nu,0,0\right).\end{aligned}$$

Summarizing, we have found parameters for which that the families ${}^{\mathrm{I}}P$, ${}^{\mathrm{III}}P$, and ${}^{\mathrm{VIII}}P$, ${}^{\mathrm{VI}}P$ satisfy decomposability equations $\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{III}}P$ and ${}^{\mathrm{III}}P={}^{\mathrm{VIII}}P+{}^{\mathrm{VI}}P$; for these parameters,

(18) $$\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{VIII}}P+{}^{\mathrm{VI}}P.$$

Clearly, condition (14) excludes parameter κ, and partition (18) holds for all $\nu,\mu\in\mathbf{R}$, where the natural projectors on the right-hand side are determined by (15), (16), and (17).

4. It remains to show that the natural partitions $\{{}^{\mathrm{I}}P,{}^{\mathrm{V}}P,{}^{\mathrm{X}}P\}$ and $\{{}^{\mathrm{I}}P,{}^{\mathrm{VI}}P,{}^{\mathrm{VIII}}P\}$ are primitive. The natural projectors ${}^{\mathrm{I}}P$, ${}^{\mathrm{X}}P$ and ${}^{\mathrm{VI}}P$ are primitive, but the families ${}^{\mathrm{V}}P$ and ${}^{\mathrm{VIII}}P$ are decomposable. Existence of a natural refinement of the natural partition $\{{}^{\mathrm{I}}P,{}^{\mathrm{V}}P,{}^{\mathrm{X}}P\}$ thus consists in existence of solutions of the system $\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{V}}P+{}^{\mathrm{X}}P$ and ${}^{\mathrm{V}}P={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P$ (Theorem 6). Similarly, existence of a natural refinement of $\{{}^{\mathrm{I}}P,{}^{\mathrm{VI}}P,{}^{\mathrm{VIII}}P\}$ is equivalent to existence of solutions of the system $\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P+{}^{\mathrm{VIII}}P$ and ${}^{\mathrm{VIII}}P={}^{\mathrm{I}}P+{}^{\mathrm{X}}P$ (Theorem 8). But these equations do not have solutions: otherwise $\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{I}}P+{}^{\mathrm{VI}}P+{}^{\mathrm{X}}P$ and $\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P+{}^{\mathrm{I}}P+{}^{\mathrm{X}}P$, so ${}^{\mathrm{I}}P+{}^{\mathrm{I}}P$ would be a natural projector, but such a natural projector does not exist (Theorem 1).

We can also verify this assertion by solving the system

(19) $$\mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{V}}P+{}^{\mathrm{X}}P,\quad {}^{\mathrm{V}}P={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P.$$

Solutions ${}^{\mathrm{I}}P$, ${}^{\mathrm{V}}P$ of the first equation are given by (2), (3),

$$\left(-\mu-\frac{1}{n-1},\nu,\mu,-\nu+\frac{1}{n-1},0,0\right),$$

$$\left(\mu-\frac{1}{2(1-n)},-\nu-\frac{1}{2(1-n)},-\mu+\frac{1}{2(1-n)},\nu+\frac{1}{2(1-n)},\frac{1}{2},\frac{1}{2}\right),$$

where $\mu,\nu\in\mathbf{R}$, while solutions ${}^{\mathrm{I}}P$, ${}^{\mathrm{V}}P$ of the second one by Theorem 6,

$$\left(-\mu'+\frac{1}{2(1+n)},-\nu'+\frac{1}{2(1+n)},\mu'+\frac{1}{2(1+n)},\nu'+\frac{1}{2(1+n)},0,0\right),$$
$$\left(-\mu',-\nu',\mu',\nu',\frac{1}{2},\frac{1}{2}\right),$$

where $\mu',\nu'\in\mathbf{R}$. If, say, ${}^{\mathrm{I}}P$ solves system (19), then the parameters satisfy

$$-\mu-\frac{1}{n-1}=-\mu'+\frac{1}{2(1+n)},$$
$$\nu=-\nu'+\frac{1}{2(1+n)},\quad \mu=\mu'+\frac{1}{2(1+n)},\quad -\nu+\frac{1}{n-1}=\nu'+\frac{1}{2(1+n)},$$

but this system has evidently no solution: condition

$$\mu=\mu'+\frac{1}{2(1+n)}=-\frac{1}{n-1}+\mu'-\frac{1}{2(1+n)},$$

is not satisfied.

Analogously, consider the system

(20) $\quad \mathrm{Id}={}^{\mathrm{I}}P+{}^{\mathrm{VI}}P+{}^{\mathrm{VIII}}P,\quad {}^{\mathrm{VIII}}P={}^{\mathrm{I}}P+{}^{\mathrm{X}}P.$

Solutions ${}^{\mathrm{I}}P$, ${}^{\mathrm{VIII}}P$ of the first equation are given by

$$\left(-\mu+\frac{1}{n+1},\nu,\mu,\frac{1}{n+1}-\nu,0,0\right),$$

and

$$\left(\mu-\frac{1}{2(1+n)},-\nu+\frac{1}{2(1+n)},-\mu+\frac{1}{2(1+n)},\nu-\frac{1}{2(1+n)},\frac{1}{2},-\frac{1}{2}\right),$$

where $\mu,\nu\in\mathbf{R}$, while solutions ${}^{\mathrm{I}}P$, ${}^{\mathrm{VIII}}P$ of the second equation are

$$\left(\mu'+\frac{1}{2(1-n)},\nu'-\frac{1}{2(1-n)},-\mu'+\frac{1}{2(1-n)},-\nu'-\frac{1}{2(1-n)},0,0\right),$$

and

$$\left(\mu',\nu',-\mu',-\nu',\frac{1}{2},-\frac{1}{2}\right),$$

where $\mu',\nu'\in\mathbf{R}$. If ${}^{\mathrm{I}}P$ solves system (20), then

$$-\mu+\frac{1}{n+1}=\mu'+\frac{1}{2(1-n)},$$
$$\nu=\nu'-\frac{1}{2(1-n)},\quad \mu=-\mu'+\frac{1}{2(1-n)},\quad \frac{1}{n+1}-\nu=-\nu'-\frac{1}{2(1-n)}$$

hence

$$\mu=-\mu'+\frac{1}{2(1-n)}=\frac{1}{n+1}-\mu'-\frac{1}{2(1-n)},$$

which leads to contradiction.

Thus, the natural partitions $\{{}^{\mathrm{I}}P,{}^{\mathrm{V}}P,{}^{\mathrm{X}}P\}$, $\{{}^{\mathrm{I}}P,{}^{\mathrm{VI}}P,{}^{\mathrm{VIII}}P\}$ are primitive.

It remains to clarify the role of parameters in the natural partitions $\{{}^{\mathrm{I}}P,{}^{\mathrm{V}}P,{}^{\mathrm{X}}P\}$ and $\{{}^{\mathrm{I}}P,{}^{\mathrm{VI}}P,{}^{\mathrm{VIII}}P\}$. To the end of this section we rewrite them in a more explicit way, by means of equations of natural projectors

$$V^i_{jk}={}^{\mathrm{M}}P^{i\ qr}_{jk\ p}U^p_{qr},$$

where $U\in T^1_2\mathbf{R}^n$, $U=U^p_{qr}$, and ${}^{\mathrm{M}}P={}^{\mathrm{M}}P^{i\ qr}_{jk\ p}$, where

$${}^{\mathrm{M}}P^{i\ qr}_{jk\ p}=a_1\delta^i_j\delta^q_k\delta^r_p+a_2\delta^i_j\delta^q_p\delta^r_k+a_3\delta^i_k\delta^q_p\delta^r_j+a_4\delta^i_k\delta^q_j\delta^r_p$$
$$+a_5\delta^i_p\delta^q_j\delta^r_k+a_6\delta^i_p\delta^q_k\delta^r_j.$$

Lemma 6 (a) *The natural partition* $\{{}^{\mathrm{I}}P,{}^{\mathrm{V}}P,{}^{\mathrm{X}}P\}$ *is formed by the natural projectors, given by*

$${}^{\mathrm{I}}P^{i\ qr}_{jk\ p}U^p_{qr}=-\mu(\delta^i_jU^p_{kp}-\delta^i_kU^p_{pj})+\nu(\delta^i_jU^p_{pk}-\delta^i_kU^p_{jp})$$
$$+\frac{1}{1-n}(\delta^i_jU^p_{kp}-\delta^i_kU^p_{jp}),$$

$${}^{\mathrm{V}}P^{i\ qr}_{jk\ p}U^p_{qr}=\mu(\delta^i_jU^p_{kp}-\delta^i_kU^p_{pj})+\nu(\delta^i_kU^p_{jp}-\delta^i_jU^p_{pk})$$
$$-\frac{1}{2(1-n)}\delta^i_j(U^p_{kp}+U^p_{pk})+\frac{1}{2(1-n)}\delta^i_k(U^p_{jp}+U^p_{pj})+\frac{1}{2}U^i_{jk}+\frac{1}{2}U^i_{kj},$$

and

$$
\begin{aligned}
&{}^{\mathrm{X}}P^{i\ qr}_{jk\ p}U^{p}_{qr}\\
&\quad = -\frac{1}{2(1-n)}(\delta^{i}_{j}(U^{p}_{kp}-U^{p}_{pk})+\delta^{i}_{k}(U^{p}_{pj}-U^{p}_{jp}))+\frac{1}{2}(U^{i}_{jk}-U^{i}_{kj}).
\end{aligned}
$$

(b) *For every* $U=U^{p}_{qr}$, *the tensor* $V=V^{i}_{jk}$, *where* $V^{i}_{jk}={}^{\mathrm{X}}P^{i\ qr}_{jk\ p}U^{p}_{qr}$, *is traceless*.

Proof (a) Lemma 6 is merely a restatement of Theorem 11, (b): The natural partition $\{{}^{\mathrm{I}}P,{}^{\mathrm{V}}P,{}^{\mathrm{X}}P\}$ is formed by the natural projectors

$$
{}^{\mathrm{I}}P^{i\ qr}_{jk\ p}=\left(-\mu+\frac{1}{1-n}\right)\delta^{i}_{j}\delta^{q}_{k}\delta^{r}_{p}+\nu\delta^{i}_{j}\delta^{q}_{p}\delta^{r}_{k}+\mu\delta^{i}_{k}\delta^{q}_{p}\delta^{r}_{j}+\left(-\nu-\frac{1}{1-n}\right)\delta^{i}_{k}\delta^{q}_{j}\delta^{r}_{p},
$$

$$
\begin{aligned}
{}^{\mathrm{V}}P^{i\ qr}_{jk\ p}&=\left(\mu-\frac{1}{2(1-n)}\right)\delta^{i}_{j}\delta^{q}_{k}\delta^{r}_{p}+\left(-\nu-\frac{1}{2(1-n)}\right)\delta^{i}_{j}\delta^{q}_{p}\delta^{r}_{k}\\
&+\left(-\mu+\frac{1}{2(1-n)}\right)\delta^{i}_{k}\delta^{q}_{p}\delta^{r}_{j}+\left(\nu+\frac{1}{2(1-n)}\right)\delta^{i}_{k}\delta^{q}_{j}\delta^{r}_{p}\\
&+\frac{1}{2}\delta^{i}_{p}\delta^{q}_{j}\delta^{r}_{k}+\frac{1}{2}\delta^{i}_{p}\delta^{q}_{k}\delta^{r}_{j},
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{\mathrm{X}}P^{i\ qr}_{jk\ p}&=\frac{1}{2(1-n)}(-\delta^{i}_{j}\delta^{q}_{k}\delta^{r}_{p}+\delta^{i}_{j}\delta^{q}_{p}\delta^{r}_{k}-\delta^{i}_{k}\delta^{q}_{p}\delta^{r}_{j}+\delta^{i}_{k}\delta^{q}_{j}\delta^{r}_{p})\\
&+\frac{1}{2}\delta^{i}_{p}(\delta^{q}_{j}\delta^{r}_{k}-\delta^{q}_{k}\delta^{r}_{j}),
\end{aligned}
$$

where $\mu,\nu\in\mathbf{R}$ are arbitrary parameters. Substituting $U=U^{p}_{qr}$ we can express these projectors as linear combinations of the components of the Kronecker δ-tensor, and linear combinations of the parameters:

$$
\begin{aligned}
{}^{\mathrm{I}}P^{i\ qr}_{jk\ p}U^{p}_{qr}&=\left(-\mu+\frac{1}{1-n}\right)\delta^{i}_{j}U^{p}_{kp}+\nu\delta^{i}_{j}U^{p}_{pk}+\mu\delta^{i}_{k}U^{p}_{pj}+\left(-\nu-\frac{1}{1-n}\right)\delta^{i}_{k}U^{p}_{jp}\\
&=\delta^{i}_{j}\left(\left(-\mu+\frac{1}{1-n}\right)U^{p}_{kp}+\nu U^{p}_{pk}\right)+\delta^{i}_{k}\left(\mu U^{p}_{pj}+\left(-\nu-\frac{1}{1-n}\right)U^{p}_{jp}\right)\\
&=-\mu\delta^{i}_{j}U^{p}_{kp}+\frac{1}{1-n}\delta^{i}_{j}U^{p}_{kp}+\nu\delta^{i}_{j}U^{p}_{pk}+\mu\delta^{i}_{k}U^{p}_{pj}-\nu\delta^{i}_{k}U^{p}_{jp}-\frac{1}{1-n}\delta^{i}_{k}U^{p}_{jp}\\
&=-\mu(\delta^{i}_{j}U^{p}_{kp}-\delta^{i}_{k}U^{p}_{pj})+\nu(\delta^{i}_{j}U^{p}_{pk}-\delta^{i}_{k}U^{p}_{jp})+\frac{1}{1-n}(\delta^{i}_{j}U^{p}_{kp}-\delta^{i}_{k}U^{p}_{jp}),
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{V}}P^{i\ qr}_{jk\ p}U^p_{qr} &= \left(\mu - \frac{1}{2(1-n)}\right)\delta^i_j U^p_{kp} + \left(-\nu - \frac{1}{2(1-n)}\right)\delta^i_j U^p_{pk} \\
&+ \left(-\mu + \frac{1}{2(1-n)}\right)\delta^i_k U^p_{pj} + \left(\nu + \frac{1}{2(1-n)}\right)\delta^i_k U^p_{jp} + \frac{1}{2}U^i_{jk} + \frac{1}{2}U^i_{kj} \\
&= \delta^i_j\left(\left(\mu - \frac{1}{2(1-n)}\right)U^p_{kp} + \left(-\nu - \frac{1}{2(1-n)}\right)U^p_{pk}\right) \\
&+ \delta^i_k\left(\left(-\mu + \frac{1}{2(1-n)}\right)U^p_{pj} + \left(\nu + \frac{1}{2(1-n)}\right)U^p_{jp}\right) \\
&+ \frac{1}{2}(U^i_{jk} + U^i_{kj}) \\
&= \mu(\delta^i_j U^p_{kp} - \delta^i_k U^p_{pj}) + \nu(\delta^i_k U^p_{jp} - \delta^i_j U^p_{pk}) - \frac{1}{2(1-n)}\delta^i_j(U^p_{kp} + U^p_{pk}) \\
&+ \frac{1}{2(1-n)}\delta^i_k(U^p_{jp} + U^p_{pj}) + \frac{1}{2}U^i_{jk} + \frac{1}{2}U^i_{kj},
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{\mathrm{X}}P^{i\ qr}_{jk\ p}U^p_{qr} \\
&= -\frac{1}{2(1-n)}(\delta^i_j(U^p_{kp} - U^p_{pk}) + \delta^i_k(U^p_{pj} - U^p_{jp})) + \frac{1}{2}(U^i_{jk} - U^i_{kj}).
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
&-\frac{1}{2(1-n)}(n(U^p_{kp} - U^p_{pk}) + U^p_{pk} - U^p_{kp}) + \frac{1}{2}(U^s_{sk} - U^s_{ks}) \\
&= \frac{1}{2}(U^p_{kp} - U^p_{pk}) + \frac{1}{2}(U^s_{sk} - U^s_{ks}) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{2(1-n)}(U^p_{jp} - U^p_{pj} + n(U^p_{pj} - U^p_{jp})) + \frac{1}{2}(U^s_{js} - U^s_{sj}) \\
&= \frac{1}{2}(U^p_{pj} - U^p_{jp}) + \frac{1}{2}(U^s_{js} - U^s_{sj}) = 0.
\end{aligned}
$$

Remark 7 If $\mu, \nu = 0$, then

$$
{}^{\mathrm{I}}P^{i\ qr}_{jk\ p}U^p_{qr} = \frac{1}{1-n}(\delta^i_j U^p_{kp} - \delta^i_k U^p_{jp}),
$$

and

$$\begin{aligned}&{}^{\mathrm{V}}P^{i\ qr}_{jk\ p}U^{p}_{qr}\\&=-\frac{1}{2(1-n)}(\delta^i_j(U^p_{kp}+U^p_{pk})-\delta^i_k(U^p_{jp}+U^p_{pj}))+\frac{1}{2}(U^i_{jk}+U^i_{kj}).\end{aligned}$$

Partition $\{{}^{\mathrm{I}}P,{}^{\mathrm{V}}P,{}^{\mathrm{X}}P\}$ reads

$$\begin{aligned}U^i_{jk}&=\frac{1}{1-n}(\delta^i_jU^p_{kp}-\delta^i_kU^p_{jp})\\&-\frac{1}{2(1-n)}\delta^i_j(U^p_{kp}+U^p_{pk})+\frac{1}{2(1-n)}\delta^i_k(U^p_{jp}+U^p_{pj})+\frac{1}{2}(U^i_{jk}+U^i_{kj})\\&-\frac{1}{2(1-n)}(\delta^i_j(U^p_{kp}-U^p_{pk})+\delta^i_k(U^p_{pj}-U^p_{jp}))+\frac{1}{2}(U^i_{jk}-U^i_{kj}),\end{aligned}$$

with ${}^{\mathrm{I}}P$, ${}^{\mathrm{V}}P$, and ${}^{\mathrm{X}}P$ in the first, second, and third lines, respectively. For general μ,ν , the corresponding partition differs from this formula in each line, however, the sum, equal to U^i_{jk} , does not change.

Lemma 7 (a) *The natural partition* $\{{}^{\mathrm{I}}P,{}^{\mathrm{VI}}P,{}^{\mathrm{VIII}}P\}$ *is formed by the natural projectors, given as*

(21)
$$\begin{aligned}{}^{\mathrm{I}}P^{i\ qr}_{jk\ p}U^{p}_{qr}&=\left(-\mu+\frac{1}{1+n}\right)\delta^i_jU^p_{kp}+\nu\delta^i_jU^p_{pk}+\mu\delta^i_kU^p_{pj}\\&+\left(-\nu+\frac{1}{1+n}\right)\delta^i_kU^p_{jp},\end{aligned}$$

(22)
$$\begin{aligned}{}^{\mathrm{VI}}P^{i\ qr}_{jk\ p}U^{p}_{qr}&=-\frac{1}{2(1+n)}(\delta^i_j(U^p_{kp}+U^p_{pk})+\delta^i_k(U^p_{pj}+U^p_{jp}))\\&+\frac{1}{2}(U^i_{jk}+U^i_{kj}),\end{aligned}$$

and

(23)
$$\begin{aligned}{}^{\mathrm{VIII}}P^{i\ qr}_{jk\ p}U^{p}_{qr}&=\left(\mu-\frac{1}{2(1+n)}\right)\delta^i_jU^p_{kp}+\left(-\nu+\frac{1}{2(1+n)}\right)\delta^i_jU^p_{pk}\\&+\left(-\mu+\frac{1}{2(1+n)}\right)\delta^i_kU^p_{pj}+\left(\nu-\frac{1}{2(1+n)}\right)\delta^i_kU^p_{jp}\\&+\frac{1}{2}(U^i_{jk}-U^i_{kj}).\end{aligned}$$

(b) *For every* $U = U^p_{qr}$, *the tensor* $V = V^i_{jk} = {}^{\mathrm{VI}}P^{i\ qr}_{jk\ p}U^p_{qr}$, *is traceless*.

Proof (a) Formulas (21), (22) and (23) follow from Theorem 11, (c):

$$
\begin{aligned}
{}^{\mathrm{I}}P^{i\ qr}_{jk\ p} &= \left(-\mu + \frac{1}{1+n}\right)\delta^i_j\delta^q_k\delta^r_p + \nu\delta^i_j\delta^q_p\delta^r_k + \mu\delta^i_k\delta^q_p\delta^r_j \\
&\quad + \left(-\nu + \frac{1}{1+n}\right)\delta^i_k\delta^q_j\delta^r_p,
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{VI}}P^{i\ qr}_{jk\ p} &= -\frac{1}{2(1+n)}(\delta^i_j(\delta^q_k\delta^r_p + \delta^q_p\delta^r_k) + \delta^i_k(\delta^q_p\delta^r_j + \delta^q_j\delta^r_p)) \\
&\quad + \frac{1}{2}\delta^i_p(\delta^q_j\delta^r_k + \delta^q_k\delta^r_j),
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{\mathrm{VIII}}P^{i\ qr}_{jk\ p} &= \left(\mu - \frac{1}{2(1+n)}\right)\delta^i_j\delta^q_k\delta^r_p + \left(-\nu + \frac{1}{2(1+n)}\right)\delta^i_j\delta^q_p\delta^r_k \\
&\quad + \left(-\mu + \frac{1}{2(1+n)}\right)\delta^i_k\delta^q_p\delta^r_j + \left(\nu - \frac{1}{2(1+n)}\right)\delta^i_k\delta^q_j\delta^r_p \\
&\quad + \frac{1}{2}\delta^i_p(\delta^q_j\delta^r_k - \delta^q_k\delta^r_j).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
{}^{\mathrm{I}}P^{i\ qr}_{jk\ p}U^p_{qr} &= \left(-\mu + \frac{1}{1+n}\right)\delta^i_j U^p_{kp} + \nu\delta^i_j U^p_{pk} + \mu\delta^i_k U^p_{pj} \\
&\quad + \left(-\nu + \frac{1}{1+n}\right)\delta^i_k U^p_{jp},
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{VI}}P^{i\ qr}_{jk\ p}U^p_{qr} &= -\frac{1}{2(1+n)}(\delta^i_j(U^p_{kp} + U^p_{pk}) + \delta^i_k(U^p_{pj} + U^p_{jp})) \\
&\quad + \frac{1}{2}(U^i_{jk} + U^i_{kj}),
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{\mathrm{VIII}}P^{i\ qr}_{jk\ p}U^p_{qr} &= \left(\mu - \frac{1}{2(1+n)}\right)\delta^i_j U^p_{kp} + \left(-\nu + \frac{1}{2(1+n)}\right)\delta^i_j U^p_{pk} \\
&\quad + \left(-\mu + \frac{1}{2(1+n)}\right)\delta^i_k U^p_{pj} + \left(\nu - \frac{1}{2(1+n)}\right)\delta^i_k U^p_{jp} + \frac{1}{2}(U^i_{jk} - U^i_{kj}).
\end{aligned}
$$

(b) Calculating traces, we get

$$-\frac{1}{2(1+n)}(n(U^p_{kp}+U^p_{pk})+U^p_{pk}+U^p_{kp})+\frac{1}{2}(U^s_{sk}+U^s_{ks})$$
$$=-\frac{1}{2(1+n)}((n+1)U^p_{kp}+(n+1)U^p_{pk}+\frac{1}{2}(U^s_{sk}+U^s_{ks})$$
$$=0,$$

and

$$-\frac{1}{2(1+n)}(U^p_{jp}+U^p_{pj}+n(U^p_{pj}+U^p_{jp}))+\frac{1}{2}(U^s_{js}+U^s_{sj})$$
$$=-\frac{1}{2(1+n)}((n+1)U^p_{pj}+(n+1)U^p_{jp})+\frac{1}{2}(U^s_{js}+U^s_{sj})$$
$$=0.$$

Remark 8 If $\mu,\nu=0$, then

$${}^{\mathrm{I}}P^{i\ qr}_{jk\ p}U^p_{qr}=\frac{1}{1+n}(\delta^i_jU^p_{kp}+\delta^i_kU^p_{jp}),$$

and

$${}^{\mathrm{VIII}}P^{i\ qr}_{jk\ p}U^p_{qr}=-\frac{1}{2(1+n)}\delta^i_jU^p_{kp}+\frac{1}{2(1+n)}\delta^i_jU^p_{pk}$$
$$+\frac{1}{2(1+n)}\delta^i_kU^p_{pj}-\frac{1}{2(1+n)}\delta^i_kU^p_{jp}+\frac{1}{2}(U^i_{jk}-U^i_{kj}).$$

Partition $\{{}^{\mathrm{I}}P,{}^{\mathrm{VI}}P,{}^{\mathrm{VIII}}P\}$ reads

$$U^i_{jk}=\frac{1}{1+n}(\delta^i_jU^p_{kp}+\delta^i_kU^p_{jp})$$
$$-\frac{1}{2(1+n)}(\delta^i_j(U^p_{kp}+U^p_{pk})+\delta^i_k(U^p_{pj}+U^p_{jp}))+\frac{1}{2}(U^i_{jk}+U^i_{kj})$$
$$-\frac{1}{2(1+n)}\delta^i_jU^p_{kp}+\frac{1}{2(1+n)}\delta^i_jU^p_{pk}$$
$$+\frac{1}{2(1+n)}\delta^i_kU^p_{pj}-\frac{1}{2(1+n)}\delta^i_kU^p_{jp}+\frac{1}{2}(U^i_{jk}-U^i_{kj})$$

with ${}^{\mathrm{I}}P$, ${}^{\mathrm{VI}}P$, and ${}^{\mathrm{VIII}}P$ in the first, second, and third lines, respectively. For general μ, ν, the corresponding partition differs from this formula in each line by some terms depending on μ *and* ν, however, the sum, equal to U^i_{jk}, does not change.

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