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Variational forces

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Abstract This research-expository paper is devoted to variational modelling of mechanical forces, depending on velocities. First we explain basic standard theory of the structure of the Euler-Lagrange mapping, assigning to a Lagrange function its Euler-Lagrange form, including the integrability problem of the system of Euler-Lagrange equations in which the unknown is the Lagrange function. Formulations and proofs of basic standard theorems on the kernel and image of the Euler-Lagrange mapping as well as explicit construction of Lagrange functions for variational systems are given. Following these theorems we then introduce the canonical decomposition of a general Lagrange function in two terms modelling its kinetic and potential energy components. Then we introduce the concept of a force, compatible with a variational principle – the variational force, and find explicit classification of variational forces.

Keywords Variational equation, Helmholtz conditions, Inverse problem of the calculus of variations, Potential energy, Force

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1 Introduction

In this paper, $\mathbf{R}$ is the field of real numbers and $\mathbf{R}^m$ is the m-dimensional (topological) Euclidean space. Our basic underlying structure is the Cartesian product $I \times U$, where I is an open interval in $\mathbf{R}$ and U is a star-shaped open set in $\mathbf{R}^m$; the canonical coordinates on $I \times U$ are denoted by t, x^i, where $i = 1, 2, \dots, m$. We also consider the first and the second prolongations of $I \times U$, the Euclidean spaces $I \times U \times \mathbf{R}^m$ and $I \times U \times \mathbf{R}^m \times \mathbf{R}^m$, with canonical coordinates $t, x^i, \dot{x}^i$ and $t, x^i, \dot{x}^i, \ddot{x}^i$; sometimes also higher prolongations are used.

Recall that in classical mechanics of particles and fields a *Lagrange function* $\mathscr{L} : I \times U \times \mathbf{R}^m \to \mathbf{R}$ of a mechanical system is usually defined to be the difference of *kinetic energy* $\mathscr{T}$ and *potential energy* $\mathscr{U}$ of a mechanical system,

$$\mathscr{L} = \mathscr{T} - \mathscr{U}. \tag{1}$$

While $\mathscr{T}$ is in a sense a *universal* function of the form

$$\mathscr{T} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j, \tag{2}$$

where g_{ij} are the components of a metric tensor on the configuration space U, potential energy $\mathcal{U}$ provides specific characteristics of mechanical systems; $\mathcal{U}$ does not depend on $\dot{x}^i$ and the first-order form $\phi = \phi_i$, where

$$\phi_i = -\frac{\partial \mathcal{U}}{\partial x^i} \tag{3}$$

is the *force*, associated with $\mathcal{U}$. The *Euler-Lagrange form* of $\mathcal{L}$ is the family $E(\mathcal{L}) = E_i(\mathcal{L})$, whose components are the *Euler-Lagrange expressions*

$$E_i(\mathcal{L}) = -\frac{\partial \mathcal{L}}{\partial x^i} + \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = -\frac{\partial \mathcal{T}}{\partial x^i} + \frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{x}^i} + \frac{\partial \mathcal{U}}{\partial x^i}.$$

The *Euler-Lagrange equations* are second-order differential equations for curves $t \to x^i(t)$ in the set U,

$$-\frac{\partial \mathcal{T}}{\partial x^i} + \frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{x}^i} = -\frac{\partial \mathcal{U}}{\partial x^i}.$$

The choice of the Lagrange functions prescribes basic properties of the corresponding variational principle, and ensures straightforward investigation of *symmetries* of the underlying mechanical systems and their consequences in terms of conservation laws.

Indeed, the simplest basic underlying space for this theory, the domain of definition $I \times U \times \mathbf{R}^m$ of $\mathcal{L}$, or $U \times \mathbf{R}^m$ for time-independent Lagrange functions $\mathcal{L}$, can be replaced in the well-known sense by more general spaces such as *smooth manifolds*, *tangent bundles* and *jet bundles* (cf. the handbook Krupka and Saunders [1]).

In this paper we study *variationally compatible extensions* of the concept of a *force* as defined by the variational principle of the Lagrange mechanics. The aim is to characterize a class of Lagrange functions $\mathcal{L} : I \times U \times \mathbf{R}^m \to \mathbf{R}$ admitting a decomposition $\mathcal{L} = \mathcal{T} - \mathcal{U}$ analogous to (1), in which "kinetic " and "potential" energy terms generalize classical concepts of kinetic and potential energy, and $\mathcal{U}$ (2) admits dependence on velocities $\dot{x}^i$. Then we introduce the concept of a *variational force*, compatible with a variational principle, and find explicit classification of variational forces. Finally, the Newton's equations of classical mechanics are discussed within the framework of Finsler metric fields.

The problem of variational compatibility of forces is not new; its elementary version was considered, probably for the first time, by Novotny [3]. This paper is based on recent research in applications of the inverse problem of the calculus of variations to differential equations (*variational completion*, Voicu, Krupka [4]), and to geometric mechanics (*variational submanifolds*, Krupka, Urban and Volna [2]). Extensive, relatively complete literature on various aspects of the inverse problem can be found Krupka and Saunders [1] and edited volume Zenkov [5].

2 Elementary differential systems

In this section we review integration formulas for some elementary differential systems on Euclidean spaces, needed for the proofs of our assertions in the theory of the inverse problem of the calculus of variations. All functions we consider are defined on a star-shaped neighbourhood U of the origin $0 \in \mathbf{R}^m$, and are supposed to be sufficiently differentiable.

Suppose we have a system of functions $A = A_k$, $1 \le k \le m$, defined on U, and consider a differential equation for an unknown function P

(1) $$A_k = \frac{\partial P}{\partial x^k}.$$

Lemma 1 (a) *Equation* (1) *has a solution P if and only if the functions* A_k *satisfy*

(2) $$\frac{\partial A_k}{\partial x^l} - \frac{\partial A_l}{\partial x^k} = 0.$$

(b) *If condition* (2) *is satisfied, then a solution P is given by*

(3) $$P = x^k \int_0^1 A_k(\tau x^l)\, d\tau.$$

Proof Necessity of condition (2) is obvious. To prove sufficiency, we differentiate P with respect to x^i. We have

$$\begin{aligned}\frac{\partial P}{\partial x^i} &= \int_0^1 A_i(\tau x^l)\, d\tau + x^k \int_0^1 \left(\frac{\partial A_k}{\partial x^i}\right)_{\tau x^l} \tau\, d\tau \\ &= \int_0^1 \frac{d}{d\tau}(A_i(\tau x^l)\tau)\, d\tau = A_i(x^l).\end{aligned}$$

Now suppose we have a system of functions $S = S_{kl}$ defined on U. Consider a system of differential equations for unknown system Q_l

(4) $$\frac{\partial Q_l}{\partial x^k} - \frac{\partial Q_k}{\partial x^l} = S_{kl}.$$

Lemma 2 (a) *Equations* (4) *have a solution* Q_l *if and only if the functions* S_{kl} *satisfy*

(5) $$S_{kl} = -S_{lk}, \quad \frac{\partial S_{ks}}{\partial x^l} + \frac{\partial S_{sl}}{\partial x^k} + \frac{\partial S_{lk}}{\partial x^s} = 0.$$

(b) *If condition* (5) *is satisfied, then every solution* Q_l *is of the form*

$$Q_l = Q_l^0 + \frac{\partial \Phi}{\partial x^l},$$

where

(6) $$Q_l^0 = x^p \int_0^1 S_{pl}(\tau x^i)\tau\, d\tau$$

and $\Phi = \Phi(x^l)$ *is an arbitrary function.*

Proof (a) Necessity of condition (5) is immediate.
To prove sufficiency, consider the function (6). Differentiating, we have

$$\frac{\partial Q_l^0}{\partial x^k} = \int_0^1 S_{kl}(\tau x^i)\tau\, d\tau + x^p \int_0^1 \left(\frac{\partial S_{pl}}{\partial x^k}\right)_{\tau x^i} \tau^2\, d\tau,$$

and

$$\begin{aligned}\frac{\partial Q_l^0}{\partial x^k} - \frac{\partial Q_k^0}{\partial x^l} &= 2\int_0^1 S_{kl}(\tau x^i)\tau\, d\tau + x^p \int_0^1 \left(\frac{\partial S_{pl}}{\partial x^k} - \frac{\partial S_{pk}}{\partial x^l}\right)_{\tau x^l} \tau^2\, d\tau \\ &= 2\int_0^1 S_{kl}(\tau x^i)\tau\, d\tau + x^p \int_0^1 \left(\frac{\partial S_{pl}}{\partial x^k} + \frac{\partial S_{kp}}{\partial x^l} + \frac{\partial S_{lk}}{\partial x^p}\right)_{\tau x^l} \tau^2\, d\tau \\ &\quad - x^p \int_0^1 \left(\frac{\partial S_{lk}}{\partial x^p}\right)_{\tau x^l} \tau^2\, d\tau.\end{aligned}$$

Using (5), this formula can also be expressed in the form

$$\begin{aligned}\frac{\partial Q_l^0}{\partial x^k} - \frac{\partial Q_k^0}{\partial x^l} &= \int_0^1 \left(\left(\frac{\partial S_{kl}}{\partial x^p}\right)_{\tau x^l} x^p \tau^2 + 2 S_{kl}(\tau x^i)\tau\right) d\tau \\ &= \int_0^1 \frac{d}{d\tau}(S_{kl}(\tau x^i)\tau^2) = S_{kl}(x^i),\end{aligned}$$

proving that Q_l^0 solves the system (5).

(b) Any two solutions Q_l and Q_l' of the system (4) satisfy

$$\frac{\partial Q_l}{\partial x^k} - \frac{\partial Q_k}{\partial x^l} = \frac{\partial Q_l'}{\partial x^k} - \frac{\partial Q_k'}{\partial x^l}$$

hence

$$\frac{\partial (Q_l - Q_l')}{\partial x^k} - \frac{\partial (Q_k - Q_k')}{\partial x^l} = 0.$$

Integrating we get

$$Q_l - Q_l' = \frac{\partial \Phi}{\partial x^l}$$

for a function $\Phi = \Phi(x^l)$.

Consider a system of differential equations for an unknown function $h_i = h_i(x^j)$

(7) $$\frac{\partial h_i}{\partial x^j} + \frac{\partial h_j}{\partial x^i} = 0.$$

Lemma 3 *Every solution of the system* (7) *is of the form*

$$h_i = A_i + B_{ij} x^j$$

for some constants $A_i, B_{ij} \in \mathbf{R}$ *such that*

$$B_{ij} = -B_{ji}.$$

Proof Differentiating (7) we get

$$\frac{\partial^2 h_i}{\partial x^j \partial x^k} = -\frac{\partial^2 h_j}{\partial x^i \partial x^k} = \frac{\partial^2 h_k}{\partial x^j \partial x^i} = -\frac{\partial^2 h_i}{\partial x^k \partial x^j} = 0.$$

Thus, solutions h_i of equations (7) must be linear functions of the form $A_i + B_{il} x^l$, where the constants B_{il} are skew-symmetric.

Suppose we have a collection of functions $g_{ij} = g_{ij}(x^k)$, defined on an open star-shaped set *U* in $\mathbf{R}^m$. Consider a system of differential equations

(8) $$g_{ij} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

for an unknown function $f = f(x^k)$; the right-hand side is the *Hessian matrix* of *f*. The following is the *Hessian matrix reconstruction lemma*.

Lemma 4 (a) *Equation* (8) *has a solution f if and only if the functions* g_{ij} *satisfy the following conditions*

(9) $$g_{ij} = g_{ij}, \quad \frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ik}}{\partial x^j}.$$

(b) *If the functions* g_{ij} *satisfy conditions* (9), *then every solution f of equation* (1) *is of the form*

(10) $$f = f_g + A + B_i x^i,$$

where

$$f_g = \frac{1}{2} h_{ij} x^i x^j, \quad h_{ij} = 2\int_0^1 \left(\int_0^1 g_{ij}(\kappa \tau x^p) d\kappa \right) \tau \, d\tau.$$

and A, B_i *are arbitrary constants. The functions* h_{ij} *satisfy*

$$h_{ij} = h_{ji}, \quad \frac{\partial h_{ij}}{\partial x^k} = \frac{\partial h_{ik}}{\partial x^j},$$

Proof 1. Conditions (9) are obviously necessary conditions for existence a solution of equation (8).

2. We show that if conditions (9) hold, then equation (8) has a solution. Integrating the second equation (9),

$$g_{ij} = \frac{\partial h_i}{\partial x^j} \tag{11}$$

for some functions h_i; h_i can be taken as

$$h_i = x^k \int_0^1 g_{ik}(\kappa x^p)\, d\kappa \tag{12}$$

(Lemma 1). Indeed, h_i obviously satisfies (11):

$$\left(\frac{\partial h_i}{\partial x^j}\right)_{(x^p)} = \int_0^1 \left(g_{ij}(\kappa x^p) + \left(\frac{\partial g_{ir}}{\partial x^j}\right)_{(\kappa x^p)} \kappa x^r \right) d\kappa$$
$$= \int_0^1 \frac{d}{d\kappa}(g_{ij}(\kappa x^p)\kappa)\, d\kappa = g_{ij}(x^p).$$

Now we apply condition $g_{ij} = g_{ij}$ (9). We get integrability condition

$$\frac{\partial h_i}{\partial x^j} = \frac{\partial h_j}{\partial x^i}$$

ensuring existence of a function f such that

$$h_i = \frac{\partial f}{\partial x^i}. \tag{13}$$

A solution may be taken as $f = f_g$, where

$$f_g = x^i \int_0^1 h_i(\tau x^p)\, d\tau.$$

Substituting from (12)

$$h_i(\tau x^p) = \tau x^r \int_0^1 g_{ir}(\kappa \tau x^p)\, d\kappa,$$

we get

$$f_g = x^i x^j \int_0^1 \left(\int_0^1 g_{ij}(\kappa \tau x^p)\, d\kappa \right) \tau\, d\tau = \frac{1}{2} h_{ij} x^i x^j, \tag{14}$$

where

$$h_{ij} = 2\int_0^1 \left(\int_0^1 g_{ij}(\kappa \tau x^p)\, d\kappa \right) \tau\, d\tau.$$

Then by constructions (11) and (13), (14), f_g satisfies

$$g_{ij} = \frac{\partial h_i}{\partial x^j} = \frac{\partial^2 f_g}{\partial x^i \partial x^j}.$$

The functions (12) satisfy

$$h_{ij} = h_{ji}, \quad \frac{\partial h_{ij}}{\partial x^k} = 2\int_0^1 \left(\int_0^1 \left(\frac{\partial g_{ij}}{\partial x^k} \right)_{(\kappa\tau x^p)} \kappa\tau \, d\kappa \right) \tau \, d\tau = \frac{\partial h_{ik}}{\partial x^j}.$$

The general solution of equation (18) is of the form $f = f_g + A + B_i x^i$, where A and B_i are arbitrary constants.

Remark 1 Given g_{ij} and setting

$$f_g = x^i x^j \int_0^1 \left(\int_0^1 g_{ij}(\kappa\tau x^p) d\kappa \right) \tau \, d\tau,$$

we can determine the second derivatives $\partial^2 f_g / \partial x^i \partial x^j$ by a straightforward calculation. We have

$$\begin{aligned} \frac{\partial f_g}{\partial x^k} &= 2x^j \int_0^1 \left(\int_0^1 g_{kj}(\kappa\tau x^p) d\kappa \right) \tau \, d\tau \\ &+ x^i x^j \frac{\partial}{\partial x^k} \int_0^1 \left(\int_0^1 g_{ij}(\kappa\tau x^p) d\kappa \right) \tau \, d\tau. \end{aligned} \tag{15}$$

Supposing that conditions (9) are fulfilled and using the chain rule, the second term in this formula can be expressed as

$$\frac{\partial}{\partial x^k} \int_0^1 \left(\int_0^1 g_{ij}(\kappa\tau x^p) d\kappa \right) \tau \, d\tau = \int_0^1 \left(\int_0^1 \left(\frac{\partial g_{ij}}{\partial x^k} \right)_{(\kappa\tau x^p)} \kappa\tau \, d\kappa \right) \tau \, d\tau.$$

On the other hand,

$$\frac{d}{d\tau}(g_{kj}(\kappa\tau x^p)\tau^2) = 2\tau g_{kj}(\kappa\tau x^p) + \tau \left(\frac{\partial g_{ij}}{\partial x^k} x^i \right)_{(\kappa\tau x^p)}.$$

Thus, returning to (15),

$$\frac{\partial f_g}{\partial x^k} = x^j \left(\int_0^1 \left(\int_0^1 \frac{d}{d\tau}(g_{kj}(\kappa\tau x^p)\tau^2) d\tau \right) d\kappa \right) = x^j \int_0^1 g_{kj}(\kappa x^p) d\kappa,$$

and

$$\frac{\partial^2 f_g}{\partial x^k \partial x^l} = \int_0^1 g_{kl}(\kappa x^p) d\kappa + x^j \int_0^1 \left(\frac{\partial g_{kl}}{\partial x^j} \right)_{(\kappa x^p)} \kappa \, d\kappa$$
$$= \int_0^1 \left(g_{kl} + \frac{\partial g_{kl}}{\partial x^j} x^j \right)_{(\kappa x^p)} d\kappa.$$

Since

$$\frac{d}{d\kappa}(g_{kl}(\kappa x^p)\kappa) = \left(g_{kl} + \frac{\partial g_{kj}}{\partial x^l} x^j \right)_{(\kappa x^p)},$$

we finally get

$$\frac{\partial^2 f_g}{\partial x^k \partial x^l} = \int_0^1 \frac{d}{d\kappa}(g_{kl}(\kappa x^p)\kappa) d\kappa = g_{kl}(x^p).$$

3 The Euler-Lagrange mapping

Our main objective in this section will be analysis of the dependence of the to *Euler-Lagrange expressions* $E_i(\mathscr{L}) : I \times U \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$, where $i = 1,2,\ldots,m$, defined by the formula

(1) $$E_i(\mathscr{L}) = -\frac{\partial \mathscr{L}}{\partial x^i} + \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{x}^i},$$

on the Lagrange functions $\mathscr{L} : I \times U \times \mathbf{R}^m \to \mathbf{R}$. The family $E(\mathscr{L}) = E_i(\mathscr{L})$ is called the *Euler-Lagrange form*. We wish to characterize the *kernel* and the *image* of the *Euler-Lagrange mapping* $\mathscr{L} \to E(\mathscr{L})$, The domain of definition of the Euler-Lagrange mapping is the vector space of C^2-functions on $I \times U \times \mathbf{R}^m$ and its image space is the vector space of *m*-tuples of C^2-functions on $I \times U \times \mathbf{R}^m \times \mathbf{R}^m$; this mapping is obviously linear.

A Lagrange function $\mathscr{L}$ is said to be *(variationally) trivial*, if

(2) $$E_i(\mathscr{L}) = 0$$

for all $i = 1,2,\ldots,m$.

Theorem 1 *A Lagrange function* $\mathscr{L} = \mathscr{L}(t, x^i, \dot{x}^i)$ *is trivial if and only if there exists a function* $f = f(t, x^i)$ *such that*

(3) $$\mathscr{L} = \frac{df}{dt}.$$

Proof 1. By a straightforward calculation

(4) $$E_i\left(\frac{df}{dt}\right) = -\frac{\partial}{\partial x^i}\frac{df}{dt} + \frac{d}{dt}\frac{\partial}{\partial \dot{x}^i}\frac{df}{dt} = -\frac{\partial}{\partial x^i}\frac{df}{dt} + \frac{d}{dt}\frac{\partial f}{\partial x^i} = 0.$$

2. Conversely, suppose that $E_i(\mathscr{L}) = 0$. Then since

(5) $$E_i(\mathscr{L}) = -\frac{\partial \mathscr{L}}{\partial x^i} + \frac{\partial^2 \mathscr{L}}{\partial t\,\partial \dot{x}^i} + \frac{\partial^2 \mathscr{L}}{\partial x^l\,\partial \dot{x}^i} x^l + \frac{\partial^2 \mathscr{L}}{\partial x^l\,\partial \dot{x}^i}\ddot{x}^l$$

and this expression is linear in $\ddot{x}^j$, hence

(6) $$\mathscr{L} = A + B_l \dot{x}^l$$

for some functions $A = A(t,x^l)$ and $B_k = B_k(t,x^l)$, and have

(7) $$-\frac{\partial A}{\partial x^i} + \frac{\partial B_i}{\partial t} = 0, \quad \frac{\partial B_l}{\partial x^i} - \frac{\partial B_i}{\partial \dot{x}^i} = 0.$$

Integrating we get for some $f_0 = f_0(t,x^j)$

(8) $$B_i = \frac{\partial f_0}{\partial x^i}, \quad \frac{\partial}{\partial x^i}\left(-A + \frac{\partial f_0}{\partial t}\right) = 0.$$

Further integration yields

(9) $$-A + \frac{\partial f_0}{\partial t} = g = \frac{dg_0}{dt}$$

for some $g = g(t)$ and some primitive g_0 of g. Then, however,

(10) $$\begin{aligned}\mathscr{L} = A + B_l \dot{x}^l &= \frac{\partial f_0}{\partial t} - \frac{dg_0}{dt} + \frac{\partial f_0}{\partial x^l}\dot{x}^l \\ &= \frac{\partial (f_0 - g_0)}{\partial t} + \frac{\partial (f_0 - g_0)}{\partial x^l}\dot{x}^l.\end{aligned}$$

Setting $f = f_0 - g_0$ we get (3).

Theorem 1 characterizes the *kernel* of the Euler-Lagrange mapping $\mathscr{L} \to E(\mathscr{L})$.

We shall now consider arbitrary systems $\varepsilon = \varepsilon_i$ of sufficiently differentiable functions $\varepsilon_i : I \times U \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$, where $i = 1,2,\dots,m$; in agreement with the calculus of variations, differential geometry and mechanics, the systems ε are covector fields, and are called *source forms*.

A source form $\varepsilon = \varepsilon_i$ is said to be *variational*, if there exists a function $\mathscr{L} : I \times U \times \mathbf{R}^m \to \mathbf{R}$ such that

(11) $$\varepsilon_i = -\frac{\partial \mathscr{L}}{\partial x^i} + \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}^i}.$$

If $\mathscr{L}$ exists, it is called a *Lagrange function for* ε. Clearly, *variationality* of a source form ε means that ε belongs to the *image* of the Euler-Lagrange mapping

$\mathscr{L} \to E(\mathscr{L})$ in the set of source forms or, in a different terminology, *integrability* of equation (11) with respect to the unknown function $\mathscr{L}$.

To find solutions of the system of partial differential equations (11) an *integrability condition* must be determined. We prove its necessity and sufficiency parts separately.

Theorem 2 *If a source form* $\varepsilon = \varepsilon_i$ *is variational, then*

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} - \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} = 0, \tag{12}$$

$$\frac{\partial \varepsilon_i}{\partial \dot{x}^j} + \frac{\partial \varepsilon_j}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} + \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} \right) = 0, \tag{13}$$

and

$$\frac{\partial \varepsilon_i}{\partial x^j} - \frac{\partial \varepsilon_j}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i} \right) = 0. \tag{14}$$

Proof 1. We show that if ε_i are expressible in the form (11), then conditions (12), (13) and (14) hold. Using explicit expressions

$$\varepsilon_i = \frac{\partial \mathscr{L}}{\partial x^i} - \frac{\partial^2 \mathscr{L}}{\partial t\, \partial \dot{x}^i} - \frac{\partial^2 \mathscr{L}}{\partial x^k\, \partial \dot{x}^i}\dot{x}^k - \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^k\, \partial \dot{x}^i}\ddot{x}^k$$

we get

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} = -\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^l\, \partial \dot{x}^i},$$

$$\begin{aligned}\frac{\partial \varepsilon_i}{\partial \dot{x}^l} = \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^l\, \partial x^i} - \frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l\, \partial t\, \partial \dot{x}^i} - \frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l\, \partial x^k\, \partial \dot{x}^i}\dot{x}^k \\ - \frac{\partial^2 \mathscr{L}}{\partial x^l\, \partial \dot{x}^i} - \frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l\, \partial \dot{x}^k\, \partial \dot{x}^i}\ddot{x}^k,\end{aligned}$$

$$\frac{\partial \varepsilon_i}{\partial x^l} = \frac{\partial^2 \mathscr{L}}{\partial x^l\, \partial x^i} - \frac{\partial^3 \mathscr{L}}{\partial x^l\, \partial t\, \partial \dot{x}^i} - \frac{\partial^3 \mathscr{L}}{\partial x^l\, \partial x^k\, \partial \dot{x}^i}\dot{x}^k - \frac{\partial^3 \mathscr{L}}{\partial x^l\, \partial \dot{x}^k\, \partial \dot{x}^i}\ddot{x}^k$$

and

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} - \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} = -\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^j\, \partial \dot{x}^i} + \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^j\, \partial \dot{x}^i} = 0.$$

Then by a direct calculation

$$\frac{\partial \varepsilon_i}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} \right) = 0$$

and

$$\frac{\partial \varepsilon_i}{\partial x^l} - \frac{\partial \varepsilon_l}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i} \right) = 0.$$

To prove the converse statment we proceed in several steps. First note that equations (12), (13) and (14) can be expressed as an equivalent system

(15)
$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} - \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} = 0,$$

(16)
$$\frac{\partial}{\partial \ddot{x}^k}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} + \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} \right) = 0,$$

(17)
$$\frac{\partial \varepsilon_i}{\partial \dot{x}^j} + \frac{\partial \varepsilon_j}{\partial \dot{x}^i} - \frac{\partial}{\partial t}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} + \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} \right) - \frac{\partial}{\partial x^l}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} + \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} \right)\dot{x}^l - \frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} + \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} \right)\ddot{x}^l = 0,$$

(18)
$$\frac{\partial}{\partial \ddot{x}^k}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i} \right) = 0,$$

(19)
$$\frac{\partial \varepsilon_i}{\partial x^j} - \frac{\partial \varepsilon_j}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i} \right) - \frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i} \right)\dot{x}^l - \frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i} \right)\ddot{x}^l = 0.$$

Lemma 5 *Let* $\varepsilon = \varepsilon_i$ *be a source form satisfying conditions* (15) – (19). *Then*

(20)
$$\varepsilon_i = P_i + Q_{il}\ddot{x}^l$$

for some functions $P_i = P_i(t, x^k, \dot{x}^k)$ *and* $Q_{ij} = Q_{ij}(t, x^k, \dot{x}^k)$ *such that*

(21)
$$Q_{ij} = Q_{ji},$$

(22)
$$\frac{\partial Q_{ik}}{\partial \dot{x}^j} - \frac{\partial Q_{jk}}{\partial \dot{x}^i} = 0,$$

and

(23) $$\frac{1}{2}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right)-\frac{\partial Q_{ij}}{\partial t}-\frac{\partial Q_{ij}}{\partial x^l}\dot{x}^l=0,$$

(24) $$\frac{\partial P_i}{\partial x^j}-\frac{\partial P_j}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial P_i}{\partial \dot{x}^j}-\frac{\partial P_j}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}-\frac{\partial P_j}{\partial \dot{x}^i}\right)\dot{x}^k=0.$$

Proof Condition (15) and (16) imply

$$\frac{\partial^2 \varepsilon_i}{\partial \ddot{x}^j\,\partial \ddot{x}^k}=0,$$

thus, ε_i must be of the form (20) with coefficients Q_{ij} satisfying (21), (22).
Substitution from (20) into (17) yields

$$\begin{aligned}&\frac{\partial \varepsilon_i}{\partial \dot{x}^j}+\frac{\partial \varepsilon_j}{\partial \dot{x}^i}-\frac{\partial}{\partial t}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j}+\frac{\partial \varepsilon_j}{\partial \ddot{x}^i}\right)\\&-\frac{\partial}{\partial x^l}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j}+\frac{\partial \varepsilon_j}{\partial \ddot{x}^i}\right)\dot{x}^l-\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j}+\frac{\partial \varepsilon_j}{\partial \ddot{x}^i}\right)\ddot{x}^l\\&=\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}+\left(\frac{\partial Q_{ik}}{\partial \dot{x}^j}+\frac{\partial Q_{jk}}{\partial \dot{x}^i}\right)\ddot{x}^k\\&-\frac{\partial (Q_{ij}+Q_{ji})}{\partial t}-\frac{\partial (Q_{ij}+Q_{ji})}{\partial x^l}\dot{x}^l-\frac{\partial (Q_{ij}+Q_{ji})}{\partial \dot{x}^l}\ddot{x}^l,\end{aligned}$$

and the vanishing of this expression implies

$$\frac{\partial Q_{ik}}{\partial \dot{x}^j}+\frac{\partial Q_{jk}}{\partial \dot{x}^i}-\frac{\partial (Q_{ij}+Q_{ij})}{\partial \dot{x}^k}=0$$

and

$$\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}-\frac{\partial (Q_{ij}+Q_{ij})}{\partial t}-\frac{\partial (Q_{ij}+Q_{ij})}{\partial x^l}\dot{x}^l=0.$$

proving (23).
Substitution from (20) into (18) proves (22) .
To substitute from (20) into (19) we use expressions

$$\begin{aligned}&\frac{\partial \varepsilon_i}{\partial x^j}-\frac{\partial \varepsilon_j}{\partial x^i}=\frac{\partial P_i}{\partial x^j}-\frac{\partial P_j}{\partial x^i}+\left(\frac{\partial Q_{il}}{\partial x^j}-\frac{\partial Q_{jl}}{\partial x^i}\right)\ddot{x}^l,\\&\frac{\partial \varepsilon_i}{\partial \dot{x}^j}-\frac{\partial \varepsilon_j}{\partial \dot{x}^i}=\frac{\partial P_i}{\partial \dot{x}^j}-\frac{\partial P_j}{\partial \dot{x}^i}+\left(\frac{\partial Q_{il}}{\partial \dot{x}^j}-\frac{\partial Q_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l\end{aligned}$$

leading to the expression

$$\frac{\partial \varepsilon_i}{\partial x^j} - \frac{\partial \varepsilon_j}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i}\right)$$
$$-\frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i}\right)\dot{x}^l - \frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i}\right)\ddot{x}^l$$
$$= \frac{\partial P_i}{\partial x^j} - \frac{\partial P_j}{\partial x^i} + \left(\frac{\partial Q_{il}}{\partial x^j} - \frac{\partial Q_{jl}}{\partial x^i}\right)\ddot{x}^l$$
$$-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial P_i}{\partial \dot{x}^j} - \frac{\partial P_j}{\partial \dot{x}^i} + \left(\frac{\partial Q_{il}}{\partial \dot{x}^j} - \frac{\partial Q_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l\right)$$
$$-\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial P_i}{\partial \dot{x}^j} - \frac{\partial P_j}{\partial \dot{x}^i} + \left(\frac{\partial Q_{il}}{\partial \dot{x}^j} - \frac{\partial Q_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l\right)\dot{x}^k$$
$$-\frac{1}{2}\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j} - \frac{\partial P_j}{\partial \dot{x}^i} + \left(\frac{\partial Q_{il}}{\partial \dot{x}^j} - \frac{\partial Q_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l\right)\ddot{x}^k.$$

Since the coefficients at $\ddot{x}^l$ and $\ddot{x}^k\ddot{x}^l$ shoud vanish separately, we have

$$\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial Q_{il}}{\partial \dot{x}^j} - \frac{\partial Q_{jl}}{\partial \dot{x}^i}\right) + \frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial Q_{ik}}{\partial \dot{x}^j} - \frac{\partial Q_{jk}}{\partial \dot{x}^i}\right) = 0,$$

$$\frac{\partial Q_{il}}{\partial x^j} - \frac{\partial Q_{jl}}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial Q_{il}}{\partial \dot{x}^j} - \frac{\partial Q_{jl}}{\partial \dot{x}^i}\right) - \frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial Q_{il}}{\partial \dot{x}^j} - \frac{\partial Q_{jl}}{\partial \dot{x}^i}\right)\dot{x}^k$$
$$-\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial P_i}{\partial \dot{x}^j} - \frac{\partial P_j}{\partial \dot{x}^i}\right) = 0,$$

and

$$\frac{\partial P_i}{\partial x^j} - \frac{\partial P_j}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial P_i}{\partial \dot{x}^j} - \frac{\partial P_j}{\partial \dot{x}^i}\right) - \frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial P_i}{\partial \dot{x}^j} - \frac{\partial P_j}{\partial \dot{x}^i}\right)\dot{x}^k = 0$$

proving (24).

The following is merely a restatement of Lemma 5.

Lemma 6 *Let* $\varepsilon = \varepsilon_i$ *be a source form satisfying conditions* (15) – (19). *Then there exist some functions* $P_i = P_i(t, x^k, \dot{x}^k)$ *and* $f = f(t, x^k, \dot{x}^k)$ *such that*

$$\varepsilon_i = P_i + \frac{\partial^2 f}{\partial \dot{x}^i \partial \dot{x}^l}\ddot{x}^l,$$

where

$$\frac{1}{2}\left(\frac{\partial P_i}{\partial \dot{x}^j} + \frac{\partial P_j}{\partial \dot{x}^i}\right) - \frac{\partial^3 f}{\partial t\, \partial \dot{x}^i\, \partial \dot{x}^j} - \frac{\partial^3 f}{\partial x^l\, \partial \dot{x}^i\, \partial \dot{x}^j}\dot{x}^l = 0 \qquad (25)$$

and

$$\frac{\partial P_i}{\partial x^j} - \frac{\partial P_j}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial P_i}{\partial \dot{x}^j} - \frac{\partial P_j}{\partial \dot{x}^i}\right) - \frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial P_i}{\partial \dot{x}^j} - \frac{\partial P_j}{\partial \dot{x}^i}\right)\dot{x}^k = 0. \tag{26}$$

Proof Only formula (25) needs proof. According to Lemma 4, conditions (21), (22) are equivalent to the existence of a function $f = f(t, x^k, \dot{x}^k)$ such that

$$Q_{ij} = \frac{\partial^2 f}{\partial \dot{x}^j \partial \dot{x}^i}. \tag{27}$$

Replacing Q_{ij} in (23) by (27) we get (25).

Now we study equation (25).

Lemma 7 *The following two conditions are equivalent:*
(a) P_i *and f satisfy*

$$\frac{1}{2}\left(\frac{\partial P_i}{\partial \dot{x}^j} + \frac{\partial P_j}{\partial \dot{x}^i}\right) - \frac{\partial^3 f}{\partial t\, \partial \dot{x}^i\, \partial \dot{x}^j} - \frac{\partial^3 f}{\partial x^l\, \partial \dot{x}^i\, \partial \dot{x}^j}\dot{x}^l = 0. \tag{28}$$

(b) *There exist unique functions* $A_{jk} = A_{jk}(t, x^i)$ *and* $R_j = R_j(t, x^i)$ *such that*

$$P_j = R_j + A_{jl}\dot{x}^l - \frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial t\, \partial \dot{x}^j} + \frac{\partial^2 f}{\partial x^l\, \partial \dot{x}^j}\dot{x}^l \tag{29}$$

and

$$A_{jk} = -A_{kj}.$$

The functions $A_{jk} = A_{jk}(t, x^i)$ *and* $R_j = R_j(t, x^i)$ *are determined by*

$$A_{jk} = \frac{1}{2}\left(\frac{\partial P_j}{\partial \dot{x}^k} - \frac{\partial P_k}{\partial \dot{x}^j}\right) - \frac{\partial^2 f}{\partial x^k\, \partial \dot{x}^j} + \frac{\partial^2 f}{\partial x^j\, \partial \dot{x}^k},$$

and

$$R_j = P_j - \left(\frac{1}{2}\left(\frac{\partial P_j}{\partial \dot{x}^k} - \frac{\partial P_k}{\partial \dot{x}^j}\right) - \frac{\partial^2 f}{\partial x^k\, \partial \dot{x}^j} + \frac{\partial^2 f}{\partial x^j\, \partial \dot{x}^k}\right)\dot{x}^l + \frac{\partial f}{\partial x^j} - \frac{\partial^2 f}{\partial t\, \partial \dot{x}^j}.$$

Proof 1. First we show that (28) implies

$$\frac{\partial P_j}{\partial \dot{x}^k} = A_{jk} + \frac{\partial^2 f}{\partial x^k\, \partial \dot{x}^j} - \frac{\partial^2 f}{\partial x^j\, \partial \dot{x}^k} + \frac{\partial^3 f}{\partial t\, \partial \dot{x}^j\, \partial \dot{x}^k} + \frac{\partial^3 f}{\partial x^l\, \partial \dot{x}^j\, \partial \dot{x}^k}\dot{x}^l \tag{30}$$

for some $A_{jk} = A_{jk}(t, x^i)$ such that

(31) $$A_{jk} = -A_{kj}.$$

Differentiating (28),

(32) $$\frac{1}{2}\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right) - \frac{\partial^4 f}{\partial t\,\partial \dot{x}^i\,\partial \dot{x}^j\,\partial \dot{x}^k} - \frac{\partial^4 f}{\partial x^l\,\partial \dot{x}^i\,\partial \dot{x}^j\,\partial \dot{x}^k}\dot{x}^l - \frac{\partial^3 f}{\partial x^k\,\partial \dot{x}^i\,\partial \dot{x}^j} = 0.$$

Cycling the indices i, j and k

(33) $$\begin{aligned}&\frac{\partial^4 f}{\partial t\,\partial \dot{x}^i\,\partial \dot{x}^j\,\partial \dot{x}^k} + \frac{\partial^4 f}{\partial x^l\,\partial \dot{x}^i\,\partial \dot{x}^j\,\partial \dot{x}^k}\dot{x}^l \\ &= \frac{1}{6}\left(\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right)+\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial P_k}{\partial \dot{x}^i}+\frac{\partial P_i}{\partial \dot{x}^k}\right)+\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial P_j}{\partial \dot{x}^k}+\frac{\partial P_k}{\partial \dot{x}^j}\right)\right) \\ &\quad -\frac{1}{3}\left(\frac{\partial^3 f}{\partial x^k\,\partial \dot{x}^i\,\partial \dot{x}^j}+\frac{\partial^3 f}{\partial x^j\,\partial \dot{x}^k\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial x^i\,\partial \dot{x}^j\,\partial \dot{x}^k}\right)\end{aligned}$$

we can eliminate expression (33) from (32). Thus

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right) \\ &-\frac{1}{6}\left(\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right)+\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial P_k}{\partial \dot{x}^i}+\frac{\partial P_i}{\partial \dot{x}^k}\right)+\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial P_j}{\partial \dot{x}^k}+\frac{\partial P_k}{\partial \dot{x}^j}\right)\right) \\ &+\frac{1}{3}\left(\frac{\partial^3 f}{\partial x^k\,\partial \dot{x}^i\,\partial \dot{x}^j}+\frac{\partial^3 f}{\partial x^j\,\partial \dot{x}^k\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial x^i\,\partial \dot{x}^j\,\partial \dot{x}^k}\right)-\frac{\partial^3 f}{\partial x^k\,\partial \dot{x}^i\,\partial \dot{x}^j} \\ &= 0,\end{aligned}$$

that is

(34) $$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right) \\ &-\frac{1}{6}\left(\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right)+\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial P_k}{\partial \dot{x}^i}+\frac{\partial P_i}{\partial \dot{x}^k}\right)+\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial P_j}{\partial \dot{x}^k}+\frac{\partial P_k}{\partial \dot{x}^j}\right)\right) \\ &=\frac{\partial^3 f}{\partial x^k\,\partial \dot{x}^i\,\partial \dot{x}^j}-\frac{1}{3}\left(\frac{\partial^3 f}{\partial x^k\,\partial \dot{x}^i\,\partial \dot{x}^j}+\frac{\partial^3 f}{\partial x^j\,\partial \dot{x}^k\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial x^i\,\partial \dot{x}^j\,\partial \dot{x}^k}\right).\end{aligned}$$

But the left-hand side and the right-hand side give

$$\frac{3}{6}\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right)-\frac{1}{6}\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^i}\right)-\frac{1}{6}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial P_k}{\partial \dot{x}^i}+\frac{\partial P_i}{\partial \dot{x}^k}\right)$$
$$-\frac{1}{6}\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial P_j}{\partial \dot{x}^k}+\frac{\partial P_k}{\partial \dot{x}^j}\right)$$
$$=\frac{1}{6}\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial P_j}{\partial \dot{x}^k}-\frac{\partial P_k}{\partial \dot{x}^j}\right)+\frac{1}{6}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial P_i}{\partial \dot{x}^k}-\frac{\partial P_k}{\partial \dot{x}^i}\right)$$

and

$$\frac{3}{3}\frac{\partial^3 f}{\partial x^k\,\partial \dot{x}^i\,\partial \dot{x}^j}-\frac{1}{3}\left(\frac{\partial^3 f}{\partial x^k\,\partial \dot{x}^i\,\partial \dot{x}^j}+\frac{\partial^3 f}{\partial x^j\,\partial \dot{x}^k\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial x^i\,\partial \dot{x}^j\,\partial \dot{x}^k}\right)$$
$$=\frac{1}{3}\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k}\right)+\frac{1}{3}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^i}-\frac{\partial^2 f}{\partial x^i\,\partial \dot{x}^k}\right)$$

so formula (34) becomes

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial P_j}{\partial \dot{x}^k}-\frac{\partial P_k}{\partial \dot{x}^j}\right)+\frac{1}{2}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial P_i}{\partial \dot{x}^k}-\frac{\partial P_k}{\partial \dot{x}^i}\right)\\ &=\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k}\right)+\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^i}-\frac{\partial^2 f}{\partial x^i\,\partial \dot{x}^k}\right).\end{aligned}\tag{35}$$

Thus,

$$\begin{aligned}&\frac{\partial}{\partial \dot{x}^i}\left(\frac{1}{2}\left(\frac{\partial P_j}{\partial \dot{x}^k}-\frac{\partial P_k}{\partial \dot{x}^j}\right)-\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}+\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k}\right)\\ &+\frac{\partial}{\partial \dot{x}^j}\left(\frac{1}{2}\left(\frac{\partial P_i}{\partial \dot{x}^k}-\frac{\partial P_k}{\partial \dot{x}^i}\right)-\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^i}+\frac{\partial^2 f}{\partial x^i\,\partial \dot{x}^k}\right)=0.\end{aligned}\tag{36}$$

For every fixed k we get a system of differential equations considered in Lemma 3. Since by hypothesis the system (28) has a solution, integrability condition for equations (36) yields

$$\frac{1}{2}\left(\frac{\partial P_j}{\partial \dot{x}^k}-\frac{\partial P_k}{\partial \dot{x}^j}\right)-\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}+\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k}=A_{jk}+B_{jkl}\dot{x}^l\tag{37}$$

for each k and for some $A_{jk}=A_{jk}(t,x^i)$ and $B_{jlk}=B_{jlk}(t,x^i)$ such that

$$B_{jkl}=-B_{lkj}\tag{38}$$

(cf. Lemma 3). Formula (37) also implies that the coefficients can be chosen in a unique way such that

$$A_{jk}=-A_{kj},\quad B_{jkl}=-B_{kjl}.\tag{39}$$

Formula (35) is a consequence of (28) obtained by derivations and subsequent integration. Writing (28) and (37) together as

$$\frac{1}{2}\left(\frac{\partial P_j}{\partial \dot{x}^k}+\frac{\partial P_k}{\partial \dot{x}^j}\right)=\frac{\partial^3 f}{\partial t\,\partial \dot{x}^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial x^l\,\partial \dot{x}^j\,\partial \dot{x}^k}\dot{x}^l,$$

$$\frac{1}{2}\left(\frac{\partial P_j}{\partial \dot{x}^k}-\frac{\partial P_k}{\partial \dot{x}^j}\right)=A_{jk}+B_{jkl}\dot{x}^l+\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k},$$

we get

(40)
$$\frac{\partial P_j}{\partial \dot{x}^k}=A_{jk}+B_{jkl}\dot{x}^l +\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial t\,\partial \dot{x}^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial x^l\,\partial \dot{x}^j\,\partial \dot{x}^k}\dot{x}^l.$$

To determine integrability condition for this equation from commutativity of the second derivatives $\partial^2 P_j/\partial \dot{x}^i\,\partial \dot{x}^k$ we calculate

$$\begin{aligned}&\frac{\partial}{\partial \dot{x}^i}\left(A_{jk}+B_{jkl}\dot{x}^l+\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial t\,\partial \dot{x}^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial x^l\,\partial \dot{x}^j\,\partial \dot{x}^k}\dot{x}^l\right)\\&=B_{jki}+\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial t\,\partial \dot{x}^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial x^l\,\partial \dot{x}^j\,\partial \dot{x}^k}\dot{x}^l\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial \dot{x}^k}\left(A_{ji}+B_{jil}\dot{x}^l+\frac{\partial^2 f}{\partial x^i\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial t\,\partial \dot{x}^j\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial x^l\,\partial \dot{x}^j\,\partial \dot{x}^i}\dot{x}^l\right)\\&=B_{jik}+\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial^2 f}{\partial x^i\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial t\,\partial \dot{x}^j\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial x^l\,\partial \dot{x}^j\,\partial \dot{x}^i}\dot{x}^l\right).\end{aligned}$$

The difference of these two expressions must be equal to 0,

$$\begin{aligned}&B_{jki}+\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial^2 f}{\partial x^k\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial t\,\partial \dot{x}^j\,\partial \dot{x}^k}+\frac{\partial^3 f}{\partial x^l\,\partial \dot{x}^j\,\partial \dot{x}^k}\dot{x}^l\right)\\&-B_{jik}-\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial^2 f}{\partial x^i\,\partial \dot{x}^j}-\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial t\,\partial \dot{x}^j\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial x^l\,\partial \dot{x}^j\,\partial \dot{x}^i}\dot{x}^l\right)\\&=B_{jki}-B_{jik}=0.\end{aligned}$$

Taking into accoung this condition together with (38) and (39) we get the following index symmetries $B_{jkl}=-B_{lkj}$, $B_{jkl}=-B_{kjl}$ and $B_{jki}=B_{jik}$. Then, however,

(41)
$$B_{ijk}=-B_{kji}=B_{jki}=-B_{ikj}=-B_{ijk}=0.$$

Summarizing, formulas (40) and (42) prove (30) and (31).

2. Suppose that P_i and f satisfy condition (a). Then also equations (30) and (49) are satisfied. But formula (30) admits integration in quadratures. Writing $\partial P_j / \partial \dot{x}^k$ as

$$\begin{aligned}\frac{\partial P_j}{\partial \dot{x}^k} &= A_{jk} + \frac{\partial^2 f}{\partial x^k \partial \dot{x}^j} - \frac{\partial^2 f}{\partial x^j \partial \dot{x}^k} + \frac{\partial^3 f}{\partial t \partial \dot{x}^j \partial \dot{x}^k} + \frac{\partial^3 f}{\partial x^l \partial \dot{x}^j \partial \dot{x}^k} \dot{x}^l \\ &= A_{jk} + \frac{\partial^2 f}{\partial x^k \partial \dot{x}^j} - \frac{\partial^2 f}{\partial x^j \partial \dot{x}^k} + \frac{\partial^3 f}{\partial t \partial \dot{x}^j \partial \dot{x}^k} + \frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial^2 f}{\partial x^l \partial \dot{x}^j} \dot{x}^l \right) - \frac{\partial^2 f}{\partial x^k \partial \dot{x}^j} \\ &= \frac{\partial A_{jl} \dot{x}^l}{\partial \dot{x}^k} - \frac{\partial^2 f}{\partial x^j \partial \dot{x}^k} + \frac{\partial^3 f}{\partial t \partial \dot{x}^j \partial \dot{x}^k} + \frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial^2 f}{\partial x^l \partial \dot{x}^j} \dot{x}^l \right),\end{aligned}$$

we have

$$\frac{\partial}{\partial \dot{x}^k}\left(P_j - A_{jl}\dot{x}^l + \frac{\partial f}{\partial x^j} - \frac{\partial^2 f}{\partial t \partial \dot{x}^j} - \frac{\partial^2 f}{\partial x^l \partial \dot{x}^j} \dot{x}^l \right) = 0$$

hence

$$P_j - A_{jl}\dot{x}^l + \frac{\partial f}{\partial x^j} - \frac{\partial^2 f}{\partial t \partial \dot{x}^j} - \frac{\partial^2 f}{\partial x^l \partial \dot{x}^j} \dot{x}^l = R_j$$

for some functions $R_j = R_j(t, x^i)$. Equation (30) determines A_{jk}; then R_j is determined by (29).

This shows that condition (a) implies (b).

3. The opposite can be proved by immediate calculation. Substitution from (29) to (28) yields

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial P_j}{\partial \dot{x}^k} + \frac{\partial P_k}{\partial \dot{x}^j} \right) - \frac{\partial^3 f}{\partial t \partial \dot{x}^j \partial \dot{x}^k} - \frac{\partial^3 f}{\partial x^l \partial \dot{x}^j \partial \dot{x}^k} \dot{x}^l \\ &= \frac{1}{2}\left(A_{jk} - \frac{\partial f}{\partial x^j \partial \dot{x}^k} + \frac{\partial^2 f}{\partial t \partial \dot{x}^j \partial \dot{x}^k} + \frac{\partial^2 f}{\partial x^l \partial \dot{x}^j \partial \dot{x}^k} \dot{x}^l + \frac{\partial^2 f}{\partial x^k \partial \dot{x}^j} \right. \\ &\quad \left. + A_{kj} - \frac{\partial f}{\partial x^k \partial \dot{x}^j} + \frac{\partial^2 f}{\partial t \partial \dot{x}^k \partial \dot{x}^j} + \frac{\partial^2 f}{\partial x^l \partial \dot{x}^k \partial \dot{x}^j} \dot{x}^l + \frac{\partial^2 f}{\partial x^j \partial \dot{x}^k} \right) \\ &\quad - \frac{\partial^3 f}{\partial t \partial \dot{x}^j \partial \dot{x}^k} - \frac{\partial^3 f}{\partial x^l \partial \dot{x}^j \partial \dot{x}^k} \dot{x}^l \equiv 0.\end{aligned}$$

Lemma 8 *The following two conditions are equivalent:*

(a) P_i *and* f *solve the system*

$$\frac{1}{2}\left(\frac{\partial P_i}{\partial \dot{x}^j} + \frac{\partial P_j}{\partial \dot{x}^i} \right) - \frac{\partial^3 f}{\partial t \partial \dot{x}^i \partial \dot{x}^j} - \frac{\partial^3 f}{\partial x^l \partial \dot{x}^i \partial \dot{x}^j} \dot{x}^l = 0, \tag{42}$$

(43) $$\frac{\partial P_i}{\partial x^j}-\frac{\partial P_j}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial P_i}{\partial \dot{x}^j}-\frac{\partial P_j}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}-\frac{\partial P_j}{\partial \dot{x}^i}\right)\dot{x}^k=0.$$

(b) P_i *is of the form*

$$P_j=R_j+A_{jl}\dot{x}^l-\frac{\partial f}{\partial x^j}+\frac{\partial^2 f}{\partial t\,\partial \dot{x}^j}+\frac{\partial^2 f}{\partial x^l\,\partial \dot{x}^j}\dot{x}^l,$$

where the functions $R_j=R_j(t,x^i)$ *and* $A_{ij}=A_{ij}(t,x^i)$ *satisfy*

$$A_{ij}=-A_{ji},\quad \frac{\partial A_{il}}{\partial x^j}+\frac{\partial A_{lj}}{\partial x^i}+\frac{\partial A_{ji}}{\partial x^l}=0$$

and

(44) $$\frac{\partial R_i}{\partial x^j}-\frac{\partial R_j}{\partial x^i}-\frac{\partial A_{ij}}{\partial t}=0.$$

Proof 1. Suppose that P_i satisfies condition (a). Then P_i also satisfies condition (b), Lemma 7; we shall determine consequences of condition (43). From formula (29) we have

$$\begin{aligned}\frac{\partial P_i}{\partial x^j}-\frac{\partial P_j}{\partial x^i}=&\frac{\partial R_i}{\partial x^j}+\frac{\partial A_{il}}{\partial x^j}\dot{x}^l-\frac{\partial^2 f}{\partial x^j\,\partial x^i}+\frac{\partial^3 f}{\partial t\,\partial x^j\,\partial \dot{x}^i}+\frac{\partial^3 f}{\partial x^l\,\partial x^j\,\partial \dot{x}^i}\dot{x}^l\\&-\frac{\partial R_j}{\partial x^i}-\frac{\partial A_{jl}}{\partial x^i}\dot{x}^l+\frac{\partial^2 f}{\partial x^i\,\partial x^j}-\frac{\partial^3 f}{\partial t\,\partial x^i\,\partial \dot{x}^j}-\frac{\partial^3 f}{\partial x^l\,\partial x^i\,\partial \dot{x}^j}\dot{x}^l\end{aligned}$$

and

$$\frac{\partial P_i}{\partial \dot{x}^j}-\frac{\partial P_j}{\partial \dot{x}^i}=2\left(A_{ij}-\frac{\partial^2 f}{\partial x^i\,\partial \dot{x}^j}+\frac{\partial^2 f}{\partial x^j\,\partial \dot{x}^i}\right).$$

Substituting

(45) $$\begin{aligned}&\frac{\partial P_i}{\partial x^j}-\frac{\partial P_j}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial P_i}{\partial \dot{x}^j}-\frac{\partial P_j}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial P_i}{\partial \dot{x}^j}-\frac{\partial P_j}{\partial \dot{x}^i}\right)\dot{x}^k\\&=\frac{\partial R_i}{\partial x^j}-\frac{\partial R_j}{\partial x^i}-\frac{\partial A_{ij}}{\partial t}+\left(\frac{\partial A_{il}}{\partial x^j}+\frac{\partial A_{lj}}{\partial x^i}+\frac{\partial A_{ji}}{\partial x^l}\right)\dot{x}^l.\end{aligned}$$

Thus, (43) implies (44) and conditions (b) are fulfilled.
2. Condition (a) follows from (b) by formula (45).

Lemma 9 *Suppose that a source form* $\varepsilon=\varepsilon_i$ *satisfies conditions* (12), (13) *and* (14). *Then*

(46) $$\varepsilon_i=P_i+Q_{il}\ddot{x}^l,$$

where

$$Q_{ij} = \frac{\partial^2 f}{\partial \dot{x}^i \partial \dot{x}^j}$$

and

$$P_j = \frac{\partial h}{\partial x^j} - \frac{\partial \eta_j}{\partial t} + \left(\frac{\partial \eta_l}{\partial x^j} - \frac{\partial \eta_j}{\partial x^l} \right) \dot{x}^l - \frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial t \partial \dot{x}^j} + \frac{\partial^2 f}{\partial x^l \partial \dot{x}^j} \dot{x}^l \tag{47}$$

for some functions $f = f(t, x^k, \dot{x}^k)$, $\eta_j = \eta_j(t, x^i)$ *and* $h = h(t, x^i)$.

Proof According to Lemma 6, P_i and f solve the system (25), (26), that is, the system (42), (43); thus by Lemma 8

$$P_j = R_j + A_{jl}\dot{x}^l - \frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial t \partial \dot{x}^j} + \frac{\partial^2 f}{\partial x^l \partial \dot{x}^j} \dot{x}^l,$$

where $R_j = R_j(t, x^i)$ and $A_{ij} = A_{ij}(t, x^i)$ are some functions such that

$$A_{ij} = -A_{ji}, \quad \frac{\partial A_{il}}{\partial x^j} + \frac{\partial A_{lj}}{\partial x^i} + \frac{\partial A_{ji}}{\partial x^l} = 0$$

and

$$\frac{\partial R_i}{\partial x^j} - \frac{\partial R_j}{\partial x^i} - \frac{\partial A_{ij}}{\partial t} = 0. \tag{48}$$

These conditions allow to construct η_j and h. We have

$$A_{ij} = \frac{\partial \eta_j}{\partial x^i} - \frac{\partial \eta_i}{\partial x^j}$$

and then (48) yields

$$\frac{\partial R_i}{\partial x^j} - \frac{\partial R_j}{\partial x^i} = \frac{\partial}{\partial t} \left(\frac{\partial \eta_j}{\partial x^i} - \frac{\partial \eta_i}{\partial x^j} \right)$$

hence

$$\frac{\partial}{\partial x^j} \left(R_i + \frac{\partial \eta_i}{\partial t} \right) - \frac{\partial}{\partial x^i} \left(R_j + \frac{\partial \eta_j}{\partial t} \right) = 0.$$

Integrating for any fixed η_j

$$R_i + \frac{\partial \eta_i}{\partial t} = \frac{\partial h}{\partial x^i}$$

for some function $h = h(t, x^i)$. Using these functions we get (47).

Our discussion now concludes into the following assertion.

Theorem 3 *Suppose that a source form* $\varepsilon = \varepsilon_i$ *satisfies conditions* (12), (13) *and* (14). *Then there exist some functions* $f = f(t, x^k, \dot{x}^k)$, $\eta_j = \eta_j(t, x^i)$ *and* $h = h(t, x^i)$ *such that* ε_i, *are the Euler-Lagrange expressions of the Lagrange function given by*

$$\mathscr{L} = f - \eta_l \dot{x}^l - h. \tag{49}$$

In particular, ε *is a variational source form.*

Proof Set $f_0 = -\eta_l \dot{x}^l - h$, $\mathscr{L} = f_0 + f$. Calculating the Euler-Lagrange expressions of the Lagrange function $\mathscr{L}$ we get

$$\begin{aligned} E_i(f_0) + E_i(f) &= -\frac{\partial f_0}{\partial x^i} + \frac{d}{dt}\frac{\partial f_0}{\partial \dot{x}^i} + E_i(f) \\ &= \frac{\partial \eta_l}{\partial x^i}\dot{x}^l + \frac{\partial h}{\partial x^i} - \frac{d\eta_i}{dt} + E_i(f) \\ &= \frac{\partial h}{\partial x^i} - \frac{\partial \eta_i}{\partial t} + \left(\frac{\partial \eta_l}{\partial x^i} - \frac{\partial \eta_i}{\partial x^l}\right)\dot{x}^l + E_i(f) \\ &= \varepsilon_i. \end{aligned}$$

Remark 2 Formula $\mathscr{L} = f - \eta_l \dot{x}^l - h$ (49) defines a (first-order) Lagrange function for the source form $\varepsilon_i = P_i + Q_{il}\ddot{x}^l$ (46), satisfying conditions (21), (22), (23) and (24). The functions *f* and η_l can be determined in an explicit form as functions of the components P_i and Q_{ij}, and *h* remains arbitrary. Indeed, according to the Hessian matrix reconstruction lemma,

$$f = x^i x^j \int_0^1 \left(\int_0^1 Q_{ij}(\kappa \tau x^p)\, d\kappa\right) \tau\, d\tau.$$

Then setting

$$\tilde{P}_j = P_j + \frac{\partial f}{\partial x^j} - \frac{\partial^2 f}{\partial t\, \partial \dot{x}^j} - \frac{\partial^2 f}{\partial x^l\, \partial \dot{x}^j}\dot{x}^l$$

formula (47) becomes

$$\tilde{P}_j = \frac{\partial h}{\partial x^j} - \frac{\partial \eta_j}{\partial t} + \left(\frac{\partial \eta_l}{\partial x^j} - \frac{\partial \eta_j}{\partial x^l}\right)\dot{x}^l. \tag{50}$$

Then

$$\frac{\partial \tilde{P}_j}{\partial \dot{x}^k} = \frac{\partial \eta_k}{\partial x^j} - \frac{\partial \eta_j}{\partial x^k} = \frac{1}{2}\left(\frac{\partial \tilde{P}_j}{\partial \dot{x}^k} - \frac{\partial \tilde{P}_k}{\partial \dot{x}^j}\right).$$

Consequently, by Lemma 2

$$\eta_l = \eta_l^0 + \frac{\partial \Phi}{\partial x^l}, \quad \eta_l^0 = \frac{1}{2} x^p \int_0^1 \left(\frac{\partial \tilde{P}_p}{\partial \dot{x}^l} - \frac{\partial \tilde{P}_l}{\partial \dot{x}^p} \right)_{(t,\kappa x^i)} \kappa \, d\kappa$$

where $\Phi = \Phi(t, x^l)$ is arbitrary. Then from (50)

$$\frac{\partial h}{\partial x^j} = \tilde{P}_j + \frac{\partial \eta_j}{\partial t} - \left(\frac{\partial \eta_l}{\partial x^j} - \frac{\partial \eta_j}{\partial x^l} \right) \dot{x}^l.$$

Substitutions for η_j and $\tilde{P}_j$ show, however, that this equation is satisfied identically.

4 First-order variational source forms

Consider a *first order* source form $\varepsilon = \varepsilon_i$, where

$$\varepsilon_i = \varepsilon_i(t, x^i, \dot{x}^i).$$

In this case Theorem 2 and Theorem 3 of Section 3 imply that the variationality of ε is equivalent to the conditions

(2)
$$\frac{\partial \varepsilon_i}{\partial \dot{x}^j} + \frac{\partial \varepsilon_j}{\partial \dot{x}^i} = 0$$

and

(3)
$$\frac{\partial}{\partial \dot{x}^k} \left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i} \right) = 0,$$

(4)
$$\frac{\partial \varepsilon_i}{\partial x^j} - \frac{\partial \varepsilon_j}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i} \right) - \frac{1}{2} \frac{\partial}{\partial x^l} \left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i} \right) \dot{x}^l = 0.$$

Note, however, that according to Section 3, Lemma 3, condition (3) follows from (2) and may be omitted.

The following two theorems provide a complete classification of the first-order source forms.

Theorem 4 *Let $\varepsilon = \varepsilon_i$ be a first-order source form. The following two conditions are equivalent:*

(a) *ε is variational.*

(b) *There exist a system of functions $\eta = \eta_i$, $\eta_i = \eta_i(t, x^j)$, and a function $h = h(t, x^j)$ such that*

(5)
$$\varepsilon_i = \frac{\partial h}{\partial x^i} - \frac{\partial \eta_i}{\partial t} + \left(\frac{\partial \eta_l}{\partial x^i} - \frac{\partial \eta_i}{\partial x^l} \right) \dot{x}^l.$$

Proof 1. Suppose that ε is variational. Then by Section 2, Lemma 3, equations (2) imply

(6) $$\varepsilon_i = R_i + S_{il}\dot{x}^l,$$

where $R_i = R_i(t,x^k)$, $S_{ij} = S_{ij}(t,x^k)$ are some functions such that $S_{ij} = -S_{ji}$. Then (3) is an identity, and equation (4) implies

$$\frac{\partial R_i}{\partial x^j} - \frac{\partial R_j}{\partial x^i} + \left(\frac{\partial S_{il}}{\partial x^j} - \frac{\partial S_{jl}}{\partial x^i}\right)\dot{x}^l - \frac{1}{2}\frac{\partial(S_{ij}-S_{ji})}{\partial t} - \frac{1}{2}\frac{\partial(S_{ij}-S_{ji})}{\partial x^l}\dot{x}^l$$
$$= \frac{\partial R_i}{\partial x^j} - \frac{\partial R_j}{\partial x^i} - \frac{\partial S_{ij}}{\partial t} + \left(\frac{\partial S_{il}}{\partial x^j} - \frac{\partial S_{jl}}{\partial x^i} - \frac{\partial S_{ij}}{\partial x^l}\right)\dot{x}^l = 0.$$

Since the coefficients in this linear expression does not depend on $\dot{x}^l$, we get

(9) $$\frac{\partial S_{li}}{\partial x^j} + \frac{\partial S_{jl}}{\partial x^i} + \frac{\partial S_{ij}}{\partial x^l} = 0$$

and

$$\frac{\partial R_i}{\partial x^j} - \frac{\partial R_j}{\partial x^i} - \frac{\partial S_{ij}}{\partial t} = 0.$$

Equation (9) can be integrated. According to Section 2, Lemma 2, there exists a system of functions $\eta = \eta_l$ such that

(11) $$S_{kl} = \frac{\partial \eta_l}{\partial x^k} - \frac{\partial \eta_k}{\partial x^l}.$$

Then condition (10) transforms to

$$\frac{\partial}{\partial x^j}\left(R_i + \frac{\partial \eta_i}{\partial t}\right) - \frac{\partial}{\partial x^i}\left(R_j + \frac{\partial \eta_j}{\partial t}\right) = 0$$

and can also be integrated. We get

(13) $$R_i + \frac{\partial \eta_i}{\partial t} = \frac{\partial h}{\partial x^i}$$

for some function $h = h(t,x^i)$. Substituting into (6) proves formula (5).

2. The converse follows by substituting from (5) into (2) – (4).

Theorem 5 *If a first-order source form* $\varepsilon = \varepsilon_i$ *is variational and is expressed by* (5), *then* ε *has a Lagrange function*

$$f = -\eta_l \dot{x}^l - h.$$

Proof The Euler-Lagrange expressions of the function (14) are

$$E_i(f) = -\frac{\partial f}{\partial x^i} + \frac{\partial^2 f}{\partial t\,\partial \dot{x}^i} + \frac{\partial^2 f}{\partial x^l\,\partial \dot{x}^i}\dot{x}^l = \frac{\partial \eta_l}{\partial x^i}\dot{x}^l + \frac{\partial h}{\partial x^i} - \frac{\partial \eta_i}{\partial t} - \frac{\partial \eta_i}{\partial x^l}\dot{x}^l$$
$$= \frac{\partial h}{\partial x^i} - \frac{\partial \eta_i}{\partial t} + \left(\frac{\partial \eta_l}{\partial x^i} - \frac{\partial \eta_i}{\partial x^l}\right)\dot{x}^l.$$

Remark 3 Formula (5) shows that the *classification parameters* for the first-order variational source forms are real-valued functions η_l and *h* of the variables t, x^i. For a given source form $\varepsilon = \varepsilon_i$ of the form (6), these functions can be determined by integration from formulas (11) and (13).

5 Lagrange functions: Canonical decomposition

In this section we consider Lagrange functions, which do not depend on *t*; this assumption simplifies calculations, but main motivation consists in possibilities of comparing of the formulas with Finsler geometry. We use an observation that the decomposition $\mathscr{L} = \mathscr{T} - \mathscr{U}$, applied in Lagrange mechanics, can be constructed for an *arbitrary* Lagrange function $\mathscr{L} : U \times \mathbf{R}^m \to \mathbf{R}$, not necessarily quadratic in the $\dot{x}^i$.

Let $\mathscr{L} : U \times \mathbf{R}^m \to \mathbf{R}$ be any Lagrange function. Setting

$$\mathscr{T} = \frac{1}{2}\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i\,\partial \dot{x}^j}\dot{x}^i \dot{x}^j,$$

we get another Lagrange function $\mathscr{T} : U \times \mathbf{R}^m \to \mathbf{R}$, called *kinetic energy*, associated with $\mathscr{L}$, and a decomposition

$$\mathscr{L} = \mathscr{T} - \mathscr{U},$$

where $\mathscr{U}$ is *potential energy*, associated with $\mathscr{L}$. The Euler-Lagrange form of $\mathscr{L}$ is

$$E_k(\mathscr{L}) = -\frac{\partial \mathscr{L}}{\partial x^k} + \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}^k}.$$

Both functions $\mathscr{T}$ and $\mathscr{U}$ are invariant with respect to coordinate transformations

$$\bar{x}^i = \bar{x}^i(x), \quad \dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^k}\dot{x}^k.$$

Indeed,

$$\frac{\partial^2 \mathscr{L}}{\partial \dot{\bar{x}}^i\,\partial \dot{\bar{x}}^j} = \frac{\partial}{\partial \dot{\bar{x}}^i}\frac{\partial \mathscr{L}}{\partial \dot{\bar{x}}^j} = \frac{\partial}{\partial \dot{\bar{x}}^j}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}^k}\frac{\partial \dot{x}^k}{\partial \dot{\bar{x}}^i}\right) = \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^k\,\partial \dot{x}^l}\frac{\partial \dot{x}^l}{\partial \dot{\bar{x}}^j}\frac{\partial \dot{x}^k}{\partial \dot{\bar{x}}^i}.$$

Set

$$h_{ij} = \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i\,\partial \dot{x}^j}.$$

In this notation

$$\mathcal{T} = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j .$$

Note that h_{ij} satisfies basic conditions of a *Finsler metric*

$$h_{ij} = h_{ji} , \quad \frac{\partial h_{ij}}{\partial \dot{x}^k} = \frac{\partial h_{ik}}{\partial \dot{x}^j} .$$

We call h_{ij} the *metric, associated* with $\mathcal{L}$.

Lemma 10 (a) *The Euler-Lagrange form of* $\mathcal{T}$ *is expressed by*

$$-\frac{\partial \mathcal{T}}{\partial x^k} + \frac{d}{dt} \frac{\partial \mathcal{T}}{\partial \dot{x}^k} = \frac{1}{2} \left(-\frac{\partial h_{ij}}{\partial x^k} + \frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j$$
$$+ \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j \dot{x}^l + \left(\frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^k \partial \dot{x}^l} \dot{x}^i \dot{x}^j + \frac{\partial h_{lj}}{\partial \dot{x}^k} \dot{x}^j + h_{lk} \right) \ddot{x}^l .$$

(b) *The Euler-Lagrange form of* $\mathcal{U}$ *is expressed as*

$$-\frac{\partial \mathcal{U}}{\partial x^k} + \frac{d}{dt} \frac{\partial \mathcal{U}}{\partial \dot{x}^k} = \frac{1}{2} \left(-\frac{\partial h_{ij}}{\partial x^k} + \frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j$$
$$+ \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j \dot{x}^l + \left(\frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^k \partial \dot{x}^l} \dot{x}^i \dot{x}^j + \frac{\partial h_{lj}}{\partial \dot{x}^k} \dot{x}^j \right) \ddot{x}^l$$
$$+ \frac{\partial \mathcal{L}}{\partial x^k} - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^k} \dot{x}^l .$$

Proof (a) Differentiating $\mathcal{T}$ we have

$$\frac{\partial \mathcal{T}}{\partial x^k} = \frac{1}{2} \frac{\partial h_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j , \quad \frac{\partial \mathcal{T}}{\partial \dot{x}^k} = \frac{1}{2} \frac{\partial h_{ij}}{\partial \dot{x}^k} \dot{x}^i \dot{x}^j + h_{ik} \dot{x}^i ,$$

and

$$-\frac{\partial \mathcal{T}}{\partial x^k} + \frac{d}{dt} \frac{\partial \mathcal{T}}{\partial \dot{x}^k} = -\frac{1}{2} \frac{\partial h_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j$$
$$+ \frac{\partial}{\partial x^l} \left(\frac{1}{2} \frac{\partial h_{ij}}{\partial \dot{x}^k} \dot{x}^i \dot{x}^j + h_{ik} \dot{x}^i \right) \dot{x}^l + \frac{\partial}{\partial \dot{x}^l} \left(\frac{1}{2} \frac{\partial h_{ij}}{\partial \dot{x}^k} \dot{x}^i \dot{x}^j + h_{ik} \dot{x}^i \right) \ddot{x}^l$$
$$= \frac{1}{2} \left(-\frac{\partial h_{ij}}{\partial x^k} + \frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j + \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j \dot{x}^l$$
$$+ \left(\frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^k \partial \dot{x}^l} \dot{x}^i \dot{x}^j + \frac{\partial h_{lj}}{\partial \dot{x}^k} \dot{x}^j + h_{lk} \right) \ddot{x}^l .$$

(b) The Euler-Lagrange form $E(\mathscr{U})$ is defined by $E(\mathscr{U}) = E(\mathscr{T}) - E(\mathscr{L})$ hence

$$-\frac{\partial \mathscr{U}}{\partial x^k} + \frac{d}{dt}\frac{\partial \mathscr{U}}{\partial \dot{x}^k} = \frac{1}{2}\left(-\frac{\partial h_{ij}}{\partial x^k} + \frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i}\right)\dot{x}^i\dot{x}^j + \frac{1}{2}\frac{\partial^2 h_{ij}}{\partial x^l\,\partial \dot{x}^k}\dot{x}^i\dot{x}^j\dot{x}^l$$
$$+\left(\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial \dot{x}^k\,\partial \dot{x}^l}\dot{x}^i\dot{x}^j + \frac{\partial h_{lj}}{\partial \dot{x}^k}\dot{x}^j\right)\ddot{x}^l + \frac{\partial \mathscr{L}}{\partial x^k} - \frac{\partial^2 \mathscr{L}}{\partial x^l\,\partial \dot{x}^k}\dot{x}^l$$

proving Lemma 10.

6 Variational forces

In this section we study Lagrange functions $\mathscr{L} : U \times \mathbf{R}^m \to \mathbf{R}$ satisfying the *metric homogeneity condition*

(1)
$$\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^i\,\partial \dot{x}^j\,\partial \dot{x}^k}\dot{x}^k = 0.$$

Using the metric h_{ij} associated with $\mathscr{L}$, this condition can equivalently be expressed as

(2)
$$\frac{\partial h_{ij}}{\partial \dot{x}^k}\dot{x}^k = 0.$$

Its meaning is explained by the following theorem.

Theorem 6 *Let* $\mathscr{L}$ *be a Lagrange function. The following two conditions are equivalent:*

(a) $\mathscr{L}$ *satisfies the metric homogeneity condition.*

(b) *The Euler-Lagrange form* $E(\mathscr{U})$ *of potential energy* $\mathscr{U}$ *is of order* 1.

Proof 1. We show that (a) implies (b). Applying condition (1) to $E_k(\mathscr{U})$ (Lemma 10, formula (15)) we get by a straightforward calculation, using formulas (13), Section 5,

(3)
$$\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial \dot{x}^k\,\partial \dot{x}^l}\dot{x}^i\dot{x}^j + \frac{\partial h_{lj}}{\partial \dot{x}^k}\dot{x}^j$$
$$= \frac{1}{2}\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial h_{ij}}{\partial \dot{x}^l}\dot{x}^i\dot{x}^j\right) - \frac{\partial h_{ik}}{\partial \dot{x}^l}\dot{x}^i + \frac{\partial h_{lj}}{\partial \dot{x}^k}\dot{x}^j = 0.$$

Consequently, $E(\mathscr{U})$ is of order 1.

2. Conversely, let $\mathscr{L}$ be a Lagrange function such that $E(\mathscr{U})$ is of order 1. Since $E(\mathscr{U})$ is a variational form, according to Theorem 5 it has a Lagrange function $\mathscr{U}' = -h - \eta_i \dot{x}^i$, where h and η_j are some functions depending on x^i only. Then, however,

(4)
$$E_i(\mathscr{U}) = E_i(\mathscr{U}'),$$

and the source form $E_i(\mathcal{U})$ has two Lagrange functions, $\mathcal{U}$ and $\mathcal{U}'$. This condition implies that the difference $\mathcal{U}-\mathcal{U}'$ belongs to the kernel of the Euler-Lagrange mapping

$$\mathcal{U}=\mathcal{U}'+\frac{df}{dt}=-h-\eta_i\dot{x}^i+\frac{df}{dt} \tag{5}$$

for some $f=f(x^i)$ (Section 4, Theorem 1). On the other hand the functions $E_i(\mathcal{U})$ are determined by equations (15), Section 5; thus, condition (4) reads

$$\begin{aligned} E_k(\mathcal{U})&=\frac{1}{2}\left(-\frac{\partial h_{ij}}{\partial x^k}+\frac{\partial h_{ik}}{\partial x^j}+\frac{\partial h_{jk}}{\partial x^i}\right)\dot{x}^i\dot{x}^j+\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial x^l\,\partial\dot{x}^k}\dot{x}^i\dot{x}^j\dot{x}^l\\ &+\left(\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial\dot{x}^k\,\partial\dot{x}^l}\dot{x}^i\dot{x}^j+\frac{\partial h_{lj}}{\partial\dot{x}^k}\dot{x}^j\right)\ddot{x}^l+\frac{\partial\mathcal{L}}{\partial x^k}-\frac{\partial^2\mathcal{L}}{\partial x^l\,\partial\dot{x}^k}\dot{x}^l\\ &=\frac{\partial h}{\partial x^k}+\left(\frac{\partial\eta_l}{\partial x^k}-\frac{\partial\eta_k}{\partial x^l}\right)\dot{x}^l. \end{aligned} \tag{6}$$

Consequently

$$\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial\dot{x}^k\,\partial\dot{x}^l}\dot{x}^i\dot{x}^j+\frac{\partial h_{jk}}{\partial\dot{x}^l}\dot{x}^j=0. \tag{7}$$

Since expressions $\partial h_{jk}/\partial\dot{x}^l$ are symmetric in j, k, l we have

$$\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial\dot{x}^k\,\partial\dot{x}^l}\dot{x}^i\dot{x}^j+\frac{\partial h_{lk}}{\partial\dot{x}^j}\dot{x}^j=\frac{\partial}{\partial\dot{x}^l}\left(\frac{1}{2}\frac{\partial h_{ij}}{\partial\dot{x}^k}\dot{x}^i\dot{x}^j\right)=0. \tag{8}$$

Integrating

$$\frac{1}{2}\frac{\partial h_{ij}}{\partial\dot{x}^k}\dot{x}^i\dot{x}^j=\boldsymbol{\Phi}_k, \tag{9}$$

where $\boldsymbol{\Phi}_k=\boldsymbol{\Phi}_k(x^i)$. This formula solves equation (7).

We use this formula to determine expression

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^k}=\frac{\partial\mathcal{T}}{\partial\dot{x}^k}-\frac{\partial\mathcal{U}}{\partial\dot{x}^k}. \tag{10}$$

Since

$$\frac{\partial}{\partial\dot{x}^k}\left(\frac{1}{2}h_{ij}\dot{x}^i\dot{x}^j\right)-h_{ik}\dot{x}^i=\frac{1}{2}\frac{\partial h_{ij}}{\partial\dot{x}^k}\dot{x}^i\dot{x}^j+h_{ik}\dot{x}^i-h_{ik}\dot{x}^i=\begin{cases}\boldsymbol{\Phi}_k,\\ \dfrac{\partial\mathcal{T}}{\partial\dot{x}^k}-h_{ik}\dot{x}^i,\end{cases} \tag{11}$$

then

(12) $$\frac{\partial \mathcal{T}}{\partial \dot{x}^k} = \Phi_k + h_{ik}\dot{x}^i .$$

On the other hand, from equations (5)

(13) $$\frac{\partial \mathcal{U}}{\partial \dot{x}^k} = -\eta_k + \frac{\partial f}{\partial x^k},$$

thus

(14) $$\frac{\partial \mathcal{L}}{\partial \dot{x}^k} = \Phi_k + h_{ik}\dot{x}^i + \eta_k - \frac{\partial f}{\partial x^k}.$$

In this expression

(15) $$\frac{\partial}{\partial \dot{x}^l}\left(\Phi_k + \eta_k - \frac{\partial f}{\partial x^k}\right) = \begin{cases} 0, \\ \dfrac{\partial}{\partial \dot{x}^l}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}^k} - h_{ik}\dot{x}^i\right) = \dfrac{\partial^2 \mathcal{L}}{\partial \dot{x}^k \partial \dot{x}^l} - \dfrac{\partial h_{ik}}{\partial \dot{x}^l}\dot{x}^i - h_{lk}, \end{cases}$$

therefore

(16) $$\frac{\partial h_{ik}}{\partial \dot{x}^l}\dot{x}^i = \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^k \partial \dot{x}^l \partial \dot{x}^i}\dot{x}^i = 0.$$

Theorem 7 *Let $\mathcal{L}$ satisfy the metric homogeneity condition. Then*

(17) $$\mathcal{U} = -h - \eta_i \dot{x}^i + \frac{df}{dt},$$

where

(18) $$\eta_k = -\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^k}\dot{x}^i + \frac{\partial \mathcal{L}}{\partial \dot{x}^k} + \frac{\partial f}{\partial x^k}, \quad h = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i}\dot{x}^j \dot{x}^i - \frac{\partial \mathcal{L}}{\partial \dot{x}^i}\dot{x}^i + \mathcal{L}$$

for some function $f = f(x^i)$. The functions η_k and h do not depend on $\dot{x}^i$.

Proof By Theorem 6, the Euler-Lagrange form of $\mathcal{U}$ is of the first order. Consequently,

(19) $$E_i(\mathcal{U}) = \frac{\partial h}{\partial x^i} + \left(\frac{\partial \eta_l}{\partial x^i} - \frac{\partial \eta_i}{\partial x^l}\right)\dot{x}^l$$

for some functions $\eta_l = \eta_l(x^i)$ and $h = h(x^i)$ (Theorem 4). $E_i(\mathcal{U})$ also admits a Lagrange function $\mathcal{U}' = -h - \eta_i \dot{x}^i$ (Theorem 5). Then $E_i(\mathcal{U})$ has two Lagrange functions, $\mathcal{U}$ and $\mathcal{U}'$ hence, using definition of $\mathcal{U}$

$$\mathcal{U} = \begin{cases} \dfrac{1}{2}\dfrac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}\dot{x}^i \dot{x}^j - \mathcal{L}, \\ -h - \eta_i \dot{x}^i + \dfrac{df}{dt} \end{cases} \tag{20}$$

for some $f = f(x^i)$. Verification of conditions (18) is now straightforward. Indeed, differentiating expressions (20) with respect to $\dot{x}^k$

$$\eta_k = -\frac{1}{2}\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k}\dot{x}^i \dot{x}^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^k}\dot{x}^i + \frac{\partial \mathcal{L}}{\partial \dot{x}^k} + \frac{\partial f}{\partial x^k} \tag{21}$$

hence

$$\begin{aligned} h &= -\left(-\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i}\dot{x}^j + \frac{\partial \mathcal{L}}{\partial \dot{x}^i} + \frac{\partial f}{\partial x^i}\right)\dot{x}^i + \frac{df}{dt} - \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}\dot{x}^i \dot{x}^j + \mathcal{L} \\ &= \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i}\dot{x}^j \dot{x}^i - \frac{\partial \mathcal{L}}{\partial \dot{x}^i}\dot{x}^i + \mathcal{L}. \end{aligned} \tag{22}$$

Then

$$\frac{\partial \eta_k}{\partial \dot{x}^l} = -\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k} + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^k \partial \dot{x}^l} = 0 \tag{23}$$

and

$$\begin{aligned} \frac{\partial h}{\partial \dot{x}^l} &= \frac{\partial}{\partial \dot{x}^l}\left(\frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}\dot{x}^i \dot{x}^j - \frac{\partial \mathcal{L}}{\partial \dot{x}^i}\dot{x}^i + \mathcal{L}\right) \\ &= -\frac{1}{2}\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^l}\dot{x}^i \dot{x}^j + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^l}\dot{x}^i - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^l}\dot{x}^i - \frac{\partial \mathcal{L}}{\partial \dot{x}^l} + \frac{\partial \mathcal{L}}{\partial \dot{x}^l} = 0. \end{aligned} \tag{24}$$

Our aim now will be to find explicit description of Lagrange functions satisfying the metric homogeneity condition. Our partial results can be summarized as follows. Given an arbitrary Lagrange function $\mathcal{L}$, we have the canonical decomposition

$$\mathcal{L} = \mathcal{T} - \mathcal{U}, \tag{25}$$

where

$$\mathcal{T} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}\dot{x}^i \dot{x}^j \tag{26}$$

Then the following conditions are equivalent:

(a) $\mathcal{L}$ satisfies the metric homogeneity condition.

(b) The Euler-Lagrange expressions $E_i(\mathcal{U})$ do not depend on $\ddot{x}^k$, that is,

$$\frac{\partial E_i(\mathscr{U})}{\partial \ddot{x}^j} = 0$$

for all i and j.

If these conditions are satisfied, then $\mathscr{U}$ is of the form

$$\mathscr{U} = -h - \eta_i \dot{x}^i + \frac{df}{dt},$$

for some function $f = f(x^i)$, and

$$\eta_k = -\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i \partial \dot{x}^k}\dot{x}^i + \frac{\partial \mathscr{L}}{\partial \dot{x}^k} + \frac{\partial f}{\partial x^k}, \quad h = \frac{1}{2}\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i \partial \dot{x}^j}\dot{x}^i \dot{x}^j - \frac{\partial \mathscr{L}}{\partial \dot{x}^i}\dot{x}^i + \mathscr{L}. \tag{29}$$

The functions h and η_i do not depend on $\dot{x}^k$ and Euler-Lagrange expressions of $\mathscr{U}$ are

$$E_k(\mathscr{U}) = \frac{\partial h}{\partial x^k} + \left(\frac{\partial \eta_l}{\partial x^k} - \frac{\partial \eta_k}{\partial x^l}\right)\dot{x}^l. \tag{30}$$

Remark 4 Equation (30) shows that for $m = 1$ (the case of mechanical systems with one degree of freedom) the Euler-Lagrange form $E(\mathscr{U})$ cannot depend on $\dot{x}$. In particular, equation of the motion of the one-dimensional *damped harmonic oscillator* cannot be variationally characterized this way.

It has already been noted in Section 5 that the kinetic energy part in the canonical decomposition of a Lagrange function has some properties of the fundamental Finsler functions in Finsler geometry. We shall now discuss these properties in more detail. We define a (possibly singular) *Finsler metric* as a system of real-valued functions $g_{ij} = g_{ij}(x^k, \dot{x}^k)$ defined on $U \times \mathbf{R}^m$, satisfying the following conditions:

(a) The matrix g_{ij} is symmetric,

$$g_{ij} = g_{ji}. \tag{31}$$

(b) The derivatives satisfy

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j}, \quad \frac{\partial g_{ij}}{\partial \dot{x}^k}\dot{x}^k = 0. \tag{32}$$

Theorem 8 *A Lagrange function* $\mathscr{L}: U \times \mathbf{R}^m \to \mathbf{R}$ *satisfies metric homogeneity condition if and only if*

$$\mathscr{L} = \frac{1}{2} g_{ij}\dot{x}^i \dot{x}^j + h + \eta_i \dot{x}^i, \tag{33}$$

where g_{ij} *is a Finsler metric and h and* η_i *are some functions depending on* x^i *only.*

Proof 1. Suppose that we have a Lagrange function $\mathscr{L}$ satisfying metric homogeneity condition. Then by Theorem 7, it is necessarily of the form

$$\mathscr{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + h + \tilde{\eta}_i \dot{x}^i + \frac{df}{dt}, \tag{34}$$

where

$$g_{ij} = \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i \partial \dot{x}^j}$$

and f, $\tilde{\eta}_k$ and h are some functions depending on x^i only. The functions g_{ij} obviously satisfy (31) and (32). Setting

$$\tilde{\eta}_i = \eta_i + \frac{\partial f}{\partial x^i}$$

we get (33).

2. Conversely, if g_{ij} is a Finsler metric, then by (31) and (32), direct differentiations of $\mathscr{L}$ (34) yield

$$\frac{\partial \mathscr{L}}{\partial \dot{x}^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^i \dot{x}^j + g_{ik} \dot{x}^i + \eta_k, \quad \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^k \partial \dot{x}^l} = \frac{1}{2} \frac{\partial g_{ik}}{\partial \dot{x}^l} \dot{x}^i + g_{lk}.$$

Hence

$$\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^j \partial \dot{x}^k \partial \dot{x}^l} \dot{x}^j = \frac{\partial g_{kl}}{\partial \dot{x}^j} \dot{x}^j = 0.$$

Theorem 9 *For every Finsler metric* g_{ij} *formula* (33) *defines a solution* $\mathscr{L}$ *of equations* (29).

Proof Suppose we have a Finsler metric g_{ij} and consider a Lagrange function $\mathscr{L}$ (33). Differentiations yield

$$\frac{\partial \mathscr{L}}{\partial \dot{x}^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^i \dot{x}^j + g_{ik} \dot{x}^i + \eta_k, \quad \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^k \partial \dot{x}^l} = \frac{\partial g_{ik}}{\partial \dot{x}^l} \dot{x}^i + g_{il}.$$

From these formulas

$$-\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i \partial \dot{x}^k} \dot{x}^i + \frac{\partial \mathscr{L}}{\partial \dot{x}^k} = -g_{ik} \dot{x}^i + g_{ik} \dot{x}^i + \eta_k = \eta_k$$

and

$$\frac{1}{2} \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i \partial \dot{x}^j} \dot{x}^i \dot{x}^j - \frac{\partial \mathscr{L}}{\partial \dot{x}^i} \dot{x}^i + \mathscr{L}$$

$$= \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - (g_{ji} \dot{x}^j + \eta_i) \dot{x}^i + \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + h + \eta_i \dot{x}^i = h.$$

Remark 5 Note that η_k can also be expressed as

$$\eta_k = -\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}^j}\dot{x}^j\right) + 2\frac{\partial \mathscr{L}}{\partial \dot{x}^k} = \frac{\partial}{\partial \dot{x}^k}\left(2\mathscr{L} - \frac{\partial \mathscr{L}}{\partial \dot{x}^j}\dot{x}^j\right).$$

Integrating

$$2\mathscr{L} - \frac{\partial \mathscr{L}}{\partial \dot{x}^j}\dot{x}^j - \eta_j \dot{x}^j = \Psi,$$

where $\Psi = \Psi(x^i)$ is an integration constant. Since

$$\begin{aligned} h &= \left(\frac{1}{2}\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i \partial \dot{x}^j}\dot{x}^j - \frac{\partial \mathscr{L}}{\partial \dot{x}^i}\right)\dot{x}^i + \mathscr{L} \\ &= \frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i \partial \dot{x}^j}\dot{x}^j - \frac{\partial \mathscr{L}}{\partial \dot{x}^i}\right)\dot{x}^i - \frac{1}{2}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}^i}\dot{x}^i - 2\mathscr{L}\right) \\ &= -\frac{1}{2}\eta_k \dot{x}^i - \frac{1}{2}(-\eta_j \dot{x}^j - \Psi) = \frac{1}{2}\Psi, \end{aligned}$$

we have

$$2\mathscr{L} - \frac{\partial \mathscr{L}}{\partial \dot{x}^j}\dot{x}^j = 2h + \eta_j \dot{x}^j.$$

Our main goal is in this section is to study variational properties of *first-order* source forms $\phi_i : U \times \mathbf{R}^m \to \mathbf{R}^m$; we call these forms *forces*. A force ϕ_i is said to be *variational*, if there exists a Lagrange function $\mathscr{L} : U \times \mathbf{R}^m \to \mathbf{R}$ such that

$$\phi_i = E_i(\mathscr{L} - \mathscr{T}), \tag{46}$$

where $\mathscr{T}$ is kinetic energy associated with $\mathscr{L}$, or, equivalently, if

$$\phi_i = E_i(\mathscr{U}),$$

where $\mathscr{U}$ is potential energy associated with $\mathscr{L}$. Thus a variational force is exactly the Euler-Lagrange form of the potential energy $\mathscr{U}$. Allowing $\mathscr{U}$ to depend on positions and velocities, then we also admit variational forces depending on x^l and $\dot{x}^l$.

The *inverse variational problem* for a force ϕ_i consists of finding integrability conditions and solutions $\mathscr{L} = \mathscr{L}(x^i, \dot{x}^i)$ of the system (46). We already known that the integrability condition is given by equations (2) and (4) of Section 4,

$$\frac{\partial \phi_i}{\partial \dot{x}^j} + \frac{\partial \phi_j}{\partial \dot{x}^i} = 0,$$

$$\frac{\partial \phi_i}{\partial x^j} - \frac{\partial \phi_j}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \phi_i}{\partial \dot{x}^j} - \frac{\partial \phi_j}{\partial \dot{x}^i}\right) - \frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial \phi_i}{\partial \dot{x}^j} - \frac{\partial \phi_j}{\partial \dot{x}^i}\right)\dot{x}^l = 0.$$

The following two theorems give a more detailed information.

Theorem 10 *The following two conditions are equivalent:*
(a) ϕ_k *is a variational force.*
(b) *There exist functions* $P = P(x^i)$ *and* $Q_k = Q_k(x^i)$ *such that*

$$\phi_k = \frac{\partial P}{\partial x^k} + \left(\frac{\partial Q_l}{\partial x^k} - \frac{\partial Q_k}{\partial x^l} \right) \dot{x}^l . \tag{50}$$

Proof 1. Suppose that ϕ_k is variational. Since it is of order 1, (b) follows from Theorem 4.
2. If ϕ_k is expressible by formula (50), then ϕ_k is variational as a first-order source form so it has a Lagrange function $\mathcal{U} = -P - Q_l \dot{x}^l$ (Theorem 4).

Remark 6 The class of variational forces admits a physical interpretation; it includes some *dissipative forces*, depending on velocities.

Theorem 10 should be completed by description of *all* Lagrange functions defining a fixed variational force ϕ_i. The following is a solution of the *inverse variational problem* for forces, depending on positions and velocities.

Theorem 11 *Let* ϕ_i *be a variational force, let* $\mathcal{U}_0 = -P - Q_i \dot{x}^i$ *be a Lagrange function for* ϕ_i. *Then the following two conditions are equivalent:*
(a) $\mathcal{L}$ *satisfies* $\phi_i = E_i(\mathcal{L} - \mathcal{T})$.
(b) $\mathcal{L}$ *is of the form* $\mathcal{L} = \mathcal{T} - \mathcal{U}_0$, *where*

$$\mathcal{T} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$$

for some Finsler metric g_{ij}.
Proof Immediate.

7 Newton's equations

In this paper the source forms $\varepsilon = \varepsilon_i$ with components

$$\varepsilon_i = g_{il} \ddot{x}^l - f_i , \tag{1}$$

where g_{il} and f_i are functions of t, x^i and $\dot{x}^i$, will be referred to as the *Newton's source forms*.

Lemma 11 *The Newton's source form* (1) *is variational if and only if*

$$g_{ij} - g_{ji} = 0, \tag{2}$$

$$\frac{\partial g_{ik}}{\partial \dot{x}^j} - \frac{\partial g_{jk}}{\partial \dot{x}^i} = 0, \tag{3}$$

(4) $$\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)+\frac{\partial g_{ij}}{\partial t}+\frac{\partial g_{ij}}{\partial x^l}\dot{x}^l=0,$$

(5) $$\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^i}+\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)=0,$$

(6) $$\frac{\partial f_i}{\partial x^j}-\frac{\partial f_j}{\partial x^i}+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)+\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^k=0.$$

Proof Integrability conditions for the source form (1) are determined by Theorem 2; we have

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^j}-\frac{\partial \varepsilon_j}{\partial \ddot{x}^i}=g_{ij}-g_{ji}=0,$$

$$\frac{\partial \varepsilon_i}{\partial \dot{x}^j}+\frac{\partial \varepsilon_j}{\partial \dot{x}^i}-\frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j}+\frac{\partial \varepsilon_j}{\partial \ddot{x}^i}\right)$$
$$=\left(\frac{\partial g_{il}}{\partial \dot{x}^j}+\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l-\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}-\frac{d(g_{ij}+g_{ji})}{dt}=0,$$

and

$$\frac{\partial \varepsilon_i}{\partial x^j}-\frac{\partial \varepsilon_j}{\partial x^i}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j}-\frac{\partial \varepsilon_j}{\partial \dot{x}^i}\right)$$
$$=\left(\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^i}\right)\ddot{x}^l-\frac{\partial f_i}{\partial x^j}+\frac{\partial f_j}{\partial x^i}-\frac{1}{2}\frac{d}{dt}\left(\left(\frac{\partial g_{il}}{\partial \dot{x}^j}-\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l-\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)=0.$$

Solving this system,

$$g_{ij}-g_{ji}=0,$$

$$\left(\frac{\partial g_{il}}{\partial \dot{x}^j}+\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l-\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}-2\frac{dg_{ij}}{dt}$$
$$=\left(\frac{\partial g_{il}}{\partial \dot{x}^j}+\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l-\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}-2\frac{\partial g_{ij}}{\partial t}-2\frac{\partial g_{ij}}{\partial x^l}\dot{x}^l-2\frac{\partial g_{ij}}{\partial \dot{x}^l}\ddot{x}^l$$
$$=\left(\frac{\partial g_{il}}{\partial \dot{x}^j}+\frac{\partial g_{jl}}{\partial \dot{x}^i}-2\frac{\partial g_{ij}}{\partial \dot{x}^l}\right)\ddot{x}^l-\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}-2\left(\frac{\partial g_{ij}}{\partial t}+\frac{\partial g_{ij}}{\partial x^l}\dot{x}^l\right)=0$$

and, after some calculation,

$$\left(\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^i}\right)\ddot{x}^l-\frac{\partial f_i}{\partial x^j}+\frac{\partial f_j}{\partial x^i}-\frac{1}{2}\frac{d}{dt}\left(\left(\frac{\partial g_{il}}{\partial \dot{x}^j}-\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l-\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)$$

$$=-\frac{\partial f_i}{\partial x^j}+\frac{\partial f_j}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(-\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^k}\left(-\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^k$$

$$+\left(\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial g_{il}}{\partial \dot{x}^j}-\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial g_{il}}{\partial \dot{x}^j}-\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(-\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)\right)\ddot{x}^l$$

$$-\frac{1}{2}\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial g_{il}}{\partial \dot{x}^j}-\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)\ddot{x}^l\ddot{x}^k-\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial \dot{x}^j}-\frac{\partial g_{jk}}{\partial \dot{x}^i}\right)\dddot{x}^k.$$

Consequently, we have a system

$$g_{ij}-g_{ji}=0,$$

$$\frac{\partial g_{il}}{\partial \dot{x}^j}+\frac{\partial g_{jl}}{\partial \dot{x}^i}-2\frac{\partial g_{ij}}{\partial \dot{x}^l}=0,$$

$$\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}+2\left(\frac{\partial g_{ij}}{\partial t}+\frac{\partial g_{ij}}{\partial x^l}\dot{x}^l\right)=0,$$

$$-\frac{\partial f_i}{\partial x^j}+\frac{\partial f_j}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(-\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^k}\left(-\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^k=0,$$

$$\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial g_{il}}{\partial \dot{x}^j}-\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial g_{il}}{\partial \dot{x}^j}-\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(-\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)=0,$$

$$\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial g_{il}}{\partial \dot{x}^j}-\frac{\partial g_{jl}}{\partial \dot{x}^i}\right)+\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial g_{ik}}{\partial \dot{x}^j}-\frac{\partial g_{jk}}{\partial \dot{x}^i}\right)=0,$$

$$\frac{\partial g_{ik}}{\partial \dot{x}^j}-\frac{\partial g_{jk}}{\partial \dot{x}^i}=0.$$

This system is equivalent to

$$g_{ij}-g_{ji}=0,$$

$$\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^i}\right)+\frac{\partial g_{ij}}{\partial t}+\frac{\partial g_{ij}}{\partial x^l}\dot{x}^l=0,$$

$$\frac{\partial f_i}{\partial x^j}-\frac{\partial f_j}{\partial x^i}+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)+\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^k=0,$$

$$\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^i}+\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)=0,$$

$$\frac{\partial g_{ik}}{\partial \dot{x}^j}-\frac{\partial g_{jk}}{\partial \dot{x}^i}=0,$$

that is to the system (2) – (6).

Supposing that integrability conditions (2) – (6) are satisfied we now find solutions $\mathscr{L}$ of the inverse problem equations

(7) $$g_{il}\ddot{x}^l - f_i = -\frac{\partial \mathscr{L}}{\partial x^i} + \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}^i}.$$

Theorem 12 *The source form* (1) *is variational if and only if*

$$\frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial f_i}{\partial \dot{x}^j} - \frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^l = 0.$$

Proof 1. First we solve the autonomous subsystem (2), (3). Its general solution $\mathscr{L}$ is, according to the Hessian matrix reconstruction lemma (Lemma 4)

$$\mathscr{L} = \mathscr{L}_0 + A + B_i\dot{x}^i,$$

where

$$\mathscr{L}_0 = \frac{1}{2}h_{ij}x^i x^j, \quad h_{ij} = 2\int_0^1\left(\int_0^1 g_{ij}(t, x^p, \kappa\tau\dot{x}^p)\,d\kappa\right)\tau\,d\tau$$

and A, B_i are arbitrary functions depending on t and x^p. Using $\mathscr{L}$, the Newton source form (1) has an expression

$$\begin{aligned}\varepsilon_i = g_{il}\ddot{x}^l - f_i &= \frac{\partial^2 \mathscr{L}_0}{\partial \dot{x}^i\,\partial \dot{x}^l}\ddot{x}^l - f_i \\ &= -\frac{\partial \mathscr{L}_0}{\partial x^i} + \frac{\partial^2 \mathscr{L}_0}{\partial t\,\partial \dot{x}^i} + \frac{\partial^2 \mathscr{L}_0}{\partial x^l\,\partial \dot{x}^i}\dot{x}^l + \frac{\partial^2 \mathscr{L}_0}{\partial \dot{x}^l\,\partial \dot{x}^i}\ddot{x}^l - f_i + \frac{\partial \mathscr{L}_0}{\partial x^i} - \frac{\partial^2 \mathscr{L}_0}{\partial t\,\partial \dot{x}^i} - \frac{\partial^2 \mathscr{L}_0}{\partial x^l\,\partial \dot{x}^i}\dot{x}^l \\ &= E_i(\mathscr{L}_0) - \phi_i,\end{aligned}$$

where

$$\phi_i = f_i - \frac{\partial \mathscr{L}_0}{\partial x^i} + \frac{\partial^2 \mathscr{L}_0}{\partial t\,\partial \dot{x}^i} + \frac{\partial^2 \mathscr{L}_0}{\partial x^l\,\partial \dot{x}^i}\dot{x}^l.$$

Thus, variationality of the Newton's source form (1) is equivalent to variationality of the (first-order) source form ϕ_i.

2. Conversely, according to Theorem 2, and Section 4, equations (2) and (4), integrability conditions for the source form (1) read

(7) $$\frac{\partial \phi_i}{\partial \dot{x}^j} + \frac{\partial \phi_j}{\partial \dot{x}^i} = 0$$

and

(8) $$\frac{\partial\phi_i}{\partial x^j}-\frac{\partial\phi_j}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\phi_i}{\partial\dot{x}^j}-\frac{\partial\phi_j}{\partial\dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial\phi_i}{\partial\dot{x}^j}-\frac{\partial\phi_j}{\partial\dot{x}^i}\right)\dot{x}^l=0.$$

Equations (7) can be integrated. We have

$$\phi_i = A_i + B_{il}\dot{x}^l,$$

where A_i and B_{il} are arbitrary functions of t and x^l such that $B_{ij}=-B_{ji}$. Thus

$$f_i = A_i + B_{il}\dot{x}^l + \frac{\partial\mathscr{L}_0}{\partial x^i}-\frac{\partial^2\mathscr{L}_0}{\partial t\,\partial\dot{x}^i}-\frac{\partial^2\mathscr{L}_0}{\partial x^l\,\partial\dot{x}^i}\dot{x}^l.$$

To determine condition (8), first calculate partial derivatives of the functions ϕ_i:

$$\frac{\partial\phi_i}{\partial x^j}=\frac{\partial f_i}{\partial x^j}-\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial x^j}+\frac{\partial^3\mathscr{L}_0}{\partial t\,\partial x^j\,\partial\dot{x}^i}+\frac{\partial^3\mathscr{L}_0}{\partial x^l\,\partial x^j\,\partial\dot{x}^i}\dot{x}^l,$$

$$\frac{\partial\phi_i}{\partial\dot{x}^j}=\frac{\partial f_i}{\partial\dot{x}^j}-\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial\dot{x}^j}+\frac{\partial^2\mathscr{L}_0}{\partial x^j\,\partial\dot{x}^i}+\frac{\partial^3\mathscr{L}_0}{\partial t\,\partial\dot{x}^i\,\partial\dot{x}^j}+\frac{\partial^3\mathscr{L}_0}{\partial x^l\,\partial\dot{x}^i\,\partial\dot{x}^j}\dot{x}^l.$$

Hence

$$\begin{aligned}\frac{\partial\phi_i}{\partial x^j}-\frac{\partial\phi_j}{\partial x^i}&=\frac{\partial f_i}{\partial x^j}-\frac{\partial f_j}{\partial x^i}\\&+\frac{\partial}{\partial t}\left(\frac{\partial^2\mathscr{L}_0}{\partial x^j\,\partial\dot{x}^i}-\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial\dot{x}^j}\right)+\frac{\partial}{\partial x^l}\left(\frac{\partial^2\mathscr{L}_0}{\partial x^j\,\partial\dot{x}^i}-\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial\dot{x}^j}\right)\dot{x}^l\end{aligned}$$

and

$$\frac{\partial\phi_i}{\partial\dot{x}^j}-\frac{\partial\phi_j}{\partial\dot{x}^i}=\frac{\partial f_i}{\partial\dot{x}^j}-\frac{\partial f_j}{\partial\dot{x}^i}-2\left(\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial\dot{x}^j}-\frac{\partial^2\mathscr{L}_0}{\partial x^j\,\partial\dot{x}^i}\right).$$

Then

$$\begin{aligned}&\frac{\partial\phi_i}{\partial x^j}-\frac{\partial\phi_j}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\phi_i}{\partial\dot{x}^j}-\frac{\partial\phi_j}{\partial\dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial\phi_i}{\partial\dot{x}^j}-\frac{\partial\phi_j}{\partial\dot{x}^i}\right)\dot{x}^l\\&=\frac{\partial f_i}{\partial x^j}-\frac{\partial f_j}{\partial x^i}+\frac{\partial}{\partial t}\left(\frac{\partial^2\mathscr{L}_0}{\partial x^j\,\partial\dot{x}^i}-\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial\dot{x}^j}\right)+\frac{\partial}{\partial x^l}\left(\frac{\partial^2\mathscr{L}_0}{\partial x^j\,\partial\dot{x}^i}-\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial\dot{x}^j}\right)\dot{x}^l\\&\quad-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial f_i}{\partial\dot{x}^j}-\frac{\partial f_j}{\partial\dot{x}^i}\right)+\frac{\partial}{\partial t}\left(\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial\dot{x}^j}-\frac{\partial^2\mathscr{L}_0}{\partial x^j\,\partial\dot{x}^i}\right)\\&\quad-\frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial f_i}{\partial\dot{x}^j}-\frac{\partial f_j}{\partial\dot{x}^i}\right)\dot{x}^l+\frac{\partial}{\partial x^l}\left(\frac{\partial^2\mathscr{L}_0}{\partial x^i\,\partial\dot{x}^j}-\frac{\partial^2\mathscr{L}_0}{\partial x^j\,\partial\dot{x}^i}\right)\dot{x}^l\\&=\frac{\partial f_i}{\partial x^j}-\frac{\partial f_j}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial f_i}{\partial\dot{x}^j}-\frac{\partial f_j}{\partial\dot{x}^i}\right)\dot{x}^l.\end{aligned}$$

Since this expression should vanish, we get formula (7) as required.

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