Part 2

The global inverse problem in fibred manifolds

In this part of the lectures we consider *variational structures* (Y,ρ), where Y is a fibred manifold over an n-dimensional, orientable base manifold X, and ρ is an n-form, defined on the r-jet prolongation J^rY of Y. Our objective is to study the *inverse problem* of the calculus of variations for integral, higher-order variational functionals, associated with the n-forms ρ.

To this purpose we first introduce the underlying geometric structures for these functionals – jet prolongations of fibred manifolds. Then, using the canonical jet projections $\pi^r : J^rY \to X$ and $\pi^{r,s} : J^rY \to J^sY$ between different order jet prolongations, we develop a canonical decomposition theory of differential forms on the jet prolongations. Of particular interest are the *contact forms*, annihilating integrable sections of the jet prolongations J^rY. We also study the decompositions defined by the fibred homotopy operators and state the corresponding fibred Poincare-Volterra lemma.

Then we introduce integral variational functionals, depending on sections of Y, defined by differential n-forms on J^rY; this general setting includes as a special case the functionals, given by the Lagrangians (considered as differential forms). To examine properties of the variational functionals, we introduce variations of sections of the fibred manifold Y as vector fields, defined on Y. Then we explain the geometric theory of the first variation and higher variations, based on the concepts of the jet prolongation of a vector field and the Lie derivative. In order to derive a main theorem, describing the global structure of the first variation, namely the first variation formula, we introduce the concept of a Lepage form, generalizing the well-known Cartan form used in classical mechanics. Lepage forms allow us to find the (global) *infinitesimal first variation formula* by means of standard geometric operations as the exterior derivative, Lie derivative and the contraction of a form by a vector field. We complete in this way the *integral* first variation formula, used in the classical calculus of variation on Euclidean spaces.

A basic concept, essential for the formulation of the *global inverse problem* of the calculus of variations, is the *Euler-Lagrange form*, an $(n+1)$-form, defined on some J^sY. As a principal part of the first variation formula, determined by ρ, it describes the extremals of the underlying variational functional; locally, the components of the Euler-Lagrange form are the Euler-Lagrange expressions.

Finally, we introduce the *source* $(n+1)$*-forms* on J^sY and study conditions for these forms to be identical with the Euler-Lagrange forms. As a natural aspect of the *global* setting of the inverse problem, we first need integrability conditions for the *local* inverse problem, ensuring that a source form can be modelled as an Euler-Lagrange form *locally*. To this purpose we derive a generalisation of the Helmholtz variationality conditions, and find the underlying local variational functionals.

1 Jet prolongations of fibred manifolds

1.1 Immersions, submersions Recall that the *rank* of a linear mapping $u: E \to F$ of vector spaces is defined to be the dimension of its image space, $\operatorname{rank} u = \dim \operatorname{Im} u$. This definition applies to tangent mappings of differentiable mappings of manifolds. Let $f: X \to Y$ be a C^r mapping of smooth manifolds, where $r \geq 1$. We define the *rank* of *f at a point* $x \in X$ to be the rank of the tangent mapping $T_x f: T_x X \to T_{f(x)} Y$. We denote $\operatorname{rank}_x f = \dim \operatorname{im} T_x f$. The function $x \to \operatorname{rank}_x f$, defined on *X*, with values in non-negative integers, is the *rank function*.

Elementary examples of real-valued functions *f* of one real variable show that the rank function is not, in general, locally constant. Our main objective in this section is to study differentiable mappings whose rank function is *locally constant*.

The proof of the following theorem is based on the rank theorem in Euclidean spaces and a standard use of charts on a smooth manifold.

Theorem 1 (Rank theorem) *Let X and Y be manifolds, let* $n = \dim X$, $m = \dim Y$, *and let q be a positive integer such that* $q \leq \min(n, m)$. *Let* $W \subset X$ *be an open set, and let* $f: W \to Y$ *be a* C^r *mapping. The following conditions are equivalent:*

(1) *f has constant rank on W equal to q.*

(2) *To every point* $x_0 \in W$ *there exist a chart* (U, φ), $\varphi = (x^i)$ *at* x_0, *an open rectangle* $P \subset \mathbf{R}^n$ *with centre* 0 *such that* $\varphi(U) = P$, $\varphi(x_0) = 0$, *a chart* (V, ψ), $\psi = (y^\sigma)$, *at* $y_0 = f(x_0)$, *and an open rectangle* $Q \subset \mathbf{R}^m$ *with centre* 0 *such that* $\psi(V) = Q$, $\psi(y_0) = 0$, *and*

$$(2) \qquad y^\sigma \circ f = \begin{cases} x^\sigma, & \sigma = 1, 2, \dots, q, \\ 0, & \sigma = q+1, q+2, \dots, m. \end{cases}$$

Proof 1. Suppose that *f* has constant rank on *W* equal to *q*. We choose a chart $(\bar{U}, \bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, at x_0, and a chart $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, at y_0, and set $g = \bar{\psi} f \bar{\varphi}^{-1}$; *g* is a C^r mapping from $\bar{\varphi}(\bar{U}) \subset \mathbf{R}^n$ into $\bar{\psi}(\bar{V}) \subset \mathbf{R}^m$. Since for every tangent vector $\xi \in T_x X$ expressed as

(3) $$\xi = \bar{\xi}^i \left(\frac{\partial}{\partial x^i} \right)_x ,$$

we have

(4) $$T_x f \cdot \xi = D_i(\bar{y}^\sigma f \bar{\varphi}^{-1})(\bar{\varphi}(x))\bar{\xi}^i \left(\frac{\partial}{\partial y^\sigma} \right)_{f(x)} ,$$

the rank of f at x is $\operatorname{rank} T_x f = \operatorname{rank} D_i(\bar{y}^\sigma f \bar{\varphi}^{-1})(\bar{\varphi}(x))$. Consequently, the rank of f is constant on the open set $\bar{\varphi}(\bar{U}) \subset \mathbf{R}^n$, and is equal to q. Shrinking $\bar{U}$ to a neighbourhood U of x_0 and $\bar{V}$ to a neighbourhood V of y_0 if necessary we may suppose that there exist an open rectangle $P \subset \mathbf{R}^n$ with centre 0, a diffeomorphism $\alpha : \bar{\varphi}(U) \to P$, an open rectangle $Q \subset \mathbf{R}^m$ with centre 0, and a diffeomorphism $\beta : \bar{\psi}(V) \to Q$, such that in the canonical coordinates on P and Q, $\beta g \alpha^{-1}(z^1, z^2, \ldots, z^n) = (z^1, z^2, \ldots, z^q, 0, 0, \ldots, 0)$. We set $\varphi = \alpha \bar{\varphi}$, $\varphi = (x^i)$, and $\psi = \beta \bar{\psi}$, $\psi = (y^\sigma)$. Then (U, φ) and (V, ψ) are charts on X and Y. In these charts, $\psi f \varphi^{-1} = \beta \bar{\psi} f \bar{\varphi}^{-1} \alpha^{-1} = \beta g \alpha^{-1}$; thus, for every $x \in U$

(5) $$\begin{aligned} \psi f(x) &= \psi f \varphi^{-1} \varphi(x) = \beta g \alpha^{-1} \varphi(x) \\ &= \beta g \alpha^{-1}(x^1(x), x^2(x), \ldots, x^n(x)) \\ &= (x^1(x), x^2(x), \ldots, x^q(x), 0, 0, \ldots, 0). \end{aligned}$$

In components,

(6) $$y^\sigma \circ f(x) = \begin{cases} x^\sigma(x), & \sigma = 1, 2, \ldots, q, \\ 0, & \sigma = q+1, q+2, \ldots, m, \end{cases}$$

proving (2).

2. Conversely, suppose that on a neighbourhood of $x_0 \in W$ the mapping f is expressed by (2). Then $\operatorname{rank} T_{x_0} f = \operatorname{rank} D_i(y^\sigma f \varphi^{-1})(\varphi(x_0)) = q$.

Let $f : X \to Y$ be a C^r mapping, and let $x_0 \in X$ be a point. We say that f is a *constant rank mapping* at x_0, if there exists a neighbourhood W of x_0 such that the rank function $x \to \operatorname{rank}_x f$ is constant on W. Then the charts (U, φ) and (V, ψ) in which the mapping f has an expression (2), are said to be *adapted* to f at x_0, or just *f-adapted*. A C^r mapping f that is a constant rank mapping at every point is called a C^r mappings of *locally constant rank*.

A C^r mapping $f : W \to Y$ such that the tangent mapping $T_{x_0} f$ is *injective* is called an *immersion at* x_0. From the definition of the rank it is immediate that *f* is an immersion at x_0 if and only if $\operatorname{rank}_{x_0} f = n \le m$. If *f* is an immersion at every point of the set *W*, we say that *f* is an *immersion*.

From the rank theorem we get the following criterion.

Theorem 2 (Immersions) *Let X and Y be manifolds,* $n = \dim X$, *and* $m = \dim Y \ge n$. *Let* $f : X \to Y$ *be a* C^r *mapping,* $x_0 \in X$ *a point, and* $y_0 = f(x_0)$. *The following two conditions are equivalent:*

(1) *f is an immersion at* x_0.

(2) *There exist a chart* (U,φ), $\varphi = (x^i)$ *at* x_0, *an open rectangle* $P \subset \mathbf{R}^n$ $P \subset \mathbf{R}^n$ *with centre* 0 *such that* $\varphi(U) = P$ *and* $\varphi(x_0) = 0$, *a chart* (V,ψ), $\psi = (y^\sigma)$ *at* $y_0 = f(x_0)$, *and an open rectangle* $Q \subset \mathbf{R}^m$ *with centre* 0 *such that* $\psi(V) = Q$ *and* $\psi(y_0) = 0$, *such that in these charts f is expressed by*

$$y^\sigma \circ f = \begin{cases} x^\sigma, & \sigma = 1,2,\dots,n, \\ 0, & \sigma = n+1, n+2,\dots,m. \end{cases} \tag{7}$$

Proof The matrix of the linear operator $T_{x_0} f$ in some charts (U,φ), $\varphi = (x^i)$, at x_0 and (V,ψ), $\psi = (y^\sigma)$, at y_0 is formed by partial derivatives $D_i(y^\sigma f \varphi^{-1})(\varphi(x_0))$, and is of dimension $n \times m$. If at x_0, $\operatorname{rank} T_{x_0} f = n$, then $\operatorname{rank} T_x f = n,$ on a neighbourhood of x_0, by continuity of the determinant function. Equivalence of conditions (1) and (2) is now an immediate consequence of Theorem 1.

Let $f : X \to Y$ be an immersion, let $x_0 \in X$ be a point, and let (U,φ) and (V,ψ) be the charts from Theorem 2, (2). Shrinking *P* and *Q* if necessary we may suppose without loss of generality that $Q = P \times R$, where *R* is an open rectangle in $\mathbf{R}^{m-n}$. Then the chart expression $\psi f \varphi^{-1} : P \to P \times R$ of *f* is the mapping $(x^1, x^2, \dots, x^n) \to (x^1, x^2, \dots, x^n, 0, 0, \dots, 0)$. The charts (U,φ), (V,ψ) with these properties are said to be *adapted* to the immersion *f* at x_0.

Example 1 (Sections) Let $s \ge r$, let $f : X \to Y$ be a surjective mapping of smooth manifolds. By a C^r *section*, or simply a *section* of *f* we mean a C^r mapping $\gamma : Y \to X$ such that

$$f \circ \gamma = \mathrm{id}_Y. \tag{8}$$

Every section is an immersion. Indeed, $T_{\gamma(y)} f \circ T_y \gamma = \mathrm{id}_{T_y Y}$ at any point $y \in Y$. Thus, for any two tangent vectors ρ satisfying $T_y \gamma \cdot \xi_1 = T_y \gamma \cdot \xi_2$, we have $T_{\gamma(y)} f \circ T_y \gamma \cdot \xi_1 = T_{\gamma(y)} f \circ T_y \gamma \cdot \xi_2$ hence $\xi_1 = \xi_2$.

A C^r mapping $f : W \to Y$ such that the tangent mapping $T_{x_0} f$ is surjective, is called a *submersion at* x_0. From the definition of the rank it is immediate that *f* is a submersion at x_0 if and only if $\mathrm{rank}_{x_0} = m \le n$. A *submersion* $f : W \to Y$ is a C^r mapping that is a submersion at every point $x \in W$.

Theorem 3 (Submersions) *Let X and Y be manifolds, let* $n = \dim X$, $m = \dim Y$. *Let* $f : X \to Y$ *be a* C^r *mapping,* x_0 *a point of X,* $y_0 = f(x_0)$. *The following conditions are equivalent:*

(1) *f is a submersion at* x_0.

(2) *There exist a chart* (U,φ), $\varphi = (x^i)$, *at* x_0, *an open rectangle* $P \subset \mathbf{R}^n$ *with centre* 0 *such that* $\varphi(U) = P$, $\varphi(x_0) = 0$, *a chart* (V,ψ), $\psi = (y^\sigma)$, *at* $y_0 = f(x_0)$, *and an open rectangle* $Q \subset \mathbf{R}^m$ *with centre* 0 *such that* $\psi(V) = Q$, $\psi(y_0) = 0$, *such that*

$$y^\sigma \circ f = x^\sigma, \quad \sigma = 1,2,\dots,m. \tag{9}$$

(3) *There exist a neighbourhood V of* y_0 *and a* C^r *section* $\gamma : V \to Y$ *such that* $\gamma(y_0) = x_0$.

Proof 1. Suppose that *f* is a submersion at x_0. Then $\mathrm{rank}\,T_x f = m$ on a neighbourhood of x_0, and equivalence of conditions (1) and (2) follows from Theorem 1.

2. Suppose that condition (2) is satisfied. Consider the chart expression $\psi f \varphi^{-1} : P \to Q$ of the submersion *f* that is equal to the Cartesian projection $(x^1,x^2,\dots,x^m,x^{m+1},x^{m+1},\dots,x^n) \to (x^1,x^2,\dots,x^m)$. $\psi f \varphi^{-1}$ admits a C^r section δ. Since $\psi f \varphi^{-1} \circ \delta = \mathrm{id}_Q$ hence $f\varphi^{-1} \circ \delta = \psi^{-1}$. Setting $\gamma = \varphi^{-1}\delta\psi$ we have $f\gamma = f\varphi^{-1}\delta\psi = \psi^{-1}\psi = \mathrm{id}_V$ proving that γ is a section of *f*. This proves (3).

3. If *f* admits a C^r section γ defined on a neighbourhood *V* of a point *y*, then $f \circ \gamma = \mathrm{id}_V$ and $T_y(f \circ \gamma) = T_x f \circ T_y \gamma = T_y\, \mathrm{id}_V = \mathrm{id}_{T_y Y}$, where $x = \gamma(y)$. Thus $T_{x_0} f$ must be surjective, proving (1).

Let π be a C^r submersion, let $x_0 \in X$ be a point, and let (U,φ) and (V,ψ) be the charts from Theorem 3, (2). Shrinking *P* and *Q* if necessary we may suppose that the rectangle *P* is of the form $P = Q \times R$, where *R* is an open rectangle in $\mathbf{R}^{n-m}$. Then the chart expression π of the submersion *f* is the mapping $(x^1,x^2,\dots,x^m,x^{m+1},x^{m+1},\dots,x^n) \to (x^1,x^2,\dots,x^m)$. The charts (U,φ), (V,ψ) with these properties are said to be *adapted* to the submersion *f* at the point x_0.

Corollary 1 *A submersion is an open mapping*.

Proof In adapted charts, a submersion is expressed as a Cartesian projection that is an open mapping. Corollary 1 now follows from the definition of the manifold topology in which the charts are homeomorphisms.

Example 2 (Cartesian projections) Cartesian projections of the product of C^∞ manifolds X and Y, $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$, are C^∞ submersions. Indeed, to show it, we can use equations of pr_1 and pr_2. If $(x,y) \in X \times Y$ is a point and (U,φ), $\varphi = (x^i)$ (resp. (V,ψ), $\psi = (y^\sigma)$) is a chart at x (resp. y), we have on $U \times V$, $T\,\mathrm{pr}_1 : T(U \times V) \to TU$; thus, since pr_1 is expressible as the mapping $(x^i, y^\sigma) \to (x^i)$, we have for every $\xi \in T_x X$, $\xi = \xi^i (\partial / \partial x^i)$, and every $\zeta \in T_y Y$,

$$T_{(x,y)}\,\mathrm{pr}_1 \cdot (\xi,\zeta) = \frac{\partial x^i}{\partial x^k} \xi^k \frac{\partial}{\partial x^i} = \xi^i \frac{\partial}{\partial x^i}. \tag{10}$$

Consequently,

$$T_{(x,y)}\,\mathrm{pr}_1 \cdot (\xi,\zeta) = \xi. \tag{11}$$

In particular, $T_{(x,y)}\,\mathrm{pr}_1$ is surjective, and pr_1 is a surjective submersion.

Example 3 The tangent bundle projection is a surjective submersion. All tensor bundle projections are surjective submersions.

With the help of Corollary 1, submersions at a point can be characterized as follows.

Corollary 2 *Let X and Y be manifolds, $n = \dim X$, $m = \dim Y \le n$. A C^r mapping $f : X \to Y$ is a submersion at a point $x_0 \in X$ if and only if there exist a neighbourhood U of x_0, an open rectangle $R \subset \mathbf{R}^{n-m}$, and a diffeomorphism $\chi : U \to Y \times \mathbf{R}^{n-m}$ such that* $\mathrm{pr}_1 \circ \chi = f$.

Proof 1. Suppose f is a submersion at x_0, and choose some adapted charts (U,φ), $\varphi = (x^i)$, at x_0 and (V,ψ), $\psi = (y^\sigma)$ at y_0. Every point $x \in U$ has the coordinates $\varphi(x) = (x^1(x), x^2(x), \dots, x^m(x), x^{m+1}(x), \dots, x^n(x))$. We define a mapping $\chi : U \to Y \times \mathbf{R}^{n-m}$ by

$$\chi(x) = (f(x), x^{m+1}(x), x^{m+2}(x), \dots, x^n(x)). \tag{12}$$

Then $\mathrm{pr}_1 \circ \chi = f$, and according to Corollary 1, $f(U)$ is an open set in Y. It remains to show that χ is a diffeomorphism. We easily find the chart expression of χ with respect to the chart (U,φ) and the chart $(V \times \mathbf{R}^{n-m}, \eta)$, $\eta = (y^1, y^2, \dots y^m, t^1, t^2, \dots, t^{n-m})$, on $Y \times \mathbf{R}^{n-m}$, where t^k are the canonical coordinates on $\mathbf{R}^{n-m}$. We have for every $x \in U$, $y^\sigma \chi(x) = y^\sigma f(x) = x^\sigma(x)$, $1 \le \sigma \le m$, and $t^k \chi(x) = x^{m+k}(x)$, $1 \le k \le n-m$, that is,

$$\begin{aligned} y^i \chi &= x^i, \quad i = 1,2,\dots,m, \\ t^k \chi &= x^{m+k}, \quad k = 1,2,\dots,n-m, \end{aligned} \tag{13}$$

i.e., $\eta \circ \chi = \varphi$. Thus $\chi = \eta^{-1}\varphi$ is a diffeomorphism.

2. Conversely, if $\mathrm{pr}_1 \circ \chi = f$, we have $T_{\chi(x_0)}\mathrm{pr}_1 \circ T_{x_0}\chi = T_{x_0} f$ and since χ is a diffeomorphism, $\mathrm{rank}\, T_{\chi(x_0)}\mathrm{pr}_1 = \mathrm{rank}\, T_{x_0} f$. But the rank of the projection pr_1 is m (Example 2).

1.2 Fibred manifolds By a *fibred manifold structure* on C^∞ manifold Y we mean a C^∞ manifold X together with a surjective submersion $\pi : Y \to X$ of class C^∞. A manifold Y endowed with a fibred manifold structure is called a *fibred manifold* of class C^∞, or just a *fibred manifold*. X is the *base*, and π is the *projection* of the fibred manifold Y.

Let Y be a fibred manifold with base X and projection π. Let $\dim X = n$, and $\dim Y = n + m$. According to Theorem 3, to every point $y \in Y$ there exist a chart adapted to the submersion π at y, a chart (V,ψ), $\psi = (u^i, y^\sigma)$, at y, where $1 \le i \le n$, $1 \le \sigma \le m$, and a chart (U,φ), $\varphi = (x^i)$, at $x = \pi(y)$, such that $U = \pi(V)$, and $x^i \circ \pi = u^i$. When using these charts, we conventionally write x^i instead of u^i. We call a chart (V,ψ) with these properties a *fibred chart* on Y. The chart (U,φ) on X is unique, and is said to be *associated* with (V,ψ). It is convenient to write (V,ψ), $\psi = (x^i, y^\sigma)$, for a fibred chart.

Lemma 1 *Every fibred manifold has an atlas consisting of fibred charts.*

Proof This is an immediate consequence of the definition of a submersion.

A C^r *section* of the fibred manifold Y, defined on an open set $W \subset X$, is by definition a C^r section $\gamma : W \to Y$ of the projection π (cf. **1.1**, Example 1). In terms of a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, and the associated chart (U,φ), $\varphi = (x^i)$, such that $U \subset W$ and $\gamma(U) \subset V$, γ has equations of the form

$$(1) \qquad x^i \circ \gamma = x^i, \quad y^\sigma \circ \gamma = f^\sigma,$$

where f^σ are real C^r functions, defined on U.

Let Y_1 (resp. Y_2) be a fibred manifold with base X_1 (resp. X_2) and projection π_1 (resp. π_2). A C^r mapping $\alpha : W \to Y_2$, where W is an open set in Y_1, is called a C^r *homomorphism* of the fibred manifold Y_1 into Y_2, if there exists a C^r mapping $\alpha_0 : W_0 \to X_2$ where $W_0 = \pi_1(W_1)$, such that

$$(2) \qquad \pi_2 \circ \alpha = \alpha_0 \circ \pi_1.$$

Note that W_0 is always an open set in X_1 (Corollary 1). If α_0 exists it is unique, and is called the *projection* of α. We also say that α is a homomorphism *over* α_0. A homomorphism of fibred manifolds $\alpha : Y_1 \to Y_2$ that is a diffeomorphism is called an *isomorphism*; the projection of an isomorphism of fibred manifolds is a diffeomorphism of their bases.

If $Y_1 = Y_2 = Y$, then a fibred homomorphism $\alpha : W \to Y$ is also called an *automorphism* of the fibred manifold *Y*.

We find the expression of a homomorphism in fibred charts. Consider a fibred chart (V_1, ψ_1), $\psi_1 = (x_1^i, y_1^\sigma)$, on Y_1 and a fibred chart (V_2, ψ_2), $\psi_2 = (x_2^p, y_2^\tau)$, on Y_2 such that $\alpha(V_1) \subset V_2$. We have the commutative diagram

(3)
$$\begin{array}{ccc} V_1 & \xrightarrow{\alpha} & V_2 \\ \downarrow & & \downarrow \\ \pi_1(V_1) & \xrightarrow{\alpha_0} & \pi_2(V_2) \end{array}$$

expressing condition (2). In terms of the charts we can write

(4)
$$\begin{aligned} \alpha_0\pi_1 &= \varphi_2^{-1} \circ \varphi_2\alpha_0\varphi_1^{-1} \circ \varphi_1\pi_1\psi_1^{-1} \circ \psi_1, \\ \pi_2\alpha &= \varphi_2^{-1} \circ \varphi_2\pi_2\psi_2^{-1} \circ \psi_2\alpha\psi_1^{-1} \circ \psi_1, \end{aligned}$$

so the commutativity yields

(5)
$$\varphi_2\alpha_0\varphi_1^{-1} \circ \varphi_1\pi_1\psi_1^{-1} = \varphi_2\pi_2\psi_2^{-1} \circ \psi_2\alpha\psi_1^{-1}.$$

Since we have fibred charts, $\varphi_1\pi_1\psi_1^{-1}$ is the Cartesian projection $(x_1^i, y_1^\sigma) \to (x_1^i)$, and $\varphi_2\pi_2\psi_2^{-1}$ is the Cartesian projection $(x_2^p, y_2^\tau) \to (x_2^p)$. Consequently, writing in components

(6)
$$\begin{aligned} &\varphi_2\alpha_0\varphi_1^{-1} \circ \varphi_1\pi_1\psi_1^{-1}(x_1^i, y_1^\sigma) = \varphi_2\alpha_0\varphi_1^{-1}(x_1^i) = (x_2^p\alpha_0\varphi_1^{-1}(x_1^i)), \\ &\varphi_2\pi_2\psi_2^{-1} \circ \psi_2\alpha\psi_1^{-1}(x_1^i, y_1^\sigma) \\ &\quad = \varphi_2\pi_2\psi_2^{-1}(x_2^p\alpha\psi_1^{-1}(x_1^i, y_1^\sigma), y_2^\tau\alpha\psi_1^{-1}(x_1^i, y_1^\sigma)) \\ &\quad = (x_2^p\alpha\psi_1^{-1}(x_1^i, y_1^\sigma)), \end{aligned}$$

we see that condition (5) implies $x_2^p\alpha_0\varphi_1^{-1}(x_1^i) = x_2^p\alpha\psi_1^{-1}(x_1^i, y_1^\sigma)$ .This shows that the right-hand side expression is independent of the coordinates y_1^σ. Therefore, we conclude that the equations of the homomorphism α in fibred charts are always of the form

(7)
$$x_2^p = f^p(x_1^i), \quad y_2^\tau = F^\tau(x_1^i, y_1^\sigma).$$

Let Y be a fibred manifold with base X and projection π. If Ξ is a tangent vector to Y at a point $y \in Y$, then the tangent vector ξ to X at $x = \pi(y) \in X$ defined by

(8) $$T_y\pi \cdot \Xi = \xi,$$

is called the π*-projection*, or simply the *projection* of Ξ. By definition of the submersion, the tangent mapping of the projection π at a point x, $T_y\pi : T_yY \to T_{\pi(x)}X$, is surjective.

A tangent vector $\Xi \in T_yY$ is said to be π*-vertical,* if

(9) $$T_y\pi \cdot \Xi = 0.$$

The vector subspace of T_yY consisted of π-vertical vectors, is denoted by VT_yY. If Ξ is expressed in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, by

(10) $$\Xi = \xi^i \left(\frac{\partial}{\partial x^i} \right)_y + \Xi^\sigma \left(\frac{\partial}{\partial y^\sigma} \right)_y,$$

then by (8)

(11) $$\xi = \xi^i \left(\frac{\partial}{\partial x^i} \right)_x = 0.$$

Thus, Ξ is π-vertical if and only if

(12) $$\Xi = \Xi^\sigma \left(\frac{\partial}{\partial y^\sigma} \right)_y.$$

If in particular, $\dim Y = n + m$ and $\dim X = n$, then $\dim VT_yY = m$. The subset VTY of the tangent bundle TY, defined by

(13) $$VTY = \bigcup_{y \in Y} VT_yY,$$

is a vector subbundle of TY, called the *vertical subbundle*.

A vector field Ξ on an open set W in Y is called π*-projectable*, if there exists a vector field ξ, defined on $\pi(W) \subset X$, such that

(14) $$T\pi \cdot \Xi = \xi \circ \pi.$$

If ξ exists, it is unique and is called the π*-projection* of Ξ; we also say that Ξ *covers* ξ. The vector field Ξ is called π*-vertical*, if

(15) $T\pi \cdot \Xi = 0.$

Let ρ be a differential *k*-form, defined on an open set *W* in *Y*. We say that ρ is π-horizontal, of just *horizontal*, if it vanishes whenever one of its vector arguments is a π-vertical vector. The same can be said in terms of the contraction of a form by a vector field requiring that ρ be π-horizontal if for every π-vertical vector field Ξ on *W*

(16) $i_\Xi \rho = 0.$

The following lemma describes chart expressions of π-horizontal forms.

Lemma 2 *The form* ρ *is* π*-horizontal if and only if in any fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$*, it has an expression*

(17) $$\rho = \frac{1}{k!}\rho_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

Proof Choose a point $y \in V$ and express the form $\rho(y)$ in the form

(18) $$\begin{aligned}\rho(y) = \frac{1}{k!}\rho_{i_1 i_2 \ldots i_k}(y) dx^{i_1}(y) \wedge dx^{i_2}(y) \wedge \ldots \wedge dx^{i_k}(y) + dy^1(y) \wedge \rho_1(y) \\ + dy^2(y) \wedge \rho_2(y) + \ldots + dy^m(y) \wedge \rho_m(y),\end{aligned}$$

where the forms $\rho_1(y)$, $\rho_2(y)$, $\ldots$, $\rho_m(y)$ do not contain $dy^1(y)$, the forms $\rho_2(y)$, $\rho_3(y)$, $\ldots$, $\rho_m(y)$ do not contain $dy^1(y)$ and $dy^2(y)$, etc. Suppose that ρ is π-horizontal. Then contracting $\rho(y)$ by the vertical vector $(\partial/\partial y^1)_y$ we get $i_{(\partial/\partial y^1)_y}\rho(y) = \rho_1(y) = 0$. Contracting $\rho(y)$ by the vertical vector $(\partial/\partial y^2)_y$ we get $i_{(\partial/\partial y^2)_y}\rho(y) = \rho_2(y) = 0$, etc. Applying this procedure several times we get (17).

Example 4 Moebius band is a fibred manifold over the circle.

A form ρ, defined on an open set *W* in *Y*, is said to be π*-projectable*, or just *projectable*, if there exists a form ρ_0, defined on $W_0 = \pi(W)$, such that

(19) $\rho = \pi^* \rho_0.$

If the form ρ_0 exists, it is unique and is called the π*-projection*, of just the *projection* of ρ.

Throughout, when using differential forms, we adopt the following *conventions*:

Conventions (a) We express a differential $(p+q)$-form on the fibred manifold *Y* as

(20)
$$\rho = \sum_{\nu_1<\nu_2<\ldots<\nu_p} \sum_{i_1<i_2<\ldots<i_q} A_{\nu_1\nu_2\ldots\nu_p\ i_1i_2\ldots i_q} dy^{\nu_1} \wedge dy^{\nu_2} \wedge \ldots \wedge dy^{\nu_p} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q},$$

or equivalently, as

(21)
$$\rho = \frac{1}{p!q!} A_{\nu_1\nu_2\ldots\nu_p\ i_1i_2\ldots i_q} dy^{\nu_1} \wedge dy^{\nu_2} \wedge \ldots \wedge dy^{\nu_p} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q}$$

with summation through all double indices and coefficients, *skew-symmetric* in $\nu_1,\nu_2,\ldots,\nu_p$ and $i_1,i_2,\ldots,i_q$, separately.

(b) π-projectable forms $\rho = \pi^*\rho_0$ on *Y* can be *canonically* (that is via π) identified with forms on *X*. To simplify notation, we sometimes denote the forms $\pi^*\rho_0$ and ρ_0 by the same symbol, ρ_0.

1.3 The contact of differentiable mappings In this section *X* and *Y* are smooth manifolds, $n = \dim X$ and $m = \dim Y$.

Let $x \in X$ be a point, W_1, W_2 neighbourhoods of *x*, and let $f_1 : W_1 \to Y$ and $f_2 : W_2 \to Y$ be two mappings. Suppose that $W_1 \cap W_2 \neq \emptyset$. We say that f_1 and f_2 have the *contact of order* 0 at *x*, if

(1)
$$f_1(x) = f_2(x).$$

Suppose that f_1 and f_2 are of class C^r, where *r* is a positive integer. We say that f_1 and f_2 have the *contact of order r* at *x*, if they have contact of order 0, and there exist a chart (U,φ), $\varphi = (x^i)$, at *x* and a chart (V,ψ), $\psi = (y^\sigma)$, at $f_1(x)$ such that $U \subset W_1 \cap W_2$, $f_1(U), f_2(U) \subset V$, and for all $k \le r$,

(2)
$$D^k(\psi f_1 \varphi^{-1})(\varphi(x)) = D^k(\psi f_2 \varphi^{-1})(\varphi(x)).$$

We say that two C^∞ mappings $f_1 : W_1 \to Y$ and $f_2 : W_2 \to Y$, have the *contact of order* ∞ at *x*, if they have the contact of order *r* for every *r*.

Writing in components $\psi f_1 \varphi^{-1} = (y^\sigma f_1 \varphi^{-1})$, $\psi f_2 \varphi^{-1} = (y^\sigma f_2 \varphi^{-1})$, we see at once that f_1 and f_2 have contact of order *r* if and only if $f_1(x) = f_2(x)$ and

(3)
$$D_{i_1} D_{i_2} \ldots D_{i_k} (y^\sigma f_1 \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \ldots D_{i_k} (y^\sigma f_2 \varphi^{-1})(\varphi(x))$$

for all $k = 1,2,\ldots,r$, $1 \le \sigma \le m$, and all $1 \le i_1, i_2, \ldots, i_k \le n$ such $i_1 \le i_2 \le \ldots \le i_k$.

We claim that if f_1, f_2 have contact of order r at a point x, then for any chart $(\bar{U},\bar{\varphi})$, $\bar{\varphi}=(\bar{x}^i)$, at x and any chart $(\bar{V},\bar{\psi})$, $\bar{\psi}=(\bar{y}^\sigma)$, at $f_1(x)=f_2(x)$,

(4) $$D^k(\bar{\psi} f_1 \bar{\varphi}^{-1})(\bar{\varphi}(x)) = D^k(\bar{\psi} f_2 \bar{\varphi}^{-1})(\bar{\varphi}(x))$$

for all $k=1,2,\dots,r$. We can verify this formula by means of the chain rule for derivatives of mappings of Euclidean spaces. Using the charts (U,φ), (V,ψ) we express the derivative

(5) $$\begin{aligned} &D_{i_1}D_{i_2}\dots D_{i_k}(\bar{y}^\sigma f_1 \bar{\varphi}^{-1})(\bar{\varphi}(x)) \\ &\quad = D_{i_1}D_{i_2}\dots D_{i_k}(\bar{y}^\sigma \psi^{-1}\circ\psi f_1\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x)) \end{aligned}$$

as a polynomial in the variables $D_{j_1}(y^\nu f_1\varphi^{-1})(\varphi(x))$, $D_{j_1}D_{j_2}(y^\nu f_1\varphi^{-1})(\varphi(x))$, $\dots$, $D_{j_1}D_{j_2}\dots D_{j_k}(y^\nu f_1\varphi^{-1})(\varphi(x))$. Then $D_{i_1}D_{i_2}\dots D_{i_k}(\bar{y}^\sigma f_2\bar{\varphi}^{-1})(\bar{\varphi}(x))$ is expressed by the same polynomial in the variables $D_{j_1}(y^\nu f_2\varphi^{-1})(\varphi(x))$ $D_{j_1}D_{j_2}(y^\nu f_2\varphi^{-1})(\varphi(x))$, $\dots$, $D_{j_1}D_{j_2}\dots D_{j_k}(y^\nu f_2\varphi^{-1})(\varphi(x))$. Clearly, equality (4) now follows from (3).

Fix two points $x\in X$, $y\in Y$, and denote by $C^r_{(x,y)}(X,Y)$ the set of C^r mappings $f:W\to Y$, where W is a neighbourhood of x and $f(x)=y$. The binary relation "*f, g have the contact of order r at x*" on $C^r_{(x,y)}(X,Y)$ is obviously *reflexive*, *transitive*, and *symmetric*, so is an equivalence relation. Equivalence classes of this equivalence relation are called *r-jets* with *source* x and *target* y. The r-jet whose representative is a mapping $f\in C^r_{(x,y)}(X,Y)$ is called the *r-jet of f at x*, and is denoted by $J^r_x f$. If there is no danger of misunderstanding, we call an r-jet with source x and target y an *r-jet*, or just a *jet*. The set of r-jets with source $x\in X$ and target $y\in Y$ is denoted by $J^r_{(x,y)}(X,Y)$.

Let X, Y, and Z be three finite-dimensional smooth manifolds. We say that two r-jets $A\in J^r_{(x,u)}(X,Y)$, $A=J^r_x f$, and $B\in J^r_{(y,z)}(Y,Z)$, $B=J^r_y g$, are *composable*, if they have representatives which are composable (as mappings), i.e., if $u=y$; this equality means that the target of A coincides with the source of B. In this case the composite $g\circ f$ of any representatives of A and B is a mapping of class C^r defined on a neighbourhood of x.

It is easily seen that the r-jet $J^r_x(g\circ f)$ is independent of the representatives of the r-jets A and B. If $\bar{f}$ and $\bar{g}$ are such that $J^r_x f = J^r_x\bar{f}$ and $J^r_x g = J^r_x\bar{g}$, then for any charts (U,φ), $\varphi=(x^i)$ at x, (V,ψ), $\psi=(y^\sigma)$, at $y=f(x)$, and (W,η), $\eta=(z^p)$, at $z=g(y)$, the derivatives $D_{i_1}D_{i_2}\dots D_{i_k}(z^p gf\varphi^{-1})(\varphi(x))$ are expressible in the form

(6) $$D_{i_1}D_{i_2}\dots D_{i_k}(z^p gf\varphi^{-1})(\varphi(x)) = D_{i_1}D_{i_2}\dots D_{i_k}(z^p g\psi^{-1}\circ\psi f\varphi^{-1})(\varphi(x)).$$

for all $k = 1,2,\dots,r$. By the chain rule for mappings of Euclidean spaces, these derivatives are polynomials in the derivatives $D_{\nu_1} D_{\nu_2} \dots D_{\nu_q} (z^p g \psi^{-1})(\psi(y))$, $D_{i_1} D_{i_2} \dots D_{i_m} (y^\nu f \varphi^{-1})(\varphi(x))$, where $m, q \le k$. The same polynomial in the derivatives $D_{\nu_1} D_{\nu_2} \dots D_{\nu_q} (z^p \bar{g} \psi^{-1})(\psi(y))$, $D_{i_1} D_{i_2} \dots D_{i_m} (y^\nu \bar{f} \varphi^{-1})(\varphi(x))$ is obtained when expressing the derivative $D_{i_1} D_{i_2} \dots D_{i_k} (z^p \overline{gf} \varphi^{-1})(\varphi(x))$ by means of the chain rule. Now since by definition

$$(7)\qquad \begin{aligned} &D_{i_1} D_{i_2} \dots D_{i_m} (y^\nu f \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \dots D_{i_m} (y^\nu \bar{f} \varphi^{-1})(\varphi(x)), \\ &D_{\nu_1} D_{\nu_2} \dots D_{\nu_q} (z^p g \psi^{-1})(\psi(y)) = D_{\nu_1} D_{\nu_2} \dots D_{\nu_q} (z^p \bar{g} \psi^{-1})(\psi(y)), \end{aligned}$$

we have

$$(8)\qquad D_{i_1} D_{i_2} \dots D_{i_k} (z^p g f \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \dots D_{i_k} (z^p \overline{gf} \varphi^{-1})(\varphi(x)).$$

This proves, that the r-jet $J^r_x (g \circ f)$ is independent of the choice of A and B.

If X, Y, and Z are three manifolds and $A \in J^r_{(x,y)}(X,Y)$, $A = J^r_x f$, and $B \in J^r_{(y,z)}(Y,Z)$, $B = J^r_y g$, are composable r-jets, i.e., $y = f(x)$, we define

$$(9)\qquad B \circ A = J^r_x (g \circ f),$$

or, explicitly, $J^r_x g \circ J^r_x f = J^r_x (g \circ f)$. The r-jet $B \circ A$ is called the *composite* of A and B, and the mapping $(A,B) \to B \circ A$ of $J^r_{(x,f(x))}(X,Y) \times J^r_{(y,g(y))}(Y,Z)$ into $J^r_{(x,z)}(X,Z)$, where $z = g(y)$, is the *composition* of r-jets.

A chart on X at the point x and a chart on Y at the point y induce a chart on the set $J^r_{(x,y)}(X,Y)$. Let (U,φ), $\varphi = (x^i)$ (resp. (V,ψ), $\psi = (x^i, y^\sigma)$), be a chart on X (resp. Y). We assign to any r-jet $J^r_x f \in J^r_{(x,y)}(X,Y)$ the numbers

$$(10)\qquad z^\sigma_{j_1 j_2 \dots j_k}(J^r_x \gamma) = D_{j_1} D_{j_2} \dots D_{j_k} (y^\sigma f \varphi^{-1})(\varphi(x)), \quad 1 \le k \le r.$$

Then the collection of functions $\chi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r})$, such that

$$(11)\qquad 1 \le j_1 \le j_2 \le \dots \le j_k \le n, \quad 1 \le \sigma \le m,$$

is a bijection of the set $J^r_{(x,y)}(X,Y)$ and the Euclidean space $\mathbf{R}^N$ of dimension

$$(12)\qquad N = n + m\left(1 + n + \binom{n+1}{2} + \binom{n+2}{3} + \dots + \binom{n+r-1}{r}\right).$$

Thus, the pair $(J^r_{(x,y)}(X,Y), \chi^r)$ is a (global) chart on $J^r_{(x,y)}(X,Y)$. This chart is said to be *associated* with the charts (U,φ) and (V,ψ).

Lemma 3 (a) *The associated charts* $(J^r_{(x,y)}(X,Y),\chi^r)$, *such that the charts* (U,φ) *and* (V,ψ) *belong to smooth structures on X and Y, form a smooth atlas on* $J^r_{(x,y)}(X,Y)$. *With this atlas,* $J^r_{(x,y)}(X,Y)$ *is a smooth manifold of dimension N* (13).

(b) *The composition of jets*

(13) $$J^r_{(x,y)}(X,Y)\times J^r_{(y,z)}(Y,Z)\ni(A,B)\to B\circ A\in J^r_{(x,z)}(X,Z)$$

is smooth.

Proof 1. One should proof that the transformation equations between the associated charts are of class C^∞. However this is obvious from (5).

2. (b) is an immediate consequence of formula (6).

1.4 Jet prolongations of fibred manifolds In this section we apply the concept of contact of differentiable mappings (Section 1.3) to C^r *sections* of fibred manifolds. We study the structure of jets of sections and mappings of these jets.

Let Y be a fibred manifold with base X and projection π, let $n=\dim X$ and $m=\dim Y-n$. We denote by J^rY, where $r\geq 0$ the set of r-jets $J^r_x\gamma$ of C^r *sections* γ of Y with source $x\in X$ and target $y=\gamma(x)\in Y$; if $r=0$, then $J^0Y=Y$. Note that the representatives of an r-jet $J^r_x\gamma$ are C^r sections $\gamma:W\to Y$, where W is an open set in X; condition that γ is a section,

(1) $$\pi\circ\gamma=\mathrm{id}_W,$$

implies that the target $y=\gamma(x)$ of the r-jet $J^r_x\gamma$ belongs to the *fibre* $\pi^{-1}(x)\subset Y$ over the source point x. For any s such that $0\leq s\leq r$ we have surjective mappings $\pi^{r,s}:J^rY\to J^sY$ and $\pi^r:J^rY\to X$, defined by

(2) $$\pi^{r,s}(J^r_x\gamma)=J^s_x\gamma,\quad \pi^r(J^r_x\gamma)=x.$$

These mappings are called the *canonical jet projections*.

The smooth structure of the fibred manifold Y induces by a canonical construction a smooth structure on the set J^rY. Let (V,ψ), $\psi=(x^i,y^\sigma)$, be a fibred chart on Y, and let (U,φ), $\varphi=(x^i)$, be the associated chart on X. We set $V^r=(\pi^{r,0})^{-1}(V)$, and introduce, for all values of the indices, a family of functions, defined on V^r,

(3) $$\begin{aligned}&x^i(J^r_x\gamma)=x^i(x),\\ &y^\sigma(J^r_x\gamma)=y^\sigma(\gamma(x)),\\ &y^\sigma_{j_1j_2\dots j_k}(J^r_x\gamma)=D_{j_1}D_{j_2}\dots D_{j_k}(y^\sigma\gamma\varphi^{-1})(\varphi(x)),\quad 1\leq k\leq r.\end{aligned}$$

Then the collection of functions $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, where the indices satisfy

(4) $\quad 1 \le i \le n, \quad 1 \le \sigma \le m, \quad 1 \le j_1 \le j_2 \le \ldots \le j_k \le n, \quad k = 2,3,\ldots,r,$

is a bijection of the set V^r onto an open subset of the Euclidean space $\mathbf{R}^N$ of dimension

(5) $$N = n + m\left(1 + n + \binom{n+1}{2} + \binom{n+2}{3} + \ldots + \binom{n+r-1}{r}\right).$$

In other words, the pair (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, is a chart on the set $J^r Y$. This chart is said to be *associated* with the fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$.

Lemma 4 (Smooth structure on the set $J^r Y$**)** *The set of associated charts* (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, *such that the charts* (V, ψ) *form an atlas on Y, is an atlas on* $J^r Y$.

Proof Let $\mathscr{A}$ be an atlas on Y whose elements are fibred charts (1.2, Lemma 1). One can easily check that $\mathscr{A}$ defines a topology on $J^r Y$ by requiring that for any fibred chart (V, ψ) from this atlas $\psi^r : V^r \to \psi^r(V^r) \subset \mathbf{R}^N$ is a homeomorphism; we consider the set $J^r Y$ with this topology.

It is clear that the associated charts with fibred charts from $\mathscr{A}$ cover the set $J^r Y$. Thus, to prove Lemma 4 it remains to check that the corresponding coordinate transformations are smooth.

Suppose we have two fibred chart on Y, (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$, such that $V \cap \bar{V} \neq \emptyset$, and consider the associated charts on $J^r Y$, (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, and $(\bar{V}^r, \bar{\psi}^r)$, $\bar{\psi}^r = (\bar{x}^i, \bar{y}^\sigma, \bar{y}^\sigma_{j_1}, \bar{y}^\sigma_{j_1 j_2}, \ldots, \bar{y}^\sigma_{j_1 j_2 \ldots j_r})$. Let $J^r_x \gamma \in V^r \cap \bar{V}^r$. Let the coordinate transformation $\bar{\psi}\psi^{-1}$ be expressed by the equations

(6) $\quad \bar{x}^i = f^i(x^j), \quad \bar{y}^\sigma = g^\sigma(x^j, y^\nu).$

Note that the functions f^i and g^σ in formula (6) are formally defined by $\bar{x}^i(x) = \bar{x}^i \varphi^{-1}(\varphi(x)) = f^i(\varphi(x))$ and $\bar{y}^\sigma(y) = \bar{y}^\sigma \psi^{-1}(\psi(y)) = g^\sigma(\psi(y))$. Thus

(7) $$\begin{aligned} &\bar{x}^i(J^r_x\gamma) = \bar{x}^i(x) = \bar{x}^i \varphi^{-1}(\varphi(x)) = \bar{x}^i \varphi^{-1}(\varphi(J^r_x \gamma)), \\ &\bar{y}^\sigma(J^r_x\gamma) = \bar{y}^\sigma(\gamma(x))) = (\bar{y}^\sigma \psi^{-1} \circ \psi)(\gamma(x)) = \bar{y}^\sigma \psi^{-1}(\psi(J^r_x\gamma)), \\ &\bar{y}^\sigma_{j_1 j_2 \ldots j_k}(J^r_x\gamma) = D_{j_1} D_{j_2} \ldots D_{j_k} (\bar{y}^\sigma \gamma \bar{\varphi}^{-1})(\bar{\varphi}(x)) \\ &\quad = D_{j_1} D_{j_2} \ldots D_{j_k} (\bar{y}^\sigma \psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\bar{\varphi}^{-1})(\bar{\varphi}(x)). \end{aligned}$$

From the chain rule it is now obvious that the right-hand sides, the coordinates of the r-jet $J_x^r\gamma$ in the chart $(\bar{V}^r,\bar{\psi}^r)$, depend smoothly on the coordinates of $J_x^r\gamma$ in the chart (V^r,ψ^r).

From now on, we consider the set J^rY with the smooth structure, defined by Lemma 4. We call J^rY the *r-jet prolongation* of the fibred manifold *Y*.

Lemma 5 *Each of the canonical jet projections* (2) *is smooth and defines a fibred manifold structure on the manifold* J^rY.

Proof Indeed, in the associated charts each of the canonical jet projections is expressed as a Cartesian projection which is smooth.

Every C^r section $\gamma : W \to Y$, where W is an open set in X, defines a mapping

$$W \ni x \to J^r\gamma(x) = J_x^r\gamma \in J^rY, \tag{8}$$

called the *r-jet prolongation* of γ.

Example 5 (Coordinate transformations on J^2Y) Consider two fibred charts on a fibred manifold Y, (V,ψ), $\psi = (x^i,y^\sigma)$, and $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{x}^i,\bar{y}^\sigma)$, such that $V \cap \bar{V} \neq \emptyset$. Suppose that the corresponding transformation equations are expressed by the equations

$$\bar{x}^i = \bar{x}^i(x^j), \quad \bar{y}^\sigma = \bar{y}^\sigma(x^j,y^\nu). \tag{9}$$

Then the induced coordinate transformation on J^2Y is expressed by the equations

$$\begin{aligned}
&\bar{x}^i = \bar{x}^i(x^j),\\
&\bar{y}^\sigma = \bar{y}^\sigma(x^j,y^\nu),\\
&\bar{y}^\sigma_{j_1} = \left(\frac{\partial \bar{y}^\sigma}{\partial x^l} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu}y^\nu_l\right)\frac{\partial x^l}{\partial \bar{x}^{j_1}},\\
&\bar{y}^\sigma_{j_1j_2} = \left(\frac{\partial^2 \bar{y}^\sigma}{\partial x^l\,\partial x^m} + \frac{\partial^2 \bar{y}^\sigma}{\partial x^l\,\partial y^\mu}y^\mu_m + \frac{\partial^2 \bar{y}^\sigma}{\partial x^m\,\partial y^\nu}y^\nu_l + \frac{\partial \bar{y}^\sigma}{\partial y^\mu\,\partial y^\nu}y^\nu_l y^\mu_m\right.\\
&\qquad \left. + \frac{\partial \bar{y}^\sigma}{\partial y^\nu}y^\nu_{lm}\right)\frac{\partial x^m}{\partial \bar{x}^{j_2}}\frac{\partial x^l}{\partial \bar{x}^{j_1}} + \left(\frac{\partial \bar{y}^\sigma}{\partial x^l} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu}y^\nu_l\right)\frac{\partial^2 x^l}{\partial \bar{x}^{j_1}\,\partial \bar{x}^{j_2}}.
\end{aligned} \tag{10}$$

To derive these equations, we use the chain rule for partial derivative operators. Let $J_x^2\gamma \in V^2 \cap \bar{V}^2$. The 2-jet $J_x^2\gamma$ has the coordinates

$$\begin{aligned}(11)\qquad &x^i(J^2_x\gamma)=x^i(x),\quad y^\sigma(J^2_x\gamma)=y^\sigma(\gamma(x)),\\ &y^\sigma_{j_1}(J^r_x\gamma)=D_{j_1}(y^\sigma\gamma\varphi^{-1})(\varphi(x)),\quad y^\sigma_{j_1j_2}(J^r_x\gamma)=D_{j_1}D_{j_2}(y^\sigma\gamma\varphi^{-1})(\varphi(x)),\end{aligned}$$

and analogously for the chart $(\bar{V},\bar{\psi})$. Then

$$\begin{aligned}
&D_{j_1}(\bar{y}^\sigma\gamma\bar{\varphi}^{-1})(\bar{\varphi}(x))=D_{j_1}(\bar{y}^\sigma\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad=D_k(\bar{y}^\sigma\psi^{-1})(\psi\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x))D_{j_1}(x^k\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad+D_\nu(\bar{y}^\sigma\psi^{-1})(\psi\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x))D_{j_1}(y^\nu\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad=D_k(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(x^k\gamma\varphi^{-1})(\varphi\bar{\varphi}^{-1}(\bar{\varphi}(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
(12)\qquad&\quad+D_\nu(\bar{y}^\sigma\psi^{-1}(\psi\gamma(x))D_l(y^\nu\gamma\varphi^{-1})(\varphi\bar{\varphi}^{-1}(\bar{\varphi}(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad=D_k(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))\delta^k_l D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad+D_\nu(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(y^\nu\gamma\bar{\varphi}^{-1})(\varphi(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad=(D_l(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))+D_\nu(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(y^\nu\gamma\bar{\varphi}^{-1})(\varphi(x)))\\
&\quad\cdot D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x)),
\end{aligned}$$

proving the third one of equations (10). To prove the fourth one, we differentiation (12) again and apply the chain rule. We can also derive the fourth equation by differentiating the third one.

1.5 The horizontalisation As before, let Y be a fibred manifold with base X and projection π, and let J^rY be the r-jet prolongation of Y. Denote $\dim X=n$ and $\dim Y=n+m$. Recall that for any open set $W\subset Y$, W^r denotes the open set $(\pi^{r,0})^{-1}(W)$ in J^rY. Ω^r_0W denotes the ring of C^r functions on W^r, and Ω^r_kW the Ω^r_0W-module of k-forms on W^r. The *exterior algebra* of W^r is denoted by Ω^rY. We show in this section that the fibred manifold structure of Y induces a canonical vector bundle homomorphism between the tangent bundles $TJ^{r+1}Y$ and TJ^rY and an exterior algebra homomorphism of Ω^rY into $\Omega^{r+1}Y$.

Let $J^{r+1}_x\gamma$ be a point of the manifold $J^{r+1}Y$. We assign to any tangent vector ξ of $J^{r+1}Y$ at $J^{r+1}_x\gamma$ a tangent vector of J^rY at $\pi^{r+1,r}(J^{r+1}_x\gamma)=J^r_x\gamma$ by

$$(1)\qquad h\xi=T_xJ^r\gamma\circ T\pi^{r+1}\cdot\xi.$$

We get a vector bundle homomorphism $h:TJ^{r+1}Y\to TJ^rY$ over the projection $\pi^{r+1,r}$, called π*-horizontalisation*, or simply the *horizontalisation*. Sometimes we call $h\xi$ the *horizontal component* of ξ (note, however, that the terminology is not standard since the vectors ξ and $h\xi$ do not belong to the same vector space). Using a complementary construction, one can also assign to every tangent vector $\xi \in TJ^{r+1}Y$ at a point $J_x^{r+1}\gamma \in J^{r+1}Y$ a tangent vector $p\xi \in TJ^rY$ at $J_x^r\gamma$ by the decomposition

$$T\pi^{r+1,r} \cdot \xi = h\xi + p\xi. \tag{2}$$

$p\xi$ is sometimes called the *contact component* of the vector ξ.

Lemma 6 *The horizontal and contact components satisfy*

$$T\pi^r \cdot h\xi = T\pi^{r+1} \cdot \xi, \quad T\pi^r \cdot p\xi = 0. \tag{3}$$

Proof The first property follows from (1). Then, however,

$$\begin{aligned} T\pi^r \cdot p\xi &= T\pi^r \cdot T\pi^{r+1,r} \cdot \xi - T\pi^r \cdot h\xi \\ &= T\pi^{r+1} \cdot \xi - T\pi^r \cdot T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi = 0. \end{aligned} \tag{4}$$

Remark 1 If $h\xi = 0$, then necessarily $T\pi^{r+1} \cdot \xi = 0$ so ξ is π^{r+1}-vertical. This observation may serve as a motivation why $h\xi$ is called the *horizontal component* of ξ.

One can easily find the chart expressions for the vectors $h\xi$ and $p\xi$. If in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, ξ has an expression

$$\xi = \xi^i \left(\frac{\partial}{\partial x^i}\right)_{J_x^{r+1}\gamma} + \sum_{k=0}^{r+1} \sum_{j_1 \le j_2 \le \ldots \le j_k} \Xi^\sigma_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}\right)_{J_x^{r+1}\gamma}, \tag{5}$$

then

$$h\xi = \xi^i \left(\left(\frac{\partial}{\partial x^i}\right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma} \right), \tag{6}$$

and

$$p\xi = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} (\Xi^\sigma_{j_1 j_2 \ldots j_k} - y^\sigma_{j_1 j_2 \ldots j_k i}\xi^i) \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma}. \tag{7}$$

The horizontalisation $h: TJ^{r+1}Y \to TJ^rY$ induces a mapping of the exterior algebra Ω^rY into $\Omega^{r+1}Y$, denoted by the same letter *h*, as follows. We set for any differential k-form ρ on W^r, any point $J^{r+1}_x\gamma \in W^{r+1}$ and any tangent vectors $\xi_1, \xi_2, \ldots, \xi_k$ to $J^{r+1}Y$ at $J^{r+1}_x\gamma$

(8) $$h\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_k) = \rho(J^r_x\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_k).$$

We extend the definition to 0-forms (functions); we set for every function $f : W^r \to \mathbf{R}$

(9) $$hf = f \circ \pi^{r+1,r}.$$

The mapping $\Omega^r W \ni \rho \to h\rho \in \Omega^{r+1}W$ is called the π*-horizontalisation*, or just the *horizontalisation* (of differential forms).

Lemma 7 (a) For all $\rho_1, \rho, \rho \in \Omega^r_k W$ and $f \in \Omega^r_0 W$

(10) $$h(\rho_1 + \rho_2) = h\rho_1 + h\rho_2 \quad h(f\rho) = (f \circ \pi^{r+1,r})h\rho.$$

(b) *For all* $\rho \in \Omega^r_p W$ *and* $\eta \in \Omega^r_q W$

(11) $$h(\rho \wedge \eta) = h\rho \wedge h\eta.$$

Proof Both assertions (a) and (b) is immediate. To prove formally (b), we use the definition of the exterior product

(12) $$\begin{aligned} &(\rho \wedge \eta)(J^r_x\gamma)(\zeta_1, \zeta_2, \ldots, \zeta_p, \zeta_{p+1}, \zeta_{p+2}, \ldots, \zeta_{p+q}) \\ &\quad = \sum_\tau \operatorname{sgn}\tau \cdot \rho(J^r_x\gamma)(\zeta_{\tau(1)}, \zeta_{\tau(2)}, \ldots, \zeta_{\tau(p)})\eta(J^r_x\gamma)(\zeta_{\tau(p+1)}, \ldots, \zeta_{\tau(p+q)}) \end{aligned}$$

(summation through the permutations τ of the set $\{1, 2, \ldots, p+q\}$ such that $\tau(1) < \tau(2) < \ldots < \tau(p)$ and $\tau(p+1) < \tau(p+2) < \ldots < \tau(p+q)$). Then

(13) $$\begin{aligned} &h(\rho \wedge \eta)(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}) \\ &\quad = (\rho \wedge \eta)(J^{r+1}_x\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_p, h\xi_{p+1}, h\xi_{p+2}, \ldots, h\xi_{p+q}) \\ &\quad = \sum_\tau \operatorname{sgn}\tau \cdot \rho(J^r_x\gamma)(h\xi_{\tau(1)}, h\xi_{\tau(2)}, \ldots, h\xi_{\tau(p)}) \\ &\quad \cdot \eta(J^r_x\gamma)(h\xi_{\tau(p+1)}, h\xi_{\tau(p+2)}, \ldots, h\xi_{\tau(p+q)}) \\ &\quad = \sum_\tau \operatorname{sgn}\tau \cdot h\rho(J^{r+1}_x\gamma)(\xi_{\tau(1)}, \xi_{\tau(2)}, \ldots, \xi_{\tau(p)}) \\ &\quad \cdot h\eta(J^{r+1}_x\gamma)(\xi_{\tau(p+1)}, \xi_{\tau(p+2)}, \ldots, \xi_{\tau(p+q)}) \\ &\quad = (h\rho(J^{r+1}_x\gamma) \wedge h\eta(J^{r+1}_x\gamma))(\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}). \end{aligned}$$

In the following lemma we summarize basic rules for computations with the horizontalisation and formal derivatives. First consider a 1-form ρ, expressed in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, by

$$\rho = A_i dx^i + \sum_{0\le k\le r} \sum_{j_1<j_2<\ldots<j_k} B_\sigma^{j_1 j_2 \ldots j_k} dy^\sigma_{j_1 j_2 \ldots j_k}. \tag{14}$$

By definition, we have at any point $J_x^{r+1}\gamma \in V^{r+1}$ and tangent vector ξ at $J_x^{r+1}\gamma$

$$\begin{aligned} h\rho(J_x^{r+1}\gamma)(\xi) &= \rho(J_x^r\gamma)(h\xi) = A_i(J_x^r\gamma)dx^i(J_x^r\gamma)(h\xi) \\ &+ \sum_{0\le k\le r} \sum_{j_1<j_2<\ldots<j_k} B_\sigma^{j_1 j_2 \ldots j_k}(J_x^r\gamma) dy^\sigma_{j_1 j_2 \ldots j_k}(J_x^r\gamma)(h\xi) \\ &= \left(A_i(J_x^r\gamma) + \sum_{0\le k\le r} \sum_{j_1<j_2<\ldots<j_k} B_\sigma^{j_1 j_2 \ldots j_k}(J_x^r\gamma) y^\sigma_{j_1 j_2 \ldots j_k i}(J_x^{r+1}\gamma) \right) \xi^i, \end{aligned} \tag{15}$$

thus, since $\xi^i = dx^i(J_x^{r+1}\gamma)(\xi)$,

$$h\rho = \left(A_i + \sum_{0\le k\le r} \sum_{j_1<j_2<\ldots<j_k} B_\sigma^{j_1 j_2 \ldots j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \right) dx^i. \tag{16}$$

In particular, for any function $f : W^r \to \mathbf{R}$

$$hdf = d_i f \cdot dx^i, \tag{17}$$

where

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{0\le k\le r} \sum_{j_1<j_2<\ldots<j_k} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i}. \tag{18}$$

The function $d_i f : V^{r+1} \to \mathbf{R}$ is called the *i-th formal derivative* of *f* with respect to the fibred chart (V,ψ). Note that formal derivatives (18) are *components* of an *invariant* object, namely the 1-form hdf.

Lemma 8 *Let* (V,ψ), $\psi = (x^i, y^\sigma)$, *be a fibred chart on Y.*
(a) *The horizontalisation h satisfies*

$$\begin{aligned} &hdx^i = dx^i, \quad hdy^\sigma = y_i^\sigma dx^i, \quad hdy^\sigma_{j_1} = y^\sigma_{j_1 i} dx^i, \\ &\quad hdy^\sigma_{j_1 j_2} = y^\sigma_{j_1 j_2 i} dx^i, \quad \ldots, \quad hdy^\sigma_{j_1 j_2 \ldots j_r} = y^\sigma_{j_1 j_2 \ldots j_r i} dx^i. \end{aligned} \tag{19}$$

(b) *The coordinate functions* $y^\nu_{j_1 j_2 \ldots j_k}$ *satisfy*

(20) $$d_i x^j = \delta_i^j, \quad d_i y^\nu_{j_1 j_2 \dots j_k} = y^\nu_{j_1 j_2 \dots j_k i}.$$

(c) *If* $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$, *is another chart on* Y *such that* $V \cap \bar{V} \neq \emptyset$, *then for every function* $f : V^r \cap \bar{V}^r \to \mathbf{R}$,

(21) $$\bar{d}_i f = d_j f \cdot \frac{\partial x^j}{\partial \bar{x}^i}.$$

(d) *For any two functions* $f, g : V^r \to \mathbf{R}$,

(22) $$d_i(f \cdot g) = g \cdot d_i f + f \cdot d_i g.$$

(e) *For every function* $f : V^r \to \mathbf{R}$ *and every section* $\gamma : U \to V \subset Y$,

(23) $$d_i f \circ J^{r+1}\gamma = \frac{\partial (f \circ J^r \gamma)}{\partial x^i}.$$

Proof (a) and (b) follow from (17) and (18). To derive (21), we write

(24) $$hdf = d_i f \cdot dx^i = d_i f \cdot \frac{\partial x^i}{\partial \bar{x}^j} d\bar{x}^j = \bar{d}_j f \cdot d\bar{x}^j.$$

(d) and (e) are immediate.

The following can be considered as a local definition of the homomorphism $h : \Omega^r W \to \Omega^{r+1} W$.

Theorem 4 (Local definition of horizontalisation) *There exists a unique linear over the ring of functions, exterior-product-preserving mapping of the exterior algebra* $\Omega^r W$ *into* $\Omega^{r+1} W$, *such that for any fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *where* $V \subset W$, *and any function* $f : W^r \to \mathbf{R}$

(25) $$hf = f \circ \pi^{r+1,r}, \quad hdf = d_i f \cdot dx^i,$$

where

(26) $$d_i f = \frac{\partial f}{\partial x^i} + \sum_{0 \le k \le r} \sum_{j_1 < j_2 < \dots < j_k} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \dots j_k}} y^\sigma_{j_1 j_2 \dots j_k i}.$$

Proof Clearly, formulas (25) and (26) locally define a unique mapping from $\Omega^r W$ to $\Omega^{r+1} W$, satisfying conditions (10) and (11) of Lemma 7 (the horizontalisation).

Remark 2 By (20), $\bar{y}^{\sigma}_{j_1 j_2 \dots j_k} = \bar{d}_{j_k} \bar{y}^{\sigma}_{j_1 j_2 \dots j_{k-1}}$. Thus, applying (21) to coordinates, we obtain the following *prolongation formula* for coordinate transformations in jet prolongations of fibred manifolds

$$(27) \qquad \bar{y}^{\sigma}_{j_1 j_2 \dots j_k} = d_i \bar{y}^{\sigma}_{j_1 j_2 \dots j_{k-1}} \cdot \frac{\partial x^i}{\partial \bar{x}^{j_k}}.$$

Remark 3 If two functions $f, g : V^r \to \mathbf{R}$ coincide along a section $J^r\gamma$, that is, $f \circ J^r\gamma = g \circ J^r\gamma$, then their formal derivatives coincide along the $(r+1)$-prolongation $J^{r+1}\gamma$,

$$(28) \qquad d_i f \circ J^{r+1}\gamma = d_i g \circ J^{r+1}\gamma.$$

This is an immediate consequence of formula (23).

1.6 Jet prolongations of automorphisms Let r be a positive integer. Consider an open set W in the fibred manifold Y and a C^r automorphism $\alpha : W \to Y$ with projection $\alpha_0 : W_0 \to X$, defined on an open set $W_0 = \pi(W)$. In this section we suppose that the projection α_0 is a C^r diffeomorphism.

Every section $\gamma : W_0 \to Y$ defines the mapping $\alpha\gamma\alpha_0^{-1} = \alpha \circ \gamma \circ \alpha_0^{-1}$; it is easily seen that this mapping is a section of Y over the open set $\alpha_0(W_0) \subset X$: indeed, using properties of homomorphisms and sections of fibred manifolds, we get $\pi \circ \alpha\gamma\alpha_0^{-1} = \alpha_0 \circ \pi \circ \gamma \circ \alpha_0^{-1} = \alpha_0 \circ \alpha_0^{-1} = \mathrm{id}_{W_0}$. Then, however, the r-jets of the section $x \to \alpha\gamma\alpha_0^{-1}(x)$ are defined and are elements of the set J^rY. Consider the r-jet $J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}$. It is immediately seen that this r-jet depends only on the r-jet $J^r_x\gamma$, that is, it is independent of the choice of a representative γ: indeed, applying the jet composition, we can write $J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1} = J^r_{\gamma(x)}\alpha \circ J^r_x\gamma \circ J^r_{\alpha_0(x)}\alpha_0^{-1}$ and, since the right-hand side depends on $J^r_x\gamma$ only, the r-jet $J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}$ does not depend on the choice of γ.

Now we denote $W^r = (\pi^{r,0})^{-1}(W)$, and set for every $J^r_x\gamma \in W^r = (\pi^{r,0})^{-1}(W)$

$$(1) \qquad J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}.$$

This formula defines a mapping $J^r\alpha : W^r \to J^rY$, called the *r-jet prolongation*, or just *prolongation* of the C^r automorphism α.

Note an immediate consequence of the definition (1). Given a C^r *section* $\gamma : V \to Y$, then we have $J^r\alpha \circ J^r\gamma = J^r\alpha\gamma\alpha_0^{-1} \circ \alpha_0$ so the r-jet prolongation $J^r\alpha\gamma\alpha_0^{-1}$ of the section $\alpha\gamma\alpha_0^{-1}$ satisfies

$$(2) \qquad J^r\alpha \circ J^r\gamma \circ \alpha_0^{-1} = J^r\alpha\gamma\alpha_0^{-1}$$

on the set $\alpha_0(V)$. In particular, this formula shows that the r-jet prolongations of automorphisms carry sections of Y into sections of J^rY (over X).

We find the chart expression of the mapping $J^r\alpha$.

Lemma 9 *Suppose that in two fibred charts on Y,* (V,ψ), $\psi=(x^i,y^\sigma)$, *and* $(\bar{V},\bar{\psi})$, $\bar{\psi}=(\bar{x}^i,\bar{y}^\sigma)$, *on Y the* C^r *automorphism* α, *restricted to V, is expressed by equations*

$$\bar{x}^i\circ\alpha(y)=f^i(x^j(x)),\quad \bar{y}^\sigma\circ\alpha(y)=F^\sigma(x^j(x),y^\nu(y)). \tag{3}$$

Then for every point $J^r_x\gamma\in V^r$, *the transformed point* $J^r\alpha(J^r_x\gamma)$ *has the coordinates*

$$\begin{aligned}
&\bar{x}^i\circ J^r\alpha(J^r_x\gamma)=f^i(x^j(x)),\\
&\bar{y}^\sigma\circ J^r\alpha(J^r_x\gamma)=F^\sigma(x^j(x),y^\nu(\gamma(x))),\\
&\bar{y}^\sigma_{j_1j_2\dots j_k}\circ J^r\alpha(J^r_x\gamma)\\
&\quad=D_{j_1}D_{j_2}\dots D_{j_k}(\bar{y}^\sigma\alpha\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))),\quad 1\le k\le r.
\end{aligned} \tag{4}$$

Proof We have

$$\begin{aligned}
&\bar{x}^i\circ J^r\alpha(J^r_x\gamma)=\bar{x}^i\circ\alpha_0(x)=\bar{x}^i\alpha_0\varphi^{-1}(\varphi(x))=f^i(x^j(x)),\\
&\bar{y}^\sigma\circ J^r\alpha(J^r_x\gamma)=\bar{y}^\sigma\circ\alpha(\gamma(x))=\bar{y}^\sigma\alpha\psi^{-1}(\psi(\gamma(x)))\\
&\quad=F^\sigma(x^j(x),y^\nu(\gamma(x))),
\end{aligned} \tag{5}$$

and by definition

$$\begin{aligned}
&\bar{y}^\sigma_{j_1j_2\dots j_k}\circ J^r\alpha(J^s_x\gamma)=\bar{y}^\sigma_{j_1j_2\dots j_k}(J^s_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1})\\
&\quad=D_{j_1}D_{j_2}\dots D_{j_k}(\bar{y}^\sigma\circ\alpha\gamma\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x)))\\
&\quad=D_{j_1}D_{j_2}\dots D_{j_k}(\bar{y}^\sigma\alpha\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))).
\end{aligned} \tag{6}$$

Formula (4) contains partial derivatives of the functions f^i and F^σ, and also partial derivatives of the functions g^k, representing the chart expression $\varphi\alpha_0^{-1}\bar{\varphi}^{-1}$ of the inverse diffeomorphism α_0^{-1}, and defined by

$$x^k\circ\alpha_0^{-1}(x')=g^k(\bar{x}^l(x')). \tag{7}$$

To obtain explicit dependence of the coordinate function $\bar{y}^\sigma_{j_1j_2\dots j_k}(J^r\alpha(J^r_x\gamma))$ on the coordinates of the r-jet $J^r_x\gamma$, we have to use the chain rule k times, which leads to polynomial dependence of $\bar{y}^{\sigma}_{j_1 j_2 \ldots j_k}(J^r\alpha(J^r_x\gamma))$ on the jet coordinates $y^{\nu}_{i_1}(J^r_x\gamma)$, $y^{\nu}_{i_1 i_2}(J^r_x\gamma)$, $\ldots$, $y^{\nu}_{i_1 i_2 \ldots i_k}(J^r_x\gamma)$. This shows, in particular, that if α is of class C^r, then $J^r\alpha$ is or class C^0; if α is of class C^s, where $s \geq r$, then $J^r\alpha$ is of class C^{s-r}.

Equations (4) can be viewed as recurrence formulas for the chart expression of the mapping $J^r\alpha$. Writing

$$\text{(8)} \qquad \bar{y}^{\sigma}_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha(J^r_x\gamma) = (\bar{y}^{\sigma}_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))),$$

we have

$$\text{(9)} \qquad \begin{aligned} &\bar{y}^{\sigma}_{j_1 j_2 \cdots j_k} \circ J^r\alpha(J^r_x\gamma) \\ &= D_{j_k}(\bar{y}^{\sigma}_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &= D_l(\bar{y}^{\sigma}_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1})(\varphi(x)) D_{j_k}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))). \end{aligned}$$

Thus, if we already have the functions $\bar{y}^{\sigma}_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha$, then the functions $\bar{y}^{\sigma}_{j_1 j_2 \cdots j_k} \circ J^r\alpha$ is determined by (4).

As an example we derive explicit expressions for the second jet prolongation $J^2\alpha$.

Example 6 (Second order prolongation of an automorphism) Let $r = 2$. We have from (3)

$$\text{(10)} \qquad \begin{aligned} &\bar{y}^{\sigma}_{j_1} \circ J^2\alpha(J^r_x\gamma) = D_{j_1}(\bar{y}^{\sigma}\alpha\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &= D_k(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x))\delta^k_l D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &+ D_{\lambda}(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) y^{\lambda}_l(J^r_x\gamma) D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &= (D_l(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) + D_{\lambda}(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) y^{\lambda}_l(J^s_x\gamma)) \\ &\cdot D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))), \end{aligned}$$

or, in a different notation,

$$\text{(11)} \qquad \bar{y}^{\sigma}_{j_1} \circ J^2\alpha(J^r_x\gamma) = d_l F^{\sigma}(J^r_x\gamma)\left(\frac{\partial g^l}{\partial \bar{x}^{j_1}}\right)_{\bar{\varphi}(\alpha_0(x))},$$

where d_l is the formal derivative operator. Differentiating formula (10) or (11) again we get the following equations for the 2-jet prolongation $J^2\alpha$ of the automorphism α

(12)
$$\bar{x}^i = f^i(x^i), \quad \bar{y}^\sigma = F^\sigma(x^i, y^\nu), \quad \bar{y}^\sigma_{j_1} = d_{k_1} F^\sigma \cdot \frac{\partial g^{k_1}}{\partial \bar{x}^{j_1}},$$
$$\bar{y}^\sigma_{j_1 j_2} = d_{k_1} d_{k_2} F^\sigma \cdot \frac{\partial g^{k_1}}{\partial \bar{x}^{j_1}} \frac{\partial g^{k_2}}{\partial \bar{x}^{j_2}} + d_{k_1} F^\sigma \cdot \frac{\partial^2 g^{k_1}}{\partial \bar{x}^{j_1} \partial \bar{x}^{j_2}},$$

where d_k denotes the formal derivative operator (Section 1.5).

Using our previous notation we can easily prove the following statements.

Lemma 10 (a) *For any s,* $0 \le s \le r$,

(13) $$\pi^r \circ J^r\alpha = \alpha_0 \circ \pi^r, \quad \pi^{r,s} \circ J^r\alpha = J^s\alpha \circ \pi^{r,s}.$$

(b) *If two* C^r *automorphisms* α *and* β *of the fibred manifold Y are composable, then*

(14) $$J^r\alpha \circ J^r\beta = J^r(\alpha \circ \beta).$$

(c) *For any* C^{r+1} *automorphism* α *of Y, and any differential form* ρ *on* J^rY ,

(15) $$J^{r+1}\alpha * h\rho = hJ^r\alpha * \rho.$$

Proof All these assertions are easy consequences of definitions.

Formula (13) shows that $J^r\alpha$ is an C^r automorphism of the *r*-jet prolongation J^rY of the fibred manifold *Y*, whose projection is a diffeomorphism α_0 . We call this C^r automorphism the *r-jet prolongation* of α .

1.7 Jet prolongations of vector fields Let *Y* be a fibred manifold with base *X* and projection π . In this section we apply the theory of jet prolongations of automorphisms of fibred manifolds to local flows of vector fields, defined on *Y*.

Let Ξ be a C^r vector field on *Y*, let $y_0 \in Y$ be a point, and consider a local flow $\alpha^\Xi : (-\varepsilon, \varepsilon) \times V \to Y$ of Ξ at y_0 . As usual, define the mappings α^Ξ_t and α^Ξ_y by

(1) $$\alpha^\Xi(t, y) = \alpha^\Xi_t(y) = \alpha^\Xi_y(t).$$

Then for any point $y \in V$ the mapping $t \to \alpha^\Xi_y(t)$ is an integral curve of Ξ passing through *y* at $t = 0$, i.e.,

(2) $$T_t\alpha_y^\Xi = \Xi(\alpha_y^\Xi(t)), \quad \alpha_y^\Xi(0) = y.$$

Moreover, shrinking the domain of definition $(-\varepsilon,\varepsilon)\times V$ of α^Ξ to a subset $(-\kappa,\kappa)\times W \subset (-\varepsilon,\varepsilon)\times V$, where W is a neighbourhood of the point y_0, we have

(3) $$\alpha^\Xi(s+t,y) = \alpha^\Xi(s,\alpha^\Xi(t,y)), \quad \alpha^\Xi(-t,\alpha^\Xi(t,y)) = y$$

for all $(s,t) \in (-\kappa,\kappa)$ and $y \in W$ or, which is the same.

(4) $$\alpha_{s+t}^\Xi(y) = \alpha_s^\Xi(\alpha_t^\Xi(y)), \quad \alpha_{-t}^\Xi\alpha_t^\Xi(y) = y.$$

Note that the second formula implies

(5) $$(\alpha_t^\Xi)^{-1} = \alpha_{-t}^\Xi.$$

In the following lemma we study properties of flows of a π *-projectable* vector field.

Lemma 11 *Let* Ξ *be a* C^r *vector field on* Y. *The following two conditions are equivalent:*

(1) *The local* 1*-parameter groups of* Ξ *consist of* C^r *automorphisms of the fibred manifold* Y.

(2) Ξ *is* π *-projectable*.

Proof 1. Let $y_0 \in Y$ be a point and let $x_0 = \pi(y_0)$. Choose a local flow $\alpha^\Xi : (-\varepsilon,\varepsilon)\times V \to Y$ at y_0, and suppose that the mappings $\alpha_t^\Xi : V \to Y$ are C^r automorphisms of Y. Then for each t there exists a unique C^r mapping $\alpha_t : U \to X$, where $U = \pi(V)$ is an open set, such that

(6) $$\pi \circ \alpha_t^\Xi = \alpha_t \circ \pi$$

on V. Setting $\alpha(t,x) = \alpha_t(x)$ we get a mapping $\alpha : (-\varepsilon,\varepsilon)\times U \to X$. It is easily seen that this mapping is of class C^r. Indeed, there exists a C^r section $\gamma : U \to Y$ such that $\gamma(x_0) = y_0$ (Section **1.1**, Theorem 3); using this section we can write $\alpha(t,x) = \alpha_t(x) = \pi \circ \alpha_t^\Xi \circ \gamma(x) = \pi \circ \alpha^\Xi(t,\gamma(x))$, so α can be expressed as the composite of C^r-mappings. Since α satisfies $\alpha(0,x) = x$, setting

(7) $$\xi(x) = T_0\alpha_x \cdot 1$$

we get a C^{r-1} vector field on U.

On the other hand, formula (6) implies $\pi \circ \alpha^\Xi(t,y) = \alpha(t,\pi(y))$, that is, $\pi \circ \alpha_y^\Xi = \alpha_{\pi(y)}$. Then from (2) $T_t(\pi \circ \alpha_y^\Xi) = T_{\alpha_y^\Xi(t)}\pi \cdot \Xi(\alpha_y^\Xi(t)) = T_t\alpha_{\pi(y)}$ and se have at the point $t=0$

(8) $$T_0\alpha_{\pi(y)} = T_y\pi \cdot \Xi(y).$$

Combining (7) and (8),

(9) $$\xi(\pi(y)) = T_y\pi \cdot \Xi(y).$$

This proves π-projectability of Ξ on *V*. π-projectability of Ξ (on *Y*) now follows form the uniqueness of the π-projection.

2. Suppose that Ξ is π-projectable and denote by ξ its π-projection. Then

(10) $$T_y\pi \cdot \Xi(y) = \xi(\pi(y))$$

at every point of *Y*. The local flow α^{Ξ} satisfies equation (2) $T_t\alpha_y^{\Xi} = \Xi(\alpha_y^{\Xi}(t))$. Applying the tangent mapping $T\pi$ to both sides we get

(11) $$T_t(\pi \circ \alpha_y^{\Xi}) = T_{\alpha_y^{\Xi}(t)}\pi \cdot \Xi(\alpha_y^{\Xi}(t)) = \xi(\pi(\alpha_y^{\Xi}(t))).$$

This equality means that the curve $t \to \pi(\alpha_y^{\Xi}(t)) = \alpha_{\pi(y)}^{\xi}(t)$ is an integral curve of the vector field ξ. Thus, denoting by α^{ξ} the local flow of ξ at the point $x_0 = \pi(y_0)$, we have

(12) $$\pi(\alpha^{\Xi}(t,y)) = \alpha^{\xi}(t,\pi(y))$$

as required.

Let Ξ be a π-projectable C^r vector field on *Y*, ξ its π-projection, α_t^{Ξ} (resp. α_t^{ξ}) the local 1-parameter group of Ξ (resp. ξ). Since the mappings α_t^{ξ} are C^r diffeomorphisms, for each *t* the C^r automorphism α_t^{Ξ} can be prolonged to the jet prolongation J^sY of *Y*, for any s, $0 \le s \le r$. The prolonged mapping is an automorphism of the fibred manifold J^sY over *X*, defined by

(13) $$J^s\alpha_t^{\Xi}(J_x^r\gamma) = J_{\alpha_t^{\xi}(x)}^s \alpha_t^{\Xi}\gamma\alpha_{-t}^{\xi},$$

the *s-jet prolongation* of α_t^{Ξ}.

It is easily seen that there exists a unique C^s vector field on J^sY whose integral curves are exactly the curves $t \to J^s\alpha_t^{\Xi}(J_x^r\gamma)$. This vector field is defined by

(14) $$J^s\Xi(J_x^r\gamma) = \left(\frac{d}{dt}J^s\alpha_t^{\Xi}(J_x^r\gamma)\right)_0,$$

and is called the *r-jet prolongation* of Ξ. It follows from the definition that $J^s\Xi$ is π^s-projectable (resp. $\pi^{r,s}$-projectable for any s, $0 \le s \le r$) and its π^r-projection (resp. $\pi^{r,s}$-projection) is ξ (resp. $J^s\Xi$).

The following lemma describes the local structure of the jet prolongations of projectable vector fields.

Lemma 12 *Let* Ξ *be a* π*-projectable vector field on Y,* (V,ψ), $\psi = (x^i, y^\sigma)$*, a fibred chart on Y, and let* Ξ *be expressed by*

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}. \tag{15}$$

Then $J^s\Xi$ *is expressed with respect to the associated chart* (V^s, ψ^s) *by*

$$J^s\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \sum_{k=1}^{s} \sum_{j_1 \le j_2 \le \ldots \le j_k} \Xi^\sigma_{j_1 j_2 \ldots j_k} \frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}, \tag{16}$$

where the components $\Xi^\sigma_{j_1 j_2 \ldots j_k}$ *are determined by the recurrence formula*

$$\Xi^\sigma_{j_1 j_2 \ldots j_k} = d_{j_k} \Xi^\sigma_{j_1 j_2 \ldots j_{k-1}} - y^\sigma_{j_1 j_2 \ldots j_{k-1} i} \frac{\partial \xi^i}{\partial x^{j_k}}. \tag{17}$$

Proof For sufficiently small t we can express the local 1-parameter group of Ξ in one chart only. Replacing α with α^Ξ_t, α_0 with α^ξ_t and α_0^{-1} with α^ξ_{-t} we get the following equations of the C^r automorphism α^Ξ_t:

$$x^i \circ \alpha^\Xi_t(y) = x^i \alpha^\xi_t(x), \quad y^\sigma \circ \alpha^\Xi_t(y) = y^\sigma \alpha^\Xi_t(y). \tag{18}$$

Thus the components of the vector field Ξ can be written as

$$\xi^i(y) = \left(\frac{d x^i \alpha^\xi_t(x)}{dt} \right)_0, \quad \Xi^\sigma(y) = \left(\frac{d y^\sigma \alpha^\Xi_t(y)}{dt} \right)_0. \tag{19}$$

To determine the components of $J^s\Xi$ we use 1.6, Lemma 9. The 1-parameter group of $J^s\Xi$ has the equations

$$\begin{aligned} & x^i \circ J^r \alpha^\Xi_t(y) = x^i \alpha^\xi_t(x), \\ & y^\sigma \circ J^r \alpha^\Xi_t(y) = y^\sigma \alpha^\Xi_t(y), \\ & y^\sigma_{j_1 j_2 \ldots j_k} \circ J^r \alpha^\Xi_t(J^r_x \gamma) \\ & \quad = D_{j_1} D_{j_2} \ldots D_{j_k} (y^\sigma \alpha^\Xi_t \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \alpha^\xi_{-t} \varphi^{-1})(\varphi(\alpha^\xi_t(x))), \quad 1 \le k \le s, \end{aligned} \tag{20}$$

so by (18) it is sufficient to determine $\Xi^{\sigma}_{j_1 j_2 \dots j_k}$. By definition,

(21) $$\Xi^{\sigma}_{j_1 j_2 \dots j_k}(J^r_x\gamma) = \left(\frac{d}{dt}(y^{\sigma}_{j_1 j_2 \dots j_k} \circ J^r\alpha^{\Xi}_t)(J^r_x\gamma) \right)_0 .$$

But

(22) $$\begin{aligned} &y^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^r\alpha^{\Xi}_t(J^r_x\gamma) \\ &\quad = D_{j_1} D_{j_2} \dots D_{j_{k-1}}(y^{\sigma}\alpha^{\Xi}_t\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha^{\xi}_{-t}\varphi^{-1})(\varphi(\alpha^{\xi}_t(x))) \\ &\quad = y^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^r\alpha^{\Xi}_t \circ J^r\gamma \circ \alpha^{\xi}_{-t}\varphi^{-1}(\varphi(\alpha^{\xi}_t(x))), \end{aligned}$$

thus,

(23) $$\begin{aligned} &y^{\sigma}_{j_1 j_2 \dots j_k} \circ J^r\alpha^{\Xi}_t(J^r_x\gamma) \\ &\quad = D_{j_k}(y^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^r\alpha^{\Xi}_t \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\alpha^{\xi}_{-t}\varphi^{-1})(\varphi(\alpha^{\xi}_t(x))) \\ &\quad = D_l(y^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^r\alpha^{\Xi}_t \circ J^r\gamma \circ \varphi^{-1})(\varphi(x)) D_{j_k}(x^l\alpha^{\xi}_{-t}\varphi^{-1})(\varphi(\alpha^{\xi}_t(x))). \end{aligned}$$

To obtain (21) we differentiate in this formula the function

(24) $$(t,\varphi(x)) \to (y^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^r\alpha^{\Xi}_t \circ J^r\gamma \circ \varphi^{-1})(\varphi(x)) = y^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^r\alpha^{\Xi}_t(J^r_x\gamma)$$

with respect to t and x^l. Since the partial derivatives commute, we can first differentiate with respect to t at $t = 0$. We get the expression $\Xi^{\sigma}_{j_1 j_2 \dots j_{k-1}}(J^r_x\gamma)$. Subsequent differentiation yields

(25) $$D_l(\Xi^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^r\gamma \circ \varphi^{-1})(\varphi(x)) = d_l\Xi^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^{r+1}\gamma,$$

where d_l is the formal derivative operator (Section **1.5**, Lemma 8).

We should also differentiate expression $D_{j_k}(x^l\alpha^{\xi}_{-t}\varphi^{-1})(\varphi(\alpha^{\xi}_t(x)))$ with respect to t. We write the identity $D_l(x^k\alpha^{\xi}_{-t}\varphi^{-1} \circ \varphi\alpha^{\xi}_t\varphi^{-1})(\varphi(x)) = \delta^k_l$ as

(26) $$D_j(x^k\alpha^{\xi}_{-t}\varphi^{-1})(\varphi\alpha^{\xi}_t(x)) D_l(x^j\alpha^{\xi}_t\varphi^{-1})(\varphi(x)) = \delta^k_l .$$

From this formula

(27) $$\begin{aligned} &\frac{d}{dt} D_j(x^k\alpha^{\xi}_{-t}\varphi^{-1})(\varphi\alpha^{\xi}_t(x)) \cdot D_l(x^j\alpha^{\xi}_t\varphi^{-1})(\varphi(x)) \\ &\quad + D_j(x^k\alpha^{\xi}_{-t}\varphi^{-1})(\varphi\alpha^{\xi}_t(x)) \cdot \frac{d}{dt} D_l(x^j\alpha^{\xi}_t\varphi^{-1})(\varphi(x)) = 0, \end{aligned}$$

thus, at $t=0$

$$\left(\frac{d}{dt}D_j(x^k\alpha^{\xi}_{-t}\varphi^{-1})(\varphi\alpha^{\xi}_t(x))\right)_0\cdot\delta^j_l+\delta^k_jD_l\xi^j(\varphi(x))=0, \tag{28}$$

hence

$$\left(\frac{d}{dt}D_l(x^k\alpha^{\xi}_{-t}\varphi^{-1})(\varphi\alpha^{\xi}_t(x))\right)_0=-D_l\xi^k(\varphi(x)). \tag{29}$$

Now we can complete the differentiation of formula (23) at $t=0$. We have

$$\begin{aligned}\Xi^{\sigma}_{j_1j_2\dots j_k}(J^r_x\gamma)&=\left(\frac{d}{dt}(y^{\sigma}_{j_1j_2\dots j_k}\circ J^r\alpha^{\Xi}_t)(J^r_x\gamma)\right)_0\\&=(d_l\Xi^{\sigma}_{j_1j_2\dots j_{k-1}}\circ J^r\gamma)(x)D_{j_k}(x^l\varphi^{-1})(\varphi(x))\\&\quad-D_l(y^{\sigma}_{j_1j_2\dots j_{k-1}}\circ J^r\gamma\circ\varphi^{-1})(\varphi(x)D_{j_k}\xi^l(\varphi(x))\\&=d_l\Xi^{\sigma}_{j_1j_2\dots j_{k-1}}(J^r_x\gamma)\delta^l_{j_k}-y^{\sigma}_{j_1j_2\dots j_{k-1}l}(J^r_x\gamma)D_{j_k}\xi^l(\varphi(x))\\&=d_{j_k}\Xi^{\sigma}_{j_1j_2\dots j_{k-1}}(J^r_x\gamma)-y^{\sigma}_{j_1j_2\dots j_{k-1}l}(J^r_x\gamma)D_{j_k}\xi^l(\varphi(x)),\end{aligned} \tag{30}$$

which coincides with (17).

Remark 4 Sometimes it is convenient to express tangent vectors and vector fields on J^rY with different summation convention. We can formally introduce the convention as follows. Let Ξ be a tangent vector at a point $J^r_x\gamma\in J^rY$. In a fibred chart at this point

$$\begin{aligned}\Xi&=\xi^i\left(\frac{\partial}{\partial x^i}\right)_{J^r_x\gamma}+\Xi^{\sigma}\left(\frac{\partial}{\partial y^{\sigma}}\right)_{J^r_x\gamma}\\&\quad+\sum_{k=1}^{s}\sum_{j_1\le j_2\le\dots\le j_k}\Xi^{\sigma}_{j_1j_2\dots j_k}\left(\frac{\partial}{\partial y^{\sigma}_{j_1j_2\dots j_k}}\right)_{J^r_x\gamma},\end{aligned} \tag{31}$$

where $\xi^i,\Xi^{\sigma},\Xi^{\sigma}_{j_1j_2\dots j_k}\in\mathbf{R}$. In this formula we sum through non-decreasing *k*-tuples $j_1j_2\dots j_k$; we want to extend the summation to *all* *k*-tuples. Since for any element τ of the symmetric group S_k the coordinate functions $y^{\sigma}_{j_{\tau(1)}j_{\tau(2)}\dots j_{\tau(k)}}$ are defined and are equal to $y^{\sigma}_{j_1j_2\dots j_k}$, with $j_1\le j_2\le\dots\le j_k$, setting $\Xi^{\sigma}_{j_{\tau(1)}j_{\tau(2)}\dots j_{\tau(k)}}=\Xi^{\sigma}_{j_1j_2\dots j_k}$ we can write

$$\Xi = \xi^i \left(\frac{\partial}{\partial x^i} \right)_{J^r_x \gamma} + \Xi^\sigma \left(\frac{\partial}{\partial y^\sigma} \right)_{J^r_x \gamma}$$

$$+ \sum_{k=1}^{s} \sum_{\tau \in S_k} \frac{1}{k!} \Xi^\sigma_{j_{\tau(1)} j_{\tau(2)} \cdots j_{\tau(k)}} \left(\frac{\partial}{\partial y^\sigma_{j_{\tau(1)} j_{\tau(2)} \cdots j_{\tau(k)}}} \right)_{J^r_x \gamma} \tag{32}$$

$$= \xi^i \left(\frac{\partial}{\partial x^i} \right)_{J^r_x \gamma} + \Xi^\sigma \left(\frac{\partial}{\partial y^\sigma} \right)_{J^r_x \gamma} + \sum_{k=1}^{s} \frac{1}{k!} \Xi^\sigma_{j_1 j_2 \cdots j_k} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \cdots j_k}} \right)_{J^r_x \gamma}$$

with the summation understood through all j_1, j_2, ..., j_k. This implies, in particular, to jet prolongations of vector fields; formula (16) can also be written as

$$J^s \Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \sum_{k=1}^{s} \frac{1}{k!} \Xi^\sigma_{j_1 j_2 \cdots j_k} \frac{\partial}{\partial y^\sigma_{j_1 j_2 \cdots j_k}}. \tag{33}$$

Example 7 (Second jet prolongation of a vector field) If a π-projectable vector field Ξ is expressed by

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}, \tag{34}$$

then

$$J^2 \Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \Xi^\sigma_j \frac{\partial}{\partial y^\sigma_j} + \sum_{j \le k} \Xi^\sigma_{jk} \frac{\partial}{\partial y^\sigma_{jk}}, \tag{35}$$

where

$$\Xi^\sigma_j = d_j \Xi^\sigma - y^\sigma_i \frac{\partial \xi^i}{\partial x^j}, \quad \Xi^\sigma_{jk} = d_j d_k \Xi^\sigma - y^\sigma_{ij} \frac{\partial \xi^i}{\partial x^k} - y^\sigma_{ik} \frac{\partial \xi^i}{\partial x^j} - y^\sigma_i \frac{\partial^2 \xi^i}{\partial x^j \partial x^k}. \tag{36}$$

In the following lemma we study the *Lie bracket* of *r*-jet prolongations of projectable vector fields, and the *Lie derivatives* by these vector fields.

Lemma 13 (a) *Let* Ξ *and* Z *be two* π*-projectable vector fields. Then the Lie bracket* $[\Xi, \mathrm{Z}]$ *is also* π*-projectable, and*

$$J^r [\Xi, \mathrm{Z}] = \left[J^r \Xi, J^r \mathrm{Z} \right]. \tag{37}$$

(b) *For any* π *-projectable vector field* Ξ *, and any differential form* ρ *on* $J^r Y$ *,*

$$\partial_{J^{r+1}\Xi} h\rho = h\partial_{J^r\Xi}\rho. \tag{38}$$

Proof 1. First we prove (a) for $r=1$; the proof in a fibred chart consists of checking formula (37) and is trivial. Suppose we have in a fibred chart

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}, \quad \mathrm{Z} = \zeta^k \frac{\partial}{\partial x^k} + \mathrm{Z}^\nu \frac{\partial}{\partial y^\nu}. \tag{39}$$

Then

$$\begin{aligned} J^1\Xi &= \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \Xi^\sigma_j \frac{\partial}{\partial y^\sigma_j}, \\ J^1\mathrm{Z} &= \zeta^i \frac{\partial}{\partial x^i} + \mathrm{Z}^\sigma \frac{\partial}{\partial y^\sigma} + \mathrm{Z}^\sigma_j \frac{\partial}{\partial y^\sigma_j}, \end{aligned} \tag{40}$$

where

$$\Xi^\sigma_j = d_j \Xi^\sigma - y^\sigma_i \frac{\partial \xi^i}{\partial x^j}, \quad \mathrm{Z}^\sigma_j = d_j \mathrm{Z}^\sigma - y^\sigma_i \frac{\partial \zeta^i}{\partial x^j}, \tag{41}$$

and

$$\begin{aligned} [J^1\Xi, J^1\mathrm{Z}] &= \left(\frac{\partial \zeta^i}{\partial x^l}\xi^l - \frac{\partial \xi^i}{\partial x^l}\zeta^l \right) \frac{\partial}{\partial x^i} \\ &+ \left(\frac{\partial \mathrm{Z}^\sigma}{\partial x^l}\xi^l + \frac{\partial \mathrm{Z}^\sigma}{\partial y^\nu}\Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^l}\zeta^l - \frac{\partial \Xi^\sigma}{\partial y^\nu}\mathrm{Z}^\nu \right) \frac{\partial}{\partial y^\sigma} \\ &+ \left(\frac{\partial \mathrm{Z}^\sigma_j}{\partial x^l}\xi^l + \frac{\partial \mathrm{Z}^\sigma_j}{\partial y^\nu}\Xi^\nu + \frac{\partial \mathrm{Z}^\sigma_j}{\partial y^\nu_l}\Xi^\nu_l - \frac{\partial \Xi^\sigma_j}{\partial x^l}\zeta^l - \frac{\partial \Xi^\sigma_j}{\partial y^\nu}\mathrm{Z}^\nu - \frac{\partial \Xi^\sigma_j}{\partial y^\nu_l}\mathrm{Z}^\nu_l \right) \frac{\partial}{\partial y^\sigma_j}. \end{aligned} \tag{42}$$

On the other hand, denoting $\Theta = [\Xi, \mathrm{Z}]$ we have

$$\Theta = \vartheta^i \frac{\partial}{\partial x^i} + \Theta^\sigma \frac{\partial}{\partial y^\sigma}, \tag{43}$$

where

$$\vartheta^i = \frac{\partial \zeta^i}{\partial x^s}\xi^s - \frac{\partial \xi^i}{\partial x^s}\zeta^s, \quad \Theta^\sigma = \frac{\partial Z^\sigma}{\partial x^s}\xi^s + \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^s}\zeta^s - \frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu, \tag{44}$$

and

$$J^1\Theta = \vartheta^i \frac{\partial}{\partial x^i} + \Theta^\sigma \frac{\partial}{\partial y^\sigma} + \Theta^\sigma_j \frac{\partial}{\partial y^\sigma_j}, \tag{45}$$

where

$$\Theta^\sigma_j = d_j\Theta^\sigma - y^\sigma_i \frac{\partial \vartheta^i}{\partial x^j}. \tag{46}$$

Comparing formulas (35) and (45) we see that to prove assertion (a) for $r=1$ it is sufficient to show that

$$\begin{aligned} &d_j\left(\frac{\partial Z^\sigma}{\partial x^s}\xi^s + \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^s}\zeta^s - \frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu\right) \\ &- y^\sigma_i \frac{\partial}{\partial x^j}\left(\frac{\partial \zeta^i}{\partial x^s}\xi^s - \frac{\partial \xi^i}{\partial x^s}\zeta^s\right) \\ &= \frac{\partial Z^\sigma_j}{\partial x^l}\xi^l + \frac{\partial Z^\sigma_j}{\partial y^\nu}\Xi^\nu + \frac{\partial Z^\sigma_j}{\partial y^\nu_l}\Xi^\nu_l - \frac{\partial \Xi^\sigma_j}{\partial x^l}\zeta^l - \frac{\partial \Xi^\sigma_j}{\partial y^\nu}Z^\nu - \frac{\partial \Xi^\sigma_j}{\partial y^\nu_l}Z^\nu_l. \end{aligned} \tag{47}$$

We we consider the lef-hand side and the right-hand side of this formula separately. The left-hand side can be expressed as

$$\begin{aligned} &d_j\left(\frac{\partial Z^\sigma}{\partial x^s}\xi^s + \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^s}\zeta^s - \frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu\right) \\ &- y^\sigma_i \frac{\partial}{\partial x^j}\left(\frac{\partial \zeta^i}{\partial x^s}\xi^s - \frac{\partial \xi^i}{\partial x^s}\zeta^s\right) \\ &= d_j\frac{\partial Z^\sigma}{\partial x^s}\xi^s + \frac{\partial Z^\sigma}{\partial x^s}\frac{\partial \xi^s}{\partial x^j} + d_j\frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu + \frac{\partial Z^\sigma}{\partial y^\nu}d_j\Xi^\nu \\ &- d_j\frac{\partial \Xi^\sigma}{\partial x^s}\zeta^s - \frac{\partial \Xi^\sigma}{\partial x^s}\frac{\partial \zeta^s}{\partial x^j} - d_j\frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu - \frac{\partial \Xi^\sigma}{\partial y^\nu}d_jZ^\nu \\ &- y^\sigma_i\left(\frac{\partial^2 \zeta^i}{\partial x^j\partial x^s}\xi^s + \frac{\partial \zeta^i}{\partial x^s}\frac{\partial \xi^s}{\partial x^j} - \frac{\partial^2 \xi^i}{\partial x^j\partial x^s}\zeta^s - \frac{\partial \xi^i}{\partial x^s}\frac{\partial \zeta^s}{\partial x^j}\right). \end{aligned} \tag{48}$$

The right-hand side of (47) is

$$\frac{\partial Z_j^\sigma}{\partial x^l}\xi^l + \frac{\partial Z_j^\sigma}{\partial y^\nu}\Xi^\nu + \frac{\partial Z_j^\sigma}{\partial y_l^\nu}\Xi_l^\nu - \frac{\partial \Xi_j^\sigma}{\partial x^l}\zeta^l - \frac{\partial \Xi_j^\sigma}{\partial y^\nu}Z^\nu - \frac{\partial \Xi_j^\sigma}{\partial y_l^\nu}Z_l^\nu$$

$$= \left(d_j \frac{\partial Z^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \zeta^i}{\partial x^l \partial x^j} \right)\xi^l + d_j \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu + \frac{\partial}{\partial y_l^\nu}\left(d_j Z^\sigma - y_i^\sigma \frac{\partial \zeta^i}{\partial x^j} \right)\Xi_l^\nu$$

$$- \left(d_j \frac{\partial \Xi^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \xi^i}{\partial x^l \partial x^j} \right)\zeta^l - d_j \frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu - \frac{\partial}{\partial y_l^\nu}\left(d_j \Xi^\sigma - y_i^\sigma \frac{\partial \xi^i}{\partial x^j} \right)Z_l^\nu$$

$$= \left(d_j \frac{\partial Z^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \zeta^i}{\partial x^l \partial x^j} \right)\xi^l + d_j \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu + \left(d_j \Xi^\nu - y_i^\nu \frac{\partial \xi^i}{\partial x^j} \right)\frac{\partial Z^\sigma}{\partial y^\nu} \tag{49}$$

$$- \left(d_l \Xi^\sigma - y_i^\sigma \frac{\partial \xi^i}{\partial x^l} \right)\frac{\partial \zeta^l}{\partial x^j}$$

$$- \left(d_j \frac{\partial \Xi^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \xi^i}{\partial x^l \partial x^j} \right)\zeta^l - d_j \frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu - \left(d_j Z^\nu - y_i^\nu \frac{\partial \zeta^i}{\partial x^j} \right)\frac{\partial \Xi^\sigma}{\partial y^\nu}$$

$$+ \left(d_l Z^\sigma - y_i^\sigma \frac{\partial \zeta^i}{\partial x^l} \right)\frac{\partial \xi^l}{\partial x^j}.$$

In this formula

$$d_l Z^\sigma \frac{\partial \xi^l}{\partial x^j} - y_i^\nu \frac{\partial \xi^i}{\partial x^j}\frac{\partial Z^\sigma}{\partial y^\nu}$$

$$= \frac{\partial Z^\sigma}{\partial x^l}\frac{\partial \xi^l}{\partial x^j} + \frac{\partial Z^\sigma}{\partial y^\nu} y_l^\nu \frac{\partial \xi^l}{\partial x^j} - y_i^\nu \frac{\partial \xi^i}{\partial x^j}\frac{\partial Z^\sigma}{\partial y^\nu} \tag{50}$$

$$= \frac{\partial Z^\sigma}{\partial x^l}\frac{\partial \xi^l}{\partial x^j},$$

and

$$-d_l \Xi^\sigma \frac{\partial \zeta^l}{\partial x^j} + y_i^\nu \frac{\partial \zeta^i}{\partial x^j}\frac{\partial \Xi^\sigma}{\partial y^\nu}$$

$$= -\frac{\partial \Xi^\sigma}{\partial x^l}\frac{\partial \zeta^l}{\partial x^j} - \frac{\partial \Xi^\sigma}{\partial y^\nu} y_l^\nu \frac{\partial \zeta^l}{\partial x^j} + y_i^\nu \frac{\partial \zeta^i}{\partial x^j}\frac{\partial \Xi^\sigma}{\partial y^\nu} \tag{51}$$

$$= -\frac{\partial \Xi^\sigma}{\partial x^l}\frac{\partial \zeta^l}{\partial x^j},$$

thus,

$$\frac{\partial Z_j^\sigma}{\partial x^l}\xi^l+\frac{\partial Z_j^\sigma}{\partial y^\nu}\Xi^\nu+\frac{\partial Z_j^\sigma}{\partial y_l^\nu}\Xi_l^\nu-\frac{\partial \Xi_j^\sigma}{\partial x^l}\zeta^l-\frac{\partial \Xi_j^\sigma}{\partial y^\nu}Z^\nu-\frac{\partial \Xi_j^\sigma}{\partial y_l^\nu}Z_l^\nu$$

$$(52)\qquad \begin{aligned}&=\left(d_j\frac{\partial Z^\sigma}{\partial x^l}-y_i^\sigma\frac{\partial^2\zeta^i}{\partial x^l\partial x^j}\right)\xi^l+d_j\frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu+d_j\Xi^\nu\frac{\partial Z^\sigma}{\partial y^\nu}+y_i^\sigma\frac{\partial \xi^i}{\partial x^l}\frac{\partial \zeta^l}{\partial x^j}\\&-\left(d_j\frac{\partial \Xi^\sigma}{\partial x^l}-y_i^\sigma\frac{\partial^2\xi^i}{\partial x^l\partial x^j}\right)\zeta^l-d_j\frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu-d_jZ^\nu\frac{\partial \Xi^\sigma}{\partial y^\nu}-y_i^\sigma\frac{\partial \zeta^i}{\partial x^l}\frac{\partial \xi^l}{\partial x^j}.\\&+\frac{\partial Z^\sigma}{\partial x^l}\frac{\partial \xi^l}{\partial x^j}-\frac{\partial \Xi^\sigma}{\partial x^l}\frac{\partial \zeta^l}{\partial x^j}.\end{aligned}$$

This is, however, exactly expression (47), proving (a) for $r=1$.

2. In this part of the proof we consider the r-jet prolongation $J^{r-1}Y$ as a fibred manifold with base X and projection $\pi^{r-1}:J^{r-1}Y\to X$, and the 1-jet prolongation of this fibred manifold, $J^1J^{r-1}Y$. Namely, we study the *canonical injection*

$$(53)\qquad J^rY\ni J_x^r\gamma\to\iota(J_x^r\gamma)=J_x^1J^{r-1}\gamma\in J^1J^{r-1}Y.$$

Obviously, ι is compatible with jet prolongations of automorphisms α of Y in the sense that

$$(54)\qquad \iota\circ J^r\alpha=(J^1J^{r-1}\alpha)\circ\iota.$$

Indeed, we have for any point $J_x^r\gamma$ from the domain of $J^r\alpha$

$$(55)\qquad \iota(J^r\alpha(J_x^r\gamma))=\iota(J_{\alpha_0(x)}^r\alpha\gamma\alpha_0^{-1})=J_{\alpha_0(x)}^1(J^{r-1}\alpha\gamma\alpha_0^{-1}),$$

and also

$$(56)\qquad J^1J^{r-1}\alpha(\iota(J_x^r\gamma))=J^1J^{r-1}\alpha(J_x^1J^{r-1}\gamma)=J_{\alpha_0(x)}^1(J^{r-1}\alpha\circ J^{r-1}\gamma\circ\alpha_0^{-1}).$$

Thus (54) follows from **1.6**, (2).

Then, however, applying (56) to local 1-parameter groups of a π-projectable vector field Ξ, we get ι-compatibility of $J^1J^{r-1}\Xi$ and $J^r\Xi$,

$$(57)\qquad J^1J^{r-1}\Xi\circ\iota=T\iota\cdot J^r\Xi.$$

Since for any two π-projectable vector fields Ξ and Z the vector fields $J^1J^{r-1}\Xi$ $J^r\Xi$ and $J^1J^{r-1}Z$ and J^rZ are ι-compatible, the corresponding Lie brackets are also ι-compatible and we have

(58) $\qquad [J^1J^{r-1}\Xi, J^1J^{r-1}\mathrm{Z}]\circ\iota = T\iota\cdot[J^r\Xi, J^r\mathrm{Z}].$

3. Using Part 1 of this proof, we now express the vector field on the left-hand side of (58) in a different way. First note that

(59) $\qquad [J^1J^{r-1}\Xi, J^1J^{r-1}\mathrm{Z}] = J^1[J^{r-1}\Xi, J^{r-1}\mathrm{Z}].$

But we may suppose for induction that $[J^{r-1}\Xi, J^{r-1}\mathrm{Z}] = J^{r-1}[\Xi,\mathrm{Z}]$, thus

(60) $\qquad [J^1J^{r-1}\Xi, J^1J^{r-1}\mathrm{Z}] = J^1J^{r-1}[\Xi,\mathrm{Z}]..$

Restricting both sides by ι and applying (54),

(61) $\qquad [J^1J^{r-1}\Xi, J^1J^{r-1}\mathrm{Z}]\circ\iota = J^1J^{r-1}[\Xi,\mathrm{Z}]\circ\iota = T\iota\cdot J^r[\Xi,\mathrm{Z}].$

Now from (59) and (61) we conclude that $T\iota\cdot([J^r\Xi, J^r\mathrm{Z}] - J^r[\Xi,\mathrm{Z}]) = 0$. This implies, however, $[J^r\Xi, J^r\mathrm{Z}] - J^r[\Xi,\mathrm{Z}] = 0$ because $T\iota$ is at every point injective.

This completes the proof of assertion (a).

4. (b) follows from 1.6, Lemma 10.

Now we consider restrictions of jet prolongations of projectable vector fields to jet prolongations of sections.

Remark 5 We find the chart expression of the canonical injection ι (50). Any fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y induces a fibred chart (V^r,ψ^r), $\psi = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2},\dots, y^\sigma_{j_1j_2\dots j_r})$, on J^rY. We also have a fibred chart on $J^1J^{r-1}Y$, induced by the fibred chart (V^{r-1},ψ^{r-1}), $\psi = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2},\dots, y^\sigma_{j_1j_2\dots j_{r-1}})$, on $J^{r-1}Y$. We denote this fibred chart by (W,Ψ), where the coordinate functions are denoted as

(63) $\qquad \Psi = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2},\dots, y^\sigma_{j_1j_2\dots j_{r-1}}, y^\sigma_{\ ,k}, y^\sigma_{j_1,k}, y^\sigma_{j_1j_2,k},\dots, y^\sigma_{j_1j_2\dots j_{r-1},k}).$

Then by definition

(64)
$$\begin{aligned} y^\sigma_{j_1j_2\dots j_s,k}\circ\iota(J^r_x\gamma) &= D_k(y^\sigma_{j_1j_2\dots j_s}\circ J^{r-1}\gamma\circ\varphi^{-1})(\varphi(x)) \\ &= D_kD_{j_1}D_{j_2}\dots D_{j_s}(y^\sigma\gamma\varphi^{-1})(\varphi(x)) = y^\sigma_{j_1j_2\dots j_sk}(J^r_x\gamma) \end{aligned}$$

for all $s = 1,2,\dots,r-1$, so the canonical injection ι is expressed by

(65)
$$\begin{aligned} &x^i\circ\iota = x^i, \quad y^\sigma\circ\iota = y^\sigma, \quad y^\sigma_{j_1j_2\dots j_s}\circ\iota = y^\sigma_{j_1j_2\dots j_s}, \quad 1\le s\le r-1, \\ &y^\sigma_{j_1j_2\dots j_s,k}\circ\iota = y^\sigma_{j_1j_2\dots j_sk}, \quad 1\le s\le r-1. \end{aligned}$$