The inverse problem of the calculus of variations

Introduction

Consider a system of second-order ordinary differential equations, solved with respect to second derivatives of the unknown curve $x^i = x^i(t)$,

(1) $$\ddot{x}^j - F^j = 0,$$

where $i, j = 1, 2, \ldots, m$, and $F = F^j(x^i, \dot{x}^i)$ are given functions. Any collection of functions $g_{jk} = g_{jk}(x^i, \dot{x}^i)$, such that $\det g_{ij} \neq 0$, defines an equivalent system $g_{ij}(\ddot{x}^j - F^j) = 0$. The goal is to study the problem of existence of a function $\mathscr{L} = \mathscr{L}(x^i, \dot{x}^i)$ such that

(2) $$g_{ij}(\ddot{x}^j - F^j) = -\frac{\partial \mathscr{L}}{\partial x^i} + \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}^i},$$

known as the *inverse problem of the calculus of variations* for the system (1). For historical reasons we also refer to this problem as the *Sonin-Douglas's variationality problem*, and call equations (2) the *Sonin-Douglas's equation*. Having in mind the correspondence with classical mechanics and differential geometry, we sometimes call the system *F* (resp. *g*) the *force* (resp. *metric*), the components g_{jk} are also called *variational multipliers*. If the function $\mathscr{L}$ exists, it is called the *Lagrangian* for the pair (F, g). Denoting (conventionally with the minus sign)

(3) $$\varepsilon_i = -g_{ij}(\ddot{x}^j - F^j),$$

we can equivalently say that the functions ε_i are the *Euler-Lagrange expressions* of $\mathscr{L}$, or that the system of functions $\varepsilon = \varepsilon_i$ is *variational*.

First ideas related with variational origin of differential equations appeared in Sonin 1866 [4], who studied the inverse problem for *one* second order equation and proved that *all* second order equations admit a Lagrangian; for English translation of his work see

http://www.lepageri.eu/publications.html

The author is indebted to Professor Skarzhinski for this reference, and for the discussions during the International Conference on Differential Geometry and its Applications, Brno, August 24 – 30, 1986. The same idea and approach has later appeared in Darboux (*Lecons sur la Theorie Generale des Surfaces*, Paris, 1894). In 1941 Douglas [2] obtained a complete classification of the systems (2) for *two* equations, and provided numerous examples of *non-variational* systems; he already studied the same subject in 1939 and 1940, but regarded these papers as preliminary notes (see References and Notes (1) and (3) in [2]). The results of Douglas have been further developed from geometrical point of view by Sarlet, Crampin and Martinez [9], Anderson and Thompson [5], Krupkova and Prince [25] and others (further references can be found in the handbook D. Krupka, D. Saunders [1]).

The Sonin-Douglas's problem is a special case of the *problem of Helmholtz*, formulated for general systems of ordinary second order equations in an implicit form

$$\varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j) = 0, \tag{4}$$

where $1 \le i, j \le n$ (Helmholtz 1887 [3]). For historical remarks and generalisations of the *Helmholtz variationality conditions* we refer to Krupkova, Prince 2008 [25] and D. Krupka, O. Krupkova, G. Prince and W. Sarlet 2007 [22].

Remark 1 We do not consider in this work the systems of differential equations (3) and (4), such that the functions ε_i depend explicitly on the parameter t of the curves $t \to x^i(t)$).

The Helmholtz conditions have been generalised to systems of higher order partial differential equations by Anderson, Duchamp 1980 [17], and Krupka 1981 [6]. As an illustrative example consider a system of second-order equations of the form

$$\varepsilon_\sigma(x^i, y^\nu, y^\nu_j, y^\nu_{jk}) = 0, \tag{5}$$

where $1 \le i \le n$, $1 \le \sigma, \nu \le m$, x^i are independent variables, y^ν, dependent variables, y^ν_j y^ν_{jk} the derivative variables, and $\varepsilon_\sigma = \varepsilon_\sigma(x^i, y^\nu, y^\nu_j, y^\nu_{jk})$ is a given system of differentiable functions. The variationality conditions for this system read

$$
\begin{aligned}
&\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{qr}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{qr}} = 0,\\
(6)\qquad &\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{q}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q}} - 2d_p \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} = 0,\\
&\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + d_p \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p}} - 2d_p d_q \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} = 0.
\end{aligned}
$$

If these conditions are satisfied, then a Lagrangian for the system ε_σ can be constructed by

$$(7)\qquad \mathscr{L}(x^i, y^\sigma, y^\sigma_p, y^\sigma_{pq}) = y^\sigma \int_0^1 \varepsilon_\sigma(x^i, \tau y^\nu, \tau y^\nu_j, \tau y^\nu_{jk}) d\tau,$$

and is known as the *Vainberg-Tonti Lagrangian* (cf. [1]).

The problem of Helmholtz was extended to second-order systems of *homogeneous* ordinary differential equations by Urban, Krupka 2013 [10] by means of combination of the Helmholtz and the Zermelo (positive homogeneity) conditions.

The *global* inverse problem as considered in these lectures, is concerned with equations for extremals in the theory of *integral variational functionals* on fibred manifolds. Let Y be a fibred manifold over the base manifold X, where $n = \dim X$, and let $J^r Y$ denote the r-jet prolongation of Y. Consider for simplicity a 1st order Lagrangian λ, that is, an n-form on $J^1 Y$ such that in *any* fibred coordinates (x^i, y^σ) on Y,

$$(8)\qquad \lambda = \mathscr{L}\omega_0,$$

where $\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, and $\mathscr{L} = \mathscr{L}(x^i, y^\sigma, y^\sigma_j)$ in the associated coordinates $(x^i, y^\sigma, y^\sigma_j)$ on $J^1 Y$ (the *local Lagrange function*). The *Euler-Lagrange form* of λ is a globally well-defined $(n+1)$-form on $J^2 Y$, defined in the associated coordinates $(x^i, y^\sigma, y^\sigma_j, y^\sigma_{jk})$ as

$$(9)\qquad E(\lambda) = E_\sigma(\mathscr{L}) dy^\sigma \wedge \omega_0,$$

where $E_\sigma(\mathscr{L})$ are the *Euler-Lagrange expressions*,

$$(10)\qquad E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - d_k \frac{\partial \mathscr{L}}{\partial y^\sigma_k}.$$

These concepts define the (global) *Euler-Lagrange mapping* $\lambda \to E(\lambda)$ between the corresponding Abelian groups of differential forms, assigning to the Lagrangians the corresponding Euler-Lagrange forms.

On the other hand, on the 2nd jet prolongation J^2Y we also have the *source forms*, the $(n+1)$-forms ε, locally expressible as

$$\varepsilon = \varepsilon_\sigma dy^\sigma \wedge \omega_0. \tag{11}$$

Clearly, the Euler-Lagrange forms belong to the set of source forms. The (*global*) *inverse problem* consists in finding conditions ensuring that a given source form ε is an Euler-Lagrange form, that is, solves the equation

$$\varepsilon = E(\lambda) \tag{12}$$

with the Euler-Lagrange mapping on the right-hand side. A *necessary condition* for existence of a solution can be written as the system

$$\varepsilon_\sigma = \frac{\partial \mathscr{L}}{\partial y^\sigma} - d_k \frac{\partial \mathscr{L}}{\partial y_k^\sigma} \tag{13}$$

for an unknown $\mathscr{L} = \mathscr{L}(x^i, y^\sigma, y_j^\sigma)$, for eny fibred coordinates (x^i, y^σ). Its solvability is equivalent with the Helmholtz conditions (6); if (6) are satisfied for some fibred coordinates at *any* point of *Y*, we say that the source form (11) is *locally variational*.

The global inverse problem, however, is to find a *global* Lagrangian λ solving (12). To this purpose we construct a sequence of classes of differential forms, the *variational sequence*, derived from the de Rham sequence of sheaves of forms on the domain J^1Y of λ, in which the Euler-Lagrange mapping represents one arrow (Krupka 1990 [21]). Then the cohomology of the complex of global sections of the variational sequence determines global properties of the Euler-Lagrange mapping, namely its *image* and *kernel*. In particular, we get as a consequence that for fibred manifolds *Y* such that the De Rham cohomology group $H^{n+1}Y$ is trivial, that is,

$$H^{n+1}Y = 0, \tag{14}$$

locally variational source forms are necessarily (globally) variational. The cohomology group H^nY is then responsible for the freedom in the choice of global Lagrangians. In this way we get a complete description of the solutions of equation (12) on *Y*.

Main motivations for the variational sequence theory came from the theory of Lepage forms (see e.g. Krupka 1975 [14]), the work of Takens 1979 [24] and the variational bicomplex theory (see references in [1]).

Part 1

The inverse problem for second-order ordinary differential equations

In this part of the lectures systems of k second-order ordinary differential equations for k unknown functions are considered. We study the conditions ensuring that such a system be expressible as the Euler-Lagrange equations for extremals of some integral variational functional. The problem how this variational functional can be recovered is also considered and the corresponding Lagrangian is constructed. The systems of the Sonin-Douglas type (solved with respect to the second derivatives of the unknown functions) and of the Helmholtz type (in an implicit form) are considered separately.

1 The Sonin's inverse problem

Given a function $F = F(t,x,\dot{x})$, the *Sonin's problem* consists in finding a *nonzero* function $g = g(t,x,\dot{x})$ for which there exists a solution $\mathscr{L} = \mathscr{L}(t,x,\dot{x})$ of the equation

(1) $$g(\ddot{x} - F) = -\frac{\partial \mathscr{L}}{\partial x} + \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}}.$$

Then if we have a solution, considering $\mathscr{L}$ as the *Lagrange function* of a variational principle, the corresponding *Euler-Lagrange equation* is

(2) $$\frac{\partial \mathscr{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}} = 0,$$

and is equivalent with the equation

(3) $$\ddot{x} - F = 0.$$

Theorem 1 *The Sonin's problem has always a solution* g.

Proof Since g is supposed to be different from zero on its domain of definition, equation (1) is equivalent with the system

$$g = \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^2}, \quad -gF = -\frac{\partial \mathscr{L}}{\partial x} + \frac{\partial^2 \mathscr{L}}{\partial t \, \partial \dot{x}} + \frac{\partial^2 \mathscr{L}}{\partial x \, \partial \dot{x}} \dot{x}. \tag{4}$$

The first equation can be solved immediately on any star-shaped domain with centre 0 in the variable $\dot{x}$. We first solve the equation

$$g = \frac{\partial h}{\partial \dot{x}}. \tag{5}$$

The solution is

$$h = \dot{x} \int_0^1 g(x, \kappa \dot{x}) d\kappa. \tag{6}$$

Indeed, we have

$$\begin{aligned} \left(\frac{\partial h}{\partial \dot{x}}\right)_{(x^p, \dot{x}^p)} &= \int_0^1 g(x, \kappa \dot{x}) d\kappa + \dot{x} \int_0^1 \left(\frac{\partial g}{\partial \dot{x}}\right)_{(x, \kappa \dot{x})} \kappa \, d\kappa \\ &= \int_0^1 \left(g(x, \kappa \dot{x}) + \left(\frac{\partial g}{\partial \dot{x}}\right)_{(x, \kappa \dot{x})} \kappa \dot{x} \right) d\kappa = \int_0^1 \frac{d}{d\kappa} (g(x, \kappa \dot{x}) \kappa) d\kappa \\ &= g(x, \dot{x}). \end{aligned} \tag{7}$$

Then we solve the equation

$$h = \frac{\partial \mathscr{L}}{\partial \dot{x}}. \tag{8}$$

We have a solution L, defined by

$$L = \dot{x}^i \int_0^1 h(x, \tau \dot{x}) d\tau. \tag{9}$$

Substituting

$$h(x, \tau \dot{x}) = \tau \dot{x} \int_0^1 g(x, \kappa \tau \dot{x}) d\kappa, \tag{10}$$

we get

$$L = \dot{x}\int_0^1\left(\tau\dot{x}\int_0^1 g(x,\kappa\tau\dot{x})\,d\kappa\right)d\tau = \dot{x}^2\int_0^1\left(\int_0^1 g(x,\kappa\tau\dot{x})\,d\kappa\right)\tau\,d\tau. \tag{11}$$

The general solution of the first equation (4) is

$$\mathscr{L} = \frac{1}{2}\dot{x}^2\int_0^1\left(\int_0^1 g(x,\kappa\tau\dot{x})\,d\kappa\right)\tau\,d\tau + A\dot{x} + B, \tag{12}$$

where the functions A and B do not depend on $\dot{x}$.

From (12) it is now sufficient to prove that the second equation (4) has a solution g. Following Sonin, we differentiate (4) with respect $\dot{x}$. We get

$$\begin{aligned} &-gF + \frac{\partial \mathscr{L}}{\partial x} - \frac{\partial^2 \mathscr{L}}{\partial t\,\partial \dot{x}} - \frac{\partial^2 \mathscr{L}}{\partial x\,\partial \dot{x}}\dot{x} \\ &\quad = -\frac{\partial g}{\partial \dot{x}}F - g\frac{\partial F}{\partial \dot{x}} + \cancel{\frac{\partial^2 \mathscr{L}}{\partial x\,\partial \dot{x}}} - \frac{\partial^3 \mathscr{L}}{\partial t\,\partial \dot{x}^2} - \frac{\partial^3 \mathscr{L}}{\partial x\,\partial \dot{x}^2}\dot{x} - \cancel{\frac{\partial^2 \mathscr{L}}{\partial x\,\partial \dot{x}}} = 0 \end{aligned} \tag{13}$$

hence g must satisfy

$$\frac{\partial g}{\partial \dot{x}}F + g\frac{\partial F}{\partial \dot{x}} + \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}\dot{x} = 0. \tag{14}$$

Then, provided $g > 0$, we get an equation

$$\frac{\partial f}{\partial \dot{x}}F + \frac{\partial F}{\partial \dot{x}} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\dot{x} = 0 \tag{15}$$

for a function $f = \ln g$. (14) is a partial differential equation for g; such equations always have solutions, and can be solved by standard methods.

2 Energy Lagrangians

Suppose we have a system of functions $h = h_{jk}(x^i, \dot{x}^i)$, such that $h_{jk} = h_{kj}$, defined on an open set $U \times \mathbf{R}^m$, where U is an open set in $\mathbf{R}^m$; when no misunderstanding can arise we call g a *metric* on $U \times \mathbf{R}^m$. Consider a variational principle for curves in $\mathbf{R}^m$, defined by the Lagrangian

$$\mathscr{L}_h = \frac{1}{2}h_{ij}\dot{x}^i\dot{x}^j \tag{1}$$

We call $\mathscr{L}_h$ the *energy Lagrangian*, associated with the metric h. We introduce a system of functions C_{ijk} by

$$C_{ijk} = \frac{1}{3}\left(\frac{\partial h_{ij}}{\partial \dot{x}^k} + \frac{\partial h_{jk}}{\partial \dot{x}^i} + \frac{\partial h_{ki}}{\partial \dot{x}^j}\right). \tag{2}$$

The system of functions $C = C_{ijk}$ is called the *Cartan tensor*, associated with h (or with the energy Lagrangian $\mathscr{L}_h$). C_{ijk} is defined by the decomposition

$$\frac{\partial h_{ij}}{\partial \dot{x}^k} = C_{ijk} - \frac{1}{3}\left(\frac{\partial h_{jk}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^k}\right) - \frac{1}{3}\left(\frac{\partial h_{ki}}{\partial \dot{x}^j} - \frac{\partial h_{ij}}{\partial \dot{x}^k}\right). \tag{3}$$

Lemma 1 *The Euler-Lagrange expressions of the Lagrangian* (1) *are*

$$\begin{aligned}
\frac{\partial \mathscr{L}_h}{\partial x^k} - \frac{d}{dt}\frac{\partial \mathscr{L}_h}{\partial \dot{x}^k} = {} & -\frac{1}{2}\left(\frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k}\right)\dot{x}^i \dot{x}^j \\
& - \frac{1}{2}\frac{\partial C_{ijk}}{\partial x^s}\dot{x}^s \dot{x}^i \dot{x}^j - \frac{1}{2}\frac{\partial C_{ijk}}{\partial \dot{x}^s}\dot{x}^i \dot{x}^j \ddot{x}^s - 2C_{isk}\dot{x}^i \ddot{x}^s \\
& + \frac{1}{3}\frac{\partial}{\partial x^s}\left(\frac{\partial h_{jk}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^k}\right)\dot{x}^s \dot{x}^i \dot{x}^j + \frac{1}{3}\frac{\partial}{\partial \dot{x}^s}\left(\frac{\partial h_{jk}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^k}\right)\dot{x}^i \dot{x}^j \ddot{x}^s \\
& - \frac{1}{3}\left(\frac{\partial h_{is}}{\partial \dot{x}^k} - \frac{\partial h_{sk}}{\partial \dot{x}^i} + \frac{\partial h_{ik}}{\partial \dot{x}^s} - \frac{\partial h_{sk}}{\partial \dot{x}^i}\right)\dot{x}^i \ddot{x}^s - h_{sk}\ddot{x}^s.
\end{aligned} \tag{4}$$

Proof Differentiating (1) we have

$$\frac{\partial \mathscr{L}_h}{\partial x^k} = \frac{1}{2}\frac{\partial h_{ij}}{\partial x^k}\dot{x}^i \dot{x}^j, \quad \frac{\partial \mathscr{L}_h}{\partial \dot{x}^k} = \frac{1}{2}\frac{\partial h_{ij}}{\partial \dot{x}^k}\dot{x}^i \dot{x}^j + h_{ik}\dot{x}^i, \tag{5}$$

and

$$\begin{aligned}
\frac{\partial \mathscr{L}_h}{\partial x^k} - \frac{d}{dt}\frac{\partial \mathscr{L}_h}{\partial \dot{x}^k} = {} & \frac{1}{2}\frac{\partial h_{ij}}{\partial x^k}\dot{x}^i \dot{x}^j - \frac{1}{2}\left(\frac{\partial^2 h_{ij}}{\partial x^s \partial \dot{x}^k}\dot{x}^s + \frac{\partial^2 h_{ij}}{\partial \dot{x}^s \partial \dot{x}^k}\ddot{x}^s\right)\dot{x}^i \dot{x}^j \\
& - \frac{\partial h_{ij}}{\partial \dot{x}^k}\dot{x}^i \ddot{x}^j - \left(\frac{\partial h_{ik}}{\partial x^s}\dot{x}^s + \frac{\partial h_{ik}}{\partial \dot{x}^s}\ddot{x}^s\right)\dot{x}^i - h_{ik}\ddot{x}^i \\
= {} & -\frac{1}{2}\left(\frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k}\right)\dot{x}^i \dot{x}^j - \frac{1}{2}\frac{\partial^2 h_{ij}}{\partial x^s \partial \dot{x}^k}\dot{x}^s \dot{x}^i \dot{x}^j \\
& - \left(\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial \dot{x}^s \partial \dot{x}^k}\dot{x}^i \dot{x}^j + \left(\frac{\partial h_{is}}{\partial \dot{x}^k} + \frac{\partial h_{ik}}{\partial \dot{x}^s}\right)\dot{x}^i + h_{sk}\right)\ddot{x}^s.
\end{aligned} \tag{6}$$

Since from (3)

$$
\begin{aligned}
&\frac{\partial h_{is}}{\partial \dot{x}^k}+\frac{\partial h_{ik}}{\partial \dot{x}^s}=C_{isk}-\frac{1}{3}\left(\frac{\partial h_{sk}}{\partial \dot{x}^i}-\cancel{\frac{\partial h_{is}}{\partial \dot{x}^k}}\right)-\frac{1}{3}\left(\cancel{\frac{\partial h_{ki}}{\partial \dot{x}^s}}-\frac{\partial h_{is}}{\partial \dot{x}^k}\right)\\
(7)\qquad &+C_{iks}-\frac{1}{3}\left(\frac{\partial h_{ks}}{\partial \dot{x}^i}-\frac{\partial h_{ik}}{\partial \dot{x}^s}\right)-\frac{1}{3}\left(\cancel{\frac{\partial h_{si}}{\partial \dot{x}^k}}-\cancel{\frac{\partial h_{ik}}{\partial \dot{x}^s}}\right)\\
&=2C_{isk}+\frac{1}{3}\left(\frac{\partial h_{ik}}{\partial \dot{x}^s}+\frac{\partial h_{is}}{\partial \dot{x}^k}-2\frac{\partial h_{ks}}{\partial \dot{x}^i}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{\partial \mathscr{L}_h}{\partial x^k}-\frac{d}{dt}\frac{\partial \mathscr{L}_h}{\partial \dot{x}^k}=-\frac{1}{2}\left(\frac{\partial h_{ik}}{\partial x^j}+\frac{\partial h_{jk}}{\partial x^i}-\frac{\partial h_{ij}}{\partial x^k}\right)\dot{x}^i\dot{x}^j\\
&-\frac{1}{2}\frac{\partial}{\partial x^s}\left(C_{ijk}-\frac{1}{3}\left(\frac{\partial h_{jk}}{\partial \dot{x}^i}-\frac{\partial h_{ij}}{\partial \dot{x}^k}\right)-\frac{1}{3}\left(\frac{\partial h_{ki}}{\partial \dot{x}^j}-\frac{\partial h_{ij}}{\partial \dot{x}^k}\right)\right)\dot{x}^s\dot{x}^i\dot{x}^j\\
&-\frac{1}{2}\frac{\partial}{\partial \dot{x}^s}\left(C_{ijk}-\frac{1}{3}\left(\frac{\partial h_{jk}}{\partial \dot{x}^i}-\frac{\partial h_{ij}}{\partial \dot{x}^k}\right)-\frac{1}{3}\left(\frac{\partial h_{ki}}{\partial \dot{x}^j}-\frac{\partial h_{ij}}{\partial \dot{x}^k}\right)\right)\dot{x}^i\dot{x}^j\ddot{x}^s\\
&-\left(\left(2C_{isk}+\frac{1}{3}\left(\frac{\partial h_{ik}}{\partial \dot{x}^s}+\frac{\partial h_{is}}{\partial \dot{x}^k}-2\frac{\partial h_{ks}}{\partial \dot{x}^i}\right)\right)\dot{x}^i\right)\ddot{x}^s-h_{sk}\ddot{x}^s\\
(8)\qquad &=-\frac{1}{2}\left(\frac{\partial h_{ik}}{\partial x^j}+\frac{\partial h_{jk}}{\partial x^i}-\frac{\partial h_{ij}}{\partial x^k}\right)\dot{x}^i\dot{x}^j\\
&-2C_{isk}\dot{x}^i\ddot{x}^s-\frac{1}{2}\frac{\partial C_{ijk}}{\partial x^s}\dot{x}^s\dot{x}^i\dot{x}^j-\frac{1}{2}\frac{\partial C_{ijk}}{\partial \dot{x}^s}\dot{x}^i\dot{x}^j\ddot{x}^s\\
&+\frac{1}{3}\frac{\partial}{\partial x^s}\left(\frac{\partial h_{ki}}{\partial \dot{x}^j}-\frac{\partial h_{ij}}{\partial \dot{x}^k}\right)\dot{x}^s\dot{x}^i\dot{x}^j+\frac{1}{3}\frac{\partial}{\partial \dot{x}^s}\left(\frac{\partial h_{jk}}{\partial \dot{x}^i}-\frac{\partial h_{ij}}{\partial \dot{x}^k}\right)\dot{x}^i\dot{x}^j\ddot{x}^s\\
&-\frac{1}{3}\left(\frac{\partial h_{ik}}{\partial \dot{x}^s}+\frac{\partial h_{is}}{\partial \dot{x}^k}-2\frac{\partial h_{ks}}{\partial \dot{x}^i}\right)\dot{x}^i\ddot{x}^s-h_{sk}\ddot{x}^s.
\end{aligned}
$$

3 Integrability conditions

In this section we recall elementary theorems on integration of differential equations, appearing in this paper; essentially, we need simple systems of Frobenius type in Euclidean spaces $\mathbf{R}^n$. All functions we consider are defined on a star-shaped neighbourhood U of the origin $0\in\mathbf{R}^n$.

Suppose we have a system of functions $A=A_k$, $1\le k\le n$ defined on U, and consider the differential equations

$$A_k = \frac{\partial P}{\partial x^k} \tag{1}$$

for an unknown function P.

Lemma 2 (a) *Equation* (1) *has a solution P if and only if the functions* A_k *satisfy*

$$\frac{\partial A_k}{\partial x^l} - \frac{\partial A_l}{\partial x^k} = 0. \tag{2}$$

(b) *If condition* (2) *is satisfied, then a solution P is given by*

$$P = x^k \int_0^1 A_k(\tau x^l)\, d\tau. \tag{3}$$

Proof Necessity of condition (2) is obvious. To prove sufficiency, we differentiate P with respect to x^i. We have

$$\begin{aligned} \frac{\partial P}{\partial x^p} &= \int_0^1 A_p(\tau x^l)\, d\tau + x^k \int_0^1 \left(\frac{\partial A_k}{\partial x^p} \right)_{\tau x^l} \tau\, d\tau \\ &= \int_0^1 A_p(\tau x^l)\, d\tau + x^k \int_0^1 \left(\frac{\partial A_p}{\partial x^k} \right)_{\tau x^l} \tau\, d\tau \\ &= \int_0^1 \frac{d}{d\tau}(A_p(\tau x^l)\tau)\, d\tau = A_p(x^l). \end{aligned} \tag{4}$$

Remark 2 In case we have a *system* of differential equations of the form

$$A_{(\alpha)k} = \frac{\partial P_{(\alpha)}}{\partial x^k}, \tag{5}$$

criterion (2) applies to each equation separately; we have

$$\frac{\partial A_{(\alpha)k}}{\partial x^l} - \frac{\partial A_{(\alpha)l}}{\partial x^k} = 0. \tag{6}$$

Now suppose we have a system of functions $A = A_{kl}$ defined on U, such that

(7) $A_{kl} = -A_{lk}$.

Consider the differential equations

(8) $$A_{kl} = \frac{\partial Q_l}{\partial x^k} - \frac{\partial Q_k}{\partial x^l}$$

for unknown system of functions $Q = Q_l$.

Lemma 3 (a) *Equation* (8) *has a solution* Q *if and only if the functions* A_{kl} *satisfy*

(9) $$\frac{\partial A_{ks}}{\partial x^l} + \frac{\partial A_{sl}}{\partial x^k} + \frac{\partial A_{lk}}{\partial x^s} = 0.$$

(b) *If condition* (7) *is satisfied, then a solution Q is given by*

(10) $$Q_l = x^p \int_0^1 A_{pl}(\tau x^i)\tau \, d\tau.$$

Proof Necessity of condition (9) is immediate. To prove sufficiency, we differentiate Q_l with respect to x^k. We have

(11) $$\frac{\partial Q_l}{\partial x^k} = \int_0^1 A_{kl}(\tau x^i)\tau \, d\tau + x^p \int_0^1 \left(\frac{\partial A_{pl}}{\partial x^k} \right)_{\tau x^i} \tau^2 \, d\tau,$$

and

(12) $$\begin{aligned} \frac{\partial Q_l}{\partial x^k} - \frac{\partial Q_k}{\partial x^l} &= \int_0^1 A_{kl}(\tau x^i)\tau \, d\tau + x^p \int_0^1 \left(\frac{\partial A_{pl}}{\partial x^k} \right)_{\tau x^l} \tau^2 \, d\tau \\ &\quad - \int_0^1 A_{lk}(\tau x^i)\tau \, d\tau - x^p \int_0^1 \left(\frac{\partial A_{pk}}{\partial x^l} \right)_{\tau x^l} \tau^2 \, d\tau \\ &= 2\int_0^1 A_{kl}(\tau x^i)\tau \, d\tau + x^p \int_0^1 \left(\frac{\partial A_{pl}}{\partial x^k} - \frac{\partial A_{pk}}{\partial x^l} \right)_{\tau x^l} \tau^2 \, d\tau \\ &= 2\int_0^1 A_{kl}(\tau x^i)\tau \, d\tau + x^p \int_0^1 \left(\frac{\partial A_{pl}}{\partial x^k} + \frac{\partial A_{kp}}{\partial x^l} + \frac{\partial A_{lk}}{\partial x^p} \right)_{\tau x^l} \tau^2 \, d\tau \\ &\quad - x^p \int_0^1 \left(\frac{\partial A_{lk}}{\partial x^p} \right)_{\tau x^l} \tau^2 \, d\tau. \end{aligned}$$

This formula can also be expressed in the form

$$\begin{aligned}\frac{\partial Q_l}{\partial x^k}-\frac{\partial Q_k}{\partial x^l}&=\int_0^1\left(\left(\frac{\partial A_{kl}}{\partial x^p}\right)_{\tau x^i}x^p\tau^2+2A_{kl}(\tau x^i)\tau\right)d\tau\\&=\int_0^1\frac{d}{d\tau}(A_{kl}(\tau x^i)\tau^2)=A_{kl}(x^i).\end{aligned}\tag{13}$$

4 Variational systems of differential equations and the Helmholtz conditions

Let $\varepsilon=\varepsilon_i$ be a system of functions $\varepsilon_i=\varepsilon_i(t,x^j,\dot{x}^j,\ddot{x}^j)$. We shall say that this system is *variational*, if there exists a function $\mathscr{L}=\mathscr{L}(t,x^j,\dot{x}^j)$ such that

$$\varepsilon_i=\frac{\partial\mathscr{L}}{\partial x^i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}^i}.\tag{1}$$

We give a straightforward proof of the following necessary and sufficient conditions for ε to be variational.

Lemma 4 *The system* ε *is variational if and only if*

$$\frac{\partial\varepsilon_i}{\partial\ddot{x}^l}-\frac{\partial\varepsilon_l}{\partial\ddot{x}^i}=0,\tag{2}$$

$$\frac{\partial\varepsilon_i}{\partial\dot{x}^l}+\frac{\partial\varepsilon_l}{\partial\dot{x}^i}-\frac{d}{dt}\left(\frac{\partial\varepsilon_i}{\partial\ddot{x}^l}+\frac{\partial\varepsilon_l}{\partial\ddot{x}^i}\right)=0,\tag{3}$$

and

$$\frac{\partial\varepsilon_i}{\partial x^l}-\frac{\partial\varepsilon_l}{\partial x^i}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial\varepsilon_i}{\partial\dot{x}^l}-\frac{\partial\varepsilon_l}{\partial\dot{x}^i}\right)=0.\tag{4}$$

Proof 1. We show that if ε_i are expressible in the form (1), then conditions (2), (3), and (4) hold. Using explicit expressions

$$\varepsilon_i=\frac{\partial\mathscr{L}}{\partial x^i}-\frac{\partial^2\mathscr{L}}{\partial t\,\partial\dot{x}^i}-\frac{\partial^2\mathscr{L}}{\partial x^k\,\partial\dot{x}^i}\dot{x}^k-\frac{\partial^2\mathscr{L}}{\partial\dot{x}^k\,\partial\dot{x}^i}\ddot{x}^k\tag{5}$$

we get

$$
\begin{aligned}
&\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} = -\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^l \,\partial \dot{x}^i}, \\
&\frac{\partial \varepsilon_i}{\partial \dot{x}^l} = \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^l \,\partial x^i} - \frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l \,\partial t \,\partial \dot{x}^i} - \frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l \,\partial x^k \,\partial \dot{x}^i}\dot{x}^k - \frac{\partial^2 \mathscr{L}}{\partial x^l \,\partial \dot{x}^i} \\
&\qquad - \frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l \,\partial \dot{x}^k \,\partial \dot{x}^i}\ddot{x}^k, \\
&\frac{\partial \varepsilon_i}{\partial x^l} = \frac{\partial^2 \mathscr{L}}{\partial x^l \,\partial x^i} - \frac{\partial^3 \mathscr{L}}{\partial x^l \,\partial t \,\partial \dot{x}^i} - \frac{\partial^3 \mathscr{L}}{\partial x^l \,\partial x^k \,\partial \dot{x}^i}\dot{x}^k - \frac{\partial^3 \mathscr{L}}{\partial x^l \,\partial \dot{x}^k \,\partial \dot{x}^i}\ddot{x}^k.
\end{aligned} \tag{6}
$$

Hence

$$
\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} - \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} = -\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^j \,\partial \dot{x}^i} + \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^j \,\partial \dot{x}^i} = 0, \tag{7}
$$

and

$$
\begin{aligned}
&\frac{\partial \varepsilon_i}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i}\right) \\
&\quad = \cancel{\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^l \,\partial x^i}} - \bcancel{\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l \,\partial t \,\partial \dot{x}^i}} - \bcancel{\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l \,\partial x^k \,\partial \dot{x}^i}}\dot{x}^k - \cancel{\frac{\partial^2 \mathscr{L}}{\partial x^l \,\partial \dot{x}^i}} \\
&\quad - \xcancel{\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^l \,\partial \dot{x}^k \,\partial \dot{x}^i}}\ddot{x}^k + \cancel{\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^i \,\partial x^l}} - \bcancel{\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^i \,\partial t \,\partial \dot{x}^l}} \\
&\quad - \bcancel{\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^i \,\partial x^k \,\partial \dot{x}^l}}\dot{x}^k - \cancel{\frac{\partial^2 \mathscr{L}}{\partial x^i \,\partial \dot{x}^l}} - \xcancel{\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^i \,\partial \dot{x}^k \,\partial \dot{x}^l}}\ddot{x}^k \\
&\quad + 2\left(\bcancel{\frac{\partial^3 \mathscr{L}}{\partial t \,\partial \dot{x}^l \,\partial \dot{x}^i}} + \bcancel{\frac{\partial^3 \mathscr{L}}{\partial x^k \,\partial \dot{x}^l \,\partial \dot{x}^i}}\dot{x}^k + \xcancel{\frac{\partial^3 \mathscr{L}}{\partial \dot{x}^k \,\partial \dot{x}^l \,\partial \dot{x}^i}}\ddot{x}^k\right) = 0.
\end{aligned} \tag{8}
$$

Analogously

$$
\begin{aligned}
&\frac{\partial \varepsilon_i}{\partial x^l} - \frac{\partial \varepsilon_l}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i}\right) \\
&\quad = \frac{\partial^2 \mathscr{L}}{\partial x^l \,\partial x^i} - \frac{\partial^3 \mathscr{L}}{\partial x^l \,\partial t \,\partial \dot{x}^i} - \frac{\partial^3 \mathscr{L}}{\partial x^l \,\partial x^k \,\partial \dot{x}^i}\dot{x}^k - \frac{\partial^3 \mathscr{L}}{\partial x^l \,\partial \dot{x}^k \,\partial \dot{x}^i}\ddot{x}^k \\
&\quad - \frac{\partial^2 \mathscr{L}}{\partial x^i \,\partial x^l} + \frac{\partial^3 \mathscr{L}}{\partial x^i \,\partial t \,\partial \dot{x}^l} + \frac{\partial^3 \mathscr{L}}{\partial x^i \,\partial x^k \,\partial \dot{x}^l}\dot{x}^k + \frac{\partial^3 \mathscr{L}}{\partial x^i \,\partial \dot{x}^k \,\partial \dot{x}^l}\ddot{x}^k
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\Bigg(\Bigg(\boxed{\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i}} - \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial t \partial \dot{x}^i}} - \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial x^k \partial \dot{x}^i}\dot{x}^k} - \boxed{\frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i}} \\
&- \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k \partial \dot{x}^i}\ddot{x}^k} - \boxed{\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial x^l}} + \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial t \partial \dot{x}^l}} \\
&+ \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial x^k \partial \dot{x}^l}\dot{x}^k} + \boxed{\frac{\partial^2 \mathcal{L}}{\partial x^i \partial \dot{x}^l}} + \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^k \partial \dot{x}^l}\ddot{x}^k}\Bigg) \\
&- \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k \partial \dot{x}^i}\ddot{x}^k} - \boxed{\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial x^l}} + \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial t \partial \dot{x}^l}} \\
&+ \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial x^k \partial \dot{x}^l}\dot{x}^k} + \boxed{\frac{\partial^2 \mathcal{L}}{\partial x^i \partial \dot{x}^l}} + \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^k \partial \dot{x}^l}\ddot{x}^k}\Bigg) \\
&= \cancel{\frac{\partial^2 \mathcal{L}}{\partial x^l \partial x^i}} - \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial t \partial \dot{x}^i}} - \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial x^k \partial \dot{x}^i}\dot{x}^k} - \boxed{\frac{\partial^3 \mathcal{L}}{\partial x^l \partial \dot{x}^k \partial \dot{x}^i}\ddot{x}^k} \\
&- \cancel{\frac{\partial^2 \mathcal{L}}{\partial x^i \partial x^l}} + \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial t \partial \dot{x}^l}} + \cancel{\frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial x^k \partial \dot{x}^l}\dot{x}^k} + \boxed{\frac{\partial^3 \mathcal{L}}{\partial x^i \partial \dot{x}^k \partial \dot{x}^l}\ddot{x}^k} \\
&- \frac{\partial}{\partial t}\left(\cancel{\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i}} - \cancel{\frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i}}\right) - \frac{\partial}{\partial x^k}\left(\cancel{\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i}} - \cancel{\frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i}}\right)\dot{x}^k \\
&- \frac{\partial}{\partial \dot{x}^k}\left(\boxed{\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i}} - \boxed{\frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i}}\right)\ddot{x}^k = 0.
\end{aligned}
$$

2. Conversely, we know that conditions (2), (3) and (4) ensure existence of a second-order Lagrangian $\mathcal{H} = \mathcal{H}(t, x^j, \dot{x}^j, \ddot{x}^j)$ such that

$$
\varepsilon_i = \frac{\partial \mathcal{H}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{H}}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial \mathcal{H}}{\partial \ddot{x}^i} \tag{10}
$$

(the *Vainberg-Tonti Lagrangian*, see e.g. Krupka, Saunders 2008 [1]); the right-hand side is polynomial in the variables $\ddot{x}^j$, $\dddot{x}^j$, but ε_i depends on $x^j, \dot{x}^j, \ddot{x}^j$ only. Since

$$
\begin{aligned}
\varepsilon_i = {} & \frac{\partial \mathcal{H}}{\partial x^i} - \frac{\partial^2 \mathcal{H}}{\partial x^j \partial \dot{x}^i}\dot{x}^j - \frac{\partial^2 \mathcal{H}}{\partial \dot{x}^j \partial \dot{x}^i}\ddot{x}^j - \frac{\partial^2 \mathcal{H}}{\partial \ddot{x}^j \partial \dot{x}^i}\dddot{x}^j \\
& + \frac{d}{dt}\left(\frac{\partial^2 \mathcal{H}}{\partial x^j \partial \ddot{x}^i}\dot{x}^j + \frac{\partial^2 \mathcal{H}}{\partial \dot{x}^j \partial \ddot{x}^i}\ddot{x}^j + \frac{\partial^2 \mathcal{H}}{\partial \ddot{x}^j \partial \ddot{x}^i}\dddot{x}^j\right)
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&= \frac{\partial \mathcal{K}}{\partial x^i} - \frac{\partial^2 \mathcal{K}}{\partial x^j \partial \dot{x}^i}\dot{x}^j - \frac{\partial^2 \mathcal{K}}{\partial \dot{x}^j \partial \dot{x}^i}\ddot{x}^j - \frac{\partial^2 \mathcal{K}}{\partial \ddot{x}^j \partial \dot{x}^i}\dddot{x}^j \\
&+ \frac{d}{dt}\frac{\partial^2 \mathcal{K}}{\partial x^j \partial \ddot{x}^i}\dot{x}^j + \frac{\partial^2 \mathcal{K}}{\partial x^j \partial \ddot{x}^i}\ddot{x}^j + \frac{d}{dt}\frac{\partial^2 \mathcal{K}}{\partial \dot{x}^j \partial \ddot{x}^i}\ddot{x}^j + \frac{\partial^2 \mathcal{K}}{\partial \dot{x}^j \partial \ddot{x}^i}\dddot{x}^j \\
&+ \xcancel{\frac{d}{dt}\frac{\partial^2 \mathcal{K}}{\partial \ddot{x}^j \partial \ddot{x}^i}\dddot{x}^j + \frac{\partial^2 \mathcal{K}}{\partial \ddot{x}^j \partial \ddot{x}^i}\ddddot{x}^j}
\end{aligned}
$$

then

$$\text{(12)} \qquad \frac{\partial^2 \mathcal{K}}{\partial \ddot{x}^j \partial \ddot{x}^i} = 0,$$

hence

$$\text{(13)} \qquad -\frac{\partial^2 \mathcal{K}}{\partial \ddot{x}^j \partial \dot{x}^i} + \frac{\partial^2 \mathcal{K}}{\partial \dot{x}^j \partial \ddot{x}^i} = 0.$$

Conditions (12) and (13) imply

$$\text{(14)} \qquad \mathcal{K} = A + B_i \ddot{x}^i, \quad -\frac{\partial B_j}{\partial \dot{x}^i} + \frac{\partial B_i}{\partial \dot{x}^j} = 0, \quad B_i = \frac{\partial C}{\partial \dot{x}^i},$$

hence

$$
\text{(15)} \qquad
\begin{aligned}
\mathcal{K} &= A + \frac{\partial C}{\partial \dot{x}^i}\ddot{x}^i = A - \frac{\partial C}{\partial x^i}\dot{x}^i + \frac{\partial C}{\partial x^i}\dot{x}^i + \frac{\partial C}{\partial \dot{x}^i}\ddot{x}^i \\
&= A - \frac{\partial C}{\partial x^i}\dot{x}^i + \frac{dC}{dt} = \mathcal{K}_0 + \frac{dC}{dt},
\end{aligned}
$$

where

$$\text{(16)} \qquad \mathcal{K}_0 = A - \frac{\partial C}{\partial x^i}\dot{x}^i.$$

Then, however, since on total derivatives the Euler-Lagrange expressions vanish,

$$\text{(17)} \qquad \varepsilon_i = \frac{\partial \mathcal{K}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial \mathcal{K}}{\partial \ddot{x}^i} = \frac{\partial \mathcal{K}_0}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{K}_0}{\partial \dot{x}^i},$$

and $\mathcal{K}_0$ is a first order Lagrangian for ε.

Equations (2), (3), (4) are called the *Helmholtz conditions*.

Notice a special case when ε_i does not depend on $\ddot{x}^s$. Applying Lemma 4, we get the following assertion.

Lemma 5 *Let* $\varepsilon = \varepsilon_i(t, x^j, \dot{x}^j)$ *be a system of functions. The following three conditions are equivalent:*

(a) *The system* ε *is variational.*

(b) *The functions* ε_i *satisfy*

$$\frac{\partial \varepsilon_i}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^i} = 0, \tag{18}$$

and

$$\frac{\partial \varepsilon_i}{\partial x^l} - \frac{\partial \varepsilon_l}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i}\right) = 0. \tag{19}$$

(c) *The functions* ε_i *are of the form*

$$\varepsilon_k = \frac{\partial P}{\partial x^k} + \left(\frac{\partial Q_l}{\partial x^k} - \frac{\partial Q_k}{\partial x^l}\right)\dot{x}^l, \tag{20}$$

where $P = P(x^l)$ *and* $Q_k = Q_k(x^l)$ *are arbitrary functions. The Lagrangian for these Euler-Lagrange expressions is*

$$\mathscr{L} = P + Q_l \dot{x}^l. \tag{21}$$

Proof 1. Equivalence of (a) and (b) follows from Lemma 4.

2. We show that (b) implies (c). Equations (18) and (19) reduce to the subsystems

$$\begin{gathered}\frac{\partial \varepsilon_k}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^k} = 0, \quad \frac{\partial \varepsilon_k}{\partial x^l} - \frac{\partial \varepsilon_l}{\partial x^k} - \frac{1}{2}\frac{\partial}{\partial x^s}\left(\frac{\partial \varepsilon_k}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^k}\right)\dot{x}^s = 0,\\ \frac{\partial}{\partial \dot{x}^s}\left(\frac{\partial \varepsilon_k}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^k}\right) = 0.\end{gathered} \tag{22}$$

The first subsystem yields

$$\frac{\partial^2 \varepsilon_k}{\partial \dot{x}^j \partial \dot{x}^l} = -\frac{\partial^2 \varepsilon_l}{\partial \dot{x}^j \partial \dot{x}^k} = \frac{\partial^2 \varepsilon_j}{\partial \dot{x}^l \partial \dot{x}^k} = -\frac{\partial^2 \varepsilon_k}{\partial \dot{x}^l \partial \dot{x}^j} = 0, \tag{23}$$

hence $\varepsilon_k = A_k + A_{ks}\dot{x}^s$, where $A_{kl} + A_{lk} = 0$. Then the second subsystem reads

$$
\begin{aligned}
(24)\qquad & \frac{\partial A_k}{\partial x^l}+\frac{\partial A_{ks}}{\partial x^l}\dot{x}^s-\frac{\partial A_l}{\partial x^k}-\frac{\partial A_{ls}}{\partial x^k}\dot{x}^s-\frac{1}{2}\frac{\partial}{\partial x^s}(A_{kl}-A_{lk})\dot{x}^s \\
& =\frac{\partial A_k}{\partial x^l}-\frac{\partial A_l}{\partial x^k}+\left(\frac{\partial A_{ks}}{\partial x^l}-\frac{\partial A_{ls}}{\partial x^k}-\frac{\partial A_{kl}}{\partial x^s}\right)\dot{x}^s=0,
\end{aligned}
$$

hence

$$(25)\qquad \frac{\partial A_k}{\partial x^l}-\frac{\partial A_l}{\partial x^k}=0,\quad \frac{\partial A_{ks}}{\partial x^l}+\frac{\partial A_{sl}}{\partial x^k}+\frac{\partial A_{lk}}{\partial x^s}=0.$$

These equations ensure existence of functions P and Q_k such that

$$(26)\qquad A_k=\frac{\partial P}{\partial x^k},$$

and

$$(27)\qquad A_{ks}=\frac{\partial Q_s}{\partial x^k}-\frac{\partial Q_k}{\partial x^s}$$

(Lemma 2 and Lemma 3).

5 The Douglas's problem

Suppose we are given two systems of functions $g=g_{ij}(x^k,\dot{x}^k)$ and $F=F^j(x^k,\dot{x}^k)$, defined on a set $U\times V\subset\mathbf{R}^n\times\mathbf{R}^n$, where V is a star-shaped neighbourhood of the origin $0\in\mathbf{R}^n$. Consider the Sonin-Douglas's system of differential equations

$$(1)\qquad g_{ij}(\ddot{x}^j-F^j)=-\frac{\partial\mathscr{L}}{\partial x^i}+\frac{\partial^2\mathscr{L}}{\partial x^j\,\partial\dot{x}^i}\dot{x}^j+\frac{\partial^2\mathscr{L}}{\partial\dot{x}^j\,\partial\dot{x}^i}\ddot{x}^j$$

as a system for an unknown function $\mathscr{L}$. Clearly, in general, this system need not have a solution for given g and F; existence of a solution implies *integrability conditions*, satisfied by g_{ij} and F^j, and *vice versa*. Our main objective in this section is to determine integrability conditions for the pair (g,F), ensuring existence of $\mathscr{L}$.

Equation (1) is equivalent with two equations

$$(2)\qquad g_{ij}=\frac{\partial^2\mathscr{L}}{\partial\dot{x}^j\,\partial\dot{x}^i}$$

and

(3) $$g_{ij}F^j = \frac{\partial \mathscr{L}}{\partial x^i} - \frac{\partial^2 \mathscr{L}}{\partial x^j \partial \dot{x}^i}\dot{x}^j .$$

First we solve the system (2).

Lemma 6 (a) *Equation* (2) *has a solution* $\mathscr{L}$ *if and only if the functions* g_{ij} *satisfy*

(4) $$g_{ij} = g_{ij}, \quad \frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j} .$$

(b) *If the functions* g_{ij} *satisfy conditions* (4), *then every solution* $\mathscr{L}$ *of equation* (2) *is of the form*

(5) $$\mathscr{L} = \mathscr{L}_h + \mathscr{L}_0 ,$$

where

(6) $$\mathscr{L}_h = \frac{1}{2}h_{ij}\dot{x}^i\dot{x}^j , \quad h_{ij} = 2\int_0^1 \left(\int_0^1 g_{ij}(x^p, \kappa\tau\dot{x}^p)\,d\kappa\right)\tau\,dt ,$$

the functions h_{ij} *satisfy*

(7) $$h_{ij} = h_{ij}, \quad \frac{\partial h_{ij}}{\partial \dot{x}^k} = \frac{\partial h_{ik}}{\partial \dot{x}^j} ,$$

and

(8) $$\mathscr{L}_0 = A + B_i\dot{x}^i ,$$

where $A = A(x^k)$, $B_i = B_i(x^k)$.

Proof 1. If (4) holds, one can easily determine all solutions $\mathscr{L}$ of (2). (4) implies that

(9) $$g_{ij} = \frac{\partial h_i}{\partial \dot{x}^j}$$

for some functions h_i; h_i can be taken as

(10) $$h_i = \dot{x}^r \int_0^1 g_{ir}(x^p, \kappa\dot{x}^p)\,d\kappa .$$

Indeed, h_i obviously satisfies (9):

$$
(11)\qquad
\begin{aligned}
\left(\frac{\partial h_i}{\partial \dot{x}^j}\right)_{(x^p,\dot{x}^p)} &= \int_0^1 g_{ij}(x^p,\kappa\dot{x}^p)\,d\kappa + \dot{x}^r\int_0^1\left(\frac{\partial g_{ir}}{\partial \dot{x}^j}\right)_{(x^p,\kappa\dot{x}^p)}\kappa\,d\kappa \\
&= \int_0^1\left(g_{ij}(x^p,\kappa\dot{x}^p) + \left(\frac{\partial g_{ir}}{\partial \dot{x}^j}\right)_{(x^p,\kappa\dot{x}^p)}\kappa\dot{x}^r\right)d\kappa \\
&= \int_0^1\left(g_{ij}(x^p,\kappa\dot{x}^p) + \left(\frac{\partial g_{ij}}{\partial \dot{x}^r}\right)_{(x^p,\kappa\dot{x}^p)}\kappa\dot{x}^r\right)d\kappa \\
&= \int_0^1 \frac{d}{d\kappa}(g_{ij}(x^p,\kappa\dot{x}^p)\kappa)\,d\kappa = g_{ij}(x^p,\dot{x}^p).
\end{aligned}
$$

Now we apply condition $g_{ij} = g_{ij}$ (4). We get the integrability condition

$$(12)\qquad \frac{\partial h_i}{\partial \dot{x}^j} = \frac{\partial h_j}{\partial \dot{x}^i},$$

ensuring existence of a function L such that

$$(13)\qquad h_i = -\frac{\partial L}{\partial \dot{x}^i}$$

(with minus sign for convenience). A solution may be taken as

$$(14)\qquad L = \dot{x}^i\int_0^1 h_i(x^p,\tau\dot{x}^p)\,d\tau.$$

Substituting from (10)

$$(15)\qquad h_i(x^p,\tau\dot{x}^p) = \tau\dot{x}^r\int_0^1 g_{ir}(x^p,\kappa\tau\dot{x}^p)\,d\kappa,$$

we get

$$
(16)\qquad
\begin{aligned}
L &= \dot{x}^i\int_0^1 h_i(x^p,\tau\dot{x}^p)\,d\tau = \dot{x}^i\int_0^1\left(\tau\dot{x}^r\int_0^1 g_{ir}(x^p,\kappa\tau\dot{x}^p)\,d\kappa\right)d\tau \\
&= \dot{x}^i\dot{x}^j\int_0^1\left(\int_0^1 g_{ij}(x^p,\kappa\tau\dot{x}^p)\,d\kappa\right)\tau\,d\tau \\
&= \frac{1}{2}h_{ij}\dot{x}^i\dot{x}^j,
\end{aligned}
$$

where

(17) $$h_{ij} = 2\int_0^1 \left(\int_0^1 g_{ij}(x^p, \kappa\tau\dot{x}^p)\, d\kappa \right) \tau\, d\tau.$$

By construction (16), L coincides with the energy Lagrangian $\mathscr{L}_h$ of the metric $h = h_{ij}$, and satisfies

(18) $$g_{ij} = \frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^i\, \partial \dot{x}^j},$$

and is equal to the energy Lagrangian $\mathscr{L}_h$. The metric (17) satisfies

(19) $$h_{ij} = h_{ji}, \quad \frac{\partial h_{ij}}{\partial \dot{x}^k} = 2\int_0^1 \left(\int_0^1 \left(\frac{\partial g_{ij}}{\partial \dot{x}^k} \right)_{(x^p, st\dot{x}^p)} \kappa\tau\, d\kappa \right) \tau\, d\tau = \frac{\partial h_{ik}}{\partial \dot{x}^j}.$$

The general solution of equation (2) is

(20) $$\mathscr{L} = \mathscr{L}_h + A + B_i \dot{x}^i,$$

where $A = A(x^j)$, $B_i = B_i(x^j)$ are arbitrary functions.

Remark 3 Suppose that conditions (4) are satisfied,

(21) $$g_{ij} = g_{ij}, \quad \frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j}.$$

Then the Euler-Lagrange expressions of the Lagrangian $\mathscr{L}$ (5) are determined by Lemma 6, (4). Computing from this formula the expression

(22) $$\frac{\partial \mathscr{L}}{\partial x^k} - \frac{\partial^2 \mathscr{L}}{\partial x^j\, \partial \dot{x}^k} \dot{x}^j$$

entering equation (3), we get a *first order* expression

(23) $$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x^k} - \frac{\partial^2 \mathscr{L}}{\partial x^j\, \partial \dot{x}^k} \dot{x}^j &= \frac{\partial \mathscr{L}_h}{\partial x^k} - \frac{\partial^2 \mathscr{L}_h}{\partial x^j\, \partial \dot{x}^k} \dot{x}^j + \frac{\partial A}{\partial x^k} + \left(\frac{\partial B_i}{\partial x^k} - \frac{\partial B_k}{\partial x^i} \right) \dot{x}^i \\ &= -\frac{1}{2} \left(\frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k} \right) \dot{x}^i \dot{x}^j - \frac{1}{2} \frac{\partial C_{ijk}}{\partial x^s} \dot{x}^s \dot{x}^i \dot{x}^j \\ &\quad + \frac{\partial A}{\partial x^k} + \left(\frac{\partial B_i}{\partial x^k} - \frac{\partial B_k}{\partial x^i} \right) \dot{x}^i, \end{aligned}$$

where C_{ijk} is the Cartan tensor of the metric $h = h_{ij}$.

Lemma 6 shows that any solution of the Sonin-Douglas's problem must be of the form

$$\mathscr{L} = \mathscr{L}_h + \mathscr{L}_0, \tag{24}$$

where $\mathscr{L}_h$ is completely determined by the metric g_{ij}, satisfying

$$g_{ij} = g_{ij}, \quad \frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j}. \tag{25}$$

It remains to determine the second summand

$$\mathscr{L}_0 = A + B_i \dot{x}^i, \tag{26}$$

with unknown functions $A = A(x^l)$ and $B_i = B_i(x^l)$. $\mathscr{L}_0$ should be determined from equation (3), which is now of the form

$$g_{ij} F^j = \frac{\partial \mathscr{L}_h}{\partial x^i} - \frac{\partial^2 \mathscr{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j + \frac{\partial \mathscr{L}_0}{\partial x^i} - \frac{\partial^2 \mathscr{L}_0}{\partial x^j \partial \dot{x}^i} \dot{x}^j. \tag{27}$$

Setting

$$f_i = g_{ij} F^j, \quad P_i = f_i - \frac{\partial \mathscr{L}_h}{\partial x^i} + \frac{\partial^2 \mathscr{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j, \tag{28}$$

and substituting into (27) we get an equivalent equation

$$P_i = \frac{\partial A}{\partial x^i} + \left(\frac{\partial B_j}{\partial x^i} - \frac{\partial B_i}{\partial x^j} \right) \dot{x}^j. \tag{29}$$

Lemma 7 *The following conditions are equivalent:*
(a) *Equation* (27) *has a solution* $\mathscr{L}_0$.
(b) *Equation* (29) *has a solution* $A,\ B_l$.
(c) *The system* $P = P_l$ *satisfies*

$$\frac{\partial P_k}{\partial \dot{x}^j} + \frac{\partial P_j}{\partial \dot{x}^k} = 0, \tag{30}$$

and

$$\frac{\partial P_k}{\partial x^l} - \frac{\partial P_l}{\partial x^k} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial P_k}{\partial \dot{x}^l} - \frac{\partial P_l}{\partial \dot{x}^k} \right) \dot{x}^j = 0. \tag{31}$$

(d) *The function* f_i *and* g_{ij} *satisfy*

$$\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^l}+\frac{\partial f_l}{\partial \dot{x}^i}\right)+\frac{\partial g_{il}}{\partial x^j}\dot{x}^j=0, \tag{32}$$

$$\frac{\partial f_i}{\partial x^l}-\frac{\partial f_l}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)\dot{x}^j=0, \tag{33}$$

$$\frac{1}{2}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)+\frac{\partial g_{ij}}{\partial x^l}-\frac{\partial g_{lj}}{\partial x^i}=0. \tag{34}$$

Proof 1. Suppose that condition (a) is satisfied. Then since (27) has a solution $\mathscr{L}_0$, the system $P=P_l$ is variational. But expressions P_l are of the first order, thus by Lemma 2, $\mathscr{L}_0$ may be of the form (26), proving condition (b).

2. Suppose that (b) is satisfied and consider a solution A, B_l of equation (29). Then by a direct computation

$$\frac{\partial P_k}{\partial \dot{x}^l}+\frac{\partial P_l}{\partial \dot{x}^k}=\frac{\partial B_l}{\partial x^k}-\frac{\partial B_k}{\partial x^l}+\frac{\partial B_k}{\partial x^l}-\frac{\partial B_l}{\partial x^k}=0, \tag{35}$$

and

$$\begin{aligned}
&\frac{\partial P_k}{\partial x^l}-\frac{\partial P_l}{\partial x^k}-\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial P_k}{\partial \dot{x}^l}-\frac{\partial P_l}{\partial \dot{x}^k}\right)\dot{x}^j\\
&=\frac{\partial^2 \cancel{A}}{\cancel{\partial x^l}\,\partial x^k}+\frac{\partial}{\partial x^l}\left(\cancel{\frac{\partial B_j}{\partial x^k}}-\frac{\partial B_k}{\partial x^j}\right)\dot{x}^j-\frac{\partial^2 \cancel{A}}{\cancel{\partial x^l}\,\partial x^k}\\
&-\frac{\partial}{\partial x^k}\left(\cancel{\frac{\partial B_j}{\partial x^l}}-\frac{\partial B_l}{\partial x^j}\right)\dot{x}^j-\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial B_l}{\partial x^k}-\frac{\partial B_k}{\partial x^l}-\frac{\partial B_k}{\partial x^l}+\frac{\partial B_l}{\partial x^k}\right)\dot{x}^j\\
&=-\frac{\partial^2 B_k}{\partial x^l\,\partial x^j}\dot{x}^j+\frac{\partial^2 B_l}{\partial x^k\,\partial x^j}\dot{x}^j-\frac{\partial}{\partial x^j}\left(\frac{\partial B_l}{\partial x^k}-\frac{\partial B_k}{\partial x^l}\right)\dot{x}^j=0.
\end{aligned} \tag{36}$$

3. Suppose that (c) holds. Then (30) implies

$$\frac{\partial^2 P_i}{\partial \dot{x}^j\,\partial \dot{x}^k}=-\frac{\partial^2 P_j}{\partial \dot{x}^k\,\partial \dot{x}^i}=\frac{\partial^2 P_k}{\partial \dot{x}^i\,\partial \dot{x}^j}=-\frac{\partial^2 P_i}{\partial \dot{x}^j\,\partial \dot{x}^k}=0, \tag{37}$$

and a straightforward computation shows that (30) and (31) are exactly the Helmholtz conditions for the functions P_i :

(38) $$\frac{\partial P_k}{\partial \dot{x}^j}+\frac{\partial P_j}{\partial \dot{x}^k}=0,$$

and

(39) $$\begin{aligned}&\frac{\partial P_i}{\partial x^l}-\frac{\partial P_l}{\partial x^i}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial P_i}{\partial \dot{x}^l}-\frac{\partial P_l}{\partial \dot{x}^i}\right)\\ &=\frac{\partial P_i}{\partial x^l}-\frac{\partial P_l}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial P_i}{\partial \dot{x}^l}-\frac{\partial P_l}{\partial \dot{x}^i}\right)\dot{x}^j-\frac{1}{2}\cancel{\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial P_i}{\partial \dot{x}^l}-\frac{\partial P_l}{\partial \dot{x}^i}\right)}\ddot{x}^j\\ &=0.\end{aligned}$$

Computing from (28)

(40) $$\begin{aligned}\frac{\partial P_i}{\partial \dot{x}^l}&=\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^l\,\partial x^i}+\frac{\partial^3 \mathscr{L}_h}{\partial \dot{x}^l\,\partial x^j\,\partial \dot{x}^i}\dot{x}^j+\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i},\\ \frac{\partial P_l}{\partial \dot{x}^i}&=\frac{\partial f_l}{\partial \dot{x}^i}-\frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^i\,\partial x^l}+\frac{\partial^3 \mathscr{L}_h}{\partial \dot{x}^i\,\partial x^j\,\partial \dot{x}^l}\dot{x}^j+\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l},\end{aligned}$$

we get

(41) $$\begin{aligned}\frac{\partial P_i}{\partial \dot{x}^l}+\frac{\partial P_l}{\partial \dot{x}^i}&=\frac{\partial f_i}{\partial \dot{x}^l}-\cancel{\frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^l\,\partial x^i}}+\frac{\partial^3 \mathscr{L}_h}{\partial \dot{x}^l\,\partial x^j\,\partial \dot{x}^i}\dot{x}^j+\cancel{\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}}\\ &\quad+\frac{\partial f_l}{\partial \dot{x}^i}-\cancel{\frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^i\,\partial x^l}}+\frac{\partial^3 \mathscr{L}_h}{\partial \dot{x}^i\,\partial x^j\,\partial \dot{x}^l}\dot{x}^j+\cancel{\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l}}\\ &=\frac{\partial f_i}{\partial \dot{x}^l}+\frac{\partial f_l}{\partial \dot{x}^i}+2\frac{\partial g_{il}}{\partial x^j}\dot{x}^j=0,\end{aligned}$$

proving (32).

Since

(42) $$\begin{aligned}\frac{\partial P_i}{\partial \dot{x}^l}-\frac{\partial P_l}{\partial \dot{x}^i}&=\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^l\,\partial x^i}+\cancel{\frac{\partial^3 \mathscr{L}_h}{\partial \dot{x}^l\,\partial x^j\,\partial \dot{x}^i}}\dot{x}^j+\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}\\ &\quad-\frac{\partial f_l}{\partial \dot{x}^i}+\frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^i\,\partial x^l}-\cancel{\frac{\partial^3 \mathscr{L}_h}{\partial \dot{x}^i\,\partial x^j\,\partial \dot{x}^l}}\dot{x}^j-\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l}\\ &=\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}-2\left(\frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^l\,\partial x^i}-\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}\right),\end{aligned}$$

and

$$
\begin{aligned}
(43)\qquad &\frac{\partial P_i}{\partial x^l}-\frac{\partial P_l}{\partial x^i}=\frac{\partial}{\partial x^l}\left(f_i-\cancel{\frac{\partial \mathscr{L}_h}{\partial x^i}}+\frac{\partial^2 \mathscr{L}_h}{\partial x^j\,\partial \dot{x}^i}\dot{x}^j\right)\\
&-\frac{\partial}{\partial x^i}\left(f_l-\cancel{\frac{\partial \mathscr{L}_h}{\partial x^l}}+\frac{\partial^2 \mathscr{L}_h}{\partial x^j\,\partial \dot{x}^l}\dot{x}^j\right)\\
&=\frac{\partial f_i}{\partial x^l}-\frac{\partial f_l}{\partial x^i}+\frac{\partial}{\partial x^j}\left(\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}-\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l}\right)\dot{x}^j,
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{\partial P_i}{\partial x^l}-\frac{\partial P_l}{\partial x^i}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial P_i}{\partial \dot{x}^l}-\frac{\partial P_l}{\partial \dot{x}^i}\right)\\
&=\frac{\partial f_i}{\partial x^l}-\frac{\partial f_l}{\partial x^i}+\frac{\partial}{\partial x^j}\left(\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}-\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l}\right)\dot{x}^j\\
&-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}-2\left(\frac{\partial^2 \mathscr{L}_h}{\partial \dot{x}^l\,\partial x^i}-\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}\right)\right)\\
&=\frac{\partial f_i}{\partial x^l}-\frac{\partial f_l}{\partial x^i}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)+\cancel{\frac{\partial}{\partial x^j}\left(\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}-\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l}\right)\dot{x}^j}\\
(44)\qquad &-\cancel{\frac{\partial}{\partial x^j}\left(\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}-\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l}\right)}\dot{x}^j-\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}-\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l}\right)\ddot{x}^j\\
&=\frac{\partial f_i}{\partial x^l}-\frac{\partial f_l}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)\dot{x}^j-\frac{1}{2}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)\ddot{x}^j\\
&-\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial^2 \mathscr{L}_h}{\partial x^l\,\partial \dot{x}^i}-\frac{\partial^2 \mathscr{L}_h}{\partial x^i\,\partial \dot{x}^l}\right)\ddot{x}^j\\
&=\frac{\partial f_i}{\partial x^l}-\frac{\partial f_l}{\partial x^i}-\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)\dot{x}^j-\frac{1}{2}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)\ddot{x}^j\\
&-\left(\frac{\partial g_{ij}}{\partial x^l}-\frac{\partial g_{lj}}{\partial x^i}\right)\ddot{x}^j=0.
\end{aligned}
$$

This proves (33) and (34).

4. (c) is obviously equivalent with (d). Thus, (32), (33) and (34) are exactly the Helmholtz conditions for the expressions P_i, so there exists a Lagrangian $\mathscr{L}_0$ for P_i hence for (27); we may take for $\mathscr{L}_0$ the Vainberg-Tonti Lagrangian

(45) $$\mathscr{L}_0 = x^i \int_0^1 P_i(\tau x^j, \tau \dot{x}^j) d\tau.$$

Now we are in a position to formulate main results of this section. Recall that by (28),

(46) $$f_l = g_{kl} F^k,$$

where g_{kl} are variational integrating factors and $F = F^k$ is the force entering the Sonin – Douglas's equation (1). The following two theorems give us a solution of the Sonin – Douglas's problem in terms of a system of differential equations.

Theorem 2 *The following conditions are equivalent:*
(a) *The Sonin-Douglas's equation* (1) *has a solution* $\mathscr{L}$.
(b) *The function* g_{ij} *and* f_i *satisfy*

(47) $$g_{ij} = g_{ij}, \quad \frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j},$$

and

(48) $$\frac{\partial}{\partial \dot{x}^l}\left(f_i - \frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^j} - \frac{\partial f_j}{\partial \dot{x}^i} \right)\dot{x}^j \right) + \frac{\partial g_{lj}}{\partial x^i}\dot{x}^j = 0,$$

(49) $$\frac{\partial f_i}{\partial x^l} - \frac{\partial f_l}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial f_i}{\partial \dot{x}^l} - \frac{\partial f_l}{\partial \dot{x}^i} \right)\dot{x}^j = 0,$$

(50) $$\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial f_i}{\partial \dot{x}^j} - \frac{\partial f_j}{\partial \dot{x}^i} \right) + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{lj}}{\partial x^i} = 0.$$

Proof 1. We show that condition (34) is a consequence of (32). Consider equation (32)

(51) $$\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^l} + \frac{\partial f_l}{\partial \dot{x}^i} \right) + \frac{\partial g_{il}}{\partial x^j}\dot{x}^j = 0.$$

Differentiating

(52) $$\frac{1}{2}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial f_i}{\partial \dot{x}^l} + \frac{\partial f_l}{\partial \dot{x}^i} \right) + \frac{\partial^2 g_{il}}{\partial \dot{x}^j \partial x^s}\dot{x}^s + \frac{\partial g_{il}}{\partial x^j} = 0.$$

Write these equations as

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial f_i}{\partial \dot{x}^l}+\frac{\partial f_l}{\partial \dot{x}^i}\right)+\frac{\partial^2 g_{il}}{\partial \dot{x}^j\,\partial x^s}\dot{x}^s+\frac{\partial g_{il}}{\partial x^j}=0,\\
(53)\qquad &\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial f_j}{\partial \dot{x}^i}+\frac{\partial f_i}{\partial \dot{x}^j}\right)+\frac{\partial^2 g_{ji}}{\partial \dot{x}^l\,\partial x^s}\dot{x}^s+\frac{\partial g_{ji}}{\partial x^l}=0,\\
&\frac{1}{2}\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial f_l}{\partial \dot{x}^j}+\frac{\partial f_j}{\partial \dot{x}^l}\right)+\frac{\partial^2 g_{lj}}{\partial \dot{x}^i\,\partial x^s}\dot{x}^s+\frac{\partial g_{lj}}{\partial x^i}=0.
\end{aligned}$$

Combining these formulas we have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial f_i}{\partial \dot{x}^l}+\cancel{\frac{\partial f_l}{\partial \dot{x}^i}}\right)+\frac{\partial^2 g_{il}}{\partial \dot{x}^j\,\partial x^s}\dot{x}^s+\frac{\partial g_{il}}{\partial x^j}+\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\cancel{\frac{\partial f_j}{\partial \dot{x}^i}}+\frac{\partial f_i}{\partial \dot{x}^j}\right)\\
(54)\qquad &+\cancel{\frac{\partial^2 g_{ji}}{\partial \dot{x}^l\,\partial x^s}}\dot{x}^s+\frac{\partial g_{ji}}{\partial x^l}\\
&-\frac{1}{2}\frac{\partial}{\partial \dot{x}^i}\left(\cancel{\frac{\partial f_l}{\partial \dot{x}^j}}+\cancel{\frac{\partial f_j}{\partial \dot{x}^l}}\right)-\cancel{\frac{\partial^2 g_{lj}}{\partial \dot{x}^i\,\partial x^s}}\dot{x}^s-\frac{\partial g_{lj}}{\partial x^i}=0,
\end{aligned}$$

and, with different indexing,

$$\begin{aligned}
&\frac{\partial^2 f_i}{\partial \dot{x}^j\,\partial \dot{x}^l}+\frac{\partial^2 g_{il}}{\partial \dot{x}^j\,\partial x^s}\dot{x}^s+\frac{\partial g_{il}}{\partial x^j}+\frac{\partial g_{ji}}{\partial x^l}-\frac{\partial g_{lj}}{\partial x^i}=0,\\
(55)\qquad &\frac{\partial^2 f_j}{\partial \dot{x}^i\,\partial \dot{x}^l}+\frac{\partial^2 g_{jl}}{\partial \dot{x}^i\,\partial x^s}\dot{x}^s+\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{ji}}{\partial x^l}-\frac{\partial g_{li}}{\partial x^j}=0.
\end{aligned}$$

Subtracting the second equation from the first one we get

$$\begin{aligned}
&\frac{\partial^2 f_i}{\partial \dot{x}^j\,\partial \dot{x}^l}+\cancel{\frac{\partial^2 g_{il}}{\partial \dot{x}^j\,\partial x^s}\dot{x}^s}+\frac{\partial g_{il}}{\partial x^j}+\cancel{\frac{\partial g_{ji}}{\partial x^l}}-\frac{\partial g_{lj}}{\partial x^i}\\
(56)\qquad &-\frac{\partial^2 f_j}{\partial \dot{x}^i\,\partial \dot{x}^l}-\cancel{\frac{\partial^2 g_{jl}}{\partial \dot{x}^i\,\partial x^s}\dot{x}^s}-\frac{\partial g_{jl}}{\partial x^i}-\cancel{\frac{\partial g_{ji}}{\partial x^l}}+\frac{\partial g_{li}}{\partial x^j}\\
&=\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)+\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{lj}}{\partial x^i}=0.
\end{aligned}$$

which is exactly condition (34).

Substituting from (56) back to (51)

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^l}+\frac{\partial f_l}{\partial \dot{x}^i}\right)+\frac{\partial g_{il}}{\partial x^j}\dot{x}^j \\
&\quad=\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^l}+\frac{\partial f_l}{\partial \dot{x}^i}\right)-\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^j+\frac{\partial g_{lj}}{\partial x^i}\dot{x}^j \\
&\quad=\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^l}+\cancel{\frac{\partial f_l}{\partial \dot{x}^i}}\right)-\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^j\right) \qquad (57)\\
&\qquad+\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\cancel{\frac{\partial f_l}{\partial \dot{x}^i}}\right)+\frac{\partial g_{lj}}{\partial x^i}\dot{x}^j \\
&\quad=\frac{\partial}{\partial \dot{x}^l}\left(f_i-\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^j\right)+\frac{\partial g_{lj}}{\partial x^i}\dot{x}^j=0.
\end{aligned}
$$

Thus (a) implies (50) (i.e. (34)) and (48).

2. Conversely, we show that (50) and (48) imply (32). Indeed, we have from (50)

$$
\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^j+\frac{\partial g_{il}}{\partial x^j}\dot{x}^j-\frac{\partial g_{lj}}{\partial x^i}\dot{x}^j=0. \tag{58}
$$

Substituting in (48)

$$
\begin{aligned}
&\frac{\partial}{\partial \dot{x}^l}\left(f_i-\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^j\right)+\frac{\partial g_{lj}}{\partial x^i}\dot{x}^j \\
&\quad=\frac{\partial f_i}{\partial \dot{x}^l}-\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)\dot{x}^j-\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial f_i}{\partial \dot{x}^j}-\frac{\partial f_j}{\partial \dot{x}^i}\right)\dot{x}^j+\frac{\partial g_{lj}}{\partial x^i}\dot{x}^j \qquad (59)\\
&\quad=\frac{\partial f_i}{\partial \dot{x}^l}-\frac{1}{2}\left(\frac{\partial f_i}{\partial \dot{x}^l}-\frac{\partial f_l}{\partial \dot{x}^i}\right)\dot{x}^j+\frac{\partial g_{il}}{\partial x^j}\dot{x}^j=0.
\end{aligned}
$$

Theorem 3 *For any metric* g_{ij} *such that*

$$
g_{ij}=g_{ij},\quad \frac{\partial g_{ij}}{\partial \dot{x}^k}=\frac{\partial g_{ik}}{\partial \dot{x}^j}, \tag{60}
$$

equations (48) *and* (50) *have a solution* f_i .

Proof Let g_{ij} be given. Equation (48) is equivalent with the system

$$
\frac{\partial \Phi_i}{\partial \dot{x}^l}+\frac{\partial g_{jl}}{\partial x^i}\dot{x}^j=0, \tag{61}
$$

$$\Phi_i = f_i + \frac{1}{2}\left(\frac{\partial f_j}{\partial \dot{x}^i} - \frac{\partial f_i}{\partial \dot{x}^j}\right)\dot{x}^j. \tag{62}$$

Since

$$\begin{aligned} &\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial g_{jl}}{\partial x^i}\dot{x}^j\right) - \frac{\partial}{\partial \dot{x}^l}\left(\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\right) \\ &\quad = \frac{\partial^2 g_{jl}}{\partial \dot{x}^k \partial x^i}\dot{x}^j + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial^2 g_{jk}}{\partial \dot{x}^l \partial x^i}\dot{x}^j - \frac{\partial g_{lk}}{\partial x^i} = 0, \end{aligned} \tag{63}$$

there exist functions Φ_i satisfying (61). Equation (62) becomes an equation for the functions f_i.

Equation (50) can be written as a system

$$\frac{1}{2}\frac{\partial \Phi_{il}}{\partial \dot{x}^j} + \frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} = 0, \tag{64}$$

$$\Phi_{il} = \frac{\partial f_i}{\partial \dot{x}^l} - \frac{\partial f_l}{\partial \dot{x}^i}. \tag{65}$$

From (60),

$$\frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i}\right) = \frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial g_{ik}}{\partial x^l} - \frac{\partial g_{lk}}{\partial x^i}\right). \tag{66}$$

the integrability condition for equation (64) is satisfied. Thus, (64) has a solution Φ_{il} and (65) becomes an equation for f_i.

Thus, given Φ_{il} and Φ_i, satisfying (61) and (64), we search for a solution f_i of the system (62), (65). We set

$$f_i = \Phi_i - \frac{1}{2}\Phi_{ji}\dot{x}^j. \tag{69}$$

Then

$$\begin{aligned} \frac{\partial f_i}{\partial \dot{x}^l} - \frac{\partial f_l}{\partial \dot{x}^i} &= \frac{\partial \Phi_i}{\partial \dot{x}^l} - \frac{\partial \Phi_l}{\partial \dot{x}^i} - \frac{1}{2}\frac{\partial \Phi_{ji}}{\partial \dot{x}^l}\dot{x}^j - \frac{1}{2}\Phi_{il} + \frac{1}{2}\frac{\partial \Phi_{jl}}{\partial \dot{x}^i}\dot{x}^j + \frac{1}{2}\Phi_{il} \\ &= \frac{\partial \Phi_i}{\partial \dot{x}^l} - \frac{\partial \Phi_l}{\partial \dot{x}^i} - \frac{1}{2}\left(\frac{\partial \Phi_{si}}{\partial \dot{x}^l} - \frac{\partial \Phi_{sl}}{\partial \dot{x}^i}\right)\dot{x}^s + \Phi_{il} \end{aligned} \tag{70}$$

$$= \left(-\frac{\partial g_{sl}}{\partial x^i} + \frac{\partial g_{si}}{\partial x^l} \right) \dot{x}^s - \left(-\frac{\partial g_{sl}}{\partial x^i} + \cancel{\frac{\partial g_{il}}{\partial x^s}} + \frac{\partial g_{si}}{\partial x^l} - \cancel{\frac{\partial g_{il}}{\partial x^s}} \right) \dot{x}^s + \Phi_{il}$$

$$= \cancel{\left(-\frac{\partial g_{sl}}{\partial x^i} + \frac{\partial g_{si}}{\partial x^l} \right) \dot{x}^s - \left(-\frac{\partial g_{sl}}{\partial x^i} + \frac{\partial g_{si}}{\partial x^l} \right) \dot{x}^s} + \Phi_{il}$$

$$= \Phi_{il}.$$

Thus, f_i satisfies (62) and (65).

Example 1 (Geodesic equations in Riemannian geometry) Given a regular metric $g = g_{ij}$, we set, using standard notation,

$$\Gamma^i_{rs} = \frac{1}{2} g^{ip} \left(\frac{\partial g_{rp}}{\partial x^s} + \frac{\partial g_{sp}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^p} \right),$$

(71)

$$\Gamma_{jrs} = g_{ij} \Gamma^i_{rs} = \frac{1}{2} \left(\frac{\partial g_{rj}}{\partial x^s} + \frac{\partial g_{sj}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^j} \right).$$

We prove by a direct verification of conditions (47) – (50) of Theorem 2 that the system of functions,

(72) $$\varepsilon_l = -g_{li} (\ddot{x}^i + \Gamma^i_{rs} \dot{x}^r \dot{x}^s)$$

defining the geodesic equations in Riemann geometry, is variational.

We set

(73) $$f_j = \Gamma_{jrs} \dot{x}^r \dot{x}^s.$$

Conditions (47) are satisfied trivially. Substituting for Γ_{jrs} to (48), (49) and (50) yields

$$\frac{\partial}{\partial \dot{x}^l} \left(f_i + \frac{1}{2} \left(\frac{\partial f_j}{\partial \dot{x}^i} - \frac{\partial f_i}{\partial \dot{x}^j} \right) \dot{x}^j \right) + \frac{\partial g_{jl}}{\partial x^i} \dot{x}^j$$

$$= -\frac{\partial}{\partial \dot{x}^l} \left(\Gamma_{irs} \dot{x}^r \dot{x}^s + \frac{1}{2} \left(\frac{\partial \Gamma_{jrs} \dot{x}^r \dot{x}^s}{\partial \dot{x}^i} - \frac{\partial \Gamma_{irs} \dot{x}^r \dot{x}^s}{\partial \dot{x}^j} \right) \dot{x}^j \right) + \frac{\partial g_{jl}}{\partial x^i} \dot{x}^j$$

(74)

$$= -\frac{\partial}{\partial \dot{x}^l} (\cancel{\Gamma_{irs} \dot{x}^r \dot{x}^s} + (\Gamma_{jri} \dot{x}^r - \cancel{\Gamma_{irj} \dot{x}^r}) \dot{x}^j) + \frac{\partial g_{jl}}{\partial x^i} \dot{x}^j$$

$$= -\frac{\partial}{\partial \dot{x}^l} (\Gamma_{jri} \dot{x}^r \dot{x}^j) + \frac{\partial g_{jl}}{\partial x^i} \dot{x}^j = -\frac{1}{2} \frac{\partial}{\partial \dot{x}^l} ((\Gamma_{jri} + \Gamma_{rji}) \dot{x}^r \dot{x}^j) + \frac{\partial g_{jl}}{\partial x^i} \dot{x}^j$$

$$
\begin{aligned}
&= -(\Gamma_{lri} + \Gamma_{rli})\dot{x}^r + \frac{\partial g_{rl}}{\partial x^i}\dot{x}^r \\
&= \left(-\frac{1}{2}\left(\frac{\partial g_{rl}}{\partial x^i} + \cancel{\frac{\partial g_{il}}{\partial x^r}} - \cancel{\frac{\partial g_{ri}}{\partial x^l}} \right) - \frac{1}{2}\left(\frac{\partial g_{rl}}{\partial x^i} + \cancel{\frac{\partial g_{ir}}{\partial x^l}} - \cancel{\frac{\partial g_{li}}{\partial x^r}} \right) + \frac{\partial g_{rl}}{\partial x^i} \right)\dot{x}^r \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial f_i}{\partial x^l} - \frac{\partial f_l}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial f_i}{\partial \dot{x}^l} - \frac{\partial f_l}{\partial \dot{x}^i} \right)\dot{x}^j \\
&\quad = -\frac{\partial \Gamma_{irs}}{\partial x^l}\dot{x}^r\dot{x}^s + \frac{\partial \Gamma_{lrs}}{\partial x^i}\dot{x}^r\dot{x}^s + \frac{\partial}{\partial x^j}(\Gamma_{irl}\dot{x}^r - \Gamma_{lri}\dot{x}^r)\dot{x}^j \\
&\quad = \left(-\frac{\partial \Gamma_{irs}}{\partial x^l} + \frac{\partial \Gamma_{lrs}}{\partial x^i} + \frac{\partial \Gamma_{irl}}{\partial x^s} - \frac{\partial \Gamma_{lri}}{\partial x^s} \right)\dot{x}^r\dot{x}^s \\
&\quad = \frac{1}{2}\left(-\frac{\partial}{\partial x^l}\left(\cancel{\frac{\partial g_{ri}}{\partial x^s}} + \cancel{\frac{\partial g_{si}}{\partial x^r}} - \cancel{\frac{\partial g_{rs}}{\partial x^i}} \right) + \frac{\partial}{\partial x^i}\left(\cancel{\frac{\partial g_{rl}}{\partial x^s}} + \cancel{\frac{\partial g_{sl}}{\partial x^r}} - \cancel{\frac{\partial g_{rs}}{\partial x^l}} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial x^s}\left(\cancel{\frac{\partial g_{ri}}{\partial x^l}} + \frac{\partial g_{li}}{\partial x^r} - \cancel{\frac{\partial g_{rl}}{\partial x^i}} \right) - \frac{\partial}{\partial x^s}\left(\cancel{\frac{\partial g_{rl}}{\partial x^i}} + \frac{\partial g_{il}}{\partial x^r} - \cancel{\frac{\partial g_{ri}}{\partial x^l}} \right) \right)\dot{x}^r\dot{x}^s \\
&\quad = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial f_i}{\partial \dot{x}^l} - \frac{\partial f_l}{\partial \dot{x}^i} \right) + \frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \\
&\quad = -\frac{\partial}{\partial \dot{x}^j}(\Gamma_{irl}\dot{x}^r - \Gamma_{lri}\dot{x}^r) + \frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \\
&\quad = -(\Gamma_{ijl} - \Gamma_{lji}) + \frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \qquad (75) \\
&\quad = -\frac{1}{2}\left(\cancel{\frac{\partial g_{ji}}{\partial x^l}} + \cancel{\frac{\partial g_{li}}{\partial x^j}} - \cancel{\frac{\partial g_{jl}}{\partial x^i}} - \cancel{\frac{\partial g_{jl}}{\partial x^i}} - \cancel{\frac{\partial g_{il}}{\partial x^j}} + \cancel{\frac{\partial g_{ji}}{\partial x^l}} \right) + \cancel{\frac{\partial g_{ij}}{\partial x^l}} - \cancel{\frac{\partial g_{lj}}{\partial x^i}} \\
&\quad = 0.
\end{aligned}
$$