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The Kruskal-Szekeres globalization

Contents

In this lecture we describe all metric tensor fields on the manifold $\mathbf{R}^3\setminus\{(0,0,0)\}$ and $\mathbf{R}\times(\mathbf{R}^3\setminus\{(0,0,0)\})$, invariant with respect to rotations. The rotations are elements of the group $\mathrm{SO}(3)$ of 3×3 matrices τ such that the transposed of τ is equal to the inverse, ${}^t\tau=\tau^{-1}$, and $\det\tau=1$. We consider the standard left action of $\mathrm{SO}(3)$ on $\mathbf{R}^3\setminus\{(0,0,0)\}$, and call a metric tensor field *g*-*invariant* with respect to rotations, if $\tau^*g=g$ for all $\tau\in\mathrm{SO}(3)$; this definition also implies to the manifold $\mathbf{R}\times(\mathbf{R}^3\setminus\{(0,0,0)\})$ via the second Cartesian factor.

Then we consider the standard Schwartzschild solution of the Einstein equations. We give a geometric construction of a 2-dimensional manifold X, diffeomorphic with $\mathbf{R}^2$, equipped with a metric g of Lorentz type, and satisfying the following two conditions:

(a) X is the union of mutually disjoint open submanifolds P and Q and a 1-dimensional closed submanifold S, and

(b) The restrictions of g to P and Q coincide with the Schwarzschild solution of the Einstein equations.

This construction provides a new interpretation of the well-known "Kruskal-Szekeres extension" as the globalization of local coordinate expressions of the Schwarzschild metric field. The metric g on X, extending the Schwarzschild metric, is given explicitly.

1 Introduction

Our aim in this lecture is to analyse the well-known relativistic concepts, related with spherically symmetric solutions of the Einstein equations, namely the Schwartzschild solution

$$(1) \qquad g_m = -\frac{r-2m}{r} dt \otimes dt + \frac{r}{r-2m} dr \otimes dr + r^2 (d\vartheta \otimes d\vartheta + \sin^2 \vartheta \cdot d\varphi \otimes d\varphi),$$

the "Kruskal-Szekeres coordinates", and the "Kruskal-Szekeres extension".

We use notations and standard terminology that allows us to restrict our considerations to 2-dimensional underlying manifolds. In our constructions we take over the well-known Schwarzschild, Kruskal and Szekeres coordinate expressions, giving them, however, a different geometric meaning. However, our basic results - an explicit manifold description of what could be called the *Kruskal-Szekeres spacetime*, as well as the methods of its construction - differ from the usual coordinate settings; formally, they seem to be closest to Kriele [4].

We consider the 2-dimensional manifold $M = \mathbf{R} \times (0,\infty)$, its open submanifolds $P = \mathbf{R} \times (0,2m)$ and $Q = \mathbf{R} \times (2m,\infty)$, and a metric field g_m, defined on $P \cup Q$ by

$$(2) \qquad g_m = -\frac{r-2m}{r} dt \otimes dt + \frac{r}{r-2m} dr \otimes dr.$$

We call g_m the *Schwartzschild metric field*. Our main objective is the discuss properties of the points $(t,2m) \in M$ that do not belong to the open set $P \cup Q$.

Historically (indeed in the context of 4-dimensional relativistic spacetimes), many authors regard the points $(t,2m)$ as "singular points" of the Schwartzschild metric. There has been a permanent effort to find a way, how to include these points into the domain of definition of g_m or "remove" them from the underlying spacetime. The most well known tools have become the Kruskal-Szekeres "coordinates"; the result is known as the "Kruskal-Szekeres extension" of the underlying manifold and "extension of the Schwarzschild metric field" (see e.g. De Felice and Clarke [3], and Kriele [4]). However, in spite of these discussions it is obvious from (1) that g_m *cannot* be extended to a continuous metric field, defined everywhere on M, and the "singularity" cannot be "removed".

Many questions concerned with these contradictory ideas, have been presented by different authors with different understanding and different opinions or criticism (see for instance Crothers [2] and references therein).

In this article we do not follow these lines. We regard g_m to be a *chart expression* of a metric field g, defined on another manifold X, not on M. We construct a 2-dimensional manifold X, diffeomorphic with the Euclidean space $\mathbf{R}^2$, and a metric field g of Lorentz type on X, with the following two properties:

(a) X is the union of mutually disjoint open submanifolds P and Q and a 1-dimensional closed submanifold S, and

(b) the restrictions of g to P and Q coincide with the Schwarzschild solution (1) of the Einstein equations.

Clearly, X will be determined up to a diffeomorphism. A concrete model for X, P, and Q can easily be recognized from a diagram describing the Kruskal-Szekeres coordinates (see [4]). The metric tensor field g, whose restrictions to P and Q coincide with g_m, is given by

$$(3) \qquad g = 16m^2 g_0,$$

where g_0 is a metric field on an open subset of $\mathbf{R}^2$, defined by

$$(4) \qquad g_0 = \frac{1}{1+W((U^2-V^2)/\mathrm{e})}\mathrm{e}^{-1-W((U^2-V^2)/\mathrm{e})}(dU\otimes dU - dV\otimes dV),$$

U and V are the canonical coordinates on $\mathbf{R}^2$ and W is the Lambert function.

Note that the metric field g_0 is independent of the mass m. Two metric fields, corresponding with different masses, are in conformal correspondence.

Our analysis has also some general aspects, namely, what kind of data we have when we are given a solution of the Einstein equations. To give a sense to these local data (metric fields defined in terms of coordinates), we need to embed them in a concrete spacetime manifold, and then to globalize them in this manifold. In this sense the method we use in this paper can also be regarded as an example of globalization of the (local) Schwarzschild metric fields. The topology of the resulting manifold is in this case the topology of $\mathbf{R}^2$.

From the nature of these results we prefer to use the phrase *Kruskal-Szekeres globalization* instead of *Kruskal-Szekeres extension*.

2 Rotations in Euler angles

Consider a rotation τ of the Euclidean space $\mathbf{R}^3$, given in the canonical coordinates as a mapping

(1) $\bar{x} = x \circ \tau, \quad \bar{y} = y \circ \tau, \quad \bar{z} = z \circ \tau.$

τ can be characterized explicitly by means of three real parameters, the *Euler angles* Φ_1, Φ_2 and Θ; equations of the rotation τ are

(2)
$$\begin{aligned}
\bar{x} &= (\cos\Phi_1 \cos\Phi_2 - \sin\Phi_1 \cos\Theta \sin\Phi_2) \cdot x \\
&\quad - (\cos\Phi_1 \sin\Phi_2 + \sin\Phi_1 \cos\Theta \cos\Phi_2) \cdot y + \sin\Phi_1 \sin\Theta \cdot z, \\
\bar{y} &= (\sin\Phi_1 \cos\Phi_2 + \cos\Phi_1 \cos\Theta \sin\Phi_2) \cdot x \\
&\quad + (-\sin\Phi_1 \sin\Phi_2 + \cos\Phi_1 \cos\Theta \cos\Phi_2) \cdot y - \cos\Phi_1 \sin\Theta \cdot z, \\
\bar{z} &= \sin\Phi_2 \sin\Theta \cdot x + \cos\Phi_2 \sin\Theta \cdot y + \cos\Theta \cdot z.
\end{aligned}$$

The domain of definition of the mapping (2) is $\mathbf{R}^3$, and the mapping is composed of periodic functions. We may restrict the domain of definition if suitable to the sets where (2) is a three-parameter family of diffeomorphism. If we take for the domain of definition of the parameters the set

(3) $-\pi < \Phi_1 < \pi, \quad -\pi < \Phi_2 < \pi, \quad -\frac{\pi}{2} < \Theta < \frac{\pi}{2},$

then the point $(\Phi_1, \Phi_2, \Theta) = (0,0,0)$ defines the identity rotation. The parameters $(\Phi_1, \Phi_2, \Theta) = (0, \Phi_2, 0)$, $(\Phi_1, \Phi_2, \Theta) = (0,0,\Theta)$, $(\Phi_1, \Phi_2, \Theta) = (\Phi_1, 0, 0)$ define one-parameter families of rotations

(4)
$$\begin{aligned}
\bar{x} &= \cos\Phi_2 \cdot x - \sin\Phi_2 \cdot y, \\
\bar{y} &= \sin\Phi_2 \cdot x + \cos\Phi_2 \cdot y, \\
\bar{z} &= z,
\end{aligned}$$

and

(5)
$$\begin{aligned}
\bar{x} &= x, \\
\bar{y} &= \cos\Theta \cdot y - \sin\Theta \cdot z, \\
\bar{z} &= \sin\Theta \cdot y + \cos\Theta \cdot z,
\end{aligned}$$

and

$$\begin{aligned}&\bar{x} = \cos\Phi_1 \cdot x - \sin\Phi_1 \cdot y,\\ &\bar{y} = \sin\Phi_1 \cdot x + \cos\Phi_1 \cdot y,\\ &\bar{z} = z.\end{aligned} \tag{6}$$

Note that multiplying these three matrices,

$$\begin{pmatrix} \cos\Phi_1 & -\sin\Phi_1 & 0 \\ \sin\Phi_1 & \cos\Phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta \\ 0 & \sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} \cos\Phi_2 & -\sin\Phi_2 & 0 \\ \sin\Phi_2 & \cos\Phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

we get the matrix of the transformation equations (2).

One can directly verify that the rotation matrices do not commute. The inverse of the rotation, given by the parameters (Φ_1, Φ_2, Θ), is the rotation given by $(-\Phi_2, -\Phi_1, -\Theta)$. Indeed, we have from (2), replacing the triple (Φ_1, Φ_2, Θ) with $(-\Phi_2, -\Phi_1, -\Theta)$,

$$\begin{aligned}&\bar{\bar{x}} = (\cos\Phi_2 \cos\Phi_1 - \sin\Phi_2 \cos\Theta \sin\Phi_1) \cdot \bar{x}\\ &\quad + (\cos\Phi_2 \sin\Phi_1 + \sin\Phi_2 \cos\Theta \cos\Phi_1) \cdot \bar{y} + \sin\Phi_2 \sin\Theta \cdot \bar{z},\\ &\bar{\bar{y}} = -(\sin\Phi_2 \cos\Phi_1 + \cos\Phi_2 \cos\Theta \sin\Phi_1) \cdot \bar{x}\\ &\quad + (-\sin\Phi_2 \sin\Phi_1 + \cos\Phi_2 \cos\Theta \cos\Phi_1) \cdot \bar{y} + \cos\Phi_2 \sin\Theta \cdot \bar{z},\\ &\bar{\bar{z}} = \sin\Phi_1 \sin\Theta \cdot \bar{x} - \cos\Phi_1 \sin\Theta \cdot \bar{y} + \cos\Theta \cdot \bar{z}.\end{aligned} \tag{8}$$

The matrix of this transformation is transposed to the matrix of (2) so it coincides with the inverse matrix of equations (2). On the other hand, it is by definition the *inverse* of the matrix of (2), because the rotation (Φ_1, Φ_2, Θ) followed by the rotation $(-\Phi_2, -\Phi_1, -\Theta)$ is the identity rotation.

For geometrical meaning of the Euler angles see e.g. I.M. Gelfand, P.A. Minlos and Z.J. Shapiro, *Representations of the rotation group and the Lorentz group*, GIFML, Moscow, 1958 (Russian).

Note that the choice of parameters $\Phi_1 = \pi/2$, $\Phi_2 = -\pi/2$ defines a one-parameter family of rotations around the *y*-axix,

$$\begin{aligned}&\bar{x} = \cos\Theta \cdot x + \sin\Theta \cdot z,\\ &\bar{y} = y,\\ &\bar{z} = -\sin\Theta \cdot x + \cos\Theta \cdot z.\end{aligned} \tag{9}$$

Another useful example of a rotation, which is the composition of the rotation α around the *z*-axis and the rotation β around the *x*-axis. The matrix is given by (7),

$$(10)\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \cos\beta\sin\alpha & \cos\beta\cos\alpha & -\sin\beta \\ \sin\beta\sin\alpha & \sin\beta\cos\alpha & \cos\beta \end{pmatrix}.$$

Let ν be the rotation, defined by $\alpha = \pi$ and $\beta = \pi/2$. Then ν has a matrix

$$(11)\quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

and the equations

$$(12)\quad x\circ\nu = -x, \quad y\circ\nu = -z, \quad z\circ\nu = -y.$$

Remark It follows from the decomposition (7) of any rotation that to prove that a differential form is invariant with respect to all rotations it is sufficient to prove its invariance with respect to rotations around the *x*-axis and *z*-axis.

3 Spherical coordinates

By the *first spherical chart* on the manifold $\mathbf{R}^3 \setminus \{(0,0,0)\}$ we mean a chart (U,Φ), $\Phi = (r,\varphi,\vartheta)$, where

$$(1)\quad U = \{(x,y,z) \in \mathbf{R}^3 \setminus \{(0,0,0)\} \mid x \geq 0, y = 0\}$$

and the coordinate functions are defined by

$$(2)\quad r = \sqrt{x^2+y^2+z^2}, \quad \cos\vartheta = \frac{z}{\sqrt{x^2+y^2+z^2}}, \quad \tan\varphi = \frac{y}{x}.$$

The range of the chart in $\mathbf{R}^3$ is the set where

$$(3)\quad r > 0, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi.$$

The transformation equations to the canonical (Cartesian) coordinates are

(4) $$x = r\sin\vartheta\cos\varphi, \quad y = r\sin\vartheta\sin\varphi, \quad z = r\cos\vartheta.$$

We now introduce another chart on $\mathbf{R}^3\setminus\{(0,0,0)\}$ as a modification of the first spherical chart. We use the mapping ν, defined by the equations $x\circ\nu = -x$, $y\circ\nu = -z$, $z\circ\nu = -y$ (Section 1, (12)) and set

(5) $$\begin{aligned}\bar U &= \nu^{-1}(U) = \{(x,y,z)\in\mathbf{R}^3\setminus\{(0,0,0)\} \mid \nu(x,y,z)\in U\}\\ &= \{(x,y,z)\in\mathbf{R}^3\setminus\{(0,0,0)\} \mid x\circ\nu \ge 0, y\circ\nu = 0\}\\ &= \{(x,y,z)\in\mathbf{R}^3\setminus\{(0,0,0)\} \mid x\le 0, z = 0\},\end{aligned}$$

and $\bar\Phi = \Phi\circ\nu = (\bar r,\bar\varphi,\bar\vartheta)$, where

(6) $$\begin{aligned}\bar r &= r\circ\nu = \sqrt{(x\circ\nu)^2+(y\circ\nu)^2+(z\circ\nu)^2} = \sqrt{x^2+y^2+z^2},\\ \bar\vartheta &= \vartheta\circ\nu = \left(\arccos\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\circ\nu\\ &= \arccos\left(-\frac{y}{\sqrt{x^2+y^2+z^2}}\right),\\ \bar\varphi &= \varphi\circ\nu = \left(\arctan\frac{y}{x}\right)\circ\nu = \arctan\left(\frac{z}{x}\right).\end{aligned}$$

or, in short,

(7) $$\bar r = \sqrt{x^2+y^2+z^2}, \quad \cos\bar\vartheta = -\frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \tan\bar\varphi = \frac{z}{x}.$$

Then the pair $(\bar U,\bar\Phi)$, $\bar\Phi = (\bar r,\bar\varphi,\bar\vartheta)$, is a chart on $\mathbf{R}^3\setminus\{(0,0,0)\}$, called the *second spherical chart*.

We determine the inverse transformation of the transformation (7). We have

(8) $$\begin{aligned}\sin\bar\varphi &= \sqrt{1-\cos^2\bar\varphi} = \sqrt{1-\frac{1}{1+\tan^2\bar\varphi}} = \sqrt{\frac{\tan^2\bar\varphi}{1+\tan^2\bar\varphi}}\\ &= \sqrt{\frac{\frac{z^2}{x^2}}{1+\frac{z^2}{x^2}}} = \sqrt{\frac{z^2}{x^2+z^2}},\\ \cos\bar\varphi &= \sqrt{1-\frac{z^2}{x^2+z^2}} = \sqrt{\frac{x^2}{x^2+z^2}},\end{aligned}$$

from which we have

(9) $$\begin{aligned}\cos\bar\vartheta &= -\frac{y}{\sqrt{x^2+y^2+z^2}},\\ \sin\bar\vartheta &= \sqrt{1-\frac{y^2}{x^2+y^2+z^2}} = \sqrt{\frac{x^2+z^2}{x^2+y^2+z^2}},\\ \sin\bar\vartheta\sin\bar\varphi &= \sqrt{\frac{x^2+z^2}{x^2+y^2+z^2}}\sqrt{\frac{z^2}{x^2+z^2}} = \sqrt{\frac{z^2}{x^2+y^2+z^2}} = \frac{z}{\bar r},\\ \sin\bar\vartheta\cos\bar\varphi &= \sqrt{\frac{x^2+z^2}{x^2+y^2+z^2}}\sqrt{\frac{x^2}{x^2+z^2}} = \sqrt{\frac{x^2}{x^2+y^2+z^2}} = \frac{x}{\bar r},\end{aligned}$$

hence

(10) $$x = \bar r\sin\bar\vartheta\cos\bar\varphi, \quad y = -\bar r\cos\bar\vartheta, \quad z = \bar r\sin\bar\vartheta\sin\bar\varphi.$$

In particular, the spherical charts (U,Φ) and $(\bar{U},\bar{\Phi})$ define an atlas, the *spherical atlas* on $\mathbf{R}^3\setminus\{(0,0,0)\}$. To see this, we find an explicit expression for the coordinate transformation

$$\text{(11)}\qquad \Phi(U\cap\bar{U})\ni(r,\varphi,\vartheta)\to(\bar{r},\bar{\varphi},\bar{\vartheta})=\bar{\Phi}\Phi^{-1}(r,\varphi,\vartheta)\in\bar{\Phi}(U\cap\bar{U}).$$

We also determine for further use the corresponding Jacobi matrix explicitly.

If $(x,y,z)\in U\cap\bar{U}$, we have from (4) and (11)

$$\text{(12)}\qquad \begin{aligned} &r\sin\vartheta\cos\varphi=\bar{r}\sin\bar{\vartheta}\cos\bar{\varphi},\\ &r\sin\vartheta\sin\varphi=-\bar{r}\cos\bar{\vartheta},\\ &r\cos\vartheta=\bar{r}\sin\bar{\vartheta}\sin\bar{\varphi}.\end{aligned}$$

From these equations we get the transformation law

$$\text{(13)}\qquad \bar{r}=r,\quad \cos\bar{\vartheta}=-\sin\vartheta\sin\varphi,\quad \tan\bar{\varphi}=\frac{\cos\vartheta}{\sin\vartheta\cos\varphi}.$$

Note an important consequence of formulas (12).

Theorem 1 *At every point* $(x,y,z)\in U\cap\bar{U}$

$$\text{(14)}\qquad d\vartheta\otimes d\vartheta+\sin^2\vartheta d\varphi\otimes d\varphi=d\bar{\vartheta}\otimes d\bar{\vartheta}+\sin^2\bar{\vartheta}d\bar{\varphi}\otimes d\bar{\varphi}.$$

Proof We differentiate both sides of (12) and apply to the left-hand side expression a certain tensorial construction. Then we repeat the same for the right-hand side and compare the resulting expressions.

Differentiating (12) we get

$$\text{(15)}\qquad \begin{aligned} &\cos\vartheta\cos\varphi d\vartheta-\sin\vartheta\sin\varphi d\varphi=\cos\bar{\vartheta}\cos\bar{\varphi}d\bar{\vartheta}-\sin\bar{\vartheta}\sin\bar{\varphi}d\bar{\varphi},\\ &\cos\vartheta\sin\varphi d\vartheta+\sin\vartheta\cos\varphi d\varphi=\sin\bar{\vartheta}d\bar{\vartheta},\\ &-\sin\vartheta d\vartheta=\cos\bar{\vartheta}\sin\bar{\varphi}d\bar{\vartheta}+\sin\bar{\vartheta}\cos\bar{\varphi}d\bar{\varphi}.\end{aligned}$$

The linear forms on the left-hand side define a $(0,2)$-tensor

$$
\begin{aligned}
&(\cos\vartheta\cos\varphi d\vartheta-\sin\vartheta\sin\varphi d\varphi)\otimes(\cos\vartheta\cos\varphi d\vartheta-\sin\vartheta\sin\varphi d\varphi)\\
&\quad+(\cos\vartheta\sin\varphi d\vartheta+\sin\vartheta\cos\varphi d\varphi)\otimes(\cos\vartheta\sin\varphi d\vartheta+\sin\vartheta\cos\varphi d\varphi)\\
&\quad+\sin\vartheta d\vartheta\otimes\sin\vartheta d\vartheta\\
&\quad=\cos^2\vartheta\cos^2\varphi d\vartheta\otimes d\vartheta-\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi d\vartheta\otimes d\varphi\\
&\quad+\cos^2\vartheta\sin^2\varphi d\vartheta\otimes d\vartheta+\cos\vartheta\sin\vartheta\sin\varphi\cos\varphi d\vartheta\otimes d\varphi\\
&\quad+\sin\vartheta\cos\vartheta\cos\varphi\sin\varphi d\varphi\otimes d\vartheta+\sin^2\vartheta\cos^2\varphi d\varphi\otimes d\varphi,\\
&\quad+\sin^2\vartheta d\vartheta\otimes d\vartheta\\
&\quad=\cos^2\vartheta d\vartheta\otimes d\vartheta+\sin^2\vartheta d\varphi\otimes d\varphi+\sin^2\vartheta d\vartheta\otimes d\vartheta\\
&\quad=d\vartheta\otimes d\vartheta+\sin^2\vartheta d\varphi\otimes d\varphi.
\end{aligned}\tag{16}
$$

The same construction applied to the right-hand side yields

$$
\begin{aligned}
&(\cos\bar\vartheta\cos\bar\varphi d\bar\vartheta-\sin\bar\vartheta\sin\bar\varphi d\bar\varphi)\otimes(\cos\bar\vartheta\cos\bar\varphi d\bar\vartheta-\sin\bar\vartheta\sin\bar\varphi d\bar\varphi)\\
&\quad+(\sin\bar\vartheta d\bar\vartheta)\otimes(\sin\bar\vartheta d\bar\vartheta)\\
&\quad+(\cos\bar\vartheta\sin\bar\varphi d\bar\vartheta+\sin\bar\vartheta\cos\bar\varphi d\bar\varphi)\otimes(\cos\bar\vartheta\sin\bar\varphi d\bar\vartheta+\sin\bar\vartheta\cos\bar\varphi d\bar\varphi)\\
&\quad=\cos^2\bar\vartheta\cos^2\bar\varphi d\bar\vartheta\otimes d\bar\vartheta-\cos\bar\vartheta\sin\bar\vartheta\cos\bar\varphi\sin\bar\varphi d\bar\vartheta\otimes d\bar\varphi\\
&\quad-\sin\bar\vartheta\cos\bar\vartheta\cos\bar\varphi\sin\bar\varphi d\bar\varphi)\otimes d\bar\vartheta+\sin^2\bar\vartheta\sin^2\bar\varphi d\bar\varphi)\otimes d\bar\varphi\\
&\quad+\sin^2\bar\vartheta d\bar\vartheta\otimes\sin\bar\vartheta d\bar\vartheta\\
&\quad+\cos^2\bar\vartheta\sin^2\bar\varphi d\bar\vartheta\otimes d\bar\vartheta+\cos\bar\vartheta\sin\bar\vartheta\cos\bar\varphi\sin\bar\varphi d\bar\vartheta\otimes d\bar\varphi\\
&\quad+\sin\bar\vartheta\cos\bar\vartheta\sin\bar\varphi\cos\bar\varphi d\bar\varphi\otimes d\bar\vartheta\\
&\quad+\sin^2\bar\vartheta\cos^2\bar\varphi d\bar\varphi\otimes d\bar\varphi\\
&\quad=\cos^2\bar\vartheta\cos^2\bar\varphi d\bar\vartheta\otimes d\bar\vartheta+\cos^2\bar\vartheta\sin^2\bar\varphi d\bar\vartheta\otimes d\bar\vartheta+\sin^2\bar\vartheta d\bar\vartheta\otimes d\bar\vartheta\\
&\quad+\sin^2\bar\vartheta\sin^2\bar\varphi d\bar\varphi\otimes d\bar\varphi+\sin^2\bar\vartheta\cos^2\bar\varphi d\bar\varphi\otimes d\bar\varphi\\
&\quad=d\bar\vartheta\otimes d\bar\vartheta+d\bar\vartheta\otimes d\bar\vartheta+\sin^2\bar\vartheta d\bar\varphi\otimes d\bar\varphi.
\end{aligned}\tag{17}
$$

This proves that $d\vartheta\otimes d\vartheta+\sin^2\vartheta d\varphi\otimes d\varphi=d\bar\vartheta\otimes d\bar\vartheta+\sin^2\bar\vartheta d\bar\varphi\otimes d\bar\varphi$ as required.

The spherical atlas induces an atlas on the unique sphere in $\mathbf{R}^3\setminus\{(0,0,0)\}$, $S^2=\{(x,y,z)\in\mathbf{R}^3\mid x^2+y^2+z^2=1\}$. S^2 is a submanifold of $\mathbf{R}^3\setminus\{(0,0,0)\}$, given in the spherical atlas by the equations

$$r=1,\quad \bar r=1. \tag{18}$$

Setting $V=U\cap S^2$, $\Psi=(\varphi,\vartheta)$, and $\bar V=\bar U\cap S^2$, $\bar\Psi=(\bar\varphi,\bar\vartheta)$, we get two charts (V,Ψ) and $(\bar V,\bar\Psi)$ on the sphere S^2, forming the associated atlas on S^2.

Note we have the *canonical identification* $\theta:\mathbf{R}^3\setminus\{(0,0,0)\}\to(0,\infty)\times S^2$, defined by

$$\theta(x,y,z) = \left(\sqrt{x^2+y^2+z^2},\left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right). \tag{19}$$

Denote

$$x_0=\frac{x}{\sqrt{x^2+y^2+z^2}},\quad y_0=\frac{y}{\sqrt{x^2+y^2+z^2}},\quad z_0=\frac{z}{\sqrt{x^2+y^2+z^2}}, \tag{20}$$

The first canonical coordinates of a point (x,y,z) are $r(x,y,z)$, $\varphi(x,y,z)$, and $\vartheta(x,y,z)$ while the image of this point $\theta(x,y,z)$ has the coordinates $r(x,y,z)$, 1, $\varphi(x_0,y_0,z_0)=\varphi(x,y,z)$, and $\vartheta(x_0,y_0,z_0)=\vartheta(x,y,z)$. Thus, the chart expression of θ is the mapping $(r,\varphi,\vartheta)\to(r,(1,\varphi,\vartheta))$.

Our aim now will be to find the Jacobian matrix of the coordinate transformation (13). To this purpose we first compute from (9) the differentials. We have

$$\begin{aligned}
&d\cos\bar\vartheta=\sin\varphi\cos\vartheta d\vartheta+\sin\vartheta\cos\varphi d\varphi\\
&\quad=\frac{\partial\cos\bar\vartheta}{\partial\varphi}d\varphi+\frac{\partial\cos\bar\vartheta}{\partial\vartheta}d\vartheta=-\sin\bar\vartheta\frac{\partial\bar\vartheta}{\partial\varphi}d\varphi-\sin\bar\vartheta\frac{\partial\bar\vartheta}{\partial\vartheta}d\vartheta,\\
&d\tan\bar\varphi=\frac{\partial}{\partial\varphi}\frac{\cot\vartheta}{\cos\varphi}d\varphi+\frac{\partial}{\partial\vartheta}\frac{\cot\vartheta}{\cos\varphi}d\vartheta\\
&\quad=-\frac{\cot\vartheta\sin\varphi}{\cos^2\varphi}d\varphi-\frac{1}{\cos\varphi\sin^2\vartheta}d\vartheta\\
&\quad=\frac{\partial\tan\bar\varphi}{\partial\varphi}d\varphi+\frac{\partial\tan\bar\varphi}{\partial\vartheta}d\vartheta=\frac{1}{\cos^2\bar\varphi}\frac{\partial\bar\varphi}{\partial\varphi}d\varphi+\frac{1}{\cos^2\bar\varphi}\frac{\partial\bar\varphi}{\partial\vartheta}d\vartheta,
\end{aligned} \tag{21}$$

from which we conclude that

$$\begin{aligned}
&\sin\varphi\cos\vartheta=-\sin\bar\vartheta\frac{\partial\bar\vartheta}{\partial\vartheta},\quad \sin\vartheta\cos\varphi=-\sin\bar\vartheta\frac{\partial\bar\vartheta}{\partial\varphi},\\
&-\frac{\cot\vartheta\sin\varphi}{\cos^2\varphi}=\frac{1}{\cos^2\bar\varphi}\frac{\partial\bar\varphi}{\partial\varphi},\quad -\frac{1}{\cos\varphi\sin^2\vartheta}=\frac{1}{\cos^2\bar\varphi}\frac{\partial\bar\varphi}{\partial\vartheta}.
\end{aligned} \tag{22}$$

In these formulas

$$\sin\overline{\vartheta} = \sqrt{1-\sin^2\vartheta\sin^2\varphi},$$

(23) $$\cos^2\overline{\varphi} = \frac{1}{\sqrt{1+\tan^2\overline{\varphi}}} = \frac{1}{\sqrt{1+\dfrac{\cot^2\vartheta}{\cos^2\varphi}}} = \frac{\cos\varphi}{\sqrt{\cos^2\varphi+\cot^2\vartheta}},$$

hence

(24) $$\frac{\partial\overline{\vartheta}}{\partial\vartheta} = -\frac{\sin\varphi\cos\vartheta}{\sqrt{1-\sin^2\vartheta\sin^2\varphi}}, \quad \frac{\partial\overline{\vartheta}}{\partial\varphi} = -\frac{\sin\vartheta\cos\varphi}{\sqrt{1-\sin^2\vartheta\sin^2\varphi}},$$
$$\frac{\partial\overline{\varphi}}{\partial\varphi} = -\frac{\cot\vartheta\sin\varphi}{\cos\varphi}\frac{1}{\sqrt{\cos^2\varphi+\cot^2\vartheta}}, \quad \frac{\partial\overline{\varphi}}{\partial\vartheta} = -\frac{1}{\sin^2\vartheta}\frac{1}{\sqrt{\cos^2\varphi+\cot^2\vartheta}},$$

These formulas define the Jacobi matrix of the coordinate transformation (13).

4 Generators of rotations

The generators of rotations around the coordinate axes are expressed by the vector fields (4), (5), (9), Section 1

(1) $$\xi = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \zeta = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \lambda = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$

Our aim now will be to find their expressions in the spherical coordinates. We want to show that

(2) $$\xi = \frac{\partial}{\partial\varphi}, \quad \zeta = -\sin\varphi\frac{\partial}{\partial\vartheta} - \cot\vartheta\cos\varphi\frac{\partial}{\partial\varphi},$$
$$\lambda = \cos\varphi\frac{\partial}{\partial\vartheta} - \cot\vartheta\sin\varphi\frac{\partial}{\partial\varphi},$$

We determine from the transformation formulas different derivatives:

(3) $$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \sin\vartheta\cos\varphi,$$
$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \sin\vartheta\sin\varphi,$$
$$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \cos\vartheta,$$

and

$$\begin{aligned}
&\frac{\partial \varphi}{\partial x} = -\frac{1}{1+\dfrac{y^2}{x^2}}\frac{y}{x^2} = -\frac{y}{x^2+y^2} = -\frac{r\sin\vartheta\sin\varphi}{r^2\sin^2\vartheta\cos^2\varphi + r^2\sin^2\vartheta\sin^2\varphi}\\
&\quad = -\frac{\sin\vartheta\sin\varphi}{r\sin^2\vartheta} = -\frac{\sin\varphi}{r\sin\vartheta}\\
&\frac{\partial \varphi}{\partial y} = \frac{1}{1+\dfrac{y^2}{x^2}}\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{r\sin\vartheta\cos\varphi}{r^2\sin^2\vartheta\cos^2\varphi + r^2\sin^2\vartheta\sin^2\varphi}\\
&\quad = \frac{\sin\vartheta\cos\varphi}{r\sin^2\vartheta} = \frac{\cos\varphi}{r\sin\vartheta},\\
&\frac{\partial \varphi}{\partial z} = 0,
\end{aligned} \tag{4}$$

and

$$\frac{\partial\vartheta}{\partial x}=\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{\frac{zx}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$=\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{zx}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2+z^2)}$$

$$=\frac{1}{\sqrt{x^2+y^2}}\frac{zx}{x^2+y^2+z^2}=\frac{1}{r\sin\vartheta}\frac{r^2\cos\vartheta\sin\vartheta\cos\varphi}{r^2}=\frac{1}{r}\cos\vartheta\cos\varphi,$$

$$\frac{\partial\vartheta}{\partial y}=\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{\frac{zy}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$=\frac{1}{\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}}\frac{zy}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2+z^2)} \tag{5}$$

$$=\frac{1}{\sqrt{x^2+y^2}}\frac{zy}{x^2+y^2+z^2}=\frac{\sin\vartheta\sin\varphi\cos\vartheta}{r\sin\vartheta}=\frac{1}{r}\sin\varphi\cos\vartheta,$$

$$\frac{\partial\vartheta}{\partial z}=-\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{\sqrt{x^2+y^2+z^2}-\frac{z^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$=-\frac{1}{\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}}\frac{x^2+y^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$=-\frac{1}{\sqrt{x^2+y^2}}\frac{x^2+y^2}{x^2+y^2+z^2}=-\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}=-\frac{1}{r}\sin\vartheta.$$

Altogether

$$\begin{aligned}
&\frac{\partial r}{\partial x} = \sin\vartheta\cos\varphi, \quad \frac{\partial r}{\partial y} = \sin\vartheta\sin\varphi, \quad \frac{\partial r}{\partial z} = \cos\vartheta, \\
&\frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\vartheta}, \quad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\vartheta}, \quad \frac{\partial \varphi}{\partial z} = 0, \\
&\frac{\partial \vartheta}{\partial x} = \frac{1}{r}\cos\vartheta\cos\varphi, \quad \frac{\partial \vartheta}{\partial y} = \frac{1}{r}\sin\varphi\cos\vartheta, \quad \frac{\partial \vartheta}{\partial z} = -\frac{1}{r}\sin\vartheta.
\end{aligned} \tag{6}$$

From these formulas

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} + \frac{\partial \vartheta}{\partial x}\frac{\partial}{\partial \vartheta} \\
&= \sin\vartheta\cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r\sin\vartheta}\frac{\partial}{\partial \varphi} + \frac{1}{r}\cos\vartheta\cos\varphi\frac{\partial}{\partial \vartheta}, \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} + \frac{\partial \vartheta}{\partial y}\frac{\partial}{\partial \vartheta} \\
&= \sin\vartheta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r\sin\vartheta}\frac{\partial}{\partial \varphi} + \frac{1}{r}\sin\varphi\cos\vartheta\frac{\partial}{\partial \vartheta} \\
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi} + \frac{\partial \vartheta}{\partial z}\frac{\partial}{\partial \vartheta} \\
&= \cos\vartheta\frac{\partial}{\partial r} - \frac{1}{r}\sin\vartheta\frac{\partial}{\partial \vartheta},
\end{aligned} \tag{7}$$

Now the desired vector fields are

$$\begin{aligned}
\xi &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = r\sin\vartheta\cos\varphi\sin\vartheta\sin\varphi\frac{\partial}{\partial r} + r\sin\vartheta\cos\varphi\frac{\cos\varphi}{r\sin\vartheta}\frac{\partial}{\partial \varphi} \\
&+ \frac{1}{r}r\sin\vartheta\cos\varphi\sin\varphi\cos\vartheta\frac{\partial}{\partial \vartheta} - r\sin\vartheta\sin\varphi\sin\vartheta\cos\varphi\frac{\partial}{\partial r} \\
&+ r\sin\vartheta\sin\varphi\frac{\sin\varphi}{r\sin\vartheta}\frac{\partial}{\partial \varphi} - \frac{1}{r}r\sin\vartheta\sin\varphi\cos\vartheta\cos\varphi\frac{\partial}{\partial \vartheta} \\
&= \frac{\partial}{\partial \varphi},
\end{aligned} \tag{8}$$

and

(9)
$$\begin{aligned}\zeta &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\\ &= r\sin\vartheta\sin\varphi\cos\vartheta\frac{\partial}{\partial r} - r\sin\vartheta\sin\varphi\frac{1}{r}\sin\vartheta\frac{\partial}{\partial\vartheta}\\ &\quad - r\cos\vartheta\sin\vartheta\sin\varphi\frac{\partial}{\partial r} - r\cos\vartheta\frac{\cos\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi} - \frac{1}{r}r\cos\vartheta\sin\varphi\cos\vartheta\frac{\partial}{\partial\vartheta}\\ &= -\sin\varphi\sin^2\vartheta\frac{\partial}{\partial\vartheta} - \sin\varphi\cos^2\vartheta\frac{\partial}{\partial\vartheta} - \cot\vartheta\cos\varphi\frac{\partial}{\partial\varphi}\\ &= -\sin\varphi\frac{\partial}{\partial\vartheta} - \cot\vartheta\cos\varphi\frac{\partial}{\partial\varphi},\end{aligned}$$

and

(10)
$$\begin{aligned}\lambda &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\\ &= r\cos\vartheta\sin\vartheta\cos\varphi\frac{\partial}{\partial r} - r\cos\vartheta\frac{\sin\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi} + \frac{1}{r}r\cos\vartheta\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta}\\ &\quad - r\sin\vartheta\cos\varphi\cos\vartheta\frac{\partial}{\partial r} + r\sin\vartheta\cos\varphi\frac{1}{r}\sin\vartheta\frac{\partial}{\partial\vartheta}\\ &= -\cot\vartheta\sin\varphi\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\vartheta}.\end{aligned}$$

Remark The commutators of the vector fields ξ,ζ,λ are

(11)
$$\begin{aligned}&[\xi,\zeta] = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} = -\lambda, \quad [\xi,\lambda] = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} = -\zeta,\\ &[\lambda,\zeta] = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \xi.\end{aligned}$$

5 Invariance: Killing equations

A $(0,2)$-tensor field g on a manifold X is said to be *invariant* with respect to a diffeomorphism $\tau : X \to X$, if its pull back $\tau^* g$ satisfies

(1) $$\tau^* g = g.$$

The definition is naturally extended to vector fields via its flow. We say that a vector field ξ on X is the *generator of invariance transformations* of g, if the Lie derivative of g by ξ vanishes,

(2) $$\partial_\xi g = 0.$$

Equation (2) is sometimes called the *Killing equation*.

Let (U,φ), $\varphi = (x^i)$, be a chart on X, and let

(3) $$g = g_{ij} dx^i \otimes dx^j, \quad \xi = \xi^i \frac{\partial}{\partial x^i}$$

in this chart. Let α_t be a one-parameter group of ξ. Then

(4) $$\begin{aligned}\alpha_t^* g &= (g_{ij} \circ \alpha_t) d(x^i \circ \alpha_t) \otimes d(x^j \circ \alpha_t) \\ &= (g_{ij} \circ \alpha_t) \frac{\partial (x^i \circ \alpha_t)}{\partial x^k} \frac{\partial (x^j \circ \alpha_t)}{\partial x^l} dx^k \otimes dx^k.\end{aligned}$$

Differentiating the coefficient with respect to t we get

(5) $$\begin{aligned}&\frac{d(g_{ij} \circ \alpha_t)}{dt} \frac{\partial (x^i \circ \alpha_t)}{\partial x^k} \frac{\partial (x^j \circ \alpha_t)}{\partial x^l} \\ &+ (g_{ij} \circ \alpha_t) \left(\frac{\partial}{\partial x^k} \frac{d(x^i \circ \alpha_t)}{dt} \frac{\partial (x^j \circ \alpha_t)}{\partial x^l} + \frac{\partial (x^i \circ \alpha_t)}{\partial x^k} \frac{\partial}{\partial x^l} \frac{d(x^j \circ \alpha_t)}{dt} \right) \\ &= \frac{\partial (g_{ij} \circ \alpha_t)}{\partial x^p} \frac{d(x^p \circ \alpha_t)}{dt} \frac{\partial (x^i \circ \alpha_t)}{\partial x^k} \frac{\partial (x^j \circ \alpha_t)}{\partial x^l} \\ &+ (g_{ij} \circ \alpha_t) \left(\frac{\partial}{\partial x^k} \frac{d(x^i \circ \alpha_t)}{dt} \frac{\partial (x^j \circ \alpha_t)}{\partial x^l} + \frac{\partial (x^i \circ \alpha_t)}{\partial x^k} \frac{\partial}{\partial x^l} \frac{d(x^j \circ \alpha_t)}{dt} \right)\end{aligned}$$

because the partial derivatives commute. At $t = 0$ this expression becomes

(6) $$\begin{aligned}&\frac{\partial g_{ij}}{\partial x^p} \xi^p \delta_k^i \delta_l^j + g_{ij} \left(\frac{\partial \xi^i}{\partial x^k} \delta_l^j + \delta_k^i \frac{\partial \xi^j}{\partial x^l} \right) \\ &= \frac{\partial g_{kl}}{\partial x^p} \xi^p + g_{il} \frac{\partial \xi^i}{\partial x^k} + g_{kj} \frac{\partial \xi^j}{\partial x^l}.\end{aligned}$$

Thus, the Killing equation is of the form

(7) $$\frac{\partial g_{kl}}{\partial x^p} \xi^p + g_{il} \frac{\partial \xi^i}{\partial x^k} + g_{kj} \frac{\partial \xi^j}{\partial x^l} = 0.$$

6 SO(3)-invariant (0,2)-tensors

Now consider a metric field g on the manifold $\mathbf{R}^3 \setminus \{(0,0,0)\}$. In the first spherical chart (Section 2)

$$
\begin{aligned}
g = {} & g_{rr} dr \otimes dr + g_{r\varphi} dr \otimes d\varphi + g_{r\vartheta} dr \otimes d\vartheta + g_{\varphi r} d\varphi \otimes dr + g_{\varphi\varphi} d\varphi \otimes d\varphi \\
& + g_{\varphi\vartheta} d\varphi \otimes d\vartheta + g_{\vartheta r} d\vartheta \otimes dr + g_{\vartheta\varphi} d\vartheta \otimes d\varphi + g_{\vartheta\vartheta} d\vartheta \otimes d\vartheta.
\end{aligned} \tag{1}
$$

We wish to find the solution

$$
g_{rr}, g_{r\varphi}, g_{r\vartheta}, g_{\varphi\varphi}, g_{\varphi\vartheta}, g_{\vartheta\vartheta} \tag{2}
$$

of the Killing equations for the vector fields

$$
\begin{aligned}
& \xi = \frac{\partial}{\partial\varphi}, \quad \zeta = -\sin\varphi \frac{\partial}{\partial\vartheta} - \cot\vartheta \cos\varphi \frac{\partial}{\partial\varphi}, \\
& \lambda = \cos\varphi \frac{\partial}{\partial\vartheta} - \cot\vartheta \sin\varphi \frac{\partial}{\partial\varphi},
\end{aligned} \tag{3}
$$

(a) Writing (2) explicitly for the generator ξ we get the system

$$
\begin{aligned}
& \frac{\partial g_{rr}}{\partial\vartheta}\xi^{\vartheta} + \frac{\partial g_{rr}}{\partial\varphi}\xi^{\varphi} + g_{\vartheta r}\frac{\partial\xi^{\vartheta}}{\partial r} + g_{\varphi r}\frac{\partial\xi^{\varphi}}{\partial r} + g_{r\vartheta}\frac{\partial\xi^{\vartheta}}{\partial r} + g_{r\varphi}\frac{\partial\xi^{\varphi}}{\partial r} = 0, \\
& \frac{\partial g_{r\varphi}}{\partial\vartheta}\xi^{\vartheta} + \frac{\partial g_{r\varphi}}{\partial\varphi}\xi^{\varphi} + g_{\vartheta\varphi}\frac{\partial\xi^{\vartheta}}{\partial r} + g_{\varphi\varphi}\frac{\partial\xi^{\varphi}}{\partial r} + g_{r\vartheta}\frac{\partial\xi^{\vartheta}}{\partial\varphi} + g_{r\varphi}\frac{\partial\xi^{\varphi}}{\partial\varphi} = 0, \\
& \frac{\partial g_{r\vartheta}}{\partial\vartheta}\xi^{\vartheta} + \frac{\partial g_{r\vartheta}}{\partial\varphi}\xi^{\varphi} + g_{\vartheta\vartheta}\frac{\partial\xi^{\vartheta}}{\partial r} + g_{\varphi\vartheta}\frac{\partial\xi^{\varphi}}{\partial r} + g_{r\vartheta}\frac{\partial\xi^{\vartheta}}{\partial\vartheta} + g_{r\varphi}\frac{\partial\xi^{\varphi}}{\partial\vartheta} = 0, \\
& \frac{\partial g_{\varphi\varphi}}{\partial\vartheta}\xi^{\vartheta} + \frac{\partial g_{\varphi\varphi}}{\partial\varphi}\xi^{\varphi} + g_{\vartheta\varphi}\frac{\partial\xi^{\vartheta}}{\partial\varphi} + g_{\varphi\varphi}\frac{\partial\xi^{\varphi}}{\partial\varphi} + g_{\varphi\vartheta}\frac{\partial\xi^{\vartheta}}{\partial\varphi} + g_{\varphi\varphi}\frac{\partial\xi^{\varphi}}{\partial\varphi} = 0, \\
& \frac{\partial g_{\varphi\vartheta}}{\partial\vartheta}\xi^{\vartheta} + \frac{\partial g_{\varphi\vartheta}}{\partial\varphi}\xi^{\varphi} + g_{\vartheta\vartheta}\frac{\partial\xi^{\vartheta}}{\partial\varphi} + g_{\varphi\vartheta}\frac{\partial\xi^{\varphi}}{\partial\varphi} + g_{\varphi\vartheta}\frac{\partial\xi^{\vartheta}}{\partial\vartheta} + g_{\varphi\varphi}\frac{\partial\xi^{\varphi}}{\partial\vartheta} = 0, \\
& \frac{\partial g_{\vartheta\vartheta}}{\partial\vartheta}\xi^{\vartheta} + \frac{\partial g_{\vartheta\vartheta}}{\partial\varphi}\xi^{\varphi} + g_{\vartheta\vartheta}\frac{\partial\xi^{\vartheta}}{\partial\vartheta} + g_{\varphi\vartheta}\frac{\partial\xi^{\varphi}}{\partial\vartheta} + g_{\vartheta\vartheta}\frac{\partial\xi^{\vartheta}}{\partial\vartheta} + g_{\vartheta\varphi}\frac{\partial\xi^{\varphi}}{\partial\vartheta} = 0,
\end{aligned} \tag{4}
$$

Substituting for the components of ξ we get

$$
\frac{\partial g_{rr}}{\partial\varphi} = 0, \quad \frac{\partial g_{r\varphi}}{\partial\varphi} = 0, \quad \frac{\partial g_{r\vartheta}}{\partial\varphi} = 0, \quad \frac{\partial g_{\varphi\varphi}}{\partial\varphi} = 0, \quad \frac{\partial g_{\varphi\vartheta}}{\partial\varphi} = 0, \quad \frac{\partial g_{\vartheta\vartheta}}{\partial\varphi} = 0. \tag{5}
$$

(b) The same equations for the generator ζ

$$\begin{aligned}
&\frac{\partial g_{rr}}{\partial \vartheta}\zeta^{\vartheta} + \frac{\partial g_{rr}}{\partial \varphi}\zeta^{\varphi} + g_{\vartheta r}\frac{\partial \zeta^{\vartheta}}{\partial r} + g_{\varphi r}\frac{\partial \zeta^{\varphi}}{\partial r} + g_{r\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial r} + g_{r\varphi}\frac{\partial \zeta^{\varphi}}{\partial r} = 0,\\
&\frac{\partial g_{r\varphi}}{\partial \vartheta}\zeta^{\vartheta} + \frac{\partial g_{r\varphi}}{\partial \varphi}\zeta^{\varphi} + g_{\vartheta\varphi}\frac{\partial \zeta^{\vartheta}}{\partial r} + g_{\varphi\varphi}\frac{\partial \zeta^{\varphi}}{\partial r} + g_{r\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial \varphi} + g_{r\varphi}\frac{\partial \zeta^{\varphi}}{\partial \varphi} = 0,\\
&\frac{\partial g_{r\vartheta}}{\partial \vartheta}\zeta^{\vartheta} + \frac{\partial g_{r\vartheta}}{\partial \varphi}\zeta^{\varphi} + g_{\vartheta\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial r} + g_{\varphi\vartheta}\frac{\partial \zeta^{\varphi}}{\partial r} + g_{r\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial \vartheta} + g_{r\varphi}\frac{\partial \zeta^{\varphi}}{\partial \vartheta} = 0,\\
&\frac{\partial g_{\varphi\varphi}}{\partial \vartheta}\zeta^{\vartheta} + \frac{\partial g_{\varphi\varphi}}{\partial \varphi}\zeta^{\varphi} + g_{\vartheta\varphi}\frac{\partial \zeta^{\vartheta}}{\partial \varphi} + g_{\varphi\varphi}\frac{\partial \zeta^{\varphi}}{\partial \varphi} + g_{\varphi\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial \varphi} + g_{\varphi\varphi}\frac{\partial \zeta^{\varphi}}{\partial \varphi} = 0,\\
&\frac{\partial g_{\varphi\vartheta}}{\partial \vartheta}\zeta^{\vartheta} + \frac{\partial g_{\varphi\vartheta}}{\partial \varphi}\zeta^{\varphi} + g_{\vartheta\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial \varphi} + g_{\varphi\vartheta}\frac{\partial \zeta^{\varphi}}{\partial \varphi} + g_{\varphi\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial \vartheta} + g_{\varphi\varphi}\frac{\partial \zeta^{\varphi}}{\partial \vartheta} = 0,\\
&\frac{\partial g_{\vartheta\vartheta}}{\partial \vartheta}\zeta^{\vartheta} + \frac{\partial g_{\vartheta\vartheta}}{\partial \varphi}\zeta^{\varphi} + g_{\vartheta\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial \vartheta} + g_{\varphi\vartheta}\frac{\partial \zeta^{\varphi}}{\partial \vartheta} + g_{\vartheta\vartheta}\frac{\partial \zeta^{\vartheta}}{\partial \vartheta} + g_{\vartheta\varphi}\frac{\partial \zeta^{\varphi}}{\partial \vartheta} = 0,
\end{aligned} \tag{6}$$

give

$$\begin{aligned}
&\frac{\partial g_{rr}}{\partial \vartheta} = 0,\\
&\frac{\partial g_{r\varphi}}{\partial \vartheta}\sin\varphi + g_{r\vartheta}\cos\varphi - g_{r\varphi}\cot\vartheta\sin\varphi = 0,\\
&\frac{\partial g_{r\vartheta}}{\partial \vartheta}\sin\varphi - g_{r\varphi}\frac{\cos\varphi}{\sin^2\vartheta} = 0,\\
&\frac{\partial g_{\varphi\varphi}}{\partial \vartheta}\sin\varphi + 2g_{\vartheta\varphi}\cos\varphi - 2g_{\varphi\varphi}\cot\vartheta\sin\varphi = 0,\\
&\frac{\partial g_{\varphi\vartheta}}{\partial \vartheta}\sin\varphi + g_{\vartheta\vartheta}\cos\varphi - g_{\varphi\vartheta}\cot\vartheta\sin\varphi - g_{\varphi\varphi}\frac{\cos\varphi}{\sin^2\vartheta} = 0,\\
&\frac{\partial g_{\vartheta\vartheta}}{\partial \vartheta}\sin\varphi - 2g_{\varphi\vartheta}\frac{\cos\varphi}{\sin^2\vartheta} = 0,
\end{aligned} \tag{7}$$

(c) Writing (2) for λ

(8)
$$\begin{aligned}
&\frac{\partial g_{rr}}{\partial\vartheta}\lambda^{\vartheta}+\frac{\partial g_{rr}}{\partial\varphi}\lambda^{\varphi}+g_{\vartheta r}\frac{\partial\lambda^{\vartheta}}{\partial r}+g_{\varphi r}\frac{\partial\lambda^{\varphi}}{\partial r}+g_{r\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial r}+g_{r\varphi}\frac{\partial\lambda^{\varphi}}{\partial r}=0,\\
&\frac{\partial g_{r\varphi}}{\partial\vartheta}\lambda^{\vartheta}+\frac{\partial g_{r\varphi}}{\partial\varphi}\lambda^{\varphi}+g_{\vartheta\varphi}\frac{\partial\lambda^{\vartheta}}{\partial r}+g_{\varphi\varphi}\frac{\partial\lambda^{\varphi}}{\partial r}+g_{r\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial\varphi}+g_{r\varphi}\frac{\partial\lambda^{\varphi}}{\partial\varphi}=0,\\
&\frac{\partial g_{r\vartheta}}{\partial\vartheta}\lambda^{\vartheta}+\frac{\partial g_{r\vartheta}}{\partial\varphi}\lambda^{\varphi}+g_{\vartheta\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial r}+g_{\varphi\vartheta}\frac{\partial\lambda^{\varphi}}{\partial r}+g_{r\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial\vartheta}+g_{r\varphi}\frac{\partial\lambda^{\varphi}}{\partial\vartheta}=0,\\
&\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}\lambda^{\vartheta}+\frac{\partial g_{\varphi\varphi}}{\partial\varphi}\lambda^{\varphi}+g_{\vartheta\varphi}\frac{\partial\lambda^{\vartheta}}{\partial\varphi}+g_{\varphi\varphi}\frac{\partial\lambda^{\varphi}}{\partial\varphi}+g_{\varphi\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial\varphi}+g_{\varphi\varphi}\frac{\partial\lambda^{\varphi}}{\partial\varphi}=0,\\
&\frac{\partial g_{\varphi\vartheta}}{\partial\vartheta}\lambda^{\vartheta}+\frac{\partial g_{\varphi\vartheta}}{\partial\varphi}\lambda^{\varphi}+g_{\vartheta\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial\varphi}+g_{\varphi\vartheta}\frac{\partial\lambda^{\varphi}}{\partial\varphi}+g_{\varphi\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial\vartheta}+g_{\varphi\varphi}\frac{\partial\lambda^{\varphi}}{\partial\vartheta}=0,\\
&\frac{\partial g_{\vartheta\vartheta}}{\partial\vartheta}\lambda^{\vartheta}+\frac{\partial g_{\vartheta\vartheta}}{\partial\varphi}\lambda^{\varphi}+g_{\vartheta\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial\vartheta}+g_{\varphi\vartheta}\frac{\partial\lambda^{\varphi}}{\partial\vartheta}+g_{\vartheta\vartheta}\frac{\partial\lambda^{\vartheta}}{\partial\vartheta}+g_{\vartheta\varphi}\frac{\partial\lambda^{\varphi}}{\partial\vartheta}=0,
\end{aligned}$$

we have the system

(9)
$$\begin{aligned}
&\frac{\partial g_{rr}}{\partial\vartheta}=0,\\
&\frac{\partial g_{r\varphi}}{\partial\vartheta}\cos\varphi-g_{r\vartheta}\sin\varphi-g_{r\varphi}\cot\vartheta\cos\varphi=0,\\
&\frac{\partial g_{r\vartheta}}{\partial\vartheta}\cos\varphi+g_{r\varphi}\frac{\sin\varphi}{\sin^2\vartheta}=0,\\
&\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}\cos\varphi-2g_{\vartheta\varphi}\sin\varphi-2g_{\varphi\varphi}\cot\vartheta\cos\varphi=0,\\
&\frac{\partial g_{\varphi\vartheta}}{\partial\vartheta}\cos\varphi-g_{\vartheta\vartheta}\sin\varphi-g_{\varphi\vartheta}\cot\vartheta\cos\varphi+g_{\varphi\varphi}\frac{\sin\varphi}{\sin^2\vartheta}=0,\\
&\frac{\partial g_{\vartheta\vartheta}}{\partial\vartheta}\cos\varphi+2g_{\varphi\vartheta}\frac{\sin\varphi}{\sin^2\vartheta}=0.
\end{aligned}$$

Summarizing, we have conditions (5)

(10)
$$\frac{\partial g_{rr}}{\partial\varphi}=0,\quad \frac{\partial g_{r\varphi}}{\partial\varphi}=0,\quad \frac{\partial g_{r\vartheta}}{\partial\varphi}=0,\quad \frac{\partial g_{\varphi\varphi}}{\partial\varphi}=0,\quad \frac{\partial g_{\varphi\vartheta}}{\partial\varphi}=0,\quad \frac{\partial g_{\vartheta\vartheta}}{\partial\varphi}=0,$$

condition

(11)
$$\frac{\partial g_{rr}}{\partial\vartheta}=0,$$

from (7), and the remaining equations from (7) and (9), written in four subsystems

$$(12)\quad \begin{aligned} &\frac{\partial g_{r\varphi}}{\partial\vartheta}\sin\varphi + g_{r\vartheta}\cos\varphi - g_{r\varphi}\cot\vartheta\sin\varphi = 0,\\ &\frac{\partial g_{r\varphi}}{\partial\vartheta}\cos\varphi - g_{r\vartheta}\sin\varphi - g_{r\varphi}\cot\vartheta\cos\varphi = 0,\\ &\frac{\partial g_{r\vartheta}}{\partial\vartheta}\sin\varphi - g_{r\varphi}\frac{\cos\varphi}{\sin^2\vartheta} = 0,\\ &\frac{\partial g_{r\vartheta}}{\partial\vartheta}\cos\varphi + g_{r\varphi}\frac{\sin\varphi}{\sin^2\vartheta} = 0, \end{aligned}$$

and

$$(13)\quad \begin{aligned} &\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}\sin\varphi + 2g_{\vartheta\varphi}\cos\varphi - 2g_{\varphi\varphi}\cot\vartheta\sin\varphi = 0,\\ &\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}\cos\varphi - 2g_{\vartheta\varphi}\sin\varphi - 2g_{\varphi\varphi}\cot\vartheta\cos\varphi = 0, \end{aligned}$$

and

$$(14)\quad \begin{aligned} &\frac{\partial g_{\varphi\vartheta}}{\partial\vartheta}\sin\varphi + g_{\vartheta\vartheta}\cos\varphi - g_{\varphi\vartheta}\cot\vartheta\sin\varphi - g_{\varphi\varphi}\frac{\cos\varphi}{\sin^2\vartheta} = 0,\\ &\frac{\partial g_{\varphi\vartheta}}{\partial\vartheta}\cos\varphi - g_{\vartheta\vartheta}\sin\varphi - g_{\varphi\vartheta}\cot\vartheta\cos\varphi + g_{\varphi\varphi}\frac{\sin\varphi}{\sin^2\vartheta} = 0, \end{aligned}$$

and

$$(15)\quad \begin{aligned} &\frac{\partial g_{\vartheta\vartheta}}{\partial\vartheta}\sin\varphi - 2g_{\varphi\vartheta}\frac{\cos\varphi}{\sin^2\vartheta} = 0,\\ &\frac{\partial g_{\vartheta\vartheta}}{\partial\vartheta}\cos\varphi + 2g_{\varphi\vartheta}\frac{\sin\varphi}{\sin^2\vartheta} = 0. \end{aligned}$$

The last two equation in (12) imply

$$(16)\quad \frac{\partial g_{r\vartheta}}{\partial\vartheta} = 0,\quad g_{r\varphi} = 0,$$

thus, subsystem (12) gives

$$(17)\quad g_{r\varphi} = 0,\quad g_{r\vartheta} = 0.$$

Next two equations (13) imply

(18) $$\frac{\partial g_{\varphi\varphi}}{\partial \vartheta} - 2g_{\varphi\varphi} \cot\vartheta = 0.$$

Solving this equation we have, with the help of (10), we get

(19) $$g_{\varphi\varphi} = f(r)\sin^2\vartheta.$$

(13) also implies

(20) $$\begin{aligned} &\frac{\partial g_{\varphi\varphi}}{\partial \vartheta}\sin\varphi\cos\varphi + 2g_{\vartheta\varphi}\cos^2\varphi - 2g_{\varphi\varphi}\cot\vartheta\sin\varphi\cos\varphi = 0, \\ &\frac{\partial g_{\varphi\varphi}}{\partial \vartheta}\cos\varphi\sin\varphi - 2g_{\vartheta\varphi}\sin^2\varphi - 2g_{\varphi\varphi}\cot\vartheta\cos\varphi\sin\varphi = 0, \end{aligned}$$

hence

(21) $$g_{\vartheta\varphi} = 0.$$

Substituting from (20) and (21) back to (13) we get the identities.

Next two equations (14) now reduce to

(22) $$g_{\vartheta\vartheta}\cos\varphi - g_{\varphi\varphi}\frac{\cos\varphi}{\sin^2\vartheta} = 0, \quad -g_{\vartheta\vartheta}\sin\varphi + g_{\varphi\varphi}\frac{\sin\varphi}{\sin^2\vartheta} = 0,$$

and imply

(23) $$g_{\vartheta\vartheta} - g_{\varphi\varphi}\frac{1}{\sin^2\vartheta} = 0.$$

Then from (19)

(24) $$g_{\vartheta\vartheta} = f(r).$$

Finally, we have the subsystem (15), which reduces to one equation

(25) $$\frac{\partial g_{\vartheta\vartheta}}{\partial \vartheta} = 0.$$

Summarizing (10), (11), (17), (19), (21), (24) and (25), we have the following formulas:

(26) $$\begin{aligned} &g_{rr} = P(r), \quad g_{r\varphi} = 0, \quad g_{r\vartheta} = 0, \\ &g_{\varphi\varphi} = Q(r)\sin^2\vartheta, \quad g_{\vartheta\varphi} = 0, \quad g_{\vartheta\vartheta} = Q(r). \end{aligned}$$

Consequently, we have the following theorem.

Theorem 2 *If a metric tensor field g on* $\mathbf{R}^3$ *is invariant with respect to rotations, then in the spherical coordinates it is of the form*

$$g = P(r)dr \otimes dr + Q(r)(d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi), \tag{27}$$

where P and Q are functions, depending on r only.

Proof We have from (24x)

$$\begin{aligned} g &= g_{rr}dr \otimes dr + g_{r\varphi}dr \otimes d\varphi + g_{r\vartheta}dr \otimes d\vartheta + g_{\varphi r}d\varphi \otimes dr + g_{\varphi\varphi}d\varphi \otimes d\varphi \\ &\quad + g_{\varphi\vartheta}d\varphi \otimes d\vartheta + g_{\vartheta r}d\vartheta \otimes dr + g_{\vartheta\varphi}d\vartheta \otimes d\varphi + g_{\vartheta\vartheta}d\vartheta \otimes d\vartheta \\ &= g_{rr}dr \otimes dr + g_{\varphi\varphi}d\varphi \otimes d\varphi + g_{\vartheta\vartheta}d\vartheta \otimes d\vartheta \\ &= P(r)dr \otimes dr + Q(r)(\sin^2 \vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta). \end{aligned} \tag{28}$$

This result coincides with F. De Felice and C.J.S. Clarke, *Relativity on Curved Manifolds*, Cambridge Monographs on Mathematical Physics, 1992, Section 10.1, p. 320.

7 Globalization

We now formally describe the globalization of our local results to the whole manifold $\mathbf{R}^3 \setminus \{(0,0,0)\}$. We denote by (U,Φ), $\Phi = (r,\varphi,\vartheta)$, and $(\bar{U},\bar{\Phi})$, $\bar{\Phi} = (\bar{r},\bar{\varphi},\bar{\vartheta})$, the first and the second spherical charts on $\mathbf{R}^3 \setminus \{(0,0,0)\}$; these two charts form an atlas for $\mathbf{R}^3 \setminus \{(0,0,0)\}$. The transformation equations are given by

$$\bar{r} = r, \quad \cos\bar{\vartheta} = -\sin\vartheta\sin\varphi, \quad \tan\bar{\varphi} = \frac{\cos\vartheta}{\sin\vartheta\cos\varphi}. \tag{1}$$

(Section 2, (13)).

Theorem 3 *Let*

$$g_U = P(r)dr \otimes dr + Q(r)(d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi), \tag{2}$$

be an SO(3) *-invariant metric field on U, and let*

$$\bar{g}_{\bar{U}} = \bar{P}(\bar{r})d\bar{r} \otimes d\bar{r} + \bar{Q}(\bar{r})(d\bar{\vartheta} \otimes d\bar{\vartheta} + \sin^2 \bar{\vartheta} d\bar{\varphi} \otimes d\bar{\varphi}) \tag{3}$$

be an $\mathrm{SO}(3)$ *-invariant metric field on* $\bar{U}$ *. Then* $g_U = \bar{g}_{\bar{U}}$ *on* $U \cap \bar{U}$ *if and only if*

(4) $\bar{P}(\bar{r}(x)) = P(r(x)), \quad \bar{Q}(\bar{r}(x)) = Q(r(x))$

for all $x \in U \cap \bar{U}$ *.*

If conditions (4) are satisfied, then the formula

(5) $$g(x) = \begin{cases} g_U(x), & x \in U \\ \bar{g}_{\bar{U}}(x), & x \in \bar{U} \end{cases}$$

defines an $\mathrm{SO}(3)$-invariant $(0,2)$-tensor field on $\mathbf{R}^3 \setminus \{(0,0,0)\}$. The functions $\bar{P}$, P, and Q, $\bar{Q}$ define two functions $p : \mathbf{R}^3 \setminus \{(0,0,0)\} \to \mathbf{R}$ and $q : \mathbf{R}^3 \setminus \{(0,0,0)\} \to \mathbf{R}$ by

(6) $$p(x) = \begin{cases} P(r(x)), & x \in U, \\ \bar{P}(\bar{r}(x)), & x \in \bar{U}, \end{cases} \qquad q(x) = \begin{cases} Q(r(x)), & x \in U, \\ \bar{Q}(\bar{r}(x)), & x \in \bar{U}. \end{cases}$$

Conversely, any two functions $p : \mathbf{R}^3 \setminus \{(0,0,0)\} \to \mathbf{R}$ and $q : \mathbf{R}^3 \setminus \{(0,0,0)\} \to \mathbf{R}$ define an $\mathrm{SO}(3)$-invariant $(0,2)$-tensor field by Theorem 3.

Thus, Theorem 3 constitutes a one-one correspondence between $\mathrm{SO}(3)$-invariant $(0,2)$-tensor fields on $\mathbf{R}^3 \setminus \{(0,0,0)\}$ and the pairs of functions (p,q), defined on $\mathbf{R}^3 \setminus \{(0,0,0)\}$.

8 SO(3)-invariant metric fields on $\mathbf{R}^3 \setminus \{(0,0,0)\}$

Denote $M = \mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$. *M* is the product manifold, endowed with a left $\mathrm{SO}(3)$-action

(1) $$\begin{aligned} &\mathrm{SO}(3) \times (\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})) \ni (A,(t,x)) \\ &\quad \to A \cdot (t,x) = (t, A \cdot x) \in \mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\}), \end{aligned}$$

induced by the action of $\mathrm{SO}(3)$ on $\mathbf{R}^3 \setminus \{(0,0,0)\}$. We will consider *M* with the atlas, formed by two charts, whose domains of definition are $\mathbf{R} \times U$ and $\mathbf{R} \times \bar{U}$, and whose coordinate functions are $(t,(r,\varphi,\vartheta))$ and $(t,(\bar{r},\bar{\varphi},\bar{\vartheta}))$, where *t* is the canonical coordinate on $\mathbf{R}$; these charts will be referred to as the *first* and the *second spherical charts* on *M*. Our aim in this section is to find all $\mathrm{SO}(3)$-invariant tensor fields of type $(0,2)$ on *M*.

We start with a tensor field

(2) $$g = g_{tt}dt \otimes dt + g_{tr}dt \otimes dr + g_{rt}dr \otimes dt + g_0,$$

where

(3) $$\begin{aligned} g_0 &= g_{rr}dr \otimes dr + g_{r\varphi}dr \otimes d\varphi + g_{r\vartheta}dr \otimes d\vartheta + g_{\varphi r}d\varphi \otimes dr + g_{\varphi\varphi}d\varphi \otimes d\varphi \\ &\quad + g_{\varphi\vartheta}d\varphi \otimes d\vartheta + g_{\vartheta r}d\vartheta \otimes dr + g_{\vartheta\varphi}d\vartheta \otimes d\varphi + g_{\vartheta\vartheta}d\vartheta \otimes d\vartheta. \end{aligned}$$

The following assertion is immediate.

Theorem 4 *The tensor field g* (2) *is* SO(3) *-invariant if and only if*

(4) $$g_{tt} = g_{tt}(t,r), \quad g_{tr} = g_{tr}(t,r),$$

and g_0 *is* SO(3) *-invariant.*

Proof For any diffeomorphism α, defined by the group action (1),

(5) $$\alpha^* g = (g_{tt} \circ \alpha)dt \otimes dt + (g_{tr} \circ \alpha)dr \otimes dr + (g_{rt} \circ \alpha)dr \otimes dr + \alpha^* g_0,$$

because SO(3) acts trivially on the coordinate functions *t* and *r*. Thus, the invariance condition $\alpha^* g = g$ is equivalent with

(6) $$g_{tt} \circ \alpha = g_{tt}, \quad g_{tr} \circ \alpha = g_{tr}, \quad \alpha^* g_0 = g_0$$

for all α. But from the generators of rotations, Section 3, (2),

(7) $$g_{tt} = g_{tt}(t,r), \quad g_{tr} = g_{tr}(t,r)$$

as required.

From Section 5, Theorem 3, we now conclude that any SO(3) -invariant tensor field of type (0,2) on the manifold *M* is in the first spherical coordinates expressed as

(8) $$\begin{aligned} g &= J(r,t)dt \otimes dt + K(r,t)(dt \otimes dr + dr \otimes dt) \\ &\quad + P(r,t)dr \otimes dr + Q(r,t)(d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi), \end{aligned}$$

where $J(r,t)$, $K(r,t)$, $P(r,t)$, and $Q(r,t)$ are arbitrary functions of *t* and *r* on the domain of definition of the first spherical chart.

The following is an analogue of Theorem 3, Section 6.

Theorem 5 *Let*

$$\begin{aligned} g_U &= J(r,t)dt\otimes dt + K(r,t)(dt\otimes dr + dr\otimes dt) \\ &+ P(r,t)dr\otimes dr + Q(r,t)(d\vartheta\otimes d\vartheta + \sin^2\vartheta d\varphi\otimes d\varphi) \end{aligned} \tag{9}$$

be an SO(3) *-invariant metric field on U, and let*

$$\begin{aligned} \bar g_{\bar U} &= \bar J(\bar r,\bar t)d\bar t\otimes d\bar t + K(\bar r,\bar t)(d\bar t\otimes d\bar r + d\bar r\otimes d\bar t) \\ &+ P(\bar r,\bar t)d\bar r\otimes d\bar r + Q(\bar r,\bar t)(d\tilde\vartheta\otimes d\tilde\vartheta + \sin^2\tilde\vartheta d\bar\varphi\otimes d\bar\varphi) \end{aligned} \tag{10}$$

be an SO(3) *-invariant metric field on* $\bar U$ *. Then* $g_U = \bar g_{\bar U}$ *on* $U\cap\bar U$ *if and only if*

$$\begin{aligned} &\bar J(\bar r,\bar t) = J(r,t), \quad K(\bar r,\bar t) = K(r,t), \\ &\bar P(\bar r,\bar t) = P(r,t), \quad \bar Q(\bar r,\bar t) = Q(r,t). \end{aligned} \tag{11}$$

for all $x\in U\cap\bar U$.

Proof The assertion follows from the transformation equations between the first and the second spherical charts; in particular, from the equations $\bar r = r$ and $\bar t = t$, and from Section 2, Theorem 1.

Theorem 5 implies, in particular, that an SO(3) -invariant (0,2) -invariant tensor field defines and is defined by four functions J, K, P, and Q , defined on the quotient manifold $\mathbf{R}\times(0,\infty) = M/\mathrm{SO}(3)$.

Remark Theorem 4 does not imply that the tensor field g be *regular*, or of certain *signature*. Assumptions of this kind should be applied independently.

Suppose that the matrix of the tensor field (8)

$$\begin{pmatrix} J(r,t) & K(r,t) & 0 & 0 \\ K(r,t) & P(r,t) & 0 & 0 \\ 0 & 0 & Q(r,t) & 0 \\ 0 & 0 & 0 & \sin^2\vartheta\cdot Q(r,t) \end{pmatrix} \tag{12}$$

is non-singular. Then the determinant is

$$\begin{vmatrix} J & K & 0 & 0 \\ K & P & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & \sin^2\vartheta\cdot Q \end{vmatrix} = (JP-K^2)Q^2\sin^2\vartheta \neq 0. \tag{13}$$

Thus, the components of g satisfy

(14) $JP - K^2 \neq 0, \quad Q \neq 0.$

Note that we can write in this case

(15)
$$\begin{aligned} g &= J(r,t)dt \otimes dt + K(r,t)(dt \otimes dr + dr \otimes dt) \\ &\quad + P(r,t)dr \otimes dr + Q(r,t)(d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi) \\ &= Q(r,t)\left(\frac{J(r,t)}{Q(r,t)} dt \otimes dt + \frac{K(r,t)}{Q(r,t)}(dt \otimes dr + dr \otimes dt) \right. \\ &\quad \left. + \frac{P(r,t)}{Q(r,t)} dr \otimes dr + d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi \right) \\ &= Q(r,t)(j(r,t)dt \otimes dt + k(r,t)(dt \otimes dr + dr \otimes dt) \\ &\quad + p(r,t)dr \otimes dr + d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi). \end{aligned}$$

Thus, each $\mathrm{SO}(3)$ -invariant regular tensor field of type $(0,2)$ on the manifold M is *conformal* with the metric field of the form

(16)
$$\begin{aligned} g' &= j(r,t)dt \otimes dt + k(r,t)(dt \otimes dr + dr \otimes dt) \\ &\quad + p(r,t)dr \otimes dr + d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi. \end{aligned}$$

(for conformal metric fields of any signature see e.g. G.S. Hall, *Symmetries and Curvature structure in General Relativity*, WS Lecture Notes in Physics 46, World Scientific, 2004, p. 114).

9 SO(3) -invariance and translation invariance

Consider an $\mathrm{SO}(3)$ -invariant regular metric field

(1)
$$\begin{aligned} g &= J(r,t)dt \otimes dt + K(r,t)(dt \otimes dr + dr \otimes dt) \\ &\quad + P(r,t)dr \otimes dr + Q(r,t)(d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi) \end{aligned}$$

(Section 7, (8)). By the *translation* in $M = \mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ we mean any transformation of the form

(2)
$$\begin{aligned} &\mathbf{R} \times (\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})) \ni (\mathrm{tr}_\varepsilon, (t,x)) \\ &\quad \to \mathrm{tr}_\varepsilon(t,x) = (t+\varepsilon, x) \in \mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\}). \end{aligned}$$

Clearly, translations define a left action of the additive group of real numbers $\mathbf{R}$

on $M=\mathbf{R}\times(\mathbf{R}^3\setminus\{(0,0,0)\})$. The generator of the translations is the vector field

(3) $$\xi=\frac{\partial}{\partial t}.$$

Theorem 6 *Each* SO(3) *-invariant translation invariant regular metric field on M is of the form*

(4) $$\begin{aligned} g&=J(r)dt\otimes dt+K(r)(dt\otimes dr+dr\otimes dt)\\ &+P(r)dr\otimes dr+Q(r)(d\vartheta\otimes d\vartheta+\sin^2\vartheta d\varphi\otimes d\varphi), \end{aligned}$$

where $J(r)$, $K(r)$, $P(r)$, $Q(r)$ *are arbitrary functions of the variable* r.

Proof The assertion is evident: In the Killing equation

(5) $$\frac{\partial g_{kl}}{\partial x^p}\xi^p+g_{il}\frac{\partial\xi^i}{\partial x^k}+g_{kj}\frac{\partial\xi^j}{\partial x^l}=0$$

(Section 4, (7)) we substitute from (3) from which we have from expression (1)

(6) $$\frac{\partial J(r,t)}{\partial t}=0,\quad \frac{\partial K(r,t)}{\partial t}=0,\quad \frac{\partial P(r,t)}{\partial t}=0,\quad \frac{\partial Q(r,t)}{\partial t}=0.$$

10 The Lambert function

To describe the structure of the Schwarzschild and Kruskal metric fields, we need solutions of the equation

(1) $$y=W(y)\mathrm{e}^{W(y)}$$

for an unknown real function W of one real variable y, known as the *Lambert function*. To this purpose we present in this section basic properties of this function; for proofs and further comments see e.g. R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth, *On the Lambert W function*, Advances in Computational Mathematics, Springer-Verlag, Berlin, New York, 5, 329-359), and http://mathworld.wolfram.com/LambertW-Function.html .

Clearly, if $y=0$, then $W(0)=0$. If $y>0$, then there exists a unique solution $W(y)$ which is positive. If $-1/\mathrm{e}<y<0$, then there are two solutions, $W(y)$ and $W_{-1}(y)$ that satisfy $W_{-1}(y)<W(y)$; if $y=-1/\mathrm{e}$, then $W_{-1}(-1/\mathrm{e})=W(-1/\mathrm{e})$. Equation (1) has no real solutions such that $y<-1/\mathrm{e}$. The following is a description of the solution W.

Lemma 1 *Equation* (1) *has a unique solution W, defined on the interval* $[-1/\mathrm{e},\infty)$*, such that*

(2) $$W(y) \geq -1.$$

This solution satisfies

(3) $$W\left(-\frac{1}{\mathrm{e}}\right) = -1, \quad \operatorname{Im} W = [-1,\infty).$$

Condition (2) means that we take for W the *principal branch* of the real solution of equation (1). The solution described by Lemma 1 is known as the *Lambert W-function*.

The following are elementary properties of the Lambert W-function:

(a) Equation $y = x\mathrm{e}^x$ holds for $x \in [-1,\infty)$ and $y \in [-1/\mathrm{e},\infty)$ if and only if $x = W(y)$. In other words, W is the *inverse* of a function F, defined on $(-\infty,\infty)$ by

(4) $$F(x) = x\mathrm{e}^x,$$

or more precisely, the inverse of its restriction $F|_{[-1,\infty)}$ to the interval where F is increasing. Using F we can write

(5) $$F^{-1}(y) = W(y).$$

(b) W has the following special values:

(6) $$W\left(-\frac{1}{e}\right) = -1, \quad W(0) = 0, \quad W(1) = \frac{1}{\mathrm{e}^{-W(1)}} = 0{,}567143\ldots.$$

(c) The derivative of W is given by

(7) $$\frac{dW}{dy} = \frac{W(y)}{y(1+W(y))}.$$

W is increasing and has an asymptote $x = -1/\mathrm{e}$.

(d) From the definition

(8) $$\ln y = W(y) + \ln W(y)$$

on the set $(0,\infty)$, and

(9) $$W(-y)\mathrm{e}^{W(-y)} = -W(y)\mathrm{e}^{W(y)}.$$

We usually consider W as defined on the open interval $(-1/\mathrm{e},\infty)$.

As the first application of the Lambert W-function we find the inverse of a function G defined on $(-\infty,\infty)$ by

(10) $G(x) = x\mathrm{e}^{-x}$.

Writing $y = x\mathrm{e}^{-x}$, we have to solve this equation with respect to x. We have $-y = -x\mathrm{e}^{-x}$, and this equation can be solved by means of W. We get $-x = W(-y)$ hence $x = -W(-y)$ whenever $-y$ belongs to the domain of definition $[-1/\mathrm{e},\infty)$ of W. Consequently, $G^{-1}(y) = -W(-y)$. The domain of definition of G^{-1} consists of the points y such that $-y$ belongs to the domain of definition of W, i.e., $-y \in [-1/\mathrm{e},\infty)$. Then $-x = W(-y) \in \operatorname{Im} W = [-1,\infty)$ hence $x = G^{-1}(y) \in (-\infty,1]$. Summarizing, we have the following lemma.

Lemma 2 *The restriction* $G|_{(-\infty,1]}$ *has the inverse* G^{-1} *defined by*

(11) $G^{-1}(y) = -W(-y).$

The domain of definition of G^{-1} *is the interval* $(-\infty,1/\mathrm{e}]$.

It follows from Lemma 2 that equation $y = x\mathrm{e}^{-x}$ holds for $x \in (-\infty,1]$ and $y \in (-\infty,1/\mathrm{e})$ if and only if $x = -W(-y)$.

Now consider the functions

(12) $(0,\infty) \ni x \to f(x) = x + \ln x \in \mathbf{R}$

and

(13) $(0,\infty) \ni x \to g(x) = -x + \ln x \in \mathbf{R}$,

and determine the inverse functions f^{-1} and g^{-1}.

Lemma 3 (a) *The inverse function* f^{-1} *is defined by*

(14) $f^{-1}(y) = W(\mathrm{e}^{y}).$

The domain of definition of f^{-1} *is the interval* $(-\infty,\infty)$.

(b) *The inverse function* g^{-1} *is defined by*

(15) $g^{-1}(y) = -W(-\mathrm{e}^{y}).$

The domain of definition of g^{-1} *is the interval* $(-\infty,\infty)$.

Indeed, we have $\mathrm{e}^{f(x)} = x\mathrm{e}^{x}$, so f^{-1} can be determined from the equation $\mathrm{e}^{y} = f^{-1}(y)\mathrm{e}^{f^{-1}(y)}$. But the right-hand side is equal to $F(f^{-1}(y))$, and $F^{-1} = W$ by

Lemma 1, so we have $W(\mathrm{e}^y) = f^{-1}(y)$ by Lemma 1.

Similarly, we have $\mathrm{e}^{g(x)} = x\mathrm{e}^{-x}$, so g^{-1} satisfies $\mathrm{e}^y = g^{-1}(y)\mathrm{e}^{-g^{-1}(y)}$. The right-hand side is equal to $G(g^{-1}(y))$. But by Lemma 2, $G^{-1}(z) = -W(-z)$, so for $z = \mathrm{e}^y$ we have $G^{-1}(\mathrm{e}^y) = -W(-\mathrm{e}^y) = g^{-1}(y)$.

11 The Kruskal-Szekeres embeddings

In this section we consider two manifolds, $M = \mathbf{R}\times(0,\infty)$ and $\mathbf{R}^2$. To any real number $m > 0$ (*mass*) we assign the subset $\{(t,r)\in\mathbf{R}\times(0,\infty) \mid r \neq 2m\}$ of *M* and construct its embedding into $\mathbf{R}^2$; the embedding is defined by means of the "Kruskal-Szekeres coordinates", and is indeed not canonical.

Denote by *t*, *r* the canonical coordinates on *M*, and by *U*, *V* the canonical coordinates on $\mathbf{R}^2$. *m* defines a mapping $\mathbf{R}\times(2m,\infty)\ni(t,r)\to\Phi_m^{(+)}(t,r)\in\mathbf{R}^2$ by the equations

$$(1)\qquad \begin{aligned} U\circ\Phi_m^{(+)}(t,r) &= \sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right),\\ V\circ\Phi_m^{(+)}(t,r) &= \sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right). \end{aligned}$$

Note that the sign of $U\circ\Phi_m^{(+)}(t,r)$ (resp. $V\circ\Phi_m^{(+)}(t,r)$) coincides with the sign of $\cosh(t/4m)$ (resp. $\sinh(t/4m)$); in particular, $U\circ\Phi_m^{(+)}$ is always positive, and the sign of $V\circ\Phi_m^{(+)}$ coincides with the sign of the argument *t*.

m also defines a mapping $\mathbf{R}\times(-\infty,2m)\ni(t,r)\to\Phi_m^{(-)}(t,r)\in\mathbf{R}^2$ by

$$(2)\qquad \begin{aligned} U\circ\Phi_m^{(-)}(t,r) &= \sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right),\\ V\circ\Phi_m^{(-)}(t,r) &= \sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right). \end{aligned}$$

We usually write equations (1) and (2) in a simplified form

$$(3)\qquad U = \sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right),\quad V = \sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right),$$

and

$$(4)\qquad U = \sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right),\quad V = \sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right).$$

Lemma 4 (a) *The mapping* $\Phi_m^{(+)}$ *is a bijection of the set* $\mathbf{R}\times(2m,\infty)$ *and the set*

(5) $$\operatorname{Im}\Phi_m^{(+)}=\{(U,V)\in\mathbf{R}^2\,|\,U>0,U^2-V^2>0\}.$$

The inverse mapping $(\Phi_m^{(+)})^{-1}$ *is given by the equations*

(6) $$t=4m\operatorname{arc}\tanh\left(\frac{V}{U}\right),\quad r=2m\left(1+W\left(\frac{U^2-V^2}{\mathrm{e}}\right)\right).$$

(b) *The mapping* $\Phi_m^{(-)}$ *is a bijection of the set and the set*

(7) $$\operatorname{Im}\Phi_m^{(-)}=\{(U,V)\in\mathbf{R}^2\,|\,V>0,-1<U^2-V^2<0\}.$$

The inverse mapping $(\Phi_m^{(-)})^{-1}$ *is given by the equations*

(8) $$t=4m\operatorname{arc}\tanh\left(\frac{U}{V}\right),\quad r=2m\left(1+W\left(\frac{U^2-V^2}{\mathrm{e}}\right)\right).$$

Proof (a) First we determine the image set $\operatorname{Im}\Phi_m^{(+)}$, which consists of the points $(U,V)\in\mathbf{R}^2$ such that equations (3) have a solution (t,r), where $r\in(2m,\infty)$. To this purpose it is convenient to use graphs of the hyperbolic functions $\cosh$, $\tanh$, and the Lambert function W.

Suppose we have a point $(U,V)\in\operatorname{Im}\Phi_m^{(+)}$. Then since $\cosh$ is always positive, $U>0$ from (3). Since (U,V) satisfies

(9) $$\frac{V}{U}=\tanh\left(\frac{t}{4m}\right),$$

and $\operatorname{Im}(\tanh)=(-1,1)$, the number V/U belongs to the interval $(-1,1)$ hence $V\in(-U,U)$. We want to show that for any (U,V) satisfying these two conditions $U>0$ and $V\in(-U,U)$, equations (3) have a solution (t,r), such that $r\in(2m,\infty)$; this will prove that $\operatorname{Im}\Phi_m^{(+)}=\{(U,V)\in\mathbf{R}^2\,|\,U>0,V\in(-U,U)\}$.

Condition (9) written as $\arctan(V/U)=t/4m$, determines t. Condition $V\in(-U,U)$ implies $U^2-V^2>0$; but from (3)

(10) $$\begin{aligned}\frac{U^2-V^2}{\mathrm{e}}&=\frac{1}{\mathrm{e}}\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}}\left(\cosh^2\left(\frac{t}{4m}\right)-\sinh^2\left(\frac{t}{4m}\right)\right)\\&=\frac{1}{\mathrm{e}}\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}}=\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}-1},\end{aligned}$$

hence

(11) $$\frac{r}{2m} - 1 = W\left(\frac{U^2 - V^2}{\mathrm{e}}\right).$$

Since the Lambert function is positive and increasing on $(0,\infty)$, there exists exactly one number $(r-2m)/2m$ solving (6), and $(r-2m)/2m \in (0,\infty)$; then, however, $r \in (2m,\infty)$.

Summarizing, we see that $\mathrm{Im}\,\Phi_m^{(+)} = \{(U,V) \in \mathbf{R}^2 \mid U > 0, V \in (-U,U)\}$, and in $\mathrm{Im}\,\Phi_m^{(+)}$ there exists exactly one pair (t,r) solving (3); from (9) and (11), (t,r) is given by (6).

(b) $\mathrm{Im}\,\Phi_m^{(-)}$ is the set of points $(U,V) \in \mathbf{R}^2$ such that equations (4) have a solution (t,r), where $r \in (-\infty,2m)$. If $(U,V) \in \mathrm{Im}\,\Phi_m^{(-)}$ is a point, then since $\cosh$ is always positive, we have $V > 0$. Then equations (4) imply

(12) $$\frac{U}{V} = \tanh\frac{t}{4m},$$

and since $\mathrm{Im}(\tanh) = (-1,1)$, we get $U \in (-V,V)$. It is clear that equation (12) determines t.

Condition $U \in (-V,V)$ implies $V^2 - U^2 > 0$. Computing $V^2 - U^2$ from (4),

(13) $$\begin{aligned} V^2 - U^2 &= \left(1 - \frac{r}{2m}\right)\mathrm{e}^{\frac{r}{2m}}\left(\cosh^2\left(\frac{t}{4m}\right) - \sinh^2\left(\frac{t}{4m}\right)\right) \\ &= \left(1 - \frac{r}{2m}\right)\mathrm{e}^{\frac{r}{2m}} = \mathrm{e}\left(1 - \frac{r}{2m}\right)\mathrm{e}^{\frac{r}{2m}-1}, \end{aligned}$$

hence

(14) $$\frac{r}{2m} - 1 = W\left(\frac{U^2 - V^2}{\mathrm{e}}\right).$$

But this formula restricts possible image points (U,V) because $0 < r < 2m$ and the left-hand side is always negative: we have $-1 < (r-2m)/2m < 0$. Thus, from the properties of W, $(U^2 - V^2)/\mathrm{e} \in (-1/\mathrm{e},0)$ and $V^2 - U^2 \in (0,1)$. Summarizing, if (U,V) belongs to the set $\mathrm{Im}\,\Phi_m^{(-)}$, then necessarily $V > 0$, $U \in (-V,V)$, and $0 < V^2 - U^2 < 1$. Note that the third condition $0 < V^2 - U^2$ already implies the second one, so we can say that if $(U,V) \in \mathrm{Im}\,\Phi_m^{(-)}$, then $V > 0$ and $0 < V^2 - U^2 < 1$.

We call the mappings $\Phi_m^{(+)}$ and $\Phi_m^{(-)}$ the *Kruskal-Szekeres embeddings*.

The set $\mathrm{Im}\,\Phi_m^{(+)}$ is defined by the inequalities $U > 0$, $U^2 - V^2 > 0$, i.e., $U > 0$ and $-U < V < U$, so is an open positive cone in $\mathbf{R}^2$ along the U-axis. The set $\mathrm{Im}\,\Phi_m^{(-)}$ is defined by $V > 0$ and $-1 < U^2 - V^2 < 0$. The second inequality is equivalent to two inequalities $V^2 < U^2 + 1$ and $U^2 < V^2$; altogether, $\mathrm{Im}\,\Phi_m^{(+)}$ is the intersection of three sets, defined by $V > 0$, $V < \sqrt{U^2+1}$, and $-V < U < V$. Thus $\mathrm{Im}\,\Phi_m^{(+)}$ is the subset of the open positive cone along the *V*-axis, consisting of the points under the hyperbola $V = \sqrt{U^2+1}$.

The common boundary of these two sets is the set of points (U,V) such that $U = V > 0$.

12 The Kruskal-Szekeres spacetime

Consider the canonical metric field of Lorentz type, defined on $\mathbf{R}^2$ by

(1) $$h = dU \otimes dU - dV \otimes dV.$$

We wish to compute the pull-backs $(\Phi_m^{(+)})^* h$ and $(\Phi_m^{(-)})^* h$.

Lemma 5 (a) *The pull-back* $(\Phi_m^{(+)})^* h$ *has an expression*

(2) $$(\Phi_m^{(+)})^* h = \frac{1}{16m^2}\frac{r}{2m}\mathrm{e}^{\frac{r}{2m}}\left(-\frac{r-2m}{r}dt \otimes dt + \frac{r}{r-2m}dr \otimes dr\right).$$

(b) *The pull-back* $(\Phi_m^{(-)})^* h$ *has an expression*

(3) $$(\Phi_m^{(-)})^* h = \frac{1}{16m^2}\frac{r}{2m}\mathrm{e}^{\frac{r}{2m}}\left(-\frac{r-2m}{r}dt \otimes dt + \frac{r}{r-2m}dr \otimes dr)\right).$$

Proof (a) We have from equations (1) or (3), Section 3,

$$dU = \sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\frac{\partial}{\partial t}\cosh\left(\frac{t}{4m}\right)\cdot dt + \cosh\left(\frac{t}{4m}\right)\frac{\partial}{\partial r}\left(\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\right)\cdot dr$$

$$= \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right)\cdot dt$$

$$+\cosh\left(\frac{t}{4m}\right)\cdot\left(\frac{1}{2m}\frac{1}{2\sqrt{\frac{r}{2m}-1}}\mathrm{e}^{\frac{r}{4m}} + \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\right)dr$$

(4)
$$= \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right)\cdot dt$$

$$+\frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\left(\frac{1}{\frac{r}{2m}-1}+1\right)dr$$

$$= \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right)\cdot dtdt$$

$$+\frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\frac{r}{r-2m}dr,$$

and

$$dV = \sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\frac{\partial}{\partial t}\sinh\left(\frac{t}{4m}\right)\cdot dt + \sinh\left(\frac{t}{4m}\right)\frac{\partial}{\partial r}\left(\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\right)\cdot dr$$

$$= \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\cdot dt$$

$$+\sinh\left(\frac{t}{4m}\right)\left(\frac{1}{2m}\frac{1}{2\sqrt{\frac{r}{2m}-1}}\mathrm{e}^{\frac{r}{4m}} + \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\right)dr$$

(5)
$$= \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\cdot dt$$

$$+\frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right)\left(\frac{1}{\frac{r}{2m}-1}+1\right)dr$$

$$= \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\cdot dt$$

$$+\frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right)\frac{r}{r-2m}dr,$$

hence

(6)
$$dU = \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\left(\sinh\left(\frac{t}{4m}\right)\cdot dt + \frac{r}{r-2m}\cosh\left(\frac{t}{4m}\right)dr\right),$$
$$dV = \frac{1}{4m}\sqrt{\frac{r}{2m}-1}\,\mathrm{e}^{\frac{r}{4m}}\left(\cosh\left(\frac{t}{4m}\right)\cdot dt + \frac{r}{r-2m}\sinh\left(\frac{t}{4m}\right)dr\right).$$

Now

(7)
$$\begin{aligned}
&(\Phi_m^{(+)})^* h \\
&= \frac{1}{16m^2}\mathrm{e}^{\frac{r}{2m}}\left(\frac{r}{2m}-1\right)\left(\left(\sinh\left(\frac{t}{4m}\right)\cdot dt + \frac{r}{r-2m}\cosh\left(\frac{t}{4m}\right)dr\right)\right. \\
&\otimes\left(\sinh\left(\frac{t}{4m}\right)\cdot dt + \frac{r}{r-2m}\cosh\left(\frac{t}{4m}\right)dr\right) \\
&-\left(\cosh\left(\frac{t}{4m}\right)\cdot dt + \frac{r}{r-2m}\sinh\left(\frac{t}{4m}\right)dr\right) \\
&\left.\otimes\left(\cosh\left(\frac{t}{4m}\right)\cdot dt + \frac{r}{r-2m}\sinh\left(\frac{t}{4m}\right)dr\right)\right),
\end{aligned}$$

so we get

(8)
$$\begin{aligned}
&(\Phi_m^{(+)})^* h \\
&= \frac{1}{16m^2}\mathrm{e}^{\frac{r}{2m}}\left(\frac{r}{2m}-1\right)\left(\left(\sinh^2\left(\frac{t}{4m}\right) - \cosh^2\left(\frac{t}{4m}\right)\right)dt\otimes dt\right. \\
&+\frac{r}{r-2m}\left(\sinh\left(\frac{t}{4m}\right)\cosh\left(\frac{t}{4m}\right) - \cosh\left(\frac{t}{4m}\right)\sinh\left(\frac{t}{4m}\right)\right)dt\otimes dr \\
&+\frac{r}{r-2m}\left(\cosh\left(\frac{t}{4m}\right)\sinh\left(\frac{t}{4m}\right) - \sinh\left(\frac{t}{4m}\right)\cosh\left(\frac{t}{4m}\right)\right)dr\otimes dt \\
&\left.+\frac{r^2}{(r-2m)^2}\left(\cosh^2\left(\frac{t}{4m}\right) - \sinh^2\left(\frac{t}{4m}\right)\right)dr\otimes dr\right) \\
&= \frac{1}{16m^2}\mathrm{e}^{\frac{r}{2m}}\frac{r-2m}{2m}\left(-dt\otimes dt + \frac{r^2}{(r-2m)^2}dr\otimes dr\right) \\
&= \frac{1}{32m^3}r\,\mathrm{e}^{\frac{r}{2m}}\left(-\frac{r-2m}{r}dt\otimes dt + \frac{r}{r-2m}dr\otimes dr\right).
\end{aligned}$$

This formula proves (2).

(b) From (2) or (4), Section 3

$$
\begin{aligned}
dU &= \sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\frac{\partial}{\partial t}\sinh\left(\frac{t}{4m}\right)\cdot dt+\sinh\left(\frac{t}{4m}\right)\frac{\partial}{\partial r}\left(\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\right)dr\\
&=\frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\cdot dt\\
&+\sinh\left(\frac{t}{4m}\right)\left(-\frac{1}{2m}\frac{1}{2\sqrt{1-\frac{r}{2m}}}\mathrm{e}^{\frac{r}{4m}}+\frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\right)dr\\
&=\frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\cdot dt\\
&+\frac{1}{4m}\sinh\left(\frac{t}{4m}\right)\mathrm{e}^{\frac{r}{4m}}\left(-\frac{1}{\sqrt{1-\frac{r}{2m}}}+\sqrt{1-\frac{r}{2m}}\right)dr\\
&=\frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\cdot dt\\
&-\frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\frac{r}{2m-r}\sinh\left(\frac{t}{4m}\right)\mathrm{e}^{\frac{r}{4m}}\,dr,
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
dV &= \sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\frac{\partial}{\partial t}\cosh\left(\frac{t}{4m}\right)\cdot dt+\cosh\left(\frac{t}{4m}\right)\frac{\partial}{\partial r}\left(\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\right)dr\\
&=\frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right)\cdot dt\\
&+\cosh\left(\frac{t}{4m}\right)\cdot\left(-\frac{1}{2m}\frac{1}{2\sqrt{1-\frac{r}{2m}}}\mathrm{e}^{\frac{r}{4m}}+\frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\right)dr\\
&=\frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right)\cdot dt\\
&+\frac{1}{4m}\mathrm{e}^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right)\cdot\left(-\frac{1}{\sqrt{1-\frac{r}{2m}}}+\sqrt{1-\frac{r}{2m}}\right)dr\\
&=\frac{1}{4m}\mathrm{e}^{\frac{r}{4m}}\sqrt{1-\frac{r}{2m}}\sinh\left(\frac{t}{4m}\right)\cdot dt\\
&-\frac{1}{4m}\mathrm{e}^{\frac{r}{4m}}\sqrt{1-\frac{r}{2m}}\cosh\left(\frac{t}{4m}\right)\cdot\frac{r}{2m-r}\,dr.
\end{aligned}
\tag{10}
$$

Thus

$$
(11)\qquad
\begin{aligned}
dU &= \frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\left(\cosh\left(\frac{t}{4m}\right)\cdot dt - \frac{r}{2m-r}\sinh\left(\frac{t}{4m}\right)\cdot dr\right),\\
dV &= \frac{1}{4m}\sqrt{1-\frac{r}{2m}}\,\mathrm{e}^{\frac{r}{4m}}\left(\sinh\left(\frac{t}{4m}\right)\cdot dt - \frac{r}{2m-r}\cosh\left(\frac{t}{4m}\right)\cdot dr\right).
\end{aligned}
$$

Now

$$
(12)\qquad
\begin{aligned}
&(\Phi_m^{(-)})^* h\\
&= \frac{1}{16m^2}\mathrm{e}^{\frac{r}{2m}}\left(1-\frac{r}{2m}\right)\Bigg(\left(\cosh\left(\frac{t}{4m}\right)\cdot dt - \frac{r}{2m-r}\sinh\left(\frac{t}{4m}\right)\cdot dr\right)\\
&\otimes\left(\cosh\left(\frac{t}{4m}\right)\cdot dt - \frac{r}{2m-r}\sinh\left(\frac{t}{4m}\right)\cdot dr\right)\\
&-\left(\sinh\left(\frac{t}{4m}\right)\cdot dt - \frac{r}{2m-r}\cosh\left(\frac{t}{4m}\right)\cdot dr\right)\\
&\otimes\left(\sinh\left(\frac{t}{4m}\right)\cdot dt - \frac{r}{2m-r}\cosh\left(\frac{t}{4m}\right)\cdot dr\right)\Bigg)
\end{aligned}
$$

hence

$$
(13)\qquad
\begin{aligned}
(\Phi_m^{(-)})^* h &= \frac{1}{16m^2}\left(1-\frac{r}{2m}\right)\mathrm{e}^{\frac{r}{2m}}\\
&\cdot\Bigg(\left(\cosh^2\left(\frac{t}{4m}\right)-\sinh^2\left(\frac{t}{4m}\right)\right)dt\otimes dt\\
&+\frac{r}{2m-r}\left(-\cosh\left(\frac{t}{4m}\right)\sinh\left(\frac{t}{4m}\right)+\sinh\left(\frac{t}{4m}\right)\cosh\left(\frac{t}{4m}\right)\right)dt\otimes dr\\
&+\frac{r}{2m-r}\left(-\sinh\left(\frac{t}{4m}\right)\cosh\left(\frac{t}{4m}\right)+\cosh\left(\frac{t}{4m}\right)\sinh\left(\frac{t}{4m}\right)\right)dr\otimes dt\\
&+\frac{r^2}{(2m-r)^2}\left(\sinh^2\left(\frac{t}{4m}\right)-\cosh^2\left(\frac{t}{4m}\right)\right)dr\otimes dr\Bigg)\\
&=\frac{1}{16m^2}\left(1-\frac{r}{2m}\right)\mathrm{e}^{\frac{r}{2m}}\left(dt\otimes dt - \frac{r^2}{(2m-r)^2}dr\otimes dr)\right).
\end{aligned}
$$

This expression can also be written as

$$
(14)\qquad (\Phi_m^{(-)})^* h = \frac{1}{16m^2}\frac{1}{2m}r\,\mathrm{e}^{\frac{r}{2m}}\left(-\frac{r-2m}{r}dt\otimes dt + \frac{r}{r-2m}dr\otimes dr)\right)
$$

as required.

Note that the coefficient in expressions (2) and (3), Lemma 5, can be expressed in terms of coordinates U and V on $\mathbf{R}^2$; from Lemma 4

$$(15) \qquad \frac{r}{2m}\mathrm{e}^{\frac{r}{2m}} = \left(1+W\left(\frac{U^2-V^2}{\mathrm{e}}\right)\right)\mathrm{e}^{1+W\left(\frac{U^2-V^2}{\mathrm{e}}\right)}.$$

The right-hand side has sense for each (U,V) such that $(U^2-V^2)/\mathrm{e}$ belongs to the domain of definition of W, i.e., to the interval $[-1/\mathrm{e},\infty)$. Equivalently, $U^2-V^2 \in [-1,\infty)$, or $U^2-V^2+1 \geq 0$. Thus, (U,V) should satisfy $V^2 \leq U^2+1$ or, equivalently, the inequalities $-\sqrt{U^2+1} \leq V \leq \sqrt{U^2+1}$, defining the region between two branches of the hyperbola $V^2 = U^2+1$. On the hyperbola,

$$(16) \qquad 1+W\left(\frac{U^2-V^2}{\mathrm{e}}\right) = 0$$

so the coefficient (15) becomes zero. Using these observations we set

$$(17) \qquad g = \frac{1}{1+W\left(\frac{U^2-V^2}{\mathrm{e}}\right)}\mathrm{e}^{-1-W\left(\frac{U^2-V^2}{\mathrm{e}}\right)}(dU \otimes dU - dV \otimes dV).$$

g is a Lorentz metric field on the set

$$(18) \qquad -\sqrt{U^2+1} < V < \sqrt{U^2+1}.$$

From Lemma 5 we now have the following result.

Theorem 7 (Kruskal-Szekeres spacetime) *There exists a manifold X, diffeomorphic with* $\mathbf{R}^2$, *a Lorentz metric field g on X, two diffeomorphisms* $\Phi_m^{(+)} : W_m^{(+)} \to X$ *and* $\Phi_m^{(-)} : W_m^{(-)} \to X$, *and a closed* 1*-dimensional submanifold Z of X with the following properties:*

(a) *Z is diffeomorphic with the real line* $\mathbf{R}$.

(b) *The sets* $\Psi_m^{(+)}(W_m^{(+)})$, $\Psi_m^{(-)}(W_m^{(+)})$, *and Z are mutually disjoint and*

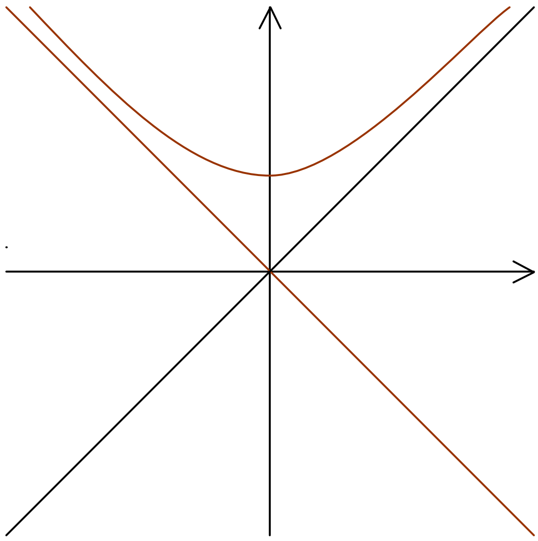
$$(19) \qquad X = \Psi_m^{(+)}(W_m^{(+)}) \cup \Psi_m^{(-)}(W_m^{(-)}) \cup Z.$$

(c) *The pull-back metric fields* $(\Phi_m^{(+)})^* g$ *and* $(\Phi_m^{(-)})^* g$ *are the Schwartzschild metric field on* $W_m^{(+)}$ *and* $W_m^{(-)}$.

Proof We take for X the connected open set in $\mathbf{R}^2$, defined by the inequalities $-U < V < \sqrt{U^2+1}$, and apply Lemma 5.

X can be represented by the region between the brown lines $V = -U$ and

$V = \sqrt{U^2 + 1}$ in the diagram



Clearly, applying a diffeomorphism, one can take for X the entire manifold $\mathbf{R}^2$.