Lepage Research Institute

Lepage Inst. Archive, No. 1 (2012), 1–42
Lepage Research Institute, Czech Republic

About determining maximal and minimal properties

N. Ja. Sonin

Dedicated to **A. Yu. Davidov**

Translated from Russian† by **R. Ya. Matsyuk** (Lepage Research Institute, Czech Republic)

The main problem of the variational calculus is known to consist in determining those curves and surfaces that possess either in profound extent or in some part the given minimum or maximum properties. Usually the property in question meets its analytical expression in that certain definite integral containing the unknown ordinate of the sought curve or surface and its derivatives of various orders takes along the sought curve a greater or a less value than along all adjacent curves. The determination of the sought curve or surface reduces to the integration of a differential equation – ordinary or with partial derivatives, the arbitrary constant or function entering in the solution being defined by means of the limiting conditions.

In this way the given maximum or minimum property together with the limiting conditions happens to become characteristic of the curve or surface and determines them definitely.

But one and the same given curve or surface may possess various characteristic properties, in that number different maximum and minimum properties. The discovering of the latter ones is thus a particular problem in the investigation of curves and surfaces, which can easily be formulated in analytic terms only if one restricts the problem to those properties, which appear in usual tasks of the calculus of variations. However even this restricted problem seems unattainable in whole generality and so one is forced to resort to still new restrictions. These will concern two issues, namely, that of the contents of the integrand the form of which should be discovered and fixed, and that of the nature of the curve or surface itself, which should be considered as belonging to this or that family.

Here we shall confine ourselves to the case when the integrand involves the derivatives of the order not greater than one and when the given curve belongs to a family, defined by the differential equation of the second order. We shall point out the methods to build up such equation, starting with the given finite equation of the curve and we subject to this investigation the following form of the second order differential equation

$$y = \frac{d^2y}{dx^2} = \Theta\left(\frac{dy}{dx}\right),$$

which can be formulated for every curve. Among others, we consider the question of whether it is possible to construct a curve through the two *given* points on the plain satisfying this equation.

† Warsaw Univ. Izvestiya, no. 1–2 (1886) pp. 1–68

It is known that in their excellent course of variational calculus Moigno and Lindelöf* noticed that the chain curve cannot always be drawn through two arbitrarily chosen points and derived a *necessary* condition for such possibility. Their considerations being somewhat muddled this gave Todhunter[1] the reason to claim that the cited authors solved the problem only in the particular case of equal ordinates; for the unequal ordinates in the end points he elaborated a new also *necessary* conditions. Our treatment of general case discovers the *necessity* of Moigno condition; moreover, we deduce the *necessary* and *sufficient* condition expressed in a form of inequality containing the root of a fourth order equation; the Todhunter condition plays no role here.

In the above mentioned work (art. 24 and 282), apart from everything else, Todhunter shows some integrals, in the investigation of which for minimum and maximum the Jacobi condition takes the as much simple geometric form as in the solution of the minimal surface of revolution problem. Investigating the before mentioned differential equation of second order we have found out that in general, for the curves that satisfy that equation, the Jacobi condition takes on the same simple geometric form. This gave us the reason to consider the general question of finding the differential equation of the second order in adaptation to which the Jacobi condition takes the above mentioned form. The answer to this problem completes this research.

1. Let's denote by x and y the coordinates with respect to some system, by p the first derivative $\dfrac{y}{x}$ and by q the second derivative $\dfrac{d^2y}{d^2x}$. Let

$$q = \phi(x, y, p) \tag{1}$$

be some differential equation of a family of curves and let the first integrals of this equation be

$$\psi(x, y, p) = \alpha, \qquad \sigma(x, y, p) = \beta, \tag{2}$$

where α and β are arbitrary constants. In the final form the equation of the family under consideration will come out after excluding p between the equations (2) and will be

$$F(x, y, \alpha, \beta) = 0. \tag{3}$$

Vice versa, from this equation by means of differentiation and exclusion the equations (2,1) might be obtained.

Let us determine the maximum and minimum properties of this family of curves that consist in that some integrals of the form

$$\int_a^b f(x, y, p)dx \tag{4}$$

gain their maximal or minimal values.

2. As known from the generally accepted theory, the integral (4) takes its maximum or minimum if the therein entering y is defined as a function of x by the equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial p} = 0,$$

or, in the developed form,

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial p} - \frac{\partial^2 f}{\partial y \partial p}p - \frac{\partial^2 f}{\partial p^2}q = 0. \tag{5}$$

* Moigno, François; Lindelöf, Lorenz Leonard. Leçons de calcul des variations, par L. Lindelöf, rédigées en collaboration avec M. l'abbé Moigno. Mallet–Bachelier, Paris, 1861, XVI–352 pp. [Texte imprimé] – R. M.

[1] Todhunter. Researches in the Calculus of Variations 1871 p. p. 56, 58.

Thus for the integral (4) to take its maximal or minimal value along the curves defined by equation (1) it is necessary that this equation coincide with the equation (5), or that the following equality holds

$$\frac{\partial f}{\partial y}-\frac{\partial^2 f}{\partial x\partial p}-\frac{\partial^2 f}{\partial y\partial p}p-\frac{\partial^2 f}{\partial p^2}\phi(x,y,p)=0. \tag{6}$$

We are faced to determine the form of the function $f(x,y,p)$ from this equation.

3. Differentiating equation (6) by p and putting $\frac{\partial^2 f}{\partial p^2}=z$ one obtains the differential equation with partial derivatives of the first order

$$\frac{\partial z}{\partial x}+p\frac{\partial z}{\partial y}+\phi(x,y,z)\frac{\partial z}{\partial p}+\frac{\partial \phi}{\partial p}x=0, \tag{7}$$

to integrate which, as is commonly known, it is necessary to solve the following system of equations

$$\frac{dx}{1}=\frac{dy}{p}=\frac{dp}{\phi(x,y,p)}=\frac{dz}{-\frac{\partial\phi}{\partial p}z}. \tag{8}$$

The two integral equations of this system obviously are (2), the third one may be presented in various gazes, out of which it suffice to focus on

$$z=\gamma.\, e^{-\int\frac{\partial\ log\ \phi}{\partial p}dp}, \tag{9}$$

where γ is an arbitrary constant. During the integration in the exponent one should substitute in the integrand the expressions x and y as functions of p from formulæ(2), or, in general, reduce $\frac{\partial\ log\ \phi}{\partial p}dp$ to the form of total differential using (2). The most general expression of z will follow while taking an arbitrary function $\Phi(\alpha,\beta)$ in place of γ and then instead of α and β inserting functions $\psi(x,y,z)$ and $\sigma(x,y,z)$. Thus

$$z=\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}, \tag{10}$$

where by putting $\frac{\partial\ log\ \phi}{\partial p}$ into parenthesis we wish to emphasize the replacing of α and β with ψ and σ in the result of the integration.

4. Once z found, it isn't difficult to determine $f(x,y,z)$. From the equation

$$\frac{\partial^2 f}{\partial p^2}=\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp} \tag{11}$$

through the integration with respect to p one gets

$$\frac{\partial f}{\partial p}=\int_A^p\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}dp+B, \tag{12}$$

where A and B are arbitrary functions of x and y, and hereat by next integration one obtains

$$f(x,y,z)=\int_A^p dp\int_A^p \Phi(\psi,\sigma)\,e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}dp+Bp+C, \tag{13}$$

or, by reduction to simple integrals,

$$\begin{aligned}f(x,y,p)=p\int_A^p \Phi(\psi,\sigma)\,e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}dp\\ -\int_A^p p\,\Phi(\psi,\sigma)\,e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}dp+Bp+C,\end{aligned} \tag{14}$$

where C is a new arbitrary function of x and y.

In this way the most general expression of $f(x,y,p)$ is obtained with the four arbitrary functions $\Phi(\psi,\sigma)$, A, B, C the last three of which depend on x and y only.

Let us mention that the arbitrary function A without loss of generality may be replaced by some definite function or even a constant; even after that the general expression $f(x,y,p)$ still will contain three arbitrary functions, whereas actually in the integration of the equation with partial derivatives of the second order (6) one can expect in the general expression $f(x,y,p)$ at most two independent arbitrary functions. So with necessity there should exist some dependency among the arbitrary functions we consider here.

5. In order to disclose this dependency let us insert the obtained expression $f(x,y,p)$ into equation (6). The differentiation of (14) produces

$$\begin{aligned}\frac{\partial f}{\partial y}=p\int_A^p\frac{\partial}{\partial y}\left[\Phi(\psi,\sigma)\,e^{-\int\left(\frac{\partial log\phi}{\partial p}\right)dp}\right]dp\\ -\int_A^p p\frac{\partial}{\partial y}\left[\Phi(\psi,\sigma)\,e^{-\int\left(\frac{\partial log\phi}{\partial p}\right)dp}\right]dp\\ +(A-p)\left[\Phi(\psi,\sigma)\,e^{-\int\left(\frac{\partial log\phi}{\partial p}\right)dp}\right]_{p=A}\frac{\partial A}{\partial y}+\frac{\partial B}{\partial y}p+\frac{\partial C}{\partial y}\,.\end{aligned}$$

Further from the formula (12) we obtain

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial p}=\int_A^p\frac{\partial}{\partial x}\left[\Phi(\psi,\sigma)\,e^{-\int\left(\frac{\partial log\phi}{\partial p}\right)dp}\right]dp\\ -\left[\Phi(\psi,\sigma)\,e^{-\int\left(\frac{\partial log\phi}{\partial p}\right)dp}\right]_{p=A}\frac{\partial A}{\partial x}+\frac{\partial B}{\partial x}\,,\end{aligned}$$

$$\frac{\partial^2 f}{\partial y \partial p} = \int_A^p \frac{\partial}{\partial y}\left[\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial log\phi}{\partial p}\right)dp}\right] dp$$

$$- \left[\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial log\phi}{\partial p}\right)dp}\right]_{p=A} \frac{\partial A}{\partial y} + \frac{\partial B}{\partial y}\,.$$

While inserting these expressions together with expression (11) into equation (6) let us notice that two integral terms with the factor p, namely, the first term in $\frac{\partial f}{\partial y}$ and the first term in $\frac{\partial^2 f}{\partial y \partial p}$ cancel out. Next, replacing in this equation the term $\frac{\partial^2 f}{\partial p^2}\phi(x,y,p)$, which is equal to

$$\phi(x,y,p)\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp},$$

with

$$\int_A^p \frac{\partial}{\partial p}\left[\phi(x,y,p)\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}\right] dp$$

$$+ \left[\phi(x,y,p)\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}\right]_{p=A}$$

and collecting all integral terms under one mutual integral sign we get the expression

$$-\int_A^p \left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + \phi(x,y,p)\frac{\partial}{\partial p} + \frac{\partial \phi}{\partial p}\right)\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp} dp\,,$$

that turns into zero by virtue of the equation (7) and of its solution (10) found before.

After carrying out some further simple groupings we ultimately come to the following conditional equation need to hold in order that the obtained function $f(x,y,p)$ satisfy equation(6):

(15)
$$\left[\frac{\partial A}{\partial x} + A\frac{\partial A}{\partial y} - \phi(x,y,A)\right]\left[\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}\right]_{p=A}$$

$$+\frac{\partial C}{\partial y} - \frac{\partial B}{\partial x} = 0.$$

6. Recall now that we did not carry out any concrete choice of the function A up to this point yet. Evidently it is most convenient to make such a choice of A that the condition (15) splits into two independent conditions, namely,

$$\left[\frac{\partial A}{\partial x} + A\frac{\partial A}{\partial y} - \phi(x,y,A)\right]\left[\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}\right]_{p=A} = 0, \tag{16}$$

$$\frac{\partial C}{\partial y} - \frac{\partial B}{\partial x} = 0. \tag{17}$$

Of these equation the first one serves to determine the A whereas the second one shows the entanglement between the arbitrary functions B and C. From this entanglement it follows that the binomial $Cdx + Bdy$ should present the exact differential of a function $f_1(x,y)$ of two variables. We obtain

$$C = \frac{\partial f_1}{\partial x}, \qquad B = \frac{\partial f_1}{\partial y}.$$

In what concerns equation (16) which determines the A, one of the next two corollaries follow, namely either

$$\frac{\partial A}{\partial x} + A\frac{\partial A}{\partial y} - \phi(x,y,A) = 0\,, \tag{18}$$

or

$$\left[\Phi(\psi,\sigma)\, e^{-\int\left(\frac{\partial\ log\ \phi}{\partial p}\right)dp}\right]_{p=A} = 0. \tag{19}$$

Condition (18) constitutes an equation with partial derivatives of the first order. Its solution is obtained by means of the integration of the system

$$\frac{dx}{1} = \frac{dy}{A} = \frac{dA}{\phi(x,y,A)}\,.$$

The integral equations of this system will certainly be

$$\psi(x,y,A) = \alpha, \qquad \sigma(x,y,A) = \beta\,,$$

in virtue of what the differential condition (18) will turn to a finite one

$$\Psi[\psi(x,y,A), \sigma(x,y,A)] = 0\,, \tag{20}$$

where character Ψ denotes an arbitrary function; the function A must be the root of the equation (20)

The condition (19) will be satisfied if either $\Phi(\psi,\sigma) = 0$, provided $\int\left(\frac{\partial log\phi}{\partial p}\right)dp$ is finite, which is the particular case of the equation (20), or $\int\left(\frac{\partial log\phi}{\partial p}\right)dp = \infty$, but at the same time $\Phi(\psi,\sigma)$ remains finite; this condition may provide a value of A not satisfying equation (20),

7. After this the ultimate result of our investigation may be formulated as follows:
The general solution of the equation (6) is expressed by the formula

$$f(x,y,p) = \int_A^p dp \int_A^p \Phi(\psi,\sigma)\, e^{-\int \left(\frac{\partial\ log\ \phi}{\partial p}\right) dp} dp + \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} p\,, \tag{21}$$

or

$$f(x,y,p) = p \int_A^p \Phi(\psi,\sigma)\, e^{-\int \left(\frac{\partial\ log\ \phi}{\partial p}\right) dp} dp - \int_A^p \Phi(\psi,\sigma)\, e^{-\int \left(\frac{\partial\ log\ \phi}{\partial p}\right) dp} dp + \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} p\,, \tag{22}$$

where A is the root of the equation (20) or (19).

Let us notice that the fact of presence of the terms including the arbitrary function $f_1(x,y)$ might with ease be expected as far as the integral

$$\int_a^b \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} p\right) dx$$

evidently depends only on the boundary values of y, and not on the shape of the function y, so that adding the terms $\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} p$ to the integrand in $\int_a^b f(x,y,p)\, dx$ will not influence the family of curves for which this integral reaches its maximum or minimum, but only will influence the choice of this or that curve of the family, depending on the boundary values. In force of this, while determining the function $f(x,y,p)$ from formulæ (21) or (22) we may ignore terms with an arbitrary function $f_1(x,y)$ as well as on the whole ignore all the terms that can undergo an integration in the undefined way, that is without assuming any special kind of dependency of the y upon the x.

8. Finally let us make one more observation of general character. The Jacobi multiplier of the equation

$$q - \phi(x,y,p) = 0$$

is known to be determined by the equation

$$\frac{\partial M}{\partial x} + p \frac{\partial M}{\partial y} + \frac{\partial M\phi}{\partial p} = 0\,,$$

which is identical with (7). This implies that

$$M = \Phi(\psi,\sigma)\, e^{-\int \left(\frac{\partial log\phi}{\partial p}\right) dp} = \frac{\partial^2 f}{\partial p^2}\,,$$

and, consequently, *after the multiplication by the Jacobi multiplier the equation* $q - \phi(x,y,p) = 0$ *reduces to*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial p} = 0\,.$$

9. Now let us imagine that we wish to determine the maximal and the minimal properties of some curve given by the equation

$$F(x,y) = 0.$$

It is possible to construct an equation of the second order in an infinite variety of ways so that the given curve would satisfy it as its partial solution. But also it is possible to point out some general approaches to the construction of such a differential equation.

Taking x and y as the rectangular coordinates, we shall just consider the given curve as a representative of the homothetic curves, the general equation of which will be

$$F\left(\frac{x}{\beta}, \frac{y}{\beta}\right) = 0,$$

where β is an arbitrary constant. Differentiating this equation twice and excluding β and x we obtain a differential equation of the second order that will be satisfied by the given curve. Because the x is eliminated from the result, it is obvious that all the curves, which are contained in the equation

$$F\left(\frac{x-\alpha}{\beta}, \frac{y}{\beta}\right) = 0, \tag{23}$$

where α is another arbitrary constant, will satisfy one and the same differential equation, the form of which may be found with no difficulty. In fact, differentiating twice the equation (23) and putting $\frac{x-\alpha}{\beta} = u$, $\frac{y}{\beta} = v$, we obtain

$$\frac{\partial F(u,v)}{\partial u} + \frac{\partial F(u,v)}{\partial v}\, p - 0,$$
$$\frac{\partial^2 F(u,v)}{\partial u^2} + 2\frac{\partial^2 F(u,v)}{\partial u \partial v}\, p + \frac{\partial^2 F(u,v)}{\partial v^2}\, p^2 + \frac{\partial F}{\partial v}\, \beta q = 0,$$

wherefrom, replacing βq with $\frac{yq}{v}$ and excluding u, v, we obtain a differential equation of the form

$$yq = \Theta(p). \tag{24}$$

This is the equation of homothetic curves having the centre of homothety in an arbitrary point x.

10. Considering this equation it is necessary to make the following important remark. As far as it was obtained by means of the exclusion of β, the latter admits to be real as well as imaginary. With real β the equation $F\left(\frac{x}{\beta}, \frac{y}{\beta}\right) = 0$ will represent a family of curves homothetic to the given one; but quite often even with the imaginary β this same equation will represent real curves whose differential equation will also be (24), but which of course will not be homothetic to the given curve. It is easy to see in which case the equation (24) will correspond to the curves of two different families; this will happen in the case when the equation of the given curve will represent a real curve even after replacing x and y in it with xi, yi. But assuming the given equation be solved with respect to y and putting

$$y = f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2},$$

we obtain after the replacement of x and y with xi, yi

$$y = \frac{f(xi) + f(-xi)}{2i} + \frac{f(xi) - f(-xi)}{2i},$$

from where one can see that for y be a real function of x it suffices and is necessary that $f(x)$ should be an odd function. In this case (24) will be the differential equation of the curves homothetic to the curves

$$y = f(x)$$

and

$$y = \frac{f(xi) - f(-xi)}{2i} = -if(xi).$$

Assuming $f(x)$ be continuous at $x = 0$ one has at $x = 0$, $y = 0$, and thus $\Theta(p) = 0$.

11. Vice versa, the differential equation (24) possesses the general integral of the kind (23), because putting in (24) $q = \dfrac{dp}{dx} = p\dfrac{dp}{dy}$ we get

$$\frac{pdp}{\Theta(p)} = \frac{dy}{y},$$

wherefrom multiplying by 2 and integrating[2] we find

$$log(y^2) = \int \frac{2p\,dp}{\Theta(p)} + log(\beta^2)\,,$$

from where

$$y^2 = \beta^2 e^{\displaystyle\int \frac{2p\,dp}{\Theta(p)}}\,,\ \text{or}\ y = \pm\beta e^{\displaystyle\int \frac{p\,dp}{\Theta(p)}}\,; \tag{25}$$

then we have

$$dx = \frac{dp}{q} = \frac{y\,dp}{\Theta(p)} = \pm\beta\, e^{\displaystyle\int \frac{p\,dp}{\Theta(p)}} \frac{dp}{\Theta(p)}$$

wherefrom

$$x - \alpha = \pm\beta\, e^{\displaystyle\int \frac{p\,dp}{\Theta(p)}} \frac{dp}{\Theta(p)}. \tag{26}$$

Eliminating p from equations (25) and (26) provides an outcome of the form (23), q.e.d.

12. One needs anyway to make the following remark here. If at some value $p = k$ the function $\Theta(p)$ vanishes then in accordance with *n*° 10 one concludes that the given equation will be satisfied by two systems of different curves having common points in which the tangents are defined by the equation $p = k$ the same for both systems of curves; the finite equations for all the curves will take the shape (23). But from these two systems it is possible to construct new systems, all elements of which will satisfy the equation (24), but which cannot be represented by means of *one* equation of the kind (23): to build up such systems it is sufficient to consider

[2] The multiplication by 2 has as its goal to introduce $log\ y^2$ instead of $log\ y$ in the integral in order to gain the possibility of considering also negative values of y keeping real values of the arbitrary constants. Accordingly, hereinafter by $e^{\int \frac{pdp}{\Theta(p)}}$ a positive function will always be understood. The equation $y = \beta\, e^{\int \frac{pdp}{\Theta(p)}}$ with real β obviously may not correspond to all the points of the curve in the case when it intersects the abscissa axis.

simultaneously the curves of the first two systems and to treat the branches of different curves that meet in a common point, as one curve. We clear this up with a simple example.

Let's consider the equation

$$yq = p^2 - 1.$$

In accordance with formula (25) it possesses the integral

$$y^2 = \beta^2 \, e^{\displaystyle\int \frac{2p\,dp}{p^2-1}} .$$

Here while caring out the integration in the exponent it is necessary to distinguish between two assumptions: $p^2 < 1$, and $p^2 > 1$.

If one takes $p^2 < 1$, one will obtain a concave curve bent down to the x axis, for which the following holds

$$y^2 = \beta^2(1-p^2), \text{ or } y = \pm\beta\sqrt{1-p^2}\,,$$
$$\pm(x-\alpha) = -\beta\int\frac{dp}{\sqrt{1-p^2}} = \beta\, arc\ cos\ p\,,$$

so that finally

$$y = \pm\beta sin\frac{x-\alpha}{\beta}\,.$$

If $p^2 > 1$, one obtains a convex curve with respect to the x axis, for which the relations

$$y^2 = \beta^2(p^2-1), \text{ or } y = \pm\beta\sqrt{p^2-1}\,,$$
$$\pm(x-\alpha) = \beta\int\frac{dp}{\sqrt{p^2-1}} = \beta\, log\,(p+\sqrt{p^2-1})\,,$$

hold, from where

$$y = \pm\frac{\beta}{2}\left(e^{\frac{x-\alpha}{\beta}} - e^{-\frac{x-\alpha}{\beta}}\right).$$

At $x = \alpha$, while $y = 0$, the curves of both systems have the point of inflexion for which $p = \pm 1$.

Let's now build a curve of parabolic type from the following three pieces

I) $x \leqq \alpha,$ $\qquad y = \dfrac{\beta}{2}\left(e^{\frac{x-\alpha}{\beta}} - e^{-\frac{x-\alpha}{\beta}}\right),$

II) $\alpha \leqq x \leqq \alpha + \pi\beta,$ $\qquad y = \beta sin\dfrac{x-\alpha}{\beta}\,,$

III) $x \geqq \alpha + \pi\beta,$ $\qquad y = \dfrac{\beta}{2}\left(e^{\frac{x-\alpha}{\beta}+\pi} - e^{-\frac{x-\alpha}{\beta}-\pi}\right).$

Each of three segments satisfies the equation (24) and in the points of junction at $x = \alpha$ and $x = \alpha + \pi\beta$, not only the coordinates, but also the first and the second derivatives of the ordinate are equal to each other. The first segment may be replaced by the strait line $y = x - \alpha$, as much as the third one may be replaced by the straight line $y = -x + \alpha + \pi\beta$. The choice of the segments was made in such a way as to keep y as a one-valued function of x; when not paying attention to this requirement, it is possible to construct other continuous curves satisfying in all their elements the equation (24), but not represented by one and the same finite equation.

These remarks on the multivaluedness of the general integral of the differential equation (24), happening in some cases, are of considerable interest in the theory of differential equations, in variational calculus, where curves are defined by differential equations and where, moreover, one often deals precisely with the equation of the same kind as (24), finally in solving the problem of determining the maximum and minimum properties of curves we are hereby occupied with. While replacing the finite equation of the given curve by its differential equation of the form (24) we must precisely point out the conditions, under which this equation corresponds exactly to the given curve.

13. Consideration of the general expressions of the coordinates x and y suggests introducing the new variable $\omega = \int \frac{dp}{\Theta(p)}$ instead of p. Denoting the integral

$$\int e^{\int \frac{p\,dp}{\Theta(p)}} \frac{dp}{\Theta(p)} = \int e^{\int p\,d\omega} d\omega$$

by $\Sigma(\omega)$ one easily finds

$$p = \frac{\Sigma''(\omega)}{\Sigma'(\omega)}, \quad x = \pm\beta\Sigma(\omega) + \alpha, \quad y = \pm\beta\Sigma'(\omega), \quad -\frac{dx}{d\omega} = y.$$

The last relation discovers the geometric meaning of the parameter ω under the accepted assumption that x and y represent the rectangular coordinates: it precisely shows that the curve under consideration presents itself as a roulette its poloid being the x axis while ω is the angle described by a straight line rigidly attached to the serpoloid.* The equation of the latter with respect to the rigidly attached to it axes with the origin in the point that draws the roulette, as known, will emerge when excluding ω from the equations

$$\xi = \frac{dx}{d\omega} sin\omega + \frac{dy}{d\omega} cos\omega,$$
$$\eta = -\frac{dx}{d\omega} cos\omega + \frac{dy}{d\omega} sin\omega.$$

14. First integrals of the equation (24) are

$$\sigma(x,y,p) = ye^{-\int \frac{pdp}{\Theta(p)}} = \pm\beta\,,$$
$$\psi(x,y,p) = x - ye^{-\int \frac{pdp}{\Theta(p)}} \int e^{\int \frac{pdp}{\Theta(p)}} \frac{dp}{\Theta(p)} = \alpha\,.$$

The function $\phi(x,y,p)$ in this case is $\frac{1}{y}\Theta(p)$, as seen from the equation (24); thus

$$\int \left(\frac{\partial log\phi}{\partial p}\right) dp = log\,\Theta(p)$$

* A curve, which is the locus of the points with which the several points of the poloid come successively in contact with the tangent plane; for the terminology see e.g.: ‘Outlines of a new theory of rotatory motion, translated from the French of Poinsot with explanatory notes, by Charles Whitley’, Cambridge, 1834; ‘Théorie nouvelle de la rotation des corps présentée a l’Institut le 19 mai 1834’, Papiers de Louis Poinsot, Manuscripts de la Bibliothèque de l’Institut de France, Ms 955; ‘First principles of mechanics: with historical and practical illustrations By William Whewell’, Cambridge, 1932. – R. M.

and we obtain

$$f(x,y,p) = \int_A^p dp \int_A^p \Phi(\psi,\sigma)\frac{dp}{\Theta(p)}.$$

Setting $\Phi(\psi,\sigma) = \sigma^n$, we get

$$f(x,y,p) = y^n \int_A^p dp \int_A^{p-n} e^{\int \frac{pdp}{\Theta(p)}} \frac{dp}{\Theta(p)},$$

where A may, for instance, be defined by the equation

$$y\, e^{\int \frac{AdA}{\Theta(A)}}$$

which will provide A as a function of y. Denoting by $\chi''(p)$ the function $e^{-n\int \frac{pdp}{\Theta(p)}} \frac{1}{\Theta(p)}$, one gets

$$f(x,y,p) = y^n \left[\chi(p) - \chi(A) - \chi'(A)p + \chi'(A)A\right],$$

where the term $y^n\chi'(A).p$ may be dropped as one integrable at y arbitrary, while the couple of terms $y^n[A\chi'(A) - \chi(A)]$ is reduced to the form $y^n \int A\chi''(A)dA$, i.e.

$$y^n \int e^{-n\int \frac{AdA}{\Theta(A)}} \frac{AdA}{\Theta(A)} = -\frac{1}{n}\left[ye^{-\int \frac{AdA}{\Theta(A)}}\right]^n = -\frac{1}{n}C^n$$

and may also be dropped, reducing the function $f(x,y,p)$ to the single term $y^n\chi(p)$.

Thus *for each curve there exists an integral of the form*

$$\int_{x_0}^{x_1} y^n\chi(p)dx,$$

that gains its maximal or minimal value.

Vice versa, the function $\chi(p)$ be given, one easily finds

$$\Theta(p) = \frac{c - n\int p\chi''(p)dp}{\chi''(p)}.$$

Thus with $\chi(p) = \sqrt{1+p^2}$ one has

$$\Theta(p) = c(1+p^2)^{\frac{3}{2}} + n(1+p^2) \text{ and so on.}$$

15. Addressing the conditions necessary for the existence of the maximum or the minimum, we must first of all assume, following Legendre, that

$$\frac{\partial^2 f}{\partial p^2} = \frac{\Phi(\psi,\sigma)}{\Theta(p)} = \frac{\Phi(\alpha,\pm\beta}{\Theta(p)}$$

preserves the sign within the limits of integration; from where it follows the function $\Theta(p)$ should preserve the sign too.

With the coordinates x and y being rectangular this will mean, according to the equation (24), that along all of the segment under consideration the curve will be either solely convex or solely concave downwards to the x axis.

[*]Further, according to the Jacobi theorem, there should exist such constant multipliers m and n that the expression

$$-m\frac{\partial y}{\partial\alpha}+n\frac{\partial y}{\partial\beta}$$

would not pass through zero within the limits of the integration and at both limit points themselves. Addressing the equations (25) and (26) and assuming x to be constant therein we obtain by differentiating the first of them along α and β

$$\frac{\partial y}{\partial\alpha}=\pm\, e^{-\int\frac{pdp}{\Theta(p)}}\frac{p}{\Theta(p)}\frac{\partial p}{\partial\alpha},$$

$$\frac{\partial y}{\partial\beta}=\pm\, e^{-\int\frac{pdp}{\Theta(p)}}\left\{1+\beta\frac{p}{\Theta(p)}\frac{\partial p}{\partial\beta}\right\},$$

and from the second one in similar way

$$-1=\pm\beta e^{-\int\frac{pdp}{\Theta(p)}}\frac{1}{\Theta(p)}\frac{\partial p}{\partial\alpha},$$

$$0=\int e^{-\int\frac{pdp}{\Theta(p)}}\frac{dp}{\Theta(p)}+\beta e^{-\int\frac{pdp}{\Theta(p)}}\frac{1}{\Theta(p)}\frac{\partial p}{\partial\beta}.$$

From here one finds

$$\frac{\partial y}{\partial\alpha}=-p$$

$$\frac{\partial y}{\partial\beta}=\pm\left[e^{-\int\frac{pdp}{\Theta(p)}}-p\int e^{-\int\frac{pdp}{\Theta(p)}}\frac{dp}{\Theta(p)}\right],$$

or, through the integration by parts, where possible,

$$\frac{\partial y}{\partial\beta}=\pm\int e^{-\int\frac{pdp}{\Theta(p)}}d\,\frac{1}{p}.$$

So, the expression

$$-m\frac{\partial y}{\partial\alpha}+n\frac{\partial y}{\partial\beta}=p\left(m\pm n\int e^{-\int\frac{pdp}{\Theta(p)}}d\,\frac{1}{p}\right)$$

should not turn into zero within the limits of integration, as much as at the both limits of integration themselves.

[*] The numbering of the paragraph *n*° 16 of the Russian original text here was moved by the translator farther below.

16. It is easy to give a geometric interpretation to the Jacobi condition in the assumption that x and y are linear coordinates.

Indeed, the first expression $\dfrac{\partial y}{\partial \beta}$ may be reduced to the form

$$\frac{\partial y}{\partial \beta} = \frac{y}{\beta} - p\frac{x-\alpha}{\beta} = -\frac{p}{\beta}\left[x - \frac{y}{p} - \alpha\right]$$

and thus

$$-m\frac{\partial y}{\partial \alpha} + n\frac{\partial y}{\partial \beta} = \frac{p}{\beta}\left[m\beta + n\alpha - n(x - \frac{y}{p})\right].$$

Now noticing that $x - \dfrac{y}{p}$ represents the abscissa of the point of intersection of the tangent to the curve with the x axis we conclude that only such a segment of the curve satisfies the Jacobi condition, to which it is not possible to draw the tangent through *each* point of the abscissa axis. In the case where the segment under consideration does not contain singular points and is concave to the x axis this condition is evidently satisfied; in the case of the same but convex segment it is both necessary and sufficient that the point of intersection of the tangents through its end points be situated between the curve itself and the abscissa axis: in this case, as in the case of a concave segment, it is not possible to draw tangents through the segment of the abscissa axis cut off by the extreme tangents.

17. Now we shall concern ourselves with finding the arbitrary constants α and β under the assumption that the extreme points through which the given curve should pass have been fixed. If denote as x_0, y_0 and x_1, x_2 the coordinates of the end points under the assumption that the equation in finite form $F\left(\dfrac{x-\alpha}{\beta}, \dfrac{y}{\beta}\right) = 0$ is known then the question in hand will reduce to the search for the real values of α and β from the two equations

$$F\left(\frac{x_0-\alpha}{\beta}, \frac{y_0}{\beta}\right) = 0, \qquad F\left(\frac{x_1-\alpha}{\beta}, \frac{y_1}{\beta}\right) = 0.$$

Assuming that the equations written above have real solutions and having them found we will have to pass further to the consideration of the minimum and maximum conditions, actually to the equation (24). Therefore we shall move on to the problem of determining the constants α and β under the assumption that the curves under consideration are defined by the differential equation (24), as it is in the calculus of variations.

If call by p_0 and p_1 the unknown values of $p = \dfrac{dy}{dx}$ at the given end points, then our problem reduces to the determination of the real values of α, β, p_0, and p_1 from the four equations

$$(27)\qquad \begin{cases} x_0 - \alpha = \pm\beta \displaystyle\int^{p_0} e^{\int \frac{pdp}{\Theta(p)}} \frac{dp}{\Theta(p)}, \quad y_0 = \pm\beta e^{\int^{p_0} \frac{pdp}{\Theta(p)}}; \\ x_1 - \alpha = \pm\beta \displaystyle\int^{p_1} e^{\int \frac{pdp}{\Theta(p)}} \frac{dp}{\Theta(p)}, \quad y_1 = \pm\beta e^{\int^{p_1} \frac{pdp}{\Theta(p)}}. \end{cases}$$

It suffices to give a brief glance at these equations to see that solving them in the above given sense needs not always be possible. It is clear for instance that if the end points lay on different sides of the abscissa axis, as well as when one or both points lay on the axis itself, it is necessary that at some values of p the integral $\displaystyle\int \frac{pdp}{\theta(p)}$ turns into $-\infty$, and this of course imposes some conditions on the function $\Theta(p)$.

We assume that $y_1 \geqq y_0 > 0$. In this case in formulæ (27) it suffices to preserve at β the positive sign alone.

18. Let us assume that the integral $\int \frac{pdp}{\Theta(p)}$, along with a certain value of the associated constant, represents a function the real values of which lay between the least of them P and the greatest of them Q. In this case we should have

$$y_0 e^{-Q} < \beta < y_0 e^{-P}, \qquad y_1 e^{-Q} < \beta < y_1 e^{-P};$$

but mutual fulfilment of the inequalities like these is evidently possible only if $y_1 e^{-Q} < y_0 e^{-P}$ and $y_0 e^{-Q} < y_1 e^{-P}$, from where

$$e^{P-Q} < \frac{y_1}{y_0} < e^{Q-P}; \tag{28}$$

incidentally, the preceding inequalities reduce to

$$y_1 e^{-Q} < \beta < y_0 e^{-P}\,. \tag{29}$$

Of course, if there exists a finite minimum or maximum of the integral $\int \frac{p\,dp}{\Theta(p)}$, this minimum or maximum may be brought to zero.

In the case when $y_1 = y_0$, the condition (28) disappears.

19. Another condition may be obtained while considering the function

$$\frac{x-\alpha}{y} = e^{-\int \frac{pdp}{\Theta(p)}} \int e^{\int \frac{pdp}{\Theta(p)}} \frac{pdp}{\Theta(p)}\,. \tag{30}$$

Let us assume that the second part can change only between the two extreme values: the lowest one N and the highest one M. In this case we must accept that

$$N < \frac{x_0 - \alpha}{y_0} < M, \qquad N < \frac{x_1 - \alpha}{y_1} < M,$$

from where

$$x_0 - N y_0 > \alpha > x_0 - M y_0\,,$$
$$x_1 - N y_1 > \alpha > x_1 - M y_1\,.$$

But the equations like these obviously can exist only under the conditions

$$x_0 - N y_0 > x_1 - M y_1, \qquad x_1 - N y_1 > x_0 - M y_0\,,$$

in other words,

$$M y_1 - N y_0 > x_1 - x_0 > N y_1 - M y_0\,. \tag{31}$$

This equation, as well as the equation (28) is to be satisfied by the coordinates of the given points in order that the posed question could have a solution.

20. The inequalities (31) or, to be more precise, the preceding ones, may be quite simply interpreted in the geometric way. Let $N = cot\phi_0$, $M = cot\phi_1$, so that

$$x_0 - y_0\, cot\phi_0 > x_1 - y_1\, cot\phi_1,$$
$$x_1 - y_1\, cot\phi_0 > x_0 - y_0\, cot\phi_1.$$

Now let us observe that $x - y\,cot\phi$ represents the abscissa of the point of intersection of the x axis with the straight line through the point (x, y) which composes the angel ϕ with the positive direction of the x axis. By this reason the first of the written inequalities, as it may be easily ensured by a simple drawing, means that the straight line drawn at the angle ϕ_0 through the point (x_0, y_0) meets above the x axis the other straight line drawn at the angle ϕ_1 through the point (x_1, y_1); the second inequality has the similar meaning. These two conditions can be replaced by the following one: if through the given point with the less value of the abscissa a straight line is drawn at the angle ϕ_0 to the positive direction of the x axis, and if through the point where it meets the x axis another straight line is drawn at the angle ϕ_1 to the x axis, then the second given point should lay within the upper angle composed by the constructed straight lines.

21. In what considers the constants P and Q representing the least and the greatest values of the integral $\int \frac{pdp}{\Theta(p)}$, their determination reduces to the finding of the real roots and of the points of discontinuity of the function $\frac{p}{\Theta(p)}$. Note that if $\Theta(0)$ takes a non zero finite value, then the integral under consideration reaches its maximum or minimum at $p = 0$, depending on whether the value $\Theta(p)$ is negative or positive.

To define the constants M and N one needs to consider the first derivative of (30).

Denoting the function (30) by the symbol $\lambda(p)$ we have

$$\int e^{\int \frac{pdp}{\Theta(p)}} \frac{dp}{\Theta(p)} = \lambda(p) \, . \, e^{\int \frac{pdp}{\Theta(p)}},$$

from where by differentiation we find

$$\frac{1}{\Theta(p)} = \lambda'(p) + \frac{p\lambda(p)}{\Theta(p)},$$

so that

$$\lambda'(p) = \frac{1 - p\lambda(p)}{\Theta(p)}, \tag{32}$$

or

$$\Theta(p) = \frac{1 - p\lambda(p)}{\lambda'(p)}. \tag{33}$$

The examination of these equations leads to the following conclusions about the zeros and the discontinuities of the function $\lambda'(p)$:

$$\text{a)} \qquad \lambda'(p) = 0,$$

1°) the case when $1 - p\lambda(p) = 0$ but $\Theta(p) \neq 0$; calculating the corresponding value of $\Theta(p)$ with the help of the equation (33) one obtains $\Theta(p) = -\frac{1}{p\lambda''(p)}$ or $\lambda''(p) = -\frac{1}{p\Theta(p)}$. From here it follows that if p' is the root of the equation $1 - p\lambda(p) = 0$ and $\Theta(p')$ represents a finite not zero quantity, then $\lambda(p') = \frac{1}{p'}$ will be the maximum or minimum of the function $\lambda(p)$, depending on whether the number $p'\Theta(p')$ is positive or negative.

2°) the case when $\Theta(p)$ turns into infinity, but $1 - p\lambda(p)$ remains finite.

3°) the case when $1-p\lambda(p) = 0$ and $\Theta(p) = 0$; in this case the second part of the equation (32) is of the type $\frac{0}{0}$ and, following the common rule, equals $-\frac{p\lambda'(p) + \lambda(p)}{\Theta'(p)}$, that is $-\frac{1}{p\Theta'(p)}$; but at the same time it has to represent the vanishing value of $\lambda'(p)$, and so $\Theta(p) = \infty$ should hold; and, simultaneously, according to 1°), $\lambda''(p) = 0$. In this case the p cannot take the infinite value because we should at the same time have $\Theta(p) = 0$ and $\Theta'(p) = \infty$, which may happen only when p is finite.

4°) the case when $1-p\lambda(p)=\infty$ and $\Theta(p)=\infty$; at finite value of p the equalities $\lambda(p)=\infty$ and $\lambda'(p)=0$ are inconsistent; at $p=\infty$ the order of the infinity $\Theta(p)$ should exceed $p\lambda(p)$.

$$\text{b)} \qquad \lambda'(p)=\infty,$$

5°) the case when $\Theta(p)=0$, but $1-p\lambda(p)$ differs from zero; in a special case $1-p\lambda(p)$ may turn into infinity; this case may be realized only at finite values of p, in order that the relation (33) might be equal to zero.

6°) the case when $\Theta(p)=0$ and $1-p\lambda(p)=0$; in this case the order of the infinitesimal $\Theta(p)$ should exceed $1-p\lambda(p)$.

7°) the case when $\Theta(p)=\infty$ and $1-p\lambda(p)=\infty$; as one spots from (33), this may be the case only at $p=\infty$.

Remark. The case when $1-p\lambda(p)=\infty$ whereas $\Theta(p)$ is finite and differs from zero, cannot take place because $p\lambda(p)$ and $\lambda'(p)$ cannot be of the same infinite order.

By virtue of these results the determination of the maximal and minimal values M and N of the function $\lambda(p)$ reduces to the consideration of the real roots and of the places of the discontinuity of the functions $1-p\lambda(p)$ and $\Theta(p)$.

22. The relation between the functions $\Theta(p)$ and $\lambda(p)$, expressed by the equations (32) and (33), permits to define each of them in terms of another[3], as much as to make conclusions about the properties of one function on the basis of the given properties of another.

For instance, let us assume that $\lambda'(p)$ has no real roots and no places of discontinuity and remains always positive. It follows then that $\lambda(p)$ represents an increasing from $-\infty$ to ∞ function and thus the condition (31) disappears.

As the function $1-p\lambda(p)$, while turning into $-\infty$ at $p=-\infty$, continuously increases until 1 at $p=0$, and then continuously decreases to $-\infty$ at $p=0$, we conclude that the function under consideration surely has two and only two real roots: one positive and one negative. Let them be $-r$ and s. By virtue of the equation (32) and of the assumption concerning $\lambda'(p)$, we must agree that the same roots do has the function $\Theta(p)$. Calculating according to the common rule the second part of the equation (32), we get

$$\lambda'(s)=-\frac{s\lambda'(s)+\lambda(s)}{\Theta(s)}=-\frac{s^2\lambda'(s)+1}{s\Theta'(s)}\,,$$

and as far as the numerator here in undoubtedly positive together with $\lambda'(s)$, we have to assume $\Theta'(s)<0$. Exactly in the same way one easily finds that $\Theta'(-r)>0$. On the strength of this we may state that s and $-r$ are simple roots of $\Theta(p)$, so that

$$\Theta(p)=(p+r)(s-p)\Theta_1(p)\,,$$

where $\Theta_1(p)$ possesses no real roots and no places of discontinuity and remains always positive.

23. Assuming the conditions (28) and (31) be met by the coordinates of the given points, we pass now to the solution of the equations (27). After eliminating α we reduce those equations to the form

$$\begin{gathered}\frac{x_1-x_0}{\beta}-\int_{p_0}^{p_1} e^{\int\frac{pdp}{\Theta(p)}}\frac{dp}{\Theta(p)}=0,\\ \frac{y_0}{\beta}=e^{\int^{p_0}\frac{pdp}{\Theta(p)}},\qquad \frac{y_1}{\beta}=e^{\int^{p_1}\frac{pdp}{\Theta(p)}}\,.\end{gathered}\tag{34}$$

[3] Evidently, the value $\frac{1}{p}$ is the only forbidden one for the function $\lambda(p)$.

If for the sake of brevity one denotes the function $e^{\int \frac{pdp}{\Theta(p)}}$ by $\psi(p)$* and the inverse as $\psi^{-1}(p)$, then, noticing that the integral entering in the first equation may be presented in the form $\int_{p_0}^{p_1} d\, e^{\int \frac{pdp}{\Theta(p)}}$ and introducing into it the variable $\frac{y}{\beta} = e^{\int \frac{pdp}{\Theta(p)}}$ in place of p, one gets by the reason of the other two equations

$$x_1 - x_0 - \int_{y_0}^{y_1} \frac{dy}{\psi^{-1}\left(\frac{y}{\beta}\right)} = 0\,. \tag{35}$$

This result, containing one arbitrary constant β, might, of course, be obtained directly from the finite equation of the curve solved with respect to x, and we would have to find real values of β enclosed between the limits established by the inequality (29) which satisfy it. But, on the strength of the remarks contained in *n*º 12, and also by the fact that once the roots of the equations (35) found, in order to find p_0 and p_1 one would nevertheless be forced to call back to the equations (34), it is useful to investigate directly the latter without reducing them to (35).

24. To this end, we shall consider the left hand side of the first equation in (34) as a function of $\frac{1}{\beta}$, presenting it in the form

$$\frac{x_1 - x_0}{\beta} - \int_{\mu}^{\nu} \psi(p)\frac{dp}{\Theta(p)} \tag{36}$$

and assuming μ and ν be defined as functions of $\frac{1}{\beta}$ by means of the equations

$$\frac{y_1}{\beta} = \psi(\mu), \qquad \frac{y_1}{\beta} = \psi(\nu). \tag{37}$$

Here μ and ν are defined as inverse functions, and, by this, generally speaking, multi–valued ones, so that actually we are forced to investigate the function (36) when all possible combinations of the values of μ and ν take place. However, sometimes this non–uniqueness may be reduced by the fact that at some systems of values of μ and ν the function (36) evidently preserves the sign and thus we shall need to investigate the function (36) only at limited number of values of μ and ν.

The first derivative of the function (36) with respect to the variable $\frac{1}{\beta}$ will be

$$x_1 - x_0 - \frac{\psi(\nu)}{\Theta(\nu)}\,\frac{\partial\nu}{\partial\frac{1}{\beta}} + \frac{\psi(\mu)}{\Theta(\mu)}\,\frac{\partial\mu}{\partial\frac{1}{\beta}}\,;$$

* This function $\psi(p)$ should not be confused with $\psi(x, y, p)$ of equation (2) – R. M.

but the differentiation of the equations (37) will produce

a)
$$\begin{cases} y_0 = \psi'(\mu)\dfrac{\partial\mu}{\dfrac{1}{\beta}} = \dfrac{\psi(\mu)}{\Theta(\mu)}\,\mu\,\dfrac{\partial\mu}{\partial\dfrac{1}{\beta}}\,, \\[2ex] y_1 = \psi'(\nu)\dfrac{\partial\nu}{\dfrac{1}{\beta}} = \dfrac{\psi(\nu)}{\Theta(\nu)}\,\nu\,\dfrac{\partial\mu}{\partial\dfrac{1}{\beta}}\,, \end{cases}$$

in consequence of what the derivative considered will take the form

(38)
$$x_1 - \frac{y_1}{\nu} - \left(x_0 - \frac{y_0}{\mu}\right).$$

25. To define the maxima and minima of the function (36), let us assume that its derivative (38) vanishes, i.e. that

(39)
$$x_1 - \frac{y_1}{\nu} = x_0 - \frac{y_0}{\mu}\,.$$

This equation, together with (37), will serve to determine the values of β, μ, and ν at which the function (36) takes its maximum or minimum. Denoting the unknown common value of the both parts of the equation (39) by ξ we will obtain

(40)
$$\mu = \frac{y_0}{x_0 - \xi}, \qquad \nu = \frac{y_1}{x_1 - \xi};$$

consequently equations (37) will turn into the following

(41)
$$\frac{1}{\beta} = \frac{1}{y_0}\psi\left(\frac{y_0}{x_0 - \xi}\right) = \frac{1}{y_1}\psi\left(\frac{y_1}{x_1 - \xi}\right)\,.$$

After determining all real roots ξ of this equation which provide β with values satisfying the inequality (29), and which provide μ, ν with values satisfying the restrictions imposed on these branches of the many–valued functions, we will readily find the values of β, μ, ν which correspond to the named roots and of the function (36). Clearly, only those roots of the equation (41) are important to us, which correspond to the actual maxima and minima of the function (36) and, consequently, do not annul its second derivative, the latter being

$$\frac{y_1}{\nu^2}\frac{\partial\nu}{\partial\dfrac{1}{\beta}} - \frac{y_0}{\mu^2}\frac{\partial\mu}{\partial\dfrac{1}{\beta}} \quad \text{or} \quad \frac{{y_1}^2}{\nu^2}\frac{1}{\psi'(\nu)} - \frac{{y_0}^2}{\mu^2}\frac{1}{\psi'(\mu)}\,.$$

But it may easily be seen that the double root of the equation (41) does not satisfy this requirement because it will at the same time be the root of the derived equation

$$\frac{1}{(x_0-\xi)^2}\psi'\left(\frac{y_0}{x_0 - \xi}\right) = \frac{1}{(x_1-\xi)^2}\psi'\left(\frac{y_1}{x_1 - \xi}\right)$$

or

$$\frac{\mu^2}{{y_0}^2}\psi'(\mu) = \frac{\nu^2}{{y_1}^2}\psi'(\nu)\,,$$

so that the second derivative of the function (36) will be equal to zero. Evidently, in general only the roots of odd multiplicity of the equation (41) provide the maxima and the minima for the function (36) and only they must be determined.

After adding to the maximal and minimal values of the function (36) also its values at the extremal values of β given by the inequalities (29), taking into account the signs of all these values, we will ensure the absence or the existence of the roots of (34); in the latter case we treat the real roots β as *separated*, so it will only remain to us to compute them together with the corresponding values of p_0 and p_1 and then turn to the application of the Legendre and Jacobi conditions.

26. We pay attention to the fact that while computing the maxima and the minima of the function (36) it is possible to avoid the calculation of ξ by defining ν and μ directly from the equations (39) and from the following equation obtained from (37),

$$\frac{1}{y_0}\psi(\mu) = \frac{1}{y_1}\psi(\nu)\,.$$

The maxima and minima of the function (36) consequently, take the guise

$$\frac{\psi(\nu)}{\nu} - \frac{\psi(\mu)}{\mu} - \int_\mu^\nu \frac{\psi(p)}{\Theta(p)}dp\,,$$

obtained from (36) with the help of the equations (39) and (37).

Anyway in some cases it is possible to avoid the computation of all the maxima and minima by virtue of the following theorem:

If $\nu > \mu > 0$, then the adjacent maximum and minimum of the function (36) will have the same sign if the minimum corresponds to a smaller value of β.

To prove this theorem, let us consider the derivative with respect to β of the function $\beta \displaystyle\int_\mu^\nu \frac{\psi(p)}{\Theta(p)}\,dp$. This derivative produces

$$\int_\mu^\nu \frac{\psi(p)}{\Theta(p)}dp + \beta\left(\frac{\psi(p)}{\Theta(p)}\frac{\partial\nu}{\partial\beta} - \frac{\psi(p)}{\Theta(p)}\frac{\partial\mu}{\partial\beta}\right),$$

or, by virtue of equations a) of *n*° 24

$$\int_\mu^\nu \frac{\psi(p)}{\Theta(p)}dp + \frac{1}{\beta}\left(\frac{y_0}{\mu} - \frac{y_1}{\nu}\right),$$

what reduces by means of equations (37) to the form

$$\int_\mu^\nu \frac{\psi(p)}{\Theta(p)}dp - \left(\frac{\psi(\nu)}{\nu} - \frac{\psi(\mu)}{\mu}\right),$$

so that for the values which correspond to the maximum or to the minimum of the function (36) the value of the derivative under consideration will with the opposite sign be equal to the maximum or to the minimum value of the function (36); generally speaking, by replacing the difference in the parentheses with the integral

$$\int_\mu^\nu d\frac{\psi(p)}{p} = \int_\mu^\nu \left(\frac{\psi'(p)}{p} - \frac{\psi(p)}{p^2}\right)dp$$

and inserting $\psi'(p) = p\dfrac{\psi(p)}{\Theta(p)}$, we reduce the derivative under consideration to the form $\displaystyle\int_\mu^\nu \frac{\psi(p)}{p^2}dp$, from which it evidently is clear that under the condition $\nu > \mu > 0$ it takes only positive values. From this it follows that $\beta\displaystyle\int_\mu^\nu \frac{\psi(p)}{\Theta(p)}dp$ represents an increasing function β. Since, on the other hand, the function (36) may be given the form

$$\frac{1}{\beta}\left(x_1 - x_0 - \beta\int_\mu^\nu \frac{\psi(p)}{\Theta(p)}dp\right),$$

and since the subtrahend in the parentheses in the case of minimum will be less than in the case of maximum, it is clear that both these values of the function (36) will have the same sign.

27. As stated above, we need to find only these roots of the equation (41) which provide β with values enclosed in the limits suggested by the inequality (29). By this reason sometimes it is possible to establish the limits of the sought roots.

Indeed, the inequalities (29) provide

$$\frac{1}{y_1}e^Q > \frac{1}{\beta} > \frac{1}{y_0}e^P,$$

and on the strength of the equations (41) the boundary values of ξ are defined by the conditions

$$e^Q > \psi\left(\frac{y_1}{x_1 - \xi}\right), \qquad e^P\psi\left(\frac{y_0}{x_0 - \xi}\right),$$

or, passing to logarithms,

(42) $$Q > \int^{\frac{y_1}{x_1-\xi}} \frac{pdp}{\Theta(p)}, \qquad P < \int^{\frac{y_0}{x_0-\xi}} \frac{pdp}{\Theta(p)}.$$

Let, for example, the minimum value P of the integral $\int \frac{pdp}{\Theta(p)}$ is attained at $p = p'$, while the minimum value is attained at $p = p''$; taking p' as the lower limit of integration we shall have $P = 0$, and if the function $\frac{p}{\Theta(p)}$ stays always positive and if P and Q represent its unique values of minimum and of maximum, then we shall have the reason to conclude that

$$\frac{y_1}{x_1 - \xi} < p'', \qquad \frac{y_0}{x_0 - \xi} > p'.$$

If $\Theta(p)$ represents a positive even function of p and in addition $p' = 0$, then the two quoted inequalities are replaced by the next one

$$\left(\frac{y_1}{x_1 - \xi}\right)^2 < p''^2$$

and so on.

In the case when $\Theta(p)$ possesses several maxima and minima the corresponding modification of the results is carried out easily.

28. What concerns the use of the Legendre conditions was already treated in the paragraph *n*° 15; but in what concerns the Jacobi condition it is worthwhile to make the following remark. If one computes the ordinate η of the intersection point of the tangents to the curve under consideration through the points (x_0, y_0) and (x_1, y_1), one easily finds

(43) $$\eta\left(\frac{1}{p_0} - \frac{1}{p_1}\right) = x_1 - \frac{y_1}{p_1} - \left(x_0 - \frac{y_0}{p_0}\right),$$

so that the sign of η will depend on the sign of the value of the function (38) at $\mu_1 = p_1$ and $\mu_0 = p_0$ as well as on the sign of the difference $\frac{1}{p_0} - \frac{1}{p_1}$. When the root ξ of the equation (41) corresponds to the minimum of the function (36), this means that as the variable $\frac{1}{\beta}$ increases (i.e. as β decreases) the function (38) changes from negative values to the positive ones; when the root ξ corresponds to the maximum of this same function (36), then as β decreases the function (38) changes from $+$ to $-$. Keeping this in mind one easily finds the sign of η and after that on the strength of n° 16 in large number of cases decides whether the Jacobi condition holds. At the positive values of y_0 and y_1 which is the case we herein consider, the ordinate η should take positive value; in view of this it is possible to classify the cases when the Jacobi condition is true into the following table, within which β_0 denotes the value of β computed along the formula (41), and β itself denotes the root of the equation (34):

A) the root of the equation (41) corresponds to the minimum:

$$\frac{1}{p_0} - \frac{1}{p_1} < 0, \qquad \beta > \beta_0,$$
$$\frac{1}{p_0} - \frac{1}{p_1} > 0, \qquad \beta < \beta_0.$$

B) the root of the equation (41) corresponds to the maximum:

$$\frac{1}{p_0} - \frac{1}{p_1} < 0, \qquad \beta < \beta_0,$$
$$\frac{1}{p_0} - \frac{1}{p_1} > 0, \qquad \beta > \beta_0.$$

29. Now we apply the obtained results on the study of the equation (24) to some special cases. Let us assume that

$$\Theta(p) = \frac{p^2 + c^2}{2k}, \tag{44}$$

from where it follows*

$$\int_0^p \frac{pdp}{\Theta(p)} = k \int_0^p \frac{2pdp}{p^2 + c^2} = log \left(1 + \frac{p^2}{c^2}\right)^k,$$
$$\psi(p) = \left(1 + \frac{p^2}{c^2}\right)^k,$$
$$y = \pm \left(1 + \frac{p^2}{c^2}\right)^k, \qquad x - \alpha = \pm \frac{2k\beta}{c^2} \int_0^p \left(1 + \frac{p^2}{c^2}\right)^{k-1} dp,$$
$$\psi(x, y, p) = x - \frac{2ky}{c^2} \left(1 + \frac{p^2}{c^2}\right)^{-k} \int_0^p \left(1 + \frac{p^2}{c^2}\right)^{k-1} dp,$$
$$\sigma(x, y, p) = y \left(1 + \frac{p^2}{c^2}\right)^{-k},$$
$$\omega = \int \frac{dp}{\Theta(p)} = 2k \int_\infty^p \frac{dp}{p^2 + c^2} = -\frac{2k}{c}\ arc\ cot\ \frac{p}{c}.$$

From this $p = -c\ cot \frac{c\,\omega}{2k}$ and therefore

$$y = \pm \beta sin^{-2k} \frac{c\omega}{2k}, \quad x - \alpha = \pm \beta \int_{\pm \frac{\pi k}{c}}^{\omega} sin^{-2k} \frac{c\omega}{2k} d\omega.$$

* See the footnote remark on page 18 – R. M.

Under special assumptions about k one obtains the following curves, setting $\pm\beta = 1$, $\alpha = 0$:

$$e^2x^2 = \left(1 - y^{\frac{2}{3}}\right)\left(2 + y^{\frac{2}{3}}\right)^2;$$

$$k = -\frac{3}{2}; \qquad x = \frac{1}{2c}(c\omega - sin\ c\omega \pm \pi), \quad y = \frac{1}{2}(1 - cos\ c\omega);$$

$$k = -1; \qquad y^2 + c^2x^2 = 1;$$

$$k = -\frac{1}{2}; \qquad y = \frac{1}{2}\left(e^{cx} + e^{-cx}\right);$$

$$k = -\frac{1}{2}; \qquad y - 1 = \frac{c^2x^2}{4}.$$

30. The function $\Theta(p)$ preserves the sign at all values of p, namely, the sign of the constant k; thus it is necessary to consider two possibilities:

I) $k > 0$, *convex cup curve.*

In this case we have $P = 0$ (at $p = 0$), $Q = \infty$; so the inequality (28) becomes redundant, but (29) produces

$$0 < \beta < y_0. \tag{45}$$

Next, one finds easily

$$\lambda(p) = \frac{2k}{c^2}\left(1 + \frac{p^2}{c^2}\right)^{-k} \int_0^p \left(1 + \frac{p^2}{c^2}\right)^{k-1} dp\,.$$

Obviously $\lambda(p)$ is an odd function of p taking positive values at $p > 0$, and $1 - p\lambda(p)$ is an even function, so if it has a positive root then there exists the negative one of the same absolute value.

Now it's easy to find

$$1 - p\lambda(p) = \frac{\left(1 + \frac{p^2}{c^2}\right)^k - \frac{2kp}{c^2}\int_0^p \left(1 + \frac{p^2}{c^2}\right)^{k-1} dp}{\left(1 + \frac{p^2}{c^2}\right)^k}\,,$$

where the numerator may be reduced to the form

$$1 - \frac{2k}{c^2}\int_0^p dp \int_0^p \left(1 + \frac{p^2}{c^2}\right)^{k-1} dp\,.$$

Evidently, this numerator takes the maximum value $+1$ at $p = 1$; on the other hand, at $p = \infty$ it goes to infinity as $-\frac{1}{2k-1}\frac{p^{2k}}{c^2k}$ at $k > \frac{1}{2}$, as $-p\ log\ p$ at $k = \frac{1}{2}$ and as $-\frac{2kp}{c^2}\int_0^\infty \left(1 + \frac{p^2}{c^2}\right)^{k-1} dp$ at $k < \frac{1}{2}$; in all cases it turns to $-\infty$. From this it follows that the numerator in hand certainly has the only positive root which we denote by ε. To isolate this root we put the numerator at hand in the guise

$$-2k\left[\frac{1}{c^2}\int_0^p dp \int_0^p \left(1 + \frac{p^2}{c^2}\right)^{k-1} dp - \frac{1}{2k}\right]$$

and noticing that the derivative with respect to k of the function in brackets preserves the positive sign we conclude that the function itself keeps growing and so if at certain values of k and p it vanishes, then at greater k it may vanish only when the corresponding p takes the less value than the previous one.

At $k = 1$ one obtains $\varepsilon = c$; at $k = 2$, $\varepsilon = c\sqrt{\sqrt{12} - 3}$, and so on. At $k = \frac{1}{2}$ one has to solve the equation

$$\frac{p}{c}\ log\ \left(\sqrt{1 + \frac{p^2}{c^2}} + \frac{p}{c}\right) - \sqrt{1 + \frac{p^2}{c^2}} = 0,$$

which by means of the change of variables $\frac{p}{c} = \frac{1}{2}\left(r - \frac{1}{r}\right)$ is being brought to the form

$$log\ r^2 = 2 + \frac{4}{r^2 - 1}$$

and is quite easily seen to take the value $\varepsilon = c\,.\,1,5088\ldots = c\,.\,tan\ 56°28'$.

31. Now we have: $N = -\frac{1}{\varepsilon}$, $M = \frac{1}{\varepsilon}$ so the inequality (31) turns into the following one:

$$\varepsilon^2(x_1 - x_0)^2 < (y_1 + y_0)^2. \tag{46}$$

Thus the equation (41) becomes

$$y_0{}^{-1}\left[1 + y_0^2 c^{-2}\big(x_0 - \xi\big)^{-2}\right]^k = y_1{}^{-1}\left[1 + y_1^2 c^{-2}\big(x_1 - \xi\big)^{-2}\right]^k,$$

or, by raising to the power $\frac{1}{k}$ and shifting terms:

$$y_0^{-\frac{1}{k}} - y_1^{-\frac{1}{k}} + y_0^{\frac{2k-1}{k}} e^{-2}(x_0 - \xi)^{-2} - y_1^{\frac{2k-1}{k}} e^{-2}(x_1 - \xi)^{-2} = 0. \tag{47}$$

The inequalities (42) impose no restrictions on ξ in this case.

Let us now consider two cases: 1) $x_0 < x_1$ and 2) $x_1 < x_0$.

32. When $x_0 < x_1$, the left side of the equation (47) changes from $y_0^{-\frac{1}{k}} - y_1^{-\frac{1}{k}}$ at $\xi = -\infty$ to $+\infty$ at $\xi = x_0$, then it turns to $-\infty$ at $\xi = x_1$, to end up again with the starting value $y_0^{-\frac{1}{k}} - y_1^{-\frac{1}{k}}$ at $\xi = +\infty$. Clearly, the equation (47) always possesses two real roots: one between x_0 and x_1, and the other one greater than x_1.

Addressing the derivative of the left side of the equation (47), which is

$$2c^{-2}\left(y_0^{\frac{2k-1}{k}}(x_0 - \xi)^{-3} - y_1^{\frac{2k-1}{k}}(x_1 - \xi)^{-3}\right),$$

we see that it possesses only one real finite root, which is

$$\xi' = \frac{y_1^{\frac{2k-1}{3k}} x_0 - y_0^{\frac{2k-1}{3k}} x_1}{y_1^{\frac{2k-1}{3k}} - y_0^{\frac{2k-1}{3k}}},$$

and for which both differences

$$x_0 - \xi' = y_0^{\frac{2k-1}{3k}} \frac{x_1 - x_0}{y_1^{\frac{2k-1}{3k}} - y_0^{\frac{2k-1}{3k}}},$$

$$x_1 - \xi' = y_1^{\frac{2k-1}{3k}} \frac{x_1 - x_0}{y_1^{\frac{2k-1}{3k}} - y_0^{\frac{2k-1}{3k}}}$$

will obviously be of the same sign, while the second derivative of the left hand side of the equation (47) will be*

$$6c^{-2}\left(y_0^{-\frac{2k-1}{3k}} - y_1^{-\frac{2k-1}{3k}}\right)\left(y_1^{\frac{2k-1}{3k}} - y_0^{\frac{2k-1}{3k}}\right)^4 (x_1 - x_0)^{-4}\,,$$

that means, surely not equal to zero. Also the value of the left side of the equation (47) will be

$$y_0^{-\frac{1}{k}} - y_1^{-\frac{1}{k}} + \left(y_0^{\frac{2k-1}{3k}} - y_1^{\frac{2k-1}{3k}}\right)^3 c^{-2}(x_1 - x_0)^{-2}. \tag{48}$$

From this it follows that the root of the first derivative will be located outside the segment with ends in x_0 and x_1, namely, will be greater than x_1 if $k < \frac{1}{2}$, and will be less than x_0 if $k > \frac{1}{2}$; also it will define the real (positive) maximum in the first case and the minimum in the second case.

If $k = \frac{1}{2}$ the derivative of the left side of the equation (47) possesses no finite root.

On the basis of these results one can affirm that *at the value $k \leqq \frac{1}{2}$ the equation (47) has only two real roots ξ_0 and ξ_1, where $x_0 < \xi_0 < x_1 < \xi_1 < \xi'$; at $k > \frac{1}{2}$ besides the two real roots ξ_0 and ξ_1 which satisfy the inequality $x_0 < \xi_0 < x_1 < \xi_1$ and which exist under all cases, there will exist, if be negative the expression (48), two more roots with values less than x_0 separated from each other by the root of the first derivative ξ'.*

33. Let us notice that the function (36) in the case under consideration will be

$$\frac{x_1 - x_0}{\beta} - \frac{2k}{c^2}\int_{\mu}^{\nu}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp\,, \tag{49}$$

where

$$\frac{y_0}{\beta} = \left(1 + \frac{\mu^2}{c^2}\right), \qquad \frac{y_1}{\beta} = \left(1 + \frac{\nu^2}{c^2}\right).$$

As far as the equations (34) obviously demand that $p_1 > p_0$ holds while $p_1 > 0$, we may constrain ourselves to the consideration of only the positive branch of the function ν. Under this assumption the extreme values of the function (49) for both branches of μ are easily seen to be: $+\infty$ at $\beta = 0$ and

$$\frac{x_1 - x_0}{y_0} - \frac{2k}{c^2}\int_0^{c\sqrt{\left(\frac{y_1}{y_0}\right)^{\frac{1}{k}} - 1}}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp \tag{50}$$

at $\beta = y_0$.

34. First we consider the negative branch of μ. On the strength of the equation (40) the conditions $0 > \mu < \nu > 0$ constrain the range of ξ to the limits $x_0 < \xi < x_1$; that's why we only need to calculate the root ξ_0 of the equation (47) and the corresponding values

$$\mu_0 = \frac{y_0}{x_0 - \xi_0}, \quad \nu_0 = \frac{y_1}{x_1 - \xi_0}, \quad \beta = y_0\left[1 + \frac{{\mu_0}^2}{c^2}\right]^{-k},$$

as well as the minimum value of the function (49)

$$\frac{x_1 - x_0}{\beta_0} - \frac{2k}{c^2}\int_{\mu_0}^{\nu_0}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp. \tag{51}$$

If this minimum is positive, the value of (50) will be positive too, and the function (48) won't possess the sought root; if the value of (50) is negative, then the minimum (51) will be negative,

---

* In Russian source we believe a misprint slipped in:
$6c^{-2}(y_0\, y_1)^{-\frac{2k-1}{3k}}\left(y_1^{\frac{2k-1}{3k}} - y_0^{\frac{2k-1}{2k}}\right)^5 (x_1 - x_0)^{-4}$. – R. M.

and the function (49) will have one root with less value than β_0; finally, if the expression (50) is positive although the minimum (51) is negative, the function (49) will have two roots separated by the value β_0.

35. Now let us consider the positive branch of μ. The conditions $0 < \mu < \nu$ provide $\xi < x_0 - y_0 \dfrac{x_1 - x_0}{y_1 - y_0}$; it should be noticed that the extreme value of ξ written above is greater than ξ' and that at this extreme value the left side of the equation (47) takes the positive sign, namely will be equal to

$$\left(y_0^{-\frac{1}{k}} - y_1^{-\frac{1}{k}}\right)\left[1 + e^{-2}(x_1 - x_0)^{-2}(y_1 - y_0)^2\right]. \tag{52}$$

When $k \leqq \frac{1}{2}$, as well as at $k > \frac{1}{2}$, then if the expression (48) is not positive the equation (47), as seen before, will have no roots less than x_0; consequently, the function (49) will decrease monotonously; and if its value (50) is positive it will have no desired root but if the value (50) is negative, then there will be one single root.

If, however, $k > 1$ and the expression (48) is negative, then in view of the positive sign of (52) one concludes that the equation (47) possesses two roots ξ_2 and ξ_3, where $\xi_3 < \xi' < \xi_2$. Let us compute according to the formulæ (40) and (41) the corresponding values of μ, ν, β, and the value of the function (49), namely

$$\frac{x_1 - x_0}{\beta_2} - \frac{2k}{c^2}\int_{\mu_2}^{\nu_2}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp \quad (minimum), \tag{53}$$

$$\frac{x_1 - x_0}{\beta_3} - \frac{2k}{c^2}\int_{\mu_3}^{\nu_3}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp \quad (maximum) \tag{54}$$

and let us add to this the extreme values ∞ and (51). Easily one recasts the minimum at hand in the form

$$\frac{y_0}{\beta_2}\left[\frac{x_1 - x_0}{y_0} - \frac{2k}{c^2}\frac{\beta_2}{y_0}\int_{\mu_2}^{\nu_2}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp\right],$$

from which it follows that whenever the expression (50) is positive, the minimum (53) is positive too and thus the function (48) won't have the desired root. Let now the expression (50) be negative. Then notice that the theorem of the paragraph *n*º 26 forces the values (53) and (54) to have the same sign. One thus concludes that the function (49) possesses one single root with the value either less than β_2 if (53) is negative, or greater than β_2 if that same expression is positive.

36. Applying the above considerations to the way we handle the equations (34) and keeping in mind the table of the paragraph *n*º 28 we make the next conclusion for the case $x_1 > x_0$.

When the expression (50) is positive, then the equations (34) will admit real solutions only under condition that (51) be negative; respectively, two systems of solutions come out: for each it holds $p_0 < 0$, $p_1 > 0$ while the roots β will be separated by the value β_0; the Jacobi condition is satisfied only by the curve which corresponds to the root $\beta > \beta_0$.

When the expression (50), and, consequently, the expression (51) as well, both are negative, then the equations (34) admit two systems of solutions; for one of them we have $p_0 < 0$, $p_1 > 0$, $\beta < \beta_0$, while for the other one $0 < p_0 < p_1$, and only this latter satisfies the Jacobi condition.

Thus in the case $x_1 > x_0$ the solution of the system (34) exists if and only if the minimum (51) is negative.

37. Let us now assume that $x_1 < x_0$. Equation (47) will necessarily possess two real roots: one less than x_1, the other between x_1 and x_0. The root of the differentiated equation ξ' will be greater than x_0 if $k > \frac{1}{2}$, and $\xi' < x_1$ if $k < \frac{1}{2}$; when the expression (48) is negative and $k > \frac{1}{2}$ there will exist two more roots greater than x_0 and separated by ξ'.

In order that the function (49) might vanish it is necessary to assume that $\nu < \mu$, and since for one and the same value of β the absolute value of the function ν is greater than μ, one should only consider the negative branch of the function ν. The extreme value of the function (49) at $\beta = y_0$ in this case will be the opposite of (50) assuming that the difference $x_0 - x_1$ is of the same absolute value as that of $x_1 - x_0$ before; the extreme value at $\beta = 0$ will be $-\infty$.

Investigating the positive branch of μ, one has $x_0 > \xi > x_1$; referring to the root ξ located between x_0 and x_1, we get the maximum of the function (36), presented by the expression (50) with the opposite sign. The conclusions of the paragraph *n*° 34 keep true also in this case, if one replaces $x_1 - x_0$ with the positive value of this difference.

Addressing the negative branch of μ one easily sees that also the conclusions of the paragraph *n*° 35 keep true so the final result may be formulated in the following way:

The existence of real solutions of the system of equations

$$\frac{x_1 - x_0}{\beta} - \frac{2k}{c^2}\int_{p_0}^{p_1}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp = 0\,,$$

$$\beta = y_0\left(1 + \frac{{p_0}^2}{c^2}\right)^{-k} = y_1\left(1 + \frac{{p_1}^2}{c^2}\right)^{-k}$$

is determined by the inequality

$$\text{(55)}\qquad \frac{2a}{\beta_0} - \frac{2k}{c^2}\int_{\mu_0}^{\nu_0}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp < 0\,,$$

where $2a$ *stands for the absolute value* $\pm(x_1 - x_0)$*; if this inequality is satisfied, then the equations under consideration have to systems of real solutions, of which only one satisfies the Jacobi condition, namely the one for which the difference* $\pm(p_1 - p_0)$ *is of the less absolute value.*

When the inequality (55) turns into equality, it provides the unique system of solutions, which, anyhow, does not satisfy the Jacobi condition, because in this case both the function (38) and the ordinate η turn to zero.

In the theoretical sense this result of cause is fully satisfactory. But it is poorly convenient for the applications, at least because to use it one is forced to solve the equation of the fourth degree (47) and to handle its root. So we shall occupy ourselves with the transformation of the condition (55) to a form, more convenient for applications.

38. We give a slightly different guise to the inequality (55)

$$\text{(56)}\qquad \frac{2\alpha}{\eta} - \frac{2k}{c^2}\int_{-\mu}^{\nu}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp < 0\,,$$

or, on the basis of the transformation of *n*° 26,

$$\text{(57)}\qquad \frac{1}{\nu}\left(1 + \frac{\nu^2}{c^2}\right)^{k} - \frac{2k}{c^2}\int_{0}^{\nu}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp + \frac{1}{\mu}\left(1 + \frac{\mu^2}{c^2}\right)^{k} - \frac{2k}{c^2}\int_{0}^{\mu}\left(1 + \frac{p^2}{c^2}\right)^{k-1} dp < 0\,,$$

where β, μ, ν are the positive roots of the equations

$$2\alpha = \frac{y_1}{\nu} + \frac{y_0}{\mu}\,,$$

$$\beta^{-\frac{1}{k}} = {y_0}^{-\frac{1}{k}}\left(1 + \frac{\mu^2}{c^2}\right) = {y_1}^{-\frac{1}{k}}\left(1 + \frac{\nu^2}{c^2}\right)$$

If we denote the product $\mu\nu$ by u, then the first equation provides straightforwardly that

$$y_1\mu + y_0\nu = 2au\,, \tag{58}$$

whereas the second equation reduces to the form

$$c^2(y_1^{\frac{1}{k}} - y_0^{\frac{1}{k}}) = y_0^{\frac{1}{k}}\nu^2 - y_1^{\frac{1}{k}}\mu^2 = (y_1\mu + y_0\nu)(y_0^{\frac{1}{k}-1}\nu - y_1^{\frac{1}{k}-1}\mu) - (y_1y_0^{\frac{1}{k}} - y_0y_1^{\frac{1}{k}})\mu\nu$$

and in turn provides

$$y_1^{\frac{1}{k}-1}\mu - y_0^{\frac{1}{k}-1}\nu = \frac{1}{2a}(y_0y_1^{\frac{1}{k}-1} - y_1y_0^{\frac{1}{k}-1}) - ({y_1}^{\frac{1}{k}} - {y_0}^{\frac{1}{k}})\frac{c^2}{2au}\,. \tag{59}$$

From the two equations (58) and (59) μ and ν are rationally expressed in terms of u as follows

$$({y_0}^2{y_1}^{\frac{1}{k}} + {y_1}^2{y_0}^{\frac{1}{k}})\mu = 2y_1y_0^{\frac{1}{k}}au - ({y_1}^2{y_0}^{\frac{1}{k}} - {y_0}^2{y_1}^{\frac{1}{k}})\frac{y_0}{2a} -({y_1}^{\frac{1}{k}} - {y_0}^{\frac{1}{k}})\frac{c^2{y_0}^2y_1}{2au}\,,$$

$$({y_0}^2{y_1}^{\frac{1}{k}} + {y_1}^2{y_0}^{\frac{1}{k}})\nu = 2y_0y_1^{\frac{1}{k}}au + ({y_1}^2{y_0}^{\frac{1}{k}} - {y_0}^2{y_1}^{\frac{1}{k}})\frac{y_1}{2a} +({y_1}^{\frac{1}{k}} - {y_0}^{\frac{1}{k}})\frac{c^2y_0{y_1}^2}{2au}\,.$$

The first of these equations by means of multiplying the left hand side by $\frac{\nu}{\nu}$ and replacing there $\mu\nu$ with u, will produce an expression for $\frac{1}{\nu}$ in terms of u. In the same way we find the expression for $\frac{1}{\mu}$ from the second equation. Taking the product of the two preceding equations gives us an equation of the fourth degree with respect to u, on the strength of which each rational function of u, and also μ and ν, may be expressed as a third or lower degree entire function of u, or, if one likes, reduced to the expression of the form

$$Au + B + \frac{C}{u+D}\,.$$

By taking square of the equation (58) it's easy to get the next relationship

$${y_1}^2\left(1 + \frac{\mu^2}{c^2}\right) + {y_0}^2\left(1 + \frac{\nu^2}{c^2}\right) = \frac{4a^2u^2}{c^2} - \frac{2y_0y_1u}{c^2} + {y_0}^2 + {y_1}^2,$$

wherefrom, with the help of the equation

$$y_1^{\frac{1}{k}}\left(1 + \frac{\mu^2}{c^2}\right) = y_0^{\frac{1}{k}}\left(1 + \frac{\nu^2}{c^2}\right)\,,$$

we find

$$y_0^{-\frac{1}{k}}\left({y_0}^2{y_1}^{\frac{1}{k}} + {y_1}^2{y_0}^{\frac{1}{k}}\right)\left(1 + \frac{\mu^2}{c^2}\right) = y_1^{-\frac{1}{k}}\left({y_0}^2{y_1}^{\frac{1}{k}} + {y_1}^2{y_0}^{\frac{1}{k}}\right)\left(1 + \frac{\nu^2}{c^2}\right) = \frac{4a^2u^2}{c^2} - \frac{2y_0y_1u}{c^2} + {y_0}^2 + {y_1}^2.$$

From the latter relationship one obtains

$$\left({y_0}^2{y_1}^{\frac{1}{k}}+{y_1}^2{y_0}^{\frac{1}{k}}\right)\sqrt{\left(1+\frac{\mu^2}{c^2}\right)\left(1+\frac{\nu^2}{c^2}\right)}$$
$$=(y_0y_1)^{\frac{1}{2k}}\left(\frac{4a^2u^2}{c^2}-\frac{2y_0y_1u}{c^2}+{y_0}^2+{y_1}^2\right),$$
$$\left({y_0}^2{y_1}^{\frac{1}{k}}+{y_1}^2{y_0}^{\frac{1}{k}}\right)\mu^2={y_0}^{\frac{1}{k}}(4a^2u^2-2y_0y_1u)-c^2{y_0}^2\left({y_1}^{\frac{1}{k}}-{y_0}^{\frac{1}{k}}\right),$$
$$\left({y_0}^2{y_1}^{\frac{1}{k}}+{y_1}^2{y_0}^{\frac{1}{k}}\right)\nu^2={y_1}^{\frac{1}{k}}(4a^2u^2-2y_0y_1u)+c^2{y_1}^2\left({y_1}^{\frac{1}{k}}-{y_0}^{\frac{1}{k}}\right).$$

Multiplication of the latter two equations one by the other provides the above mentioned fourth degree equation in the form

(60)
$$\left({y_0}^2{y_1}^{\frac{1}{k}}+{y_1}^2{y_0}^{\frac{1}{k}}\right)u^2=\left[{y_0}^{\frac{1}{k}}(4a^2u^2-2y_0y_1u)-c^2{y_0}^2\left({y_1}^{\frac{1}{k}}-{y_0}^{\frac{1}{k}}\right)\right]$$
$$\left[{y_1}^{\frac{1}{k}}(4a^2u^2-2y_0y_1u)+c^2{y_1}^2\left({y_1}^{\frac{1}{k}}-{y_0}^{\frac{1}{k}}\right)\right].$$

This equation is a complete substitute for the equation (47) and can be obtained from it. One may state on the basis of the final result of *n*º 32 that the equation (60) without doubt possesses one positive and one or three negative roots. By this reason one may affirm that *the positive root of the equation (60) should provide positive values of* μ *and* ν *satisfying the inequality (57); or vice versa, if the inequality (57) provides* $u > A$*, then at* $u = A$ *the right hand side of the equation (60) should be less than the left hand side,* because only in this case the equation (60) will have the root grater than A.

39. The left hand side of the inequalities (56) or (57) might be presented as a rational function of u in two cases, namely, 1) when k is an integer number, and 2) when k is the one half of an odd number.

When k is an integer, one readily finds

$$\frac{1}{\nu}\left(1+\frac{\nu^2}{c^2}\right)^k-\frac{2k}{c^2}\int_0^\nu\left(1+\frac{p^2}{c^2}\right)^{k-1}dp=\frac{1}{\nu}-k_1\frac{\nu}{c^2}-\frac{1}{3}k_2\frac{\nu^3}{c^4}$$
$$-\frac{1}{5}k_3\frac{\nu^5}{c^6}-\cdots,$$

where k_n is the binomial coefficient,

$$\frac{k(k-1)\cdots(k-n+1)}{1\cdot 2\cdots n}.$$

Substituting terms of the inequality (57) with this expression, together with the similar expression containing μ, one gains the opportunity to cancel its left hand side by the positive factor $\mu+\nu$ and to reduce it to the form

(61)
$$\frac{1}{u}-k_1\frac{1}{c^2}-\frac{1}{3}k_2\frac{\nu^2+\mu^2-u}{c^4}$$
$$-\frac{1}{5}k_3\frac{\nu^4+\mu^4-u(\nu^2+\mu^2)+u^2}{c^6}-\cdots<0,$$

where the sums of the even powers easily may be expressed in terms of u.

In the case of the parabola, $k=1$, one gets $u>c^2$, and in order that the right hand side of the equation (60) be less than its left hand side, there with necessity should be $a^2c^2<y_0y_1$. It is clear that for the special case under consideration this condition may be obtained in more simple way directly from the equation of the parabola.

40. In the case when k equals one half of an odd number, $\dfrac{2m+1}{2}$, one gets

$$\frac{2m+1}{c^2}\int_0^\nu \left(1+\frac{p^2}{c^2}\right)^{m-\frac{1}{2}} dp = \frac{\nu}{c^2}\left\{\frac{2m+1}{2m}\left(1+\frac{\nu^2}{c^2}\right)^{m-\frac{1}{2}} + \frac{2m+1}{2m}\frac{2m-1}{2m-2}\left(1+\frac{nu^2}{c^2}\right)^{m-\frac{3}{2}} + \cdots + \frac{2m+1}{2m}\frac{2m-1}{2m-2}\cdots \frac{3}{2}\left(1+\frac{\nu^2}{c^2}\right)^{\frac{1}{2}}\right\} + \frac{2m+1}{2m}\frac{2m-1}{2m-2}\cdots\frac{3}{2}\frac{1}{c^2}\int_0^\nu \frac{dp}{\sqrt{1+\frac{p^2}{c^2}}}\,.$$

Replacing in the second part the multiplier $\dfrac{\nu}{c^2}$ with $\dfrac{1}{\nu}\left(1+\dfrac{\nu^2}{c^2}-1\right)$ one without difficulty brings the inequality (57) to the form

$$-\frac{1}{2m}\left[\sigma_m + \frac{2m+1}{2m-2}\sigma_{m-1} + \cdots + \frac{2m+1}{2m-2}\frac{2m-1}{2m-4}\cdots\frac{5}{2}\sigma_1\right] + \frac{2m+1}{2m}\frac{2m-1}{2m-2}\cdots\frac{3}{2}\left[\sigma_0 - \frac{1}{c^2}\int_0^\nu\left(1+\frac{p^2}{c^2}\right)^{-\frac{1}{2}} dp - \frac{1}{c^2}\int_0^\mu \left(1+\frac{p^2}{c^2}\right)^{-\frac{1}{2}} dp\right] < 0\,,$$

where

$$\sigma_n = \frac{1}{\nu}\left(1+\frac{\nu^2}{c^2}\right)^{n+\frac{1}{2}} + \frac{1}{\mu}\left(1+\frac{\mu^2}{c^2}^{n+\frac{1}{2}}\right)\,.$$

It is not difficult to ensure that σ_0 divides σ_n to produce the quotient

$$c^2\left\{1+\frac{u}{c^2}\sqrt{1+\frac{\mu^2}{c^2}}\sqrt{1+\frac{\nu^2}{c^2}} - \frac{u^2}{c^4}\right\}\frac{\left(1+\frac{\nu^2}{c^2}\right)^n - \left(1+\frac{\mu^2}{c^2}\right)^n}{\nu^2-\mu^2} - c^2\left(1+\frac{\nu^2}{c^2}\right)\left(1+\frac{\mu^2}{c^2}\right)\frac{\left(1+\frac{\nu^2}{c^2}\right)^{n-1} - \left(1+\frac{\mu^2}{c^2}\right)^{n-1}}{\nu^2-\mu^2}\,,$$

and this expression, on the strength of the formulæ contained in section n^{o} 38, reduces to a rational function of u. Further, while putting

$$\frac{p}{\sqrt{1+\frac{p^2}{c^2}}} = \frac{\nu z}{\sqrt{1+\frac{\nu^2}{c^2}}}\,,$$

one obtains

$$\int_0^\nu \left(1+\frac{p^2}{c^2}\right)^{-\frac{1}{2}} dp = \int_0^1 \frac{\nu\sqrt{1+\frac{\nu^2}{c^2}}}{1+\frac{\nu^2}{c^2}-\frac{\nu^2}{c^2}z^2}dz\,.$$

In the similar way one transforms the integral with the limit μ, and, adding together both integrals, gets

$$\int_0^{\nu}\left(1+\frac{p^2}{c^2}\right)^{-\frac{1}{2}}dp+\int_0^{\mu}\left(1+\frac{p^2}{c^2}\right)^{-\frac{1}{2}}dp$$
$$=\sigma_0\int_0^1\frac{u\sqrt{1+\frac{\mu^2}{c^2}}\sqrt{1+\frac{\nu^2}{c^2}}-\frac{u^2z^2}{c^2}}{\left(1+\frac{\nu^2}{c^2}-\frac{\nu^2}{c^2}z^2\right)\left(1+\frac{\mu^2}{c^2}-\frac{\mu^2}{c^2}z^2\right)}dz\,.$$

Here the integrand is a rational function of u, and thus the left hand side of the inequality (57) after cancelling by the positive factor σ_0 becomes a rational function of u. However, it worth mentioning that the reduction of the integral to an entire function of u entails quite tedious computations.

II) $k<0$, *concave down curve.*

41. Let us replace k with k in our formulæ. The values of P and Q in this case will be $-\infty$ and 0, so that the condition (28) drops off, and the condition (29) gives

$$\beta>y_1\,.$$

The function $\lambda(p)$ will decrease monotonously from $+\infty$ to $-\infty$ and thus the condition (31) will not hold; in the very same way the inequalities (42) do not provide any restrictions for ξ. Equation (47) transforms into the following one

$$y_0{}^{\frac{1}{k}}-y_1{}^{\frac{1}{k}}+y_0{}^{\frac{2k+1}{k}}c^{-2}(x_0-\xi)^{-2}-y_1{}^{\frac{2k+1}{k}}c^{-2}(x_1-\xi)^{-2}=0 \tag{62}$$

and under the assumption $x_0<x_1$, possesses one real root ξ_0 which is less than x, and another one, ξ_1, located between x_0 and x_1. The first derivative of the left hand side of the latter equation has the only real root less than x and providing a negative minimum; so we may assert that the equation has the sole root, less than x_0, namely the one lying between ξ and x_0.

Equations (34) might be satisfied only under the assumption that $p_0>p_1$ and, moreover, $p_0>0$; thus we are forced to consider the function

$$\frac{2\alpha}{\beta}+\frac{2k}{c^2}\int_{\mu}^{\nu}\left(1+\frac{p^2}{c^2}\right)^{-k-1}dp\,, \tag{63}$$

where

$$2\alpha=x_1-x_0\,,\qquad \frac{\beta}{y_0}=\left(1+\frac{\mu^2}{c^2}\right)^k,\qquad \frac{\beta}{y_1}=\left(1+\frac{\nu^2}{c^2}\right)^k,\qquad \mu>0\,.$$

The extreme values of this function are: at $\beta=y_1$ the extreme value is

$$\frac{2\alpha}{y_1}-\frac{2k}{c^2}\int_0^{\sqrt{\left(\frac{y_1}{y_0}\right)^{\frac{1}{k}}-1}}\left(1+\frac{p^2}{c^2}\right)^{-k-1}dp\,, \tag{64}$$

while at $\beta=\infty$ the extreme value is zero if $\nu>0$ and

$$-\frac{2k}{c^2}\int_{-\infty}^{\infty}\left(1+\frac{p^2}{c^2}\right)^{k-1}dp$$

if $\nu<0$.

42. Considering the positive branch of ν, we get, from the conditions $0 < \mu > \nu$, that $x_0 > \xi > x_0 - \dfrac{x_1 - x_0}{y_1 - y_0}\, y_0$, where the lower bound is less than, greater than, or equal to ξ' respectively in case k is less than, greater than, or equal to 1.

As far as at this lower bound ξ the left hand side of the equation (62) will be negative, namely, will equal

$$\left(y_0{}^{\frac{1}{k}} - y_1{}^{\frac{1}{k}}\right)\left[1 + c^{-2}(x_1 - x_0)^{-2}(y_1 - y_0)^2\right] \,,$$

while at the higher bound, which is $\xi = x_0$, the left hand side of the equation (62) equals $+\infty$, it becomes obvious that the root ξ_0 will lie within these same limits, and that after finding the corresponding values β_0, μ_0, ν_0, we will have to compute the maximal value of the function (63), namely,

$$\frac{x_1 - x_0}{\beta_0} + \frac{2k}{c^2}\int_{\mu_0}^{\nu_0}\left(1 + \frac{p^2}{c^2}\right)^{-k-1} dp\,. \tag{65}$$

By the reason that this value is the maximum of the function that turns to zero at $\beta = \infty$, it will be positive, and thus *in order that the function (63) might vanish at some finite value of β, it is necessary that its extreme value (64) be negative.*

The presupposition $\mu > 0$, $\nu < 0$, provides the inconsistent inequalities $\xi < x_0$, $\xi > x_1$, showing that the function (63) does not possess neither maximum nor minimum, and thus in order that it might possess a root, its extreme values should be of different sign, that means the value of the function (64) should be positive.

And so, *at $x_1 > x_0$, $y_1 > y_0$, there exists one curve of the considered type, connecting the given points for which $p_0 > 0$ while $p_1 > 0$ or < 0 depending on whether the expression (64) is negative or positive one.*

43. If $x_1 < x_0$, one should assume $\nu > \mu$, along with $\mu < 0$ or $\xi > x_0$. The presupposition $\nu > 0$ entails the inequality $\xi < x_1$, which is inconsistent with the previous one. That's why we may conclude that in this case the function (63) wont possess neither maximum nor minimum and in order that it vanishes its extremal values should be of different signs. These extreme values will differ from the ones presented in *n*° 41, if under the $2a$ one means the absolute value of the difference $x_1 - x_0$; consequently, the vanishing of the function (63) will become possible only if the expression (64) is positive.

Assuming $\nu > 0$, from the condition $\nu > \mu$ one finds that $\xi < x_0 + \dfrac{x_0 - x_1}{y_1 - y_0}\, y_0$, and this extreme value will be less than, equal to, or greater than ξ' depending on whether k is greater than, equal to, or less than 1. In any case, between the limits for ξ there will lay the root of the equation (62) that provides a negative minimum of the function (63) and for that reason this function will possess a real root only if it's extreme value at $\beta = y_1$ is positive, and, as a consequence, the expression (64) is negative.

Thus *there always exists one concave curve of the type here considered, connecting the two given points.*

44. Now we set

$$\Theta(p) = \frac{p^2}{2k}\,,$$

from where

$$\int_{\pm 1}^{p} \frac{pdp}{\Theta(p)} = k \int_{\pm 1}^{p} \frac{2pdp}{p^2} = k\; log p^2\,,$$

$$\psi(p) = p^{2.k}\,,$$

$$y = \pm\beta p^{2.k}, \qquad x - \alpha = \pm\frac{2k}{2k-1}\,\beta p^{2k-1}.$$

From this

$$\pm\beta y^{2k-1} = \left(\frac{2k-1}{2k}\right)^{2k}(x-\alpha)^{2k}$$

and this equation represents parabolic curves if $k > \frac{1}{2}$ or if $k < 0$, and hyperbolic ones if $0 < k < \frac{1}{2}$. Assuming $k = \frac{1}{2}$ one obtains

$$y = \pm\beta\sqrt{p^2}, \qquad (x-\alpha) = \pm\beta log\sqrt{p^2}\,,$$

from where the equation of the logarithmic curve follows

$$y = \pm\beta e^{\pm\dfrac{x-\alpha}{\beta}}\,.$$

The equations (28), (29), and (31) in this case drop off and the equation (41) reduces to a quadratic one, namely

$$y_0^{\frac{2k-1}{k}}(x_1-\xi)^2 - y_1^{\frac{2k-1}{k}}(x_0-\xi)^2 = 0\,.$$

All integrations are curried out in finite form and all analysis is fulfilled rather simply.

45. Let now

$$\Theta(p) = \frac{p^2-c^2}{2k}\,,$$

weherefrom in two different cases regarding p, one gets

$p^2 < c^2$:
$$\int_{0}^{p} \frac{pdp}{\Theta(p)} = k\; log\left(1-\frac{p^2}{c^2}\right)\,,$$

$$y = \pm\beta\left(1-\frac{p^2}{c^2}\right)^k, \qquad x-\alpha = \mp\frac{2k\beta}{c^2}\int_{p_0}^{p}\left(1-\frac{p^2}{c^2}\right)^{k-1}dp\,;$$

$p^2 > c^2$:
$$\int_{\pm c\sqrt{2}}^{p} \frac{pdp}{\Theta(p)} = k\; log\left(\frac{p^2}{c^2}-1\right)\,,$$

$$y = \pm\beta\left(\frac{p^2}{c^2}-1\right)^k, \qquad x-\alpha_1 = \pm\frac{2k\beta}{c^2}\int_{p_1}^{p}\left(\frac{p^2}{c^2}-1\right)^{k-1}dp\,.$$

Let $k > 0$. Above the x axis the differential equation will be satisfied by the following segments which meet the x axis in two points:

1. concave segment $y = \beta\left(1-\dfrac{p^2}{c^2}\right)^k$, along which $p^2 < c^2$ and which meets the x axis in

the points $x = \alpha$ at $p = c$ and

$$x = x_1 = \alpha + \frac{2k\beta}{c^2} \int_{-c}^{c} \left(1 - \frac{p^2}{c^2}\right)^{k-1} dp \qquad \text{at } p = -c\,,$$

where we set $p_0 = c$;

2. convex segment $y = \beta \left(\frac{p^2}{c^2} - 1\right)^k$, along which $p > c$ and which, starting from the x axis at $x = \alpha$, extends to the infinity; here we set $\alpha_1 = \alpha$, $p_1 = c$;
3. convex segment $y = \beta \left(1 - \frac{p^2}{c^2}\right)^k$, along which $p < -c$ and which, starting from the x axis at $x = \alpha$, extends to the infinity; here we set $\alpha_1 = \alpha$, $p_1 = -c$;
4. convex segment $y = \beta \left(\frac{p^2}{c^2} - 1\right)^k$, along which $p < -c$ and which, starting at $x = x_1$, extends to the infinity; here $\alpha_1 = x_1$, $p_1 = -c$;
5. convex segment $y = \beta \left(\frac{p^2}{c^2} - 1\right)^k$, along which $p > c$ and which starts at $x = x_1$; here $\alpha_1 = x_1$, $p_1 = c$.

To these five segments one should add another four straight lines with the slopes $\pm c$ passing through the points $x = \alpha$ and $x = x_1$ of the x axis.

In this way through each of the two arbitrarily chosen points of the x axis it is possible to draw five branches satisfying the differential equation and lying above the x axis. The transition from one to another is accomplished keeping the continuity of the ordinate; but, if to this latter condition one adds also the conditions 1) of the continuity of p and 2) of the single-valuedness of the ordinate, then under the totality of these conditions the prolongation of each branch can be obtained only at the other side of the x axis. Thus as the prolongation of the concave segment from the point $x = \alpha$ in the negative direction of the x axis one may take either *a)* the convex segment symmetric to (3) and represented by the equation $y = -\beta \left(\frac{p^2}{c^2} - 1\right)^k$, where $p > c$, $p_1 = c$, $\alpha = \alpha_1$; or *b)* the concave segment symmetric to (5) and represented by the equation $y = -\beta \left(1 - \frac{p^2}{c^2}\right)^k$, where $p^2 < c^2$, $p_0 = c$; or still *c)* the straight line $y = c(x - a)$. As the prolongation of the same concave segment from the point $x = x_1$ one may take either *a′)* the convex segment symmetric to (5) and represented by the equation $y = -\beta \left(\frac{p^2}{c^2} - 1\right)^k$ for $p < -c$; or *b′)* the concave segment $y = -\beta \left(1 - \frac{p^2}{c^2}\right)^k$ for $p^2 < c^2$, $p_0 = -c$, when α is replaced by x_1; or still *c′)* the straight line $y = -c(x - x_1)$.

Considering q we have

$$p^2 < c^2: \qquad q = \mp \frac{c^2}{2k\beta} \left(1 - \frac{p^2}{c^2}\right)^{1-k},$$

$$p^2 > c^2: \qquad q = \pm \frac{c^2}{2k\beta} \left(\frac{p^2}{c^2} - 1\right)^{1-k}.$$

If to the above three conditions one adds the fourth one consisting in that q remains either continuous or should have a polar type singularity, then for $k < 1$ the choice of the only one among the three prolongations of the curve will remain undefined because for all the prolongations $q = 0$ for $p^2 = c^2$ holds; but for $k = 1$, in the case of parabola, only *a)* and *a′)* will serve as the prolongations, whereas for $k > 1$ in case k is a fraction with both the numerator and the denominator odd, only *a)* and *a′)* might be chosen as the prolongations; only *b)* and *b′)* might be chosen in case k is a fraction with an odd numerator and an even denominator; in case k is a fraction with an even numerator and an odd denominator, there does not exist a prolongation of the curve below the x axis. The same choice of the prolongations should be done also for $k < 1$ if one assumes that q turns to zero of an algebraic type.

Similar considerations should effect the choice of the prolongations of convex segments.

Let us notice however that in order to solve problems of the variational calculus connected with the equation $yq = \dfrac{p^2 - c^2}{2k}$, one does not need to resort to the procedure of choosing the prolongations, because on the strength of the Legendre condition, p^2 would not overcome c^2.

For $k < 0$, the equation at issue will be satisfied by the curves located between a pair of asymptotes akin to conjugated hyperbolæ which in fact appear at $k = -1$.

That we have investigated the curves satisfying the equation $yq = \dfrac{p^2 + c^2}{2k}$ in details, we thereby take the liberty of limiting ourselves to only the brief notes about the curves that satisfy the equation $yq = \dfrac{p^2 - c^2}{2k}$, given above.

46. Similar to the equation (24), it is possible to make another differential equation of the second order which the given curve will satisfy as a representative of the family of similar curves possessing the centre of similarity on the y axis. The finite equation of such family will be $F\left(\dfrac{x}{\beta}, \dfrac{y-\alpha}{\beta}\right) = 0$, and after the elimination of arbitrary constants it reduces to the differential equation of the form

$$xq = \Theta(p)\,.$$

On the way of transformation of the given curve into a similar one relative to a centre of similarity located on one of the axes, or, in general, while transforming the given curve into another one similarly arranged, it is obvious that the quantity $\dfrac{dy}{dx}$ will stay invariant because setting $x_1 = \dfrac{x-\alpha}{\beta}$, $y_1 = \dfrac{y-\alpha_1}{\beta}$, one has $\dfrac{dy_1}{dx_1} = \dfrac{dy}{dx}$. Let us note by the way, that it is easy to find the general expression for the invariant of the similar curves and the differential equation of the fourth order which the curves similar to the given one satisfy. As far as the linear dimensions of the similar curves are proportional, calling by ρ and s, ρ_1 and s_1 the radii of curvature and the perimeters of the similar curves, one has $\rho = k\rho_1$, $s - \alpha = s_1$, wherefrom $\dfrac{d\rho_1}{ds_1} = \dfrac{d\rho}{ds}$. If, moreover, one assumes that the basic equation for the given curve is $F(s_1, \rho_1)$, then for the similar curve one obtains $F\left(\dfrac{s-\alpha}{k}, \dfrac{\rho}{k}\right) = 0$, wherefrom follows, as in (24), the differential equation in the guise of

$$\rho\frac{d^2\rho}{ds^2} = \Theta\left(\frac{d\rho}{ds}\right)\,.$$

47. Referring the given curve $F(x, y) = 0$ to an arbitrary origin of the system of reference one gains its equation in the form $F(x+\alpha, y+\beta) = 0$, wherefrom one then finds the differential equation for the curve in the shape of

$$q = \Theta(p)\,.$$

48. Finally, considering the given curve as a representative of similar but nevertheless not similarly positioned curves with the centre of similarity in the origin, and presenting its equation in polar coordinates in the form $F\,(log\,r, \varphi) = 0$, one obtains the equation of the whole family in the form $F\,(log\,r + log\,k, \varphi + \alpha) = 0$, wherefrom, as in (47) one gains the differential equation in the polar coordinates

$$\frac{d^2 \log r}{d\varphi^2} = \Theta\left(\frac{d\log r}{d\varphi}\right) .$$

In this case the invariant obviously will be the angle between the tangent and the radius vector expressed by $\frac{d \log r}{d\varphi}$.

49. In general, given a finite equation of a curve $y = f(x)$, we shall consider this curve as a representative of a family of curves $y_1 = f(x_1)$, where

$$y_1 = \tau(x, y, \alpha, \beta), \qquad x_1 = \sigma(x, y, \alpha, \beta)$$

and where the functions τ and σ transform into y and x at some pair of values of α and β, say at $\alpha = 0$, $\beta = 1$. Given the function $f(x)$, it is obviously always possible to come up with a differential equation of the second order by means of excluding α and β from the equation $\tau = f(\sigma)$, and from the other two, obtained from it by means of differentiation. But, with certain special properties of the functions τ and σ, one can get differential equations of particular *form* independently of the individual properties of the function $f(x)$. This will be the case if it might be possible to construct invariants of the first and of the second order $F\left(x, y, \frac{dy}{dx}\right)$ and $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$, because out of the existence of the equations

$$F\left(x, y, \frac{dy}{dx}\right) = F\left[\sigma, f(\sigma), f'(\sigma)\right] ,$$
$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = F_1\left[\sigma, f(\sigma), f'(\sigma), f''(\sigma)\right]$$

one can infer the existence of the differential equation of the form

$$F_1\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = \Theta\left[F\left(x, y, \frac{dy}{dx}\right)\right] .$$

50. From the expressions for y_1 and x_1 one gets

$$p_1 = \frac{dy_1}{dx_1} = \left(\frac{\partial\tau}{\partial x} + \frac{\partial\tau}{\partial y}\, p\right) : \left(\frac{\partial\sigma}{\partial x} + \frac{\partial\sigma}{\partial y}\, p\right)$$

and thus infers that if an invariant exists, it has the shape

$$\frac{M + Np}{P + Qp} = \frac{M_1 + N_1 p_1}{P_1 + Q_1 p_1}$$

where M, N, P, Q are certain functions of x, y, whereas M_1, N_1, P_1, Q_1 denote the values of these functions under the replacement of x and y by x_1 and y_1.

Putting

$$M dx + N dy = \mu dv \,, \quad P dx + Q dy = \nu du \,,$$

we transform the preceding equality into the following one

$$\frac{\mu}{\nu}\frac{dv}{du} = \frac{\mu_1}{\nu_1}\frac{dv_1}{du_1} \,,$$

wherefrom we conclude that if a first order invariant exists, then by changing the coordinates x,

y into new ones, u, v, it may be put in the form $f(u,v)\dfrac{dv}{du}$, where $f(u,v)$ denotes the fraction $\dfrac{\mu}{\nu}$ brought down to the variables u, v.

Evidently, the existence of the invariant of this type is possible only under the assumption $u_1 = \psi(u,\alpha,\beta)$, $v_1 = \psi_1(v,\alpha,\beta)$, so we shall have

$$f(u,v)\,.\,\psi'(u,\alpha,\beta) = f(u_1,v_1)\,.\,\psi_1{}'(v,\alpha,\beta)\,. \tag{66}$$

Out of this functional equation it is necessary to obtain the functions f, ψ, and ψ_1 profiting by the consideration that an invariant of the form $F(u,v)$ cannot exist. Hence, having come by an equation of the form

$$F(u,v) = F(u_1,v_1)\,, \tag{67}$$

we will have to conclude from it that $F(u,v) = \text{const}$.

51. Taking logarithm of the equality (66) and then differentiating it by u and v, we gain

$$\frac{\partial^2 \, log\, f(u,v)}{\partial u \partial v} = \frac{\partial^2 \, log\, f(u_1,v_1)}{\partial u_1 \partial v_1}\,\psi'(u,\alpha,\beta)\,.\,\psi_1{}'(v,\alpha,\beta).$$

Taking logarithm of this new equality and differentiating it we find

$$\frac{\partial^2}{\partial u \partial v}\, log \frac{\partial^2 \, log\, f(u,v)}{\partial u \partial v} = \frac{\partial^2}{\partial u_1 \partial v_1}\, log \frac{\partial^2 \, log\, f(u_1,v_1)}{\partial u_1 \partial v_1}\,\psi'(u,\alpha,\beta)\,.\,\psi_1{}'(v,\alpha,\beta),$$

and, dividing this equation by the preceding one, we shall get an equation of the form (67). Hence, we obtain

$$\frac{\partial^2}{\partial u \partial v}\, log \frac{\partial^2 log\, f(u,v)}{\partial u \partial v} = 2a\,\frac{\partial^2 log\, f(u,v)}{\partial u \partial v}\,, \tag{68}$$

where a is an absolute constant quantity.

Denoting $2a\,\dfrac{\partial^2 log\, f(u,v)}{\partial u \partial v} = z$, we transform the equation (68) into the following

$$\frac{\partial^2 log\, z}{\partial u \partial v} = z\,,$$

the integral of which was found by Liouville[4] and may be given the shape

$$z = 2\,\chi'(u)\,.\,\chi_1{}'(v) : [\chi(u) + \chi_1(v)]^2\,,$$

where $\chi(u)$ and $\chi_1(v)$ are two arbitrary functions.

Replacing here z with its value, after integrating twice we get

$$f^{-a} = \chi_2(u)\,.\,\chi_3(v)\,.\,[\chi(u) + \chi_1(v)]^2\,,$$

where $\chi_2(u)$ and $\chi_3(v)$ are two new arbitrary functions.

52. After that the equation (66) will provide

$$\psi'(u)^{-a}\,\frac{\chi_2(u)}{\chi_2(u_1)}\,.\,\psi_1{}'(v)^a\,\frac{\chi_3(v)}{\chi_3(v_1)}\,[\chi(u) + \chi_1(v)] = \chi(u) + \chi_1(v)\,, \tag{69}$$

where the left hand side has the shape $\lambda(u)\,.\,\lambda_1(v) + \lambda_2(u)\,.\,\lambda_3(v)$. The differentiation of this equation by u and by v leads to the result

$$\lambda'(u)\,.\,\lambda_1{}'(v) + \lambda_2{}'(u)\,.\,\lambda_3{}'(v) = 0\,,$$

[4] Journal de mathématiques, 1 série, t. XVIII, p. 71

wherefrom one finds

$$\frac{\lambda'(u)}{\lambda_2{}'(u)} = \frac{\lambda_3{}'(v)}{\lambda_1{}'(v)} = A\,,$$

where A is a constant, in general depending on α and β. From this one finds by integration

$$\lambda(u) - A\lambda_2(u) = B, \qquad \lambda_3(v) + A\lambda_1(v) = C\,,$$

that is

$$\left\{\begin{array}{l}\psi'(u)^{-a}\chi_2(u)[\chi(u) - A] = B\chi_2(u_1)\,, \\ \psi_1{}'(v)^{a}\chi_3(v)[\chi_1(v) + A] = C\chi_3(v_1)\,.\end{array}\right. \tag{70}$$

The substitution of $\chi(u)$ and $\chi_1(v)$ in the left hand side of the equation (69) with the values obtained from (70), leads to the result

$$B\psi_1{}'(v)^a\frac{\chi_3(v)}{\chi_3(v_1)} - \chi_1(v_1) = \chi(u_1) - C\psi'(u)^{-a}\frac{\chi_2(u)}{\chi_2(u_1)} = D\,, \tag{71}$$

which, together with formulæ (70) again, provides

$$\left\{\begin{array}{l}\big[\chi(u) - A\big]\big[\chi(u_1) - D\big] = BC\,, \\ \big[\chi_1(v) + A\big]\big[\chi_1(v_1) + D\big] = BC\,.\end{array}\right. \tag{72}$$

53. In this way equation (66) splits into four equations, (70) and (71), i.e. (72). In what concerns this four equations, they involve: 1) four functions of one argument, denoted by the root character χ; some of these functions, as well as all of them may be constant, provided $\chi_2(u)$, $\chi_3(v)$ and $\chi(u) + \chi_1(u)$ differ from zero; besides, if $\chi(u) = \mathrm{const}$, then, evidently, without loss of generality of the form of function f we may at the same time also put $\chi_1(v) = \mathrm{const}$, and assume $\alpha = \pm 1$; 2) two functions $\psi(u, \alpha, \beta)$ and $\psi_1(v, \alpha, \beta)$, each depending on two or three arguments, u and v necessarily entering in that number; 3) four functions A, B, C, D of the arguments α and β; some of these functions may depend only on one argument or merely be constant. Due to these peculiarities in the structure of *ten* mentioned functions we can fetch explicit expressions of some of them despite the seeming deficiency of *four* equations.

Indeed, with respect to the variables u and v, the considered four equations evidently split into two independent pairs, so that the solution of one of them will come out from the solution of the other by mere replacement of the characters. Passing on to the pair that involves u, we find by the differentiation of (72)

$$\psi'(u, \alpha, \beta) = -\chi'(u)[\chi(u_1) - D] : \chi'(u_1)[\chi(u) - A]\,,$$

whereby (70) turns into a relation between u and u_1. Differentiating this relation and inserting in there the just found expression for $\psi'(u)$, we obtain some new relationship between u and u_1, and so on Having gained four such relationships between u and u_1 and attaching to them the equation (72) we will obtain the possibility to exclude A, B, C, D and will arrive at a relationship that will involve *solely* u and u_1. This relationship should present an identity in u and u_1 because in other case in contradiction to our assumption it would provide an expression for u_1 in terms of u without any arbitrary parameters. From this identity and, if necessary, partially differentiating it with respect to u, it possible get a number of equations containing only the variable u and defining the functions $\chi(u)$ and $\chi_2(u)$ with arbitrary absolute constants. We failed to accomplish this general computation in simple enough manner so we limit ourselves only to the statement about its feasibility. We shall consider instead a special case when the equations (72) cannot determine the derivatives $\psi'(u)$ and $\psi_1{}'(v)$.

54. Let us put $B = 0$, wherefrom on the strength of the first equation in (70), $\chi(u) = A$, so that A should stay absolutely constant. The second equation in (70) shows that C will differ from zero because otherwise we would have $\chi_1(v) = -A$ and $\chi(u)+\chi_1(v) = 0$ which is not possible; but the second equation in (72) provides $\chi_1(v) = -D$, so that D will be an absolutely constant number, different from A; nothing prevents us from choosing $D = 0$ and also, as mentioned above, $a = -1$. The second equations in (70) and (71) within present assumptions will be

$$\frac{\psi'(u)}{\chi_2(u_1)} = \frac{A}{C}\cdot\frac{1}{\chi_2(u)}\,, \qquad \chi_3(v_1)\,.\,\psi'(v_1) = \frac{A}{C}\,\chi_3(v)$$

and after the integration by introducing succinct notations

$$\beta = \frac{C}{A}\,, \quad \xi(u) = \int\frac{du}{\chi_2(u_1)}\,, \quad \xi_1(v) = \int\chi_3(v)dv$$

they will produce

$$\xi(u_1) = \frac{\xi(u)+\alpha}{\beta}\,, \qquad \xi_1(v_1) = \frac{\xi_1(v)+\gamma}{\beta}\,.$$

With the help of these equations containing arbitrary functions ξ and ξ_1 one defines u_1 and v_1.

55. Once the first order invariant $f(u,v)\dfrac{dv}{du}$ is known, one easily finds also the second order invariant.

Having taken the differential of the equation

$$f(u,v)\frac{dv}{du} = f(u_1,v_1)\frac{dv_1}{du_1}$$

and dividing it by $du = \dfrac{1}{\psi'(u)}\,.\,du_1$, one comes up with

$$\frac{\partial f}{\partial u}\frac{dv}{du} + \frac{\partial f}{\partial v}\left(\frac{dv}{du}\right)^2 + f\,.\,\frac{d^2v}{du^2} = \left[\frac{\partial f}{\partial u_1}\frac{dv_1}{du_1} + \frac{\partial f}{\partial v_1}\left(\frac{dv_1}{du_1}\right)^2 + f(u_1,v_1)\,.\,\frac{d^2v_1}{du_1{}^2}\right]\psi'(u)\,.$$

Let us now assume that there exists the following equality

$$f_1(u_1,v_1) = f_1(u,v)\,.\,\psi_1{}'(v)\,;$$

differentiating it with respect to u one finds

$$\frac{\partial f_1(u_1,v_1)}{\partial u_1}\,\psi'(u) = \frac{\partial f_1(u,v)}{\partial u}\,\psi_1{}'(v)$$

and from this and from (66) one gets

$$f(u,v)\,\frac{\partial f_1(u,v)}{\partial u} = k\,,$$

where k is an absolute constant. This equation permits to define the function $f_1(u,v)$ which, once defined, produces the sought invariant of the second order

$$\left[\frac{\partial f(u,v)}{\partial u}\frac{dv}{du} + \frac{\partial f(u,v)}{\partial v}\left(\frac{dv}{du}\right)^2 + f(u,v)\,\frac{d^2v}{du^2}\right].\,f_1(u,v)f(u,v)\,.$$

In the case we considered in *n*º 54, it is necessary to put $\gamma = 0$ and then one gets

$$\frac{\xi_1{}'()}{\xi(v_1)}\,\psi_1{}'(v) = \frac{\xi_1{}'(v)}{\xi(v)}\,,$$

so that $f_1(u,v) = \dfrac{\xi_1(v)}{\xi_1{}'(v)}$. The first order invariant will be

$$\frac{\xi_1{}'(v)}{\xi'(u)}\frac{dv}{du}\,.$$

The second order invariant will be

$$\frac{\xi_1(v)}{\xi'(u)}\frac{d}{du}\left[\frac{\xi_1{}'(v)}{\xi'(u)}\frac{dv}{du}\right]\,,$$

and the differential equation will be

$$\frac{\xi_1(v)}{\xi'(u)}\frac{d}{du}\left[\frac{\xi_1{}'(v)}{\xi'(u)}\frac{dv}{du}\right] = \Theta\left[\frac{\xi_1{}'(v)}{\xi'(u)}\frac{dv}{du}\right]\,,$$

or

$$\xi_1(v)\,\frac{d^2\xi_1(v)}{d\xi(u)^2} = \Theta\left[\frac{d\xi_1(v)}{d\xi(u)}\right]\,,$$

what in fact is the equation (24).

56. We shall complete this investigation by solving one problem tied to the application of the minimum and maximum condition given by Jacobi.

If

$$1)\qquad q = \varphi(x,y,p)$$

is a differential equation of the second order and

$$2)\qquad \psi(x,y,p) = \alpha\,,\quad \sigma(x,y,p) = \beta$$

are its first integrals, then to apply Jacobi's condition it is necessary to consider the value of the quotient $\dfrac{\partial y}{\partial\beta} : \dfrac{\partial y}{\partial\alpha}$. Treating y, p, α, β in 2) as variables and under this assumption partially differentiating mentioned equations with respect to α and β, we will have

$$\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\alpha} + \frac{\partial\psi}{\partial p}\frac{\partial p}{\partial\alpha} = 1 \qquad \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\beta} + \frac{\partial\psi}{\partial p}\frac{\partial p}{\partial\beta} = 0\,,$$
$$\frac{\partial\sigma}{\partial y}\frac{\partial y}{\partial\alpha} + \frac{\partial\sigma}{\partial p}\frac{\partial p}{\partial\alpha} = 0 \qquad \frac{\partial\sigma}{\partial y}\frac{\partial y}{\partial\beta} + \frac{\partial\sigma}{\partial p}\frac{\partial p}{\partial\beta} = 1\,,$$

wherefrom without difficulty one finds the values $\dfrac{\partial y}{\partial\alpha}$, $\dfrac{\partial y}{\partial\beta}$ and arrives at

$$\frac{\partial y}{\partial\beta} : \frac{\partial y}{\partial\alpha} = -\frac{\partial\psi}{\partial p}\frac{\partial\sigma}{\partial p}\,.$$

Using equations 2) one needs to express $\dfrac{\partial\psi}{\partial p}\dfrac{\partial\sigma}{\partial p}$ in terms of a function of one variable which might be as well as x, y, or p, any other function of these variables, as, for example, the abscissa of the intersection point of the tangent with the x axis, i. e. $x = -\dfrac{y}{p}$. Designating this abscissa with γ one has

$$\frac{\partial y}{\partial \beta} : \frac{\partial y}{\partial \alpha} = -\frac{\partial \psi}{\partial p} : \frac{\partial \sigma}{\partial p} = F(\alpha, \beta, \gamma) .$$

The limits for γ are defined by the stipulation that the function $F(\alpha, \beta, \gamma)$ cannot take all possible values.

The problem we shall occupy ourselves with consists in fixing the differential equation 1) for which $F(\alpha, \beta, \gamma)$ is a linear function of γ. In this case, conversely, γ will linearly express itself through $\frac{\partial \psi}{\partial p} : \frac{\partial \sigma}{\partial p}$, i. e.

$$\gamma = x - \frac{y}{p} = f(\alpha, \beta) + f_1(\alpha, \beta)\, \frac{\partial \psi}{\partial p} : \frac{\partial \sigma}{\partial p} ,$$

and this equation should turn into identity after replacing α and β with the functions ψ and σ, so that

$$x - \frac{y}{p} = \left[f(\psi, \sigma) \frac{\partial \sigma}{\partial p} + f_1(\psi, \sigma) \frac{\partial \psi}{\partial p} \right] : \frac{\partial \sigma}{\partial p} .$$

57. Denominating the integrating multiplier of the binomial

$$f(\psi, \sigma)\, d\sigma + f_1(\psi, \sigma)\, d\psi$$

by $\mu(\psi, \sigma)$ and designating the integral by $f_2(\psi, \sigma)$, we get

$$\mu \frac{\partial \sigma}{\partial p} \left(x - \frac{y}{p} \right) = \frac{\partial f_2(\psi, \sigma)}{\partial p} ,$$

where $f_2(\psi, \sigma)$ will be the integral of the equation 1) and thus may be simply replaced by the function ψ.

So it suffices to consider the equation

a) $$\frac{\partial \psi}{\partial p} = \mu \,.\, \left(x - \frac{y}{p} \right) . \frac{\partial \sigma}{\partial p} .$$

Noticing now that equations (2), as the integrals of the one and the same equation of the second order, must provide the same values of q, we get

$$\frac{\partial \psi}{\partial p} : \frac{\partial \sigma}{\partial p} = \left(\frac{\partial \psi}{\partial x} + p\, \frac{\partial \psi}{\partial y} \right) : \left(\frac{\partial \sigma}{\partial x} + p\, \frac{\partial \sigma}{\partial y} \right) ,$$

on the strength of what we obtain

$$\frac{\partial \psi}{\partial x} + p\, \frac{\partial \psi}{\partial y} = \mu \,.\, \left(x - \frac{y}{p} \right) \left(\frac{\partial \sigma}{\partial x} + p\, \frac{\partial \sigma}{\partial y} \right) .$$

Differentiating this equation by p, along with differentiating the equation (a) by x and by y, one gains the possibility to eliminate $\frac{\partial^2 \psi}{\partial x \partial p}$ and $\frac{\partial^2 \psi}{\partial y \partial p}$ and comes up with the following equations

b) $$\begin{cases} \dfrac{\partial \psi}{\partial x} = \mu \left[\left(x - \dfrac{2y}{p} \right) \dfrac{\partial \sigma}{\partial x} - y\, \dfrac{\partial \sigma}{\partial y} \right] , \\ \dfrac{\partial \psi}{\partial y} = \mu \left[\dfrac{y}{p^2} \dfrac{\partial \sigma}{\partial x} + x\, \dfrac{\partial \sigma}{\partial y} \right] , \\ \dfrac{\partial \psi}{\partial p} = \mu \left(x - \dfrac{y}{p} \right) \dfrac{\partial \sigma}{\partial p} . \end{cases}$$

58. From these equation it is possible to obtain two different sets of expressions for the derivatives $\frac{\partial^2\psi}{\partial y\partial p}$, $\frac{\partial^2\psi}{\partial p\partial x}$, $\frac{\partial^2\psi}{\partial x\partial y}$; comparing these expressions with each other one gets

$$\text{c)}\qquad \begin{cases} \frac{\partial^2\sigma}{\partial p\partial x} + p\,\frac{\partial^2\sigma}{\partial y\partial p} + \frac{p}{y}\frac{\partial\sigma}{\partial p} - \frac{2p}{y}\frac{\partial\sigma}{\partial x} + \frac{\partial\, log\,\mu}{\partial\sigma}\frac{\partial\sigma}{\partial p}\left(\frac{\partial\sigma}{\partial x} + p\,\frac{\partial\sigma}{\partial y}\right) = 0\,, \\ \frac{\partial^2\sigma}{\partial x^2} + 2p\,\frac{\partial^2\sigma}{\partial x\partial y} + p^2\,\frac{\partial^2\sigma}{\partial y^2} + \frac{2p}{y}\left(\frac{\partial\sigma}{\partial x} + p\,\frac{\partial\sigma}{\partial y}\right) \\ \qquad + \frac{\partial\, log\,\mu}{\partial\sigma}\left(\frac{\partial\sigma}{\partial x} + p\,\frac{\partial\sigma}{\partial y}\right)^2 = 0 \end{cases}$$

These two equations present the integrability conditions of the exact differential equation

$$d\psi - \frac{\partial\psi}{\partial x}\,dx - \frac{\partial\psi}{\partial y}\,dy - \frac{\partial\psi}{\partial p}\,dp = 0\,,$$

and they must hold no matter what the value of the function ψ be. From this it follows that if $\frac{\partial\, log\,\mu}{\partial\sigma}$ contains ψ, then with necessity

$$\frac{\partial\sigma}{\partial x} + p\,\frac{\partial\sigma}{\partial y} = 0\,,$$

which gives, together with other conditions, $\sigma =$ arbitrary$\phi\,.\,x - \frac{y}{p}$. Further on, equations (b) in this case give

$$d\psi = \mu\,.\,\phi'\left(x - \frac{y}{p}\right).\,d\left(x - \frac{y}{p}\right),$$

that presents an entanglement between ψ and σ which cannot exist. By this virtue the assumption that $\frac{\partial\, log\,\mu}{\partial\sigma}$ depends on ψ cannot be true.

Once taken that $\frac{\partial\, log\,\mu}{\partial\sigma}$ does not depend on ψ this will mean that μ appears as the product $\phi(\psi)\,.\,\phi_1(\sigma)$; if so, then replacing the integral equations (2) by some functions of themselves, one can put $\mu = 1$. After that the second condition in (c) can be integrated and provides

$$\sigma = \frac{1}{-}ypF(p,\gamma) + F_1(p,\gamma)$$

where F and F_1 are two arbitrary functions of two arguments. Inserting this value of σ in the first of the integrability conditions of (c) one finds

$$p^3\frac{\partial F_1}{\partial p} = \frac{\partial F}{\partial\gamma}\,,$$

wherefrom one concludes that taken an absolutely arbitrary function Σ, it is possible to set

$$F(p,\gamma) = p^3\frac{\partial\Sigma}{\partial p}\,,\qquad F_1(p,\gamma) = \frac{\partial\Sigma}{\partial\gamma}\,,$$

so that

$$\sigma = \frac{p^2}{y}\frac{\partial\Sigma}{\partial p} + \frac{\partial\Sigma}{\partial\gamma}\,.$$

Next from the derivatives of ψ we find the function ϕ itself, namely

$$\psi = \gamma \left(\frac{p^2}{y} \frac{\partial \Sigma}{\partial p} + \frac{\partial \Sigma}{\partial \gamma} \right) - \Sigma \,.$$

The sought differential equation of the second order will then be

$$\left[p^4 \, \frac{\partial^2 \Sigma}{\partial p^2} + 2py \, \frac{\partial^2 \Sigma}{\partial p \partial \gamma} + y^2 \, \frac{\partial^2 \Sigma}{\partial \gamma^2} + 2p^3 \, \frac{\partial \Sigma}{\partial p} \right] yq = p^5 \, \frac{\partial \Sigma}{\partial p} \,.$$

Warsaw, November 20, 1885[5]

[5] Although A. Yu. Davidov, to whom this article was dedicated on the occasion of the thirtieth anniversary of his professor's activities, passed away on December 22, 1885, it nevertheless seemed appropriate to the author to keep the dedication, for which the permission from the late had been received. February 18, 1886