2 Contact forms on jet prolongations of fibred manifolds

2.1 The trace decomposition of tensor spaces This section is devoted to a specific topic of the tensor calculus, the trace decomposition theory. As a rule, this topic does not appear in standard textbooks and monographs on tensor algebra, and needs a detailed independent introduction; our exposition follows the paper D. Krupka, Trace decompositions of tensor spaces, Linear and Multilinear Algebra 54 (2006) 235-263. In the proofs we also need the Young decomposition theory of tensor spaces. In subsequent chapters we us the trace decomposition theory for the study of the structure of differential forms on jet prolongations of fibred manifolds.

Beside the usual index notation for the components of tensors, we also use multi-indices of the form $I=(i_1 i_2 \ldots i_k)$, where r and n are positive integers, $k=0,1,2,\ldots,r$, and $1 \le i_1, i_2, \ldots, i_k \le n$. The number k is called the *length* of I and is denoted by $|I|$. We use multi-indices with different lengths. For any index j, such that $1 \le j \le n$ we denote by Ij the multi-index $(i_1 i_2 \ldots i_k j)$. The symbol $\mathrm{Alt}(i_1 i_2 \ldots i_k)$ (resp. $\mathrm{Sym}(i_1 i_2 \ldots i_k)$) denotes *alternation* (resp. *symmetrisation*) in the indices $i_1, i_2, \ldots, i_k$.

Let E be an n-dimensional vector space, E^* its dual vector space, and let r and s be two non-negative integers; suppose that at least one of these integers is non-zero. Then by a *tensor of type* (r,s) over E we mean a multilinear mapping $U: E^* \times E^* \times \ldots \times E^* \times E \times E \times \ldots \times E \to \mathbf{R}$ (r factors E^*, s factors E); r (resp. s) is called the *contravariant* (resp. *covariant*) *degree* of U. A tensor of type $(r,0)$ (resp. $(0,s)$) is called *contravariant* (*covariant*) of degree r (resp. s). The set of tensors of type (r,s) considered with its natural real vector space structure, is called the *tensor space of type* (r,s) over E, and is denoted by $T^r_s E$.

Let e_i be a basis of the vector space E, e^i the dual basis of E^*. The tensors $\mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \ldots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \ldots \otimes \mathrm{e}^{i_s}$, $1 \le j_1, j_2, \ldots, j_r, i_1, i_2, \ldots, i_s \le n$, form a *basis* of the vector space $T^r_s E$. Each tensor $u \in T^r_s E$ has a unique expression

$$(1) \qquad U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \ldots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \ldots \otimes \mathrm{e}^{i_s},$$

where the numbers $U^{i_1 i_2 \ldots i_r}{}_{i_1 i_2 \ldots i_s}$ are the *components* of U in the basis e_i.

Remark 1 If a basis of the vector space E is fixed, it is sometimes convenient to denote the tensors simply by their components; in this case a tensor U of type (r,s) over E is usually written as

(2) $$U = U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s}.$$

Remark 2 The *canonical basis* of the vector space $E = \mathbf{R}^n$ consists of the vectors $\mathrm{e}_1 = (1,0,0,\dots,0)$, $\mathrm{e}_2 = (0,1,0,0,\dots,0)$, $\dots$, $\mathrm{e}_n = (0,0,\dots,0,1)$. The basis of the tensor space $T^r_s\mathbf{R}^n$, associated with $(\mathrm{e}_1, \mathrm{e}_2, \dots, \mathrm{e}_n)$ is also called *canonical*. A tensor $U \in T^r_s\mathbf{R}^n$ can be expressed either by formula (1) or by (2); formula (2) defines the *canonical identification* of the vector space $T^r_s\mathbf{R}^n$ with the vector space $\mathbf{R}^N$ of the collections $U = U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s}$, where $N = \dim T^r_s\mathbf{R}^n = n^{rs}$.

The *Kronecker tensor* over E is a $(1,1)$-tensor δ, defined in any basis of E as

(3) $$\delta = \mathrm{e}_i \otimes \mathrm{e}^i.$$

We can also write $\delta = \delta^i_j \mathrm{e}_i \otimes \mathrm{e}^j$, where δ^i_j is the *Kronecker symbol*, $\delta^i_i = 1$ and $\delta^i_j = 0$ if $i \neq j$; in components, $\delta = \delta^i_j$. It is immediately seen that the tensor δ does not depend on the choice of the basis e_i.

This definition can be extended to tensors of type (r,s) for any positive integers r and s. Let α and β be integers such that $1 \le \alpha \le r$, $1 \le \beta \le s$, and let e_i be a basis of E. We introduce a linear mapping $\iota^\alpha_\beta : T^{r-1}_{s-1}E \to T^r_s E$ of tensor spaces as follows. For every $V \in T^{r-1}_{s-1}E$,

(4) $$V = V^{j_1 j_2 \dots j_{r-1}}{}_{i_1 i_2 \dots i_{s-1}} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \dots \otimes \mathrm{e}_{j_{r-1}} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \dots \otimes \mathrm{e}^{i_{s-1}},$$

define a tensor $\iota^\alpha_\beta V \in T^r_s E$ by

(5) $$\iota^\alpha_\beta V = W^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \dots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \dots \otimes \mathrm{e}^{i_s},$$

where

(6) $$W^{j_1 j_2 \dots j_{\alpha-1} j_\alpha j_{\alpha+1} \dots j_r}{}_{i_1 i_2 \dots i_{\beta-1} i_\beta i_{\beta+1} \dots i_s} = \delta^{j_\alpha}_{i_\beta} V^{j_1 j_2 \dots j_{\alpha-1} j_{\alpha+1} \dots j_r}{}_{i_1 i_2 \dots i_{\beta-1} i_{\beta+1} \dots i_s}.$$

Thus,

(7) $$\begin{aligned}\iota^\alpha_\beta V = {} & V^{j_1 j_2 \dots j_{r-1}}{}_{i_1 i_2 \dots i_{s-1}} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \dots \otimes \mathrm{e}_{j_{\alpha-1}} \otimes \mathrm{e}_s \otimes \mathrm{e}_{j_{\alpha+1}} \otimes \dots \otimes \mathrm{e}_{j_r} \\ & \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \dots \otimes \mathrm{e}^{i_{\beta-1}} \otimes \mathrm{e}^s \otimes \mathrm{e}^{i_{\beta+1}} \otimes \dots \otimes \mathrm{e}^{i_s}\end{aligned}$$

(summation through s on the right-hand side). It is easily verified that this tensor is independent of the choice of e_i.

The mapping ι_β^α defined by formulas (5), (6) is the (α,β)-*canonical injection*. A tensor $U \in T_s^r E$, belonging to the vector subspace generated by the subspaces $\iota_\beta^\alpha(T_{s-}^{r-}E) \subset T_s^r E$, where $1 \le \alpha \le r$ and $1 \le \beta \le s$, is called a *Kronecker tensor*, or a tensor of *Kronecker type*.

A tensor $V \in T_s^r E$, $V = V^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s}$, is a Kronecker tensor if and only if there exist tensors $V_{(q)}^{(p)} \in T_{s-1}^{r-1} E$, $V_{(q)}^{(p)} = V_{(q)}^{(p)\ k_1 k_2 \dots k_{r-1}}{}_{l_1 l_2 \dots l_{s-1}}$, where $1 \le p \le r$, $1 \le q \le s$, such that $V^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s}$ can be expressed in the form

$$
\begin{aligned}
&V^{j_1 j_2 \dots j_r}{}_{l_1 l_2 \dots l_s}\\
&= \delta_{l_1}^{j_1} V_{(1)}^{(1)\ j_2 j_3 \dots j_r}{}_{l_2 l_3 \dots l_s} + \delta_{l_2}^{j_1} V_{(2)}^{(1)\ j_2 j_3 \dots j_r}{}_{l_1 l_3 \dots l_s} + \dots + \delta_{l_s}^{j_1} V_{(s)}^{(1)\ j_2 j_3 \dots j_r}{}_{l_1 l_2 \dots l_{s-1}}\\
&+ \delta_{l_1}^{j_2} V_{(1)}^{(2)\ j_1 j_3 \dots j_r}{}_{l_2 l_3 \dots l_s} + \delta_{l_2}^{j_2} V_{(2)}^{(2)\ j_1 j_3 \dots j_r}{}_{l_1 l_3 \dots l_s} + \dots + \delta_{l_s}^{j_2} V_{(s)}^{(2)\ j_1 j_3 \dots j_r}{}_{l_1 l_2 \dots l_{s-1}}\\
&+ \dots\\
&+ \delta_{l_1}^{j_r} V_{(1)}^{(r)\ j_1 j_2 \dots j_{r-1}}{}_{l_2 l_3 \dots l_s} + \delta_{l_2}^{j_r} V_{(2)}^{(r)\ j_1 j_2 \dots j_{r-1}}{}_{l_1 l_3 \dots l_s} + \dots + \delta_{l_s}^{j_r} V_{(s)}^{(r)\ j_1 j_2 \dots j_{r-1}}{}_{l_1 l_2 \dots l_{s-1}}.
\end{aligned} \tag{8}
$$

A tensor $U \in T_s^r E$ expressed as in (1), is said to be *traceless*, if its traces are all zero,

$$
\begin{aligned}
&U^{s l_1 l_2 \dots l_{r-1}}{}_{s j_1 j_2 \dots j_{s-1}} = 0, \quad U^{l_1 s l_2 \dots l_{r-1}}{}_{s j_1 j_2 \dots j_{s-1}} = 0, \quad \dots, \quad U^{l_1 l_2 \dots l_{r-1} s}{}_{s j_1 j_2 \dots j_{s-1}} = 0,\\
&U^{s l_1 l_2 \dots l_{r-1}}{}_{j_1 s j_2 \dots j_{s-1}} = 0, \quad U^{l_1 s l_2 \dots l_{r-1}}{}_{j_1 s j_2 \dots j_{s-1}} = 0, \quad \dots, \quad U^{l_1 l_2 \dots l_{r-1} s}{}_{j_1 s j_2 \dots j_{s-1}} = 0,\\
&\dots\\
&U^{s l_1 l_2 \dots l_{r-1}}{}_{j_1 j_2 \dots j_{s-1} s} = 0, \quad U^{l_1 s l_2 \dots l_{r-1}}{}_{j_1 j_2 \dots j_{s-1} s} = 0, \quad \dots, \quad U^{l_1 l_2 \dots l_{r-1} s}{}_{j_1 j_2 \dots j_{s-1} s} = 0.
\end{aligned} \tag{9}
$$

To prove a theorem of the decomposition of the tensor space $T_s^r E$, including traceless tensors, recall that every scalar product g on E induces a scalar product on $T_s^r E$ as follows. Let g be expressed in a basis as

$$g(\xi,\zeta) = g_{ij} \xi^i \zeta^j, \tag{10}$$

where $\xi = \xi^i$, $\zeta = \zeta^i$. Let $U, V \in T_s^r E$ be any tensors, $U = U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s}$, $V = V^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_s}$. We set

$$g(U,V) = g_{j_1 k_1} g_{j_2 k_2} \dots g_{j_r k_r} g^{i_1 l_1} g^{i_2 l_2} \dots g^{i_s l_s} U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s} V^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s}. \tag{11}$$

Lemma 1 *Formula* (11) *defines a scalar product on* $T_s^r E$.

Proof Only positive definiteness needs proof. If we choose a basis of E such that $g_{jk} = \delta_{jk}$, (11) has an expression

$$g(U,V)=\sum_{k_1,k_2,\ldots,k_r}\sum_{l_1,l_2,\ldots,l_s}U^{j_1j_2\ldots j_r}{}_{l_1l_2\ldots l_s}V^{j_1j_2\ldots j_r}{}_{l_1l_2\ldots l_s}. \tag{12}$$

Obviously, this is the Euclidean scalar product, which is positive definite.

Theorem 1 (The trace decomposition theorem) *The vector space T^r_sE is the direct sum of its vector subspaces of traceless and Kronecker tensors.*

Proof We want to show that any tensor $W\in T^r_sE$, has a unique decomposition of the form $W=U+V$, where U is traceless and V is of Kronecker type.

To prove existence, consider a scalar product g (12) on T^r_sE. It is immediately seen that the orthogonal complement of the subspace of Kronecker tensors coincides with the subspace of traceless tensors. If $U\in T^r_sE$, $U=U^{i_1i_2\ldots i_r}{}_{j_1j_2\ldots j_s}$, and if a tensor $V\in T^r_sE$, $V=V^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_s}$, satisfies condition (8), then

$$\begin{aligned}
g(U,V)={}&U^{mj_2j_3\ldots j_r}{}_{ml_2l_3\ldots l_s}V^{(1)\ j_2j_3\ldots j_r}_{(1)}{}_{l_2l_3\ldots l_s}\\
&+U^{mj_2j_3\ldots j_r}{}_{i_1mi_3i_4\ldots i_s}V^{(1)\ k_2k_3\ldots k_r}_{(1)}{}_{l_1l_3l_4\ldots l_s}\\
&+\ldots+U^{mj_2j_3\ldots j_r}{}_{i_1i_2\ldots i_{s-1}m}V^{(1)\ k_2k_3\ldots k_r}_{(s)}{}_{l_1l_2\ldots l_{s-1}}\\
&+U^{j_1mj_3j_4\ldots j_r}{}_{mi_2i_3\ldots i_s}V^{(2)\ k_1k_3k_4\ldots k_r}_{(1)}{}_{l_2l_3\ldots l_s}\\
&+U^{j_1mj_3j_4\ldots j_r}{}_{i_1mi_3i_4\ldots i_s}V^{(2)\ k_1k_3k_4\ldots k_r}_{(1)}{}_{l_1l_3l_4\ldots l_s}\\
&+\ldots+U^{j_1mj_3j_4\ldots j_r}{}_{i_1i_2\ldots i_{s-1}m}V^{(2)\ k_1k_3k_4\ldots k_r}_{(s)}{}_{l_1l_2\ldots l_{s-1}}\\
&+\ldots\\
&+U^{j_1j_2\ldots j_{r-1}m}{}_{mi_2i_3\ldots i_s}V^{(r)\ k_1k_2\ldots k_{r-1}}_{(1)}{}_{l_2l_3\ldots l_s}\\
&+U^{j_1j_2\ldots j_{r-1}m}{}_{i_1mi_3i_4\ldots i_s}V^{(1)\ k_1k_2\ldots k_{r-1}}_{(1)}{}_{l_1l_3l_4\ldots l_s}\\
&+\ldots+U^{j_1j_2\ldots j_{r-1}m}{}_{i_1i_2\ldots i_{s-1}m}V^{(r)\ k_1k_2\ldots k_{r-1}}_{(s)}{}_{l_1l_2\ldots l_{s-1}}.
\end{aligned} \tag{13}$$

Thus the vector subspace of tensors U such that $g(U,V)=0$ for all V, consists of traceless tensors. The uniqueness of the direct sum follows from the orthogonality of subspaces of traceless and Kronecker tensors in T^r_sE in the scalar product g.

Theorem 1 states that every tensor $W\in T^r_sE$, $W=W^{i_1i_2\ldots i_r}{}_{l_1l_2\ldots l_s}$ is expressible in the form

$$
\begin{aligned}
(14)\qquad & W^{i_1 i_2 \ldots i_r}{}_{l_1 l_2 \ldots l_s} = U^{i_1 i_2 \ldots i_r}{}_{l_1 l_2 \ldots l_s} \\
& + \delta^{i_1}_{l_1} V^{(1)}_{(1)}{}^{i_2 i_3 \ldots i_r}{}_{l_2 l_3 \ldots l_s} + \delta^{i_1}_{l_2} V^{(1)}_{(2)}{}^{i_2 i_3 \ldots i_r}{}_{l_1 l_3 \ldots l_s} + \ldots + \delta^{i_1}_{l_s} V^{(1)}_{(s)}{}^{i_2 i_3 \ldots i_r}{}_{l_1 l_2 \ldots l_{s-1}} \\
& + \delta^{i_2}_{l_1} V^{(2)}_{(1)}{}^{i_1 i_3 \ldots i_r}{}_{l_2 l_3 \ldots l_s} + \delta^{i_2}_{l_2} V^{(2)}_{(2)}{}^{i_1 i_3 \ldots i_r}{}_{l_1 l_3 \ldots l_s} + \ldots + \delta^{i_2}_{l_s} V^{(2)}_{(s)}{}^{i_1 i_3 \ldots i_r}{}_{l_1 l_2 \ldots l_{s-1}} \\
& + \ldots \\
& + \delta^{i_r}_{l_1} V^{(r)}_{(1)}{}^{i_1 i_2 \ldots i_{r-1}}{}_{l_2 l_3 \ldots l_s} + \delta^{i_r}_{l_2} V^{(r)}_{(2)}{}^{i_1 i_2 \ldots i_{r-1}}{}_{l_1 l_3 \ldots l_s} + \ldots + \delta^{k_r}_{l_s} V^{(r)}_{(s)}{}^{i_1 i_2 \ldots i_{r-1}}{}_{l_1 l_2 \ldots l_{s-1}},
\end{aligned}
$$

where $U = U^{i_1 i_2 \ldots i_r}{}_{l_1 l_2 \ldots l_s}$ is a uniquely defined traceless tensor, and for every p and q such that $1 \le p \le r$, $1 \le q \le s$, the tensor $V^{(p)}_{(q)} = V^{(p)}_{(q)}{}^{i_1 i_2 \ldots i_{r-1}}{}_{l_1 l_2 \ldots l_{s-1}}$ belongs to the tensor space $T^{r-1}_{s-1}E$.

Remark 3 The traceless component $U^{i_1 i_2 \ldots i_r}{}_{l_1 l_2 \ldots l_s}$ and the complementary Kronecker component of the tensor W in (14) are determined uniquely. However, this does not imply, in general, that the tensors $V^{(p)}_{(q)}$ are unique. If the contravariant and covariant degrees satisfy $r + s \le n + 1$, then the tensors $V^{(p)}_{(q)}$ may not be unique.

Formula (14) is called the *trace decomposition formula*.

Denote by E^r_s the vector subspace of tensors $U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}$ in the tensor space $T^r_s E$, *symmetric* in the superscripts and *skew-symmetric* in the subscripts. We wish to find the trace decomposition formula for these. Set

$$
(15)\qquad \operatorname{tr} U = U^{k j_1 j_2 \ldots j_{r-1}}{}_{k i_1 i_2 \ldots i_{s-1}},
$$

and

$$
\begin{aligned}
(16)\qquad \mathbf{q}U = & \frac{(r+1)(s+1)}{n+r-s} \delta^{j_1}_{i_1} U^{j_2 j_3 \ldots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} \quad \operatorname{Alt}(i_1 i_2 \ldots i_{s+1}) \\
& \operatorname{Sym}(j_1 j_2 \ldots j_{r+1}).
\end{aligned}
$$

These formulas define two linear mappings $\operatorname{tr} : E^r_s \to E^{r-1}_{s-1}$ and $\mathbf{q} : E^r_s \to E^{r+1}_{s+1}$.

Theorem 2 (a) *Any tensor* $U \in E^r_s$ *has a decomposition*

$$
(17)\qquad U = \operatorname{tr} \mathbf{q} U + \mathbf{q} \operatorname{tr} U.
$$

(b) *The mappings* tr *and* $\mathbf{q}$ *satisfy*

$$
(18)\qquad \operatorname{tr} \operatorname{tr} U = 0, \quad \mathbf{q}\mathbf{q} U = 0.
$$

Proof (a) We prove formula (17). Using the definition (16) of $\mathbf{q}$ we have, with obvious notation,

$$\mathbf{q}U = \frac{r+1}{n+r-s}(\delta^{j_1}_{i_1}U^{j_2j_3\dots j_{r+1}}{}_{i_2i_3\dots i_{s+1}} - \delta^{j_1}_{i_2}U^{j_2j_3\dots j_{r+1}}{}_{i_1i_3i_4\dots i_{s+1}}$$
$$- \delta^{j_1}_{i_3}U^{j_2j_3\dots j_{r+1}}{}_{i_2i_1i_4i_5\dots i_{s+1}} - \dots - \delta^{j_1}_{i_{s+1}}U^{j_2j_3\dots j_{r+1}}{}_{i_2i_3\dots i_si_1}) \quad \mathrm{Sym}(j_1j_2\dots j_{r+1}). \tag{19}$$

Thus,

$$\mathrm{tr}\,\mathbf{q}U = \frac{1}{n+r-s}(\delta^k_kU^{j_2j_3\dots j_{r+1}}{}_{i_2i_3\dots i_{s+1}} - \delta^k_{i_2}U^{j_2j_3\dots j_{r+1}}{}_{ki_3i_4\dots i_{s+1}}$$
$$- \delta^k_{i_3}U^{j_2j_3\dots j_{r+1}}{}_{i_2ki_4i_5\dots i_{s+1}} - \dots - \delta^k_{i_{s+1}}U^{j_2j_3\dots j_{r+1}}{}_{i_2i_3\dots i_sk}$$
$$+ \delta^{j_2}_kU^{kj_3j_4\dots j_{r+1}}{}_{i_2i_3\dots i_{s+1}} - \delta^{j_2}_{i_2}U^{kj_3j_4\dots j_{r+1}}{}_{ki_3i_4\dots i_{s+1}}$$
$$- \delta^{j_2}_{i_3}U^{kj_3j_4\dots j_{r+1}}{}_{i_2ki_4i_5\dots i_{s+1}} - \dots - \delta^{j_2}_{i_{s+1}}U^{kj_3j_4\dots j_{r+1}}{}_{i_2i_3\dots i_sk}$$
$$+ \delta^{j_3}_kU^{j_2kj_4j_5\dots j_{r+1}}{}_{i_2i_3\dots i_{s+1}} - \delta^{j_3}_{i_2}U^{j_2kj_4j_5\dots j_{r+1}}{}_{ki_3i_4\dots i_{s+1}}$$
$$- \delta^{j_3}_{i_3}U^{j_2kj_4j_5\dots j_{r+1}}{}_{i_2ki_4i_5\dots i_{s+1}} - \dots - \delta^{j_3}_{i_{s+1}}U^{j_2kj_4j_5\dots j_{r+1}}{}_{i_2i_3\dots i_sk}$$
$$+ \dots + \delta^{j_{r+1}}_kU^{j_2j_3\dots j_rk}{}_{i_2i_3\dots i_{s+1}} - \delta^{j_{r+1}}_{i_2}U^{j_2j_3\dots j_rk}{}_{ki_3i_4\dots i_{s+1}}$$
$$- \delta^{j_{r+1}}_{i_3}U^{j_2j_3\dots j_rk}{}_{i_2ki_4i_5\dots i_{s+1}} - \dots - \delta^{j_{r+1}}_{i_{s+1}}U^{j_2j_3\dots j_rk}{}_{i_2i_3\dots i_sk})$$
$$= \frac{1}{n+r-s}(nU^{j_2j_3\dots j_{r+1}}{}_{i_2i_3\dots i_{s+1}} - U^{j_2j_3\dots j_{r+1}}{}_{i_2i_3i_4\dots i_{s+1}} - U^{j_2j_3\dots j_{r+1}}{}_{i_2i_3i_4i_5\dots i_{s+1}}$$
$$- \dots - U^{j_2j_3\dots j_{r+1}}{}_{i_2i_3\dots i_si_{s+1}} + U^{j_2j_3j_4\dots j_{r+1}}{}_{i_2i_3\dots i_{s+1}} - \delta^{j_2}_{i_2}U^{kj_3j_4\dots j_{r+1}}{}_{ki_3i_4\dots i_{s+1}}$$
$$- \delta^{j_2}_{i_3}U^{kj_3j_4\dots j_{r+1}}{}_{i_2ki_4i_5\dots i_{s+1}} - \dots - \delta^{j_2}_{i_{s+1}}U^{kj_3j_4\dots j_{r+1}}{}_{i_2i_3\dots i_sk}$$
$$+ U^{j_2j_3j_4j_5\dots j_{r+1}}{}_{i_2i_3\dots i_{s+1}} - \delta^{j_3}_{i_2}U^{j_2kj_4j_5\dots j_{r+1}}{}_{ki_3i_4\dots i_{s+1}}$$
$$- \delta^{j_3}_{i_3}U^{j_2kj_4j_5\dots j_{r+1}}{}_{i_2ki_4i_5\dots i_{s+1}} - \dots - \delta^{j_3}_{i_{s+1}}U^{j_2kj_4j_5\dots j_{r+1}}{}_{i_2i_3\dots i_sk}$$
$$+ \dots + U^{j_2j_3\dots j_rj_{r+1}}{}_{i_2i_3\dots i_{s+1}} - \delta^{j_{r+1}}_{i_2}U^{j_2j_3\dots j_rk}{}_{ki_3i_4\dots i_{s+1}}$$
$$- \delta^{j_{r+1}}_{i_3}U^{j_2j_3\dots j_rk}{}_{i_2ki_4i_5\dots i_{s+1}} - \dots - \delta^{j_{r+1}}_{i_{s+1}}U^{j_2j_3\dots j_rk}{}_{i_2i_3\dots i_sk})$$
$$= \frac{n+r-s}{n+r-s}U^{j_2j_3\dots j_{r+1}}{}_{i_2i_3\dots i_{s+1}} - \frac{1}{n+r-s}(\delta^{j_2}_{i_2}U^{kj_3j_4\dots j_{r+1}}{}_{ki_3i_4\dots i_{s+1}}$$
$$+ \delta^{j_2}_{i_3}U^{kj_3j_4\dots j_{r+1}}{}_{i_2ki_4i_5\dots i_{s+1}} + \dots + \delta^{j_2}_{i_{s+1}}U^{kj_3j_4\dots j_{r+1}}{}_{i_2i_3\dots i_sk}$$
$$+ \delta^{j_3}_{i_2}U^{j_2kj_4j_5\dots j_{r+1}}{}_{ki_3i_4i_5\dots i_{s+1}} + \delta^{j_3}_{i_3}U^{j_2kj_4j_5\dots j_{r+1}}{}_{i_2ki_4i_5\dots i_{s+1}}$$
$$+ \dots + \delta^{j_3}_{i_{s+1}}U^{j_2kj_4j_5\dots j_{r+1}}{}_{i_2i_3\dots i_sk} + \dots + \delta^{j_{r+1}}_{i_2}U^{j_2j_3\dots j_rk}{}_{ki_3i_4\dots i_{s+1}}$$
$$+ \delta^{j_{r+1}}_{i_3}U^{j_2j_3\dots j_rk}{}_{i_2ki_4i_5\dots i_{s+1}} + \dots + \delta^{j_{r+1}}_{i_{s+1}}U^{j_2j_3\dots j_rk}{}_{i_2i_3\dots i_sk}) \tag{20}$$

Further computations yield

$$\begin{aligned}(21)\qquad &\operatorname{tr}\mathbf{q}U = U^{j_2 j_3 \dots j_{r+1}}{}_{i_2 i_3 \dots i_{s+1}} - \frac{rs}{n+r-s}\delta^{j_2}_{i_2} U^{k j_3 j_4 \dots j_{r+1}}{}_{k i_3 i_4 \dots i_{s+1}}\\ &\operatorname{Sym}(j_2 j_3 \dots j_{r+1}) \quad \operatorname{Alt}(i_2 i_3 \dots i_{s+1}).\end{aligned}$$

But by (15), the second term is exactly $\mathbf{q}\operatorname{tr}u$, proving (17).

(b) Formulas (18) are immediate.

(17) is the *trace decomposition formula* for tensors $U \in E^r_s$.

The following assertion is an immediate consequence of Theorem 2. It states, in particular, that the decomposition (17) of a tensor $U \in E^r_s$ is unique.

Theorem 3 (a) *Equation* $\mathbf{q}V + \operatorname{tr}W = U$ *for unknown tensors* $V \in E^{r-1}_{s-1}$ *and* $W \in E^{r+1}_{s+1}$ *has a unique solution such that* $\operatorname{tr}V = 0$ *,* $\mathbf{q}W = 0$ *. This solution is given by* $V = \operatorname{tr}U$ *,* $W = \mathbf{q}U$ *.*

(b) *Let* $U \in E^r_s$ *. Equation* $\mathbf{q}X = U$ *has a solution* $X \in E^{r-1}_{s-1}$ *if and only if* $\mathbf{q}U = 0$ *. If this condition is satisfied, then* $X = \operatorname{tr}U$ *is a solution. Any other solution is of the form* $X' = X + \mathbf{q}Y$ *for some tensor* $Y \in E^{r-1}_{s-2}$ *.*

Proof (a) If $\mathbf{q}V + \operatorname{tr}W = U$, $\operatorname{tr}V = 0$ then $V = \operatorname{tr}\mathbf{q}V = \operatorname{tr}U$ because $\operatorname{tr}\operatorname{tr}W = 0$; if $\mathbf{q}W = 0$, then $W = \mathbf{q}\operatorname{tr}W = \mathbf{q}(U - \mathbf{q}V) = \mathbf{q}U$.

(b) If equation $\mathbf{q}X = U$ has a solution U, then necessarily $\mathbf{q}U = 0$. Conversely, if $\mathbf{q}U = 0$, then $U = \mathbf{q}\operatorname{tr}U$ and $X = \operatorname{tr}U$ solves equation $\mathbf{q}X = U$. Clearly, the tensors $X' = X + \mathbf{q}Y$, where $Y \in E^{r-1}_{s-2}$ also solve this equation.

Example 1 We find the trace decomposition formula (17) for $r = 1$. Writing $U = U^{j_1}{}_{i_1 i_2 \dots i_s}$, we have $\operatorname{tr}U = U^{k}{}_{k i_1 i_2 \dots i_{s-1}}$ and

$$(22)\qquad \mathbf{q}\operatorname{tr}U = \frac{1}{n+1-s}(\delta^{j_1}_{i_1} U^{k}{}_{k i_2 i_3 \dots i_s} + \delta^{j_1}_{i_2} U^{k}{}_{i_1 k i_3 i_4 \dots i_s} + \dots + \delta^{j_1}_{i_s} U^{k}{}_{i_1 i_2 \dots i_{s-1} k}).$$

Analogously

$$\begin{aligned}&\mathbf{q}U = \frac{2(s+1)}{n+1-s}\delta^{j_1}_{i_1} U^{j_2}{}_{i_2 i_3 \dots i_{s+1}} \quad \operatorname{Alt}(i_1 i_2 \dots i_{s+1}) \quad \operatorname{Sym}(j_1 j_2)\\ (23)\qquad &= \frac{1}{n+1-s}(\delta^{j_1}_{i_1} U^{j_2}{}_{i_2 i_3 \dots i_{s+1}} - \delta^{j_1}_{i_2} U^{j_2}{}_{i_1 i_3 i_4 \dots i_{s+1}} - \dots - \delta^{j_1}_{i_{s+1}} U^{j_2}{}_{i_2 i_3 \dots i_s i_1}\\ &+ \delta^{j_2}_{i_1} U^{j_1}{}_{i_2 i_3 \dots i_{s+1}} - \delta^{j_2}_{i_2} U^{j_1}{}_{i_1 i_3 i_4 \dots i_{s+1}} - \dots - \delta^{j_2}_{i_{s+1}} U^{j_1}{}_{i_2 i_3 \dots i_s i_1})\end{aligned}$$

hence

$$
\begin{aligned}
\operatorname{tr}\mathbf{q}U = {} & \frac{1}{n+1-s}(nU^{j_2}{}_{i_2i_3\ldots i_{s+1}} - (s-1)U^{j_2}{}_{i_2i_3i_4\ldots i_{s+1}} \\
& - \frac{1}{n+1-s}(\delta^{j_2}_{i_2}U^{k}{}_{ki_3i_4\ldots i_{s+1}} + \delta^{j_2}_{i_3}U^{k}{}_{i_2ki_4i_5\ldots i_{s+1}} + \ldots + \delta^{j_2}_{i_{s+1}}U^{k}{}_{i_2i_3\ldots i_sk}) \\
= {} & U^{j_2}{}_{i_2i_3\ldots i_{s+1}} - \mathbf{q}\operatorname{tr}U.
\end{aligned}
\tag{24}
$$

Formulas (24) and (26) yield $U = \operatorname{tr}\mathbf{q}U + \mathbf{q}\operatorname{tr}U$. In particular, if $r = 1$ and $s = n$, then $U = U^{j}{}_{i_1i_2\ldots i_n}$, $\operatorname{tr}U = U^{s}{}_{si_1i_2\ldots i_{n-1}}$ and $\mathbf{q}U = 0$. Thus,

$$
\begin{aligned}
U = {} & n\delta^{j}_{i_1}U^{s}{}_{si_2i_3\ldots i_n} \quad \operatorname{Alt}(i_1i_2\ldots i_n) \\
= {} & \delta^{j}_{i_1}U^{s}{}_{si_2i_3\ldots i_n} + \delta^{j}_{i_2}U^{s}{}_{i_1si_3i_4\ldots i_n} + \ldots + \delta^{j}_{i_n}U^{s}{}_{i_1i_2\ldots i_{n-1}s}.
\end{aligned}
\tag{25}
$$

Example 2 Consider the decomposition (17) for $r = 2$ and $s = n-1$, and find explicit expressions for the traceless and Kronecker components $\operatorname{tr}\mathbf{q}U$ and $\mathbf{q}\operatorname{tr}U$ of the tensor U. Writing $U = U^{j_1j_2}{}_{i_1i_2\ldots i_{n-1}}$ and using the proof of Theorem 2 we have

$$
\begin{aligned}
\operatorname{tr}\mathbf{q}U = {} & U^{j_2j_3}{}_{i_2i_3\ldots i_n} \\
& - \frac{1}{3}(\delta^{j_2}_{i_2}U^{kj_3}{}_{ki_3i_4\ldots i_n} + \delta^{j_2}_{i_3}U^{kj_3}{}_{i_2ki_4i_5\ldots i_n} + \ldots + \delta^{j_2}_{i_n}U^{kj_3}{}_{i_2i_3\ldots i_{n-1}k} \\
& + \delta^{j_3}_{i_2}U^{j_2k}{}_{ki_3i_4i_5\ldots i_n} + \delta^{j_3}_{i_3}U^{j_2k}{}_{i_2ki_4i_5\ldots i_n} + \ldots + \delta^{j_3}_{i_n}U^{j_2k}{}_{i_2i_3\ldots i_{n-1}k})
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\mathbf{q}\operatorname{tr}U = {} & \frac{1}{3}(\delta^{j_2}_{i_2}U^{kj_3}{}_{ki_3i_4\ldots i_n} + \delta^{j_2}_{i_3}U^{kj_3}{}_{i_2ki_4i_5\ldots i_n} + \ldots + \delta^{j_2}_{i_n}U^{kj_3}{}_{i_2i_3\ldots i_{n-1}k} \\
& + \delta^{j_3}_{i_2}U^{j_2k}{}_{ki_3i_4i_5\ldots i_n} + \delta^{j_3}_{i_3}U^{j_2k}{}_{i_2ki_4i_5\ldots i_n} + \ldots + \delta^{j_3}_{i_n}U^{j_2k}{}_{i_2i_3\ldots i_{n-1}k}) \\
= {} & \frac{1}{3}(\delta^{j_2}_{i_2}U^{kj_3}{}_{ki_3i_4\ldots i_n} - \delta^{j_2}_{i_3}U^{kj_3}{}_{ki_2i_4i_5\ldots i_n} - \ldots - \delta^{j_2}_{i_n}U^{kj_3}{}_{ki_3\ldots i_{n-1}i_2} \\
& + \delta^{j_3}_{i_2}U^{kj_2}{}_{ki_3i_4i_5\ldots i_n} - \delta^{j_3}_{i_3}U^{kj_2}{}_{ki_2i_4i_5\ldots i_n} - \ldots - \delta^{j_3}_{i_n}U^{kj_2}{}_{ki_3\ldots i_{n-1}i_2}) \\
= {} & \frac{2(n-1)}{3}\delta^{j_2}_{i_2}U^{kj_3}{}_{ki_3i_4\ldots i_n} \quad \operatorname{Sym}(j_2j_3) \quad \operatorname{Alt}(i_2i_3\ldots i_n).
\end{aligned}
\tag{27}
$$

Let k and j be positive integers, $j \le k \le n$. Let $X = X^{I_1I_2\ldots I_j}{}_{i_{j+1}i_{j+2}\ldots i_k}$ be a tensor indexed with multi-indices I_q , of length r_q and indices i_1 , i_2 , $\ldots$, i_j , such that $1 \le i_1, i_2, \ldots i_j \le n$; we suppose X to be *symmetric* in the superscripts entering each of the multi-indices, and *skewsymmetric* in the subscripts. Our objective will be to solve the system of homogeneous equations

$$(28)\qquad \begin{array}{l} \delta^{p_1}_{p_1}\delta^{p_2}_{p_2}\dots\delta^{p_j}_{p_j}X^{I_1I_2\dots I_j}{}_{i_{j+1}i_{j+2}\dots i_k}=0 \quad \mathrm{Alt}(p_1p_2\dots p_ji_{j+1}i_{j+2}\dots i_k) \\ \mathrm{Sym}(I_1p_1) \quad \mathrm{Sym}(I_2p_2) \quad \dots \quad \mathrm{Sym}(I_jp_j) \end{array}$$

for the unknown tensor X. In this formula, the alternation operation is applied to the subscripts, and the symmetrizations are to the superscripts.

Theorem 4 *Let n, k, j, and r be positive integers, and assume that $1\le j<k\le n$. Then a tensor $X=X^{I_1I_2\dots I_j}{}_{i_{j+1}i_{j+2}\dots i_k}$ satisfies the system* (28) *if and only if it is a Kronecker tensor.*

Proof 1. We show that condition (28) implies that X is a Kronecker tensor. Let I_1, I_2, $\dots$, I_j, and i_{j+1}, i_{j+2}, $\dots$, i_k be given. Choose p_1, p_2, $\dots$, p_j such that the s-tuple $(p_1,p_2,\dots,p_j,i_{j+1},i_{j+2},\dots,i_k)$ consists of mutually different indices, and consider the expression on the left of (30) without the summations defined by the trace operation,

$$(29)\qquad \begin{array}{l} \delta^{p_1}_{i_1}\delta^{p_2}_{i_2}\dots\delta^{p_j}_{i_j}X^{I_1I_2\dots I_j}{}_{i_{j+1}i_{j+2}\dots i_s} \quad \mathrm{Alt}(i_1i_2\dots i_ji_{j+1}\dots i_k) \\ \mathrm{Sym}(I_1p_1) \quad \mathrm{Sym}(I_2p_2) \quad \dots \quad \mathrm{Sym}(I_jp_j). \end{array}$$

Set $i_1=p_1$, $i_2=p_2$, $\dots$, $i_j=p_j$, and consider the summation prescribed by the alternation $\mathrm{Alt}(i_1i_2\dots i_ji_{j+1}\dots i_k)$. Then the sum (29) splits in two groups of summands. The first is given by the factor $\delta^{p_1}_{p_1}\delta^{p_2}_{p_2}\dots\delta^{p_j}_{p_j}$; these are the summands in which all p_1, p_2, $\dots$, p_j are covariant indices in the Kronecker δ-tensors, i.e.,

$$(30)\qquad \begin{array}{l} \frac{1}{s!}\delta^{p_1}_{p_1}\delta^{p_2}_{p_2}\dots\delta^{p_j}_{p_j}X^{I_1I_2\dots I_j}{}_{i_{j+1}i_{j+2}\dots i_k} \quad \mathrm{Sym}(I_1p_1) \\ \mathrm{Sym}(I_2p_2) \quad \dots \quad \mathrm{Sym}(I_jp_j). \end{array}$$

Further summations in this expression arise from the symmetrizations. Thus, (30) is equal to $cX^{I_1I_2\dots I_j}{}_{i_{j+1}i_{j+2}\dots i_k}$ for some $c>0$. The second group of summands consists of all the remaining terms, in which at least one covariant index in the product $\delta^{p_1}_{p_1}\delta^{p_2}_{p_2}\dots\delta^{p_j}_{p_j}$ in (28) is replaced by some of the indices i_{j+1}, i_{j+2}, $\dots$, i_k. But since all indices from the s-tuple $(p_1,p_2,\dots,p_j,i_{j+1},i_{j+2},\dots,i_k)$ are mutually different, we have $\delta^\alpha_\beta=0$ whenever $\alpha\in\{p_1,p_2,\dots,p_j\}$, $\beta\in\{i_{q+1}i_{q+2}\dots i_k\}$. Consequently, the second group consists of those terms, which are multiples of δ^α_β with $\alpha\notin\{p_1,p_2,\dots,p_j\}$, $\beta\in\{i_{j+1}i_{j+2}\dots i_k\}$. Now summations in (28) imply that $X=X^{I_1I_2\dots I_j}{}_{i_{j+1}i_{j+2}\dots i_s}$ must contain at least one factor the Kronecker δ-tensor.

2. Conversely, the alternation, and the symmetrizations in (28) imply that any Kronecker tensor $X=X^{I_1I_2\dots I_q}{}_{i_{q+1}i_{q+2}\dots i_s}$ solves (28).

Corollary 1 *Assume that in addition to the assumptions of* Theorem 4*, the tensor* $X = X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_k}$ *is traceless. Then*

(31) $$X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_k} = 0.$$

Proof This follows from Theorem 4, and from the orthogonality of traceless and Kronecker tensors.

Example 3 We solve equations (28) for $j = 2$ and $q = 3$ for the traceless tensors $X = X^{i_1 i_2}{}_{i_3}$. We have the system

(32) $$\delta^{p_1}_{p_1} \delta^{p_2}_{p_2} X^{i_1 i_2}{}_{i_3} = 0 \quad \mathrm{Alt}(p_1 p_2 i_3) \quad \mathrm{Sym}(i_1 p_1) \quad \mathrm{Sym}(i_2 p_2).$$

The sum on the left-hand side of (32) can be written explicitly. We get an expression

(33) $$\begin{aligned}
&\delta^{p_1}_{p_1} \delta^{p_2}_{p_2} X^{i_1 i_2}{}_{i_3} + \delta^{i_1}_{p_1} \delta^{p_2}_{p_2} X^{p_1 i_2}{}_{i_3} + \delta^{p_1}_{p_1} \delta^{i_2}_{p_2} X^{i_1 p_2}{}_{i_3} + \delta^{i_1}_{p_1} \delta^{i_2}_{p_2} X^{p_1 p_2}{}_{i_3} \\
&\quad - \delta^{p_1}_{p_2} \delta^{p_2}_{p_1} X^{i_1 i_2}{}_{i_3} - \delta^{i_1}_{p_2} \delta^{p_2}_{p_1} X^{p_1 i_2}{}_{i_3} - \delta^{p_1}_{p_2} \delta^{i_2}_{p_1} X^{i_1 p_2}{}_{i_3} - \delta^{i_1}_{p_2} \delta^{i_2}_{p_1} X^{p_1 p_2}{}_{i_3} \\
&\quad - \delta^{p_1}_{i_3} \delta^{p_2}_{p_2} X^{i_1 i_2}{}_{p_1} - \xcancel{\delta^{i_1}_{i_3} \delta^{p_2}_{p_2} X^{p_1 i_2}{}_{p_1}} - \delta^{p_1}_{i_3} \delta^{i_2}_{p_2} X^{i_1 p_2}{}_{p_1} - \xcancel{\delta^{i_1}_{i_3} \delta^{i_2}_{p_2} X^{p_1 p_2}{}_{p_1}} \\
&\quad + \delta^{p_1}_{p_2} \delta^{p_2}_{i_3} X^{i_1 i_2}{}_{p_1} + \xcancel{\delta^{i_1}_{p_2} \delta^{p_2}_{i_3} X^{p_1 i_2}{}_{p_1}} + \cancel{\delta^{p_1}_{p_2} \delta^{i_2}_{i_3} X^{i_1 p_2}{}_{p_1}} + \xcancel{\delta^{i_1}_{p_2} \delta^{i_2}_{i_3} X^{p_1 p_2}{}_{p_1}} \\
&\quad - \delta^{p_1}_{p_1} \delta^{p_2}_{i_3} X^{i_1 i_2}{}_{p_2} - \delta^{i_1}_{p_1} \delta^{p_2}_{i_3} X^{p_1 i_2}{}_{p_2} - \xcancel{\delta^{p_1}_{p_1} \delta^{i_2}_{i_3} X^{i_1 p_2}{}_{p_2}} - \xcancel{\delta^{i_1}_{p_1} \delta^{i_2}_{i_3} X^{p_1 p_2}{}_{p_2}} \\
&\quad + \delta^{p_1}_{i_3} \delta^{p_2}_{p_1} X^{i_1 i_2}{}_{p_2} + \cancel{\delta^{i_1}_{i_3} \delta^{p_2}_{p_1} X^{p_1 i_2}{}_{p_2}} + \xcancel{\delta^{p_1}_{i_3} \delta^{i_2}_{p_1} X^{i_1 p_2}{}_{p_2}} + \xcancel{\delta^{i_1}_{i_3} \delta^{i_2}_{p_1} X^{p_1 p_2}{}_{p_2}}.
\end{aligned}$$

The summations yield

(34) $$\begin{aligned}
&n^2 X^{i_1 i_2}{}_{i_3} + \xcancel{n X^{i_1 i_2}{}_{i_3}} + \xcancel{n X^{i_1 i_2}{}_{i_3}} + \cancel{X^{i_1 i_2}{}_{i_3}} \\
&\quad - \xcancel{n X^{i_1 i_2}{}_{i_3}} - \cancel{X^{i_1 i_2}{}_{i_3}} - X^{i_1 i_2}{}_{i_3} - X^{i_2 i_1}{}_{i_3} \\
&\quad - X^{i_1 i_2}{}_{i_3} - \cancel{X^{i_1 i_2}{}_{i_3}} + \cancel{X^{i_1 i_2}{}_{i_3}} - \xcancel{n X^{i_1 i_2}{}_{i_3}} - \cancel{X^{i_1 i_2}{}_{i_3}} + \cancel{X^{i_1 i_2}{}_{i_3}} \\
&\quad = (n^2 - 2) X^{i_1 i_2}{}_{i_3} - X^{i_2 i_1}{}_{i_3}.
\end{aligned}$$

Consequently, equation (28) implies $(n^2 - 2) X^{i_1 i_2}{}_{i_3} - X^{i_2 i_1}{}_{i_3} = 0$ so we get the solution

(35) $$X^{i_1 i_2}{}_{i_3} = 0.$$

2.2 Contact forms Let Y be a fibred manifold with base X and projection π, and let W be an open set in Y. Consider the exterior algebras $\Omega^r W$ and $\Omega^{r+1} W$ and the *horizontalisation* $\Omega^r W \ni \rho \to h\rho \in \Omega^{r+1} W$, introduced in Section 1.5. Since h is a homomorphism, the *kernel* of h,

(1) $$\operatorname{Ker} h = \{\rho \in \Omega^r W \mid h\rho = 0\},$$

is the *ideal* of the ring $\Omega^r W$. Clearly, since the dimension of the basis X of the fibred manifold Y is n, any q-form $\rho \in \Omega^r_q W$ such that $q \geq n+1$ always belongs to $\operatorname{Ker} h$. If $q \leq n$, we define a q-form $\rho \in \Omega^r_q W$ to be *contact*, if

(2) $$h\rho = 0.$$

It is easy to find the chart expression of a contact 1-form ρ (cf. 1.5, Lemma 9). Writing ρ in a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, as

(3) $$\rho = A_i dx^i + \sum_{0 \leq k \leq r} \sum_{j_1 < j_2 < \ldots < j_k} B^{j_1 j_2 \cdots j_k}_\sigma dy^\sigma_{j_1 j_2 \cdots j_k},$$

condition (2) yields

(4) $$A_i + \sum_{0 \leq k \leq r} B^{j_1 j_2 \cdots j_k}_\sigma y^\sigma_{j_1 j_2 \cdots j_k i} = 0,$$

or, equivalently,

(5) $$B^{j_1 j_2 \cdots j_r}_\sigma = 0, \quad A_i = -\sum_{0 \leq k \leq r-1} B^{j_1 j_2 \cdots j_k}_\sigma y^\sigma_{j_1 j_2 \cdots j_k i}.$$

Thus, setting for all k, $0 \leq k \leq r-1$,

(6) $$\omega^\sigma_{j_1 j_2 \cdots j_k} = dy^\sigma_{j_1 j_2 \cdots j_k} - y^\sigma_{j_1 j_2 \cdots j_k j} dx^j,$$

we see that ρ has the chart expression

(7) $$\rho = \sum_{0 \leq k \leq r-1} B^{j_1 j_2 \cdots j_k}_\sigma \omega^\sigma_{j_1 j_2 \cdots j_k i}.$$

In particular,

(8) $$h\omega^\sigma_{j_1 j_2 \cdots j_k} = 0,$$

and a contact 1-form is always a linear combination of the forms (6).

These observations lead to the following assertion.

Theorem 5 (a) *For any fibred chart* (V,ψ), $\psi=(x^i,y^\sigma)$, *on* Y *the forms*

(9) $$dx^i,\quad \omega^\sigma_{j_1j_2\dots j_k},\quad dy^\sigma_{l_1l_2\dots l_{r-1}l_r},$$

where $1\le k\le r-1$, $1\le i,j_1\le j_2\le\ldots\le j_k,l_1\le l_2\le\ldots\le l_r\le n$, *and* $1\le\sigma\le m$, *constitute a basis of linear forms on the set* V^r.

(b) *The forms* $\omega^\sigma_{j_1j_2\dots j_k}$ *satisfy*

(10) $$\begin{aligned} d\omega^\sigma_{j_1j_2\dots j_k}&=-\omega^\sigma_{j_1j_2\dots j_kl}\wedge dx^l,\quad 0\le k\le r-2,\\ d\omega^\sigma_{j_1j_2\dots j_{r-1}}&=-dy^\sigma_{j_1j_2\dots j_{r-1}l}\wedge dx^l,\end{aligned}$$

and

(11) $$hd\omega^\sigma_{j_1j_2\dots j_k}=0.$$

(c) *If* (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\bar V,\bar\psi)$, $\bar\psi=(\bar x^i,\bar y^\sigma)$, *are two fibred chart on Y such that* $V\cap\bar V\ne\emptyset$, *then*

(12) $$\omega^\lambda_{p_1p_2\dots p_k}=\sum_{j_1\le j_2\le\ldots\le j_k}\frac{\partial y^\lambda_{p_1p_2\dots p_k}}{\partial\bar y^\tau_{j_1j_2\dots j_m}}\bar\omega^\tau_{j_1j_2\dots j_m}.$$

(d) *If* (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\bar V,\bar\psi)$, $\bar\psi=(\bar x^i,\bar y^\sigma)$, *are two fibred chart on Y such that* $V\cap\bar V\ne\emptyset$, *then*

(12) $$\omega^\lambda_{p_1p_2\dots p_k}=\sum_{j_1\le j_2\le\ldots\le j_k}\frac{\partial y^\lambda_{p_1p_2\dots p_k}}{\partial\bar y^\tau_{j_1j_2\dots j_m}}\bar\omega^\tau_{j_1j_2\dots j_m}.$$

Proof (a) Clearly, the form (3) is expressible as linear combinations of the forms of the forms dx^i, $\omega^\sigma_{j_1j_2\dots j_k}$, $dy^\sigma_{l_1l_2\dots l_{r-1}l_r}$.

(b) Since *h* preserves exterior product, (11) follows from (10).

(c) For any function *f*, defined on V^r,

(13) $$\begin{aligned}(\pi^{r+1,r})^*df=hdf+pdf&=d_if\cdot dx^i+\sum_{0\le k\le r}\sum_{l_1\le l_2\le\ldots\le l_k}\frac{\partial f}{\partial y^\nu_{l_1l_2\dots l_k}}\omega^\nu_{l_1l_2\dots l_k}\\ &=\bar d_pf\cdot d\bar x^p+\sum_{0\le k\le r}\sum_{j_1\le j_2\le\ldots\le j_k}\frac{\partial f}{\partial\bar y^\tau_{j_1j_2\dots j_m}}\bar\omega^\tau_{j_1j_2\dots j_m}\\ &=\bar d_pf\frac{\partial\bar x^p}{\partial x^i}dx^i+\sum_{0\le k\le r}\sum_{j_1\le j_2\le\ldots\le j_k}\sum_{l_1\le l_2\le\ldots\le l_k}\frac{\partial f}{\partial y^\nu_{l_1l_2\dots l_k}}\frac{\partial y^\nu_{l_1l_2\dots l_k}}{\partial\bar y^\tau_{j_1j_2\dots j_m}}\bar\omega^\tau_{j_1j_2\dots j_m}.\end{aligned}$$

Setting $f = y^{\lambda}_{p_1 p_2 \dots p_k}$ we get (12).

(d) Formula (12) can be obtained by a direct computation.

The basis of 1-forms (9) is called the *contact basis*.

Now we consider sections of the fibred manifold $J^r Y$ over the base *X*. We say that a section δ of $J^r Y$, defined on an open set in *X*, is *integrable,* or *holonomic*, if there exists a section γ of *Y* such that

$$\delta = J^r \gamma. \tag{14}$$

Obviously, if γ exists, then applying the projection $\pi^{r.0}$ to both sides we get $\pi^{r.0} \circ \delta = \gamma$, thus γ is unique and is determined by

$$\gamma = \pi^{r.0} \circ \delta. \tag{15}$$

The following theorem describes the relations of the contact forms and the holonomic sections of $J^r Y$.

Theorem 6 *Let* (V, ψ), $\psi = (x^i, y^\sigma)$, *be a fibred chart on Y.*

(a) *Every* C^r *section* γ *of Y, defined on an open subset of* $\pi^r(W) \subset X$, *satisfies*

$$J^r \gamma * \omega^{\sigma}_{i_1 i_2 \dots i_k} = 0. \tag{16}$$

(b) *If a* C^r *section* $\delta : U \to J^r Y$, *where* $U \subset \pi(V)$, *satisfies*

$$\delta * \omega^{\sigma}_{i_1 i_2 \dots i_k} = 0 \tag{17}$$

for all k such that $0 \le k \le r-1$ *and all* σ *and* $i_1, i_2, \dots, i_k$, *such that* $1 \le \sigma \le m$ *and* $1 \le i_1, i_2, \dots, i_k \le n$, *then it is integrable.*

Proof (a) By definition,

$$\begin{aligned} J^r \gamma * \omega^{\sigma}_{i_1 i_2 \dots i_k} &= d(y^{\sigma}_{i_1 i_2 \dots i_k} \circ J^r \gamma) - (y^{\sigma}_{i_1 i_2 \dots i_k l} \circ J^r \gamma) dx^l \\ &= \left(\frac{\partial (y^{\sigma}_{i_1 i_2 \dots i_k} \circ J^r \gamma)}{\partial x^l} - y^{\sigma}_{i_1 i_2 \dots i_k l} \circ J^r \gamma \right) dx^l \\ &= (D_{i_1} D_{i_2} \dots D_{i_k} D_l (y^\sigma \circ \gamma) - D_{i_1} D_{i_2} \dots D_{i_k} D_l (y^\sigma \circ \gamma)) dx^l = 0. \end{aligned} \tag{18}$$

(b) Using the definition of $\omega^{\sigma}_{i_1 i_2 \dots i_k}$, we get

$$\delta^*\omega^{\sigma}_{i_1 i_2 \dots i_k} = d(y^{\sigma}_{i_1 i_2 \dots i_k} \circ \delta) - (y^{\sigma}_{i_1 i_2 \dots i_k l} \circ \delta) dx^l$$
$$= \left(\frac{\partial (y^{\sigma}_{i_1 i_2 \dots i_k} \circ \delta)}{\partial x^l} - y^{\sigma}_{i_1 i_2 \dots i_k l} \circ \delta \right) dx^l. \tag{19}$$

Thus, condition (17) implies

$$\frac{\partial (y^{\sigma}_{i_1 i_2 \dots i_k} \circ \delta)}{\partial x^l} - y^{\sigma}_{i_1 i_2 \dots i_k l} \circ \delta = 0, \tag{20}$$

which can also be written as

$$\begin{aligned}
&\frac{\partial (y^{\sigma} \circ \delta)}{\partial x^l} - y^{\sigma}_{l} \circ \delta = 0,\\
&\frac{\partial (y^{\sigma}_{i_1} \circ \delta)}{\partial x^l} - y^{\sigma}_{i_1 l} \circ \delta = \frac{\partial^2 (y^{\sigma} \circ \delta)}{\partial x^{i_1}\, \partial x^l} - y^{\sigma}_{i_1 l} \circ \delta = 0,\\
&\dots\\
&\frac{\partial (y^{\sigma}_{i_1 i_2 \dots i_{r-1}} \circ \delta)}{\partial x^l} - y^{\sigma}_{i_1 i_2 \dots i_{r-1} l} \circ \delta = \frac{\partial^r (y^{\sigma} \circ \delta)}{\partial x^{i_1}\, \partial x^{i_2} \dots \partial x^{i_{r-1}}\, \partial x^l} - y^{\sigma}_{i_1 i_2 \dots i_{r-1} l} \circ \delta = 0.
\end{aligned} \tag{21}$$

These conditions mean that the section δ is of the form $\delta = J^r(\pi^{r,0} \circ \delta)$ as required.

Theorem 7 *Let* $1 \le k \le n$ *, and let* $\rho \in \Omega^r_k W$ *. The following two conditions are equivalent:*

(a) ρ *is contact.*

(b) *In any fibred chart,* ρ *is expressible as*

$$\rho = \sum_{0 \le |I| \le r-1} \omega^{\sigma}_I \wedge \Phi^I_{\sigma} + \sum_{|I| = r-1} d\omega^{\sigma}_I \wedge \Psi^I_{\sigma}, \tag{22}$$

where Φ^J_{σ} *are* $(k-1)$*-forms, containing all exterior factors* ω^{σ}_J *, and* Ψ^I_{σ} *are* $(k-2)$*-forms, not containing any exterior factor* ω^{σ}_J *, where* $0 \le |J| \le r-1$.

Proof We show that (a) implies (b). Express ρ in the contact basis of 1-forms (9). We get

$$\rho = \sum_{0 \le |J| \le r-1} \omega^{\sigma}_J \wedge \Phi^J_{\sigma} + \rho', \tag{23}$$

where the sum includes all terms, generated by the forms ω^{σ}_J,

$0 \le |J| \le r-1$, and

$$\begin{aligned}
\rho' = {} & \frac{1}{k!0!} A_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} \\
& + \frac{1}{(k-1)!1!} A^{I_1}_{\sigma_1 \ i_2 i_3 \dots i_k} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_k} \\
& + \frac{1}{(k-2)!2!} A^{I_1 \ I_2}_{\sigma_1 \ \sigma_2 \ i_3 i_4 \dots i_k} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_k} \\
& + \dots + \frac{1}{1!(k-1)!} A^{I_1 \ I_2 \ \dots \ I_{k-1}}_{\sigma_1 \ \sigma_2 \ \dots \ \sigma_{k-1} \ i_k} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_{k-1}}_{I_{k-1}} \wedge dx^{i_k} \\
& + \frac{1}{0!k!} A^{I_1 \ I_2 \ \dots \ I_k}_{\sigma_1 \ \sigma_2 \ \dots \ \sigma_k} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_k}_{I_k},
\end{aligned} \tag{24}$$

with multi-indices of length r. Since h is an exterior algebra homomorphism, we have from 1.5, Lemma 9

$$\begin{aligned}
h\rho' = {} & \Big(\frac{1}{k!0!} A_{i_1 i_2 \dots i_k} + \frac{1}{(k-1)!1!} A^{I_1}_{\sigma_1 \ i_2 i_3 \dots i_k} y^{\sigma_1}_{I_1 i_1} + \frac{1}{(k-2)!2!} A^{I_1 \ I_2}_{\sigma_1 \ \sigma_2 \ i_3 i_4 \dots i_k} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \\
& + \dots + \frac{1}{1!(k-1)!} A^{I_1 \ I_2 \ \dots \ I_{k-1}}_{\sigma_1 \ \sigma_2 \ \dots \ \sigma_{k-1} \ i_k} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \dots y^{\sigma_{k-1}}_{I_{k-1} i_{k-1}} \\
& + \frac{1}{0!k!} A^{I_1 \ I_2 \ \dots \ I_k}_{\sigma_1 \ \sigma_2 \ \dots \ \sigma_k} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \dots y^{\sigma_k}_{I_k i_k} \Big) dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_k},
\end{aligned} \tag{25}$$

thus, because the components of $h\rho'$ are polynomials, condition (a) implies

$$\begin{aligned}
& A_{i_1 i_2 \dots i_k} = 0, \\
& A^{I_1}_{\sigma_1 \ i_2 i_3 \dots i_k} \delta^{j_1}_{i_1} = 0 \quad \mathrm{Sym}(I_1 j_1) \quad \mathrm{Alt}(i_1 i_2 \dots i_k), \\
& A^{I_1 \ I_2}_{\sigma_1 \ \sigma_2 \ i_3 i_4 \dots i_k} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} = 0 \quad \mathrm{Sym}(I_1 j_1) \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Alt}(i_1 i_2 \dots i_k), \\
& \dots \\
& A^{I_1 \ I_2 \ \dots \ I_{k-1}}_{\sigma_1 \ \sigma_2 \ \dots \ \sigma_{k-1} \ i_k} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \dots \delta^{j_{k-1}}_{i_{k-1}} = 0 \quad \mathrm{Sym}(I_1 j_1) \quad \mathrm{Sym}(I_2 j_2) \\
& \quad \dots \quad \mathrm{Sym}(I_{k-1} j_{k-1}) \quad \mathrm{Alt}(i_1 i_2 \dots i_k), \\
& A^{I_1 \ I_2 \ \dots \ I_k}_{\sigma_1 \ \sigma_2 \ \dots \ \sigma_k} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \dots \delta^{j_k}_{i_k} = 0 \quad \mathrm{Sym}(I_1 j_1) \quad \mathrm{Sym}(I_2 j_2) \\
& \quad \dots \quad \mathrm{Sym}(I_k j_k) \quad \mathrm{Alt}(i_1 i_2 \dots i_k).
\end{aligned} \tag{26}$$

These equations show that the coefficients $A^{I_1}_{\sigma_1 \ i_2 i_3 \dots i_k}$, $A^{I_1 \ I_2}_{\sigma_1 \ \sigma_2 \ i_3 i_4 \dots i_k}$, $\dots$, $A^{I_1 \ I_2 \ \dots \ I_{k-1}}_{\sigma_1 \ \sigma_2 \ \dots \ \sigma_{k-1} \ i_k}$ must be of Kronecker type (Theorem 3). Expressing them as Kronecker tensors and substituting to (24) we get (22).

Conversely, since h is an exterior algebra homomorphism, (b) implies (a) by Theorem 5.

Theorem 7 can also be restated in terms of the ideal $\operatorname{Ker} h \subset \Omega^r W$.

Theorem 8 (a) *The contact k-forms such that* $k \le n$, *are locally generated by the forms* $\omega^\sigma_{j_1 j_2 \dots j_k}$ *and* $d\omega^\sigma_{j_1 j_2 \dots j_{r-1}}$.
(b) *The ideal* $\operatorname{Ker} h$ *is closed under exterior derivative*.

Proof (a) This assertion follows from Theorem 7.
(b) This follows from Theorem 7.

Since $\Omega^r_q W \subset \operatorname{Ker} h$ for any $q \ge n+1$, the ideal $\operatorname{Ker} h$ is *not* generated by the forms $\omega^\sigma_{j_1 j_2 \dots j_k}$ nor by the forms $\omega^\sigma_{j_1 j_2 \dots j_k}$ and $d\omega^\sigma_{j_1 j_2 \dots j_{r-1}}$. On the other hand, Theorem 5, (c) implies that the 1-forms $\omega^\sigma_{j_1 j_2 \dots j_k}$, where $0 \le k \le r-1$, also define an ideal; however, this ideal is *not* integrable (see e.g. Theorem 5, (b)). Its *completion* is an ideal, closed under exterior derivative operator. This ideal is locally generated by the forms $\omega^\sigma_{j_1 j_2 \dots j_k}$ and $d\omega^\sigma_{j_1 j_2 \dots j_{r-1}}$, and is denoted by ${}^{(c)}\Omega^r W$. Clearly, ${}^{(c)}\Omega^r W \subset \operatorname{Ker} h$.

We now extend the definition of contact forms to *any* q-forms $\rho \in \Omega^r W$ (see also Theorem 7, (22)); we shall say that a form $\rho \in \Omega^r W$ is *contact*, if $\rho \in {}^{(c)}\Omega^r W$. Thus, a q-form $\rho \in \Omega^r W$ is contact if and only if for any fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, such that $V \subset W$, it is generated by the 1-forms ω^σ_J, $0 \le |J| \le r-1$, and 2-forms $d\omega^\sigma_J$, $|J| = r-1$, that is,

$$\rho = \sum_{0 \le |J| \le r-1} \omega^\sigma_J \wedge \Phi^J_\sigma + \sum_{|J| = r-1} d\omega^\sigma_J \wedge \Psi^J_\sigma \tag{27}$$

for some $(q-1)$-forms Φ^J_σ and $(q-2)$-forms Ψ^J_σ.

2.3 The first canonical decomposition In Section 1.5 we introduced a vector bundle homomorphism h of the tangent bundle $TJ^{r+1}Y$ into $TJ^r Y$ by the formula

$$h\xi = T_x J^r \gamma \circ T\pi^{r+1} \cdot \xi, \tag{1}$$

where ξ is a tangent vector to the manifold $J^{r+1}Y$ at a point $J^{r+1}_x \gamma$. h makes the following diagram

$$\begin{array}{ccc} TJ^{r+1}Y & \xrightarrow{\ h\ } & TJ^r Y \\ \downarrow & & \downarrow \\ J^{r+1}Y & \xrightarrow{\ \pi^{r+1,r}\ } & J^r Y \end{array} \tag{2}$$

commutative, and induces a decomposition of the tangent vectors $T\pi^{r+1}\cdot\xi$,

(3) $$T\pi^{r+1,r}\cdot\xi = h\xi + p\xi.$$

$h\xi$ (resp. $p\xi$) is the *horizontal* (resp. *contact*) *component* of the vector ξ. Recall that the horizontal and contact components satisfy

(4) $$T\pi^{r}\cdot h\xi = T\pi^{r+1}\cdot\xi, \quad T\pi^{r}\cdot h\xi = 0$$

(1.5, Lemma 7).

The horizontalization h also induces a decomposition of each of the modules of q-forms $\Omega_q^r W$. Suppose that $q \geq 1$. Let $\rho \in \Omega_q^r W$ be a q-form, $J_x^{r+1}\gamma \in W^{r+1}$. Consider the pull-back $(\pi^{r+1,r})^*\rho$, the form $(\pi^{r+1,r})^*\rho(J_x^{r+1}\gamma)$ at a point $J_x^{r+1}\gamma$, and the value $(\pi^{r+1,r})^*\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)$ on any tangent vectors ξ_1, ξ_2, …, ξ_q of $J^{r+1}Y$ at $J_x^{r+1}\gamma$. We write for each l,

(5) $$T\pi^{r+1,r}\cdot\xi_l = h\xi_l + p\xi_l,$$

and substitute these vectors in the pull-back $(\pi^{r+1,r})^*\rho$. Since by definitions

(6) $$\begin{aligned}&(\pi^{r+1,r})^*\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\ &\quad= \rho(J_x^r\gamma)(T\pi^{r+1,r}\cdot\xi_1, T\pi^{r+1,r}\cdot\xi_2,\ldots,T\pi^{r+1,r}\cdot\xi_q)\\ &\quad= \rho(J_x^r\gamma)(h\xi_1+p\xi_1, h\xi_2+p\xi_2,\ldots,h\xi_q+p\xi_q),\end{aligned}$$

we get, collecting together all terms homogeneous of degree k in the contact components $p\xi_1$, $p\xi_2$, …, $p\xi_q$ of the vectors ξ_1, ξ_2, … ξ_q, where $k = 0,1,2,\ldots,q$, a q-form $p_k\rho$ on W^{r+1}, defined by

(7) $$\begin{aligned}&p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\ &\quad= \sum \varepsilon^{j_1 j_2\ldots j_k j_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_{j_1},\ldots,p\xi_{j_k},h\xi_{j_{k+1}},\ldots,h\xi_{j_q}),\end{aligned}$$

(summation through $j_1 < j_2 < \ldots < j_k$ and $_{k+1} < j_{k+2} < \ldots < j_q$), or equivalently, by

(8) $$\begin{aligned}&p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\ &\quad= \frac{1}{k!(q-k)!}\varepsilon^{j_1 j_2\ldots j_k j_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_{j_1},p\xi_{j_2},\ldots,p\xi_{j_k},h\xi_{j_{k+1}},\ldots,h\xi_{j_q})\end{aligned}$$

(summation through *all* values of the indices j_1, j_2, …, j_k, j_{k+1}, …, j_q). The form $p_k\rho$ is called the *k-contact component* of the form ρ.

If $(\pi^{r+1,r})^*\rho = p_k\rho$ or, which is the same, if $p_j\rho = 0$ for all $j \neq k$, then the integer k is called the *degree of contactness* of the form ρ. The degree of contactness of the q-form $\rho = 0$ is equal to k for every $k = 0,1,2,\ldots,q$. We say that ρ is of *degree of contactness* $\geq k$, if $p_0\rho = 0$, $p_1\rho = 0$, $\ldots$, $p_{k-1}\rho = 0$.

We usually write

(9) $$p_0\rho = h\rho$$

and call this form the *horizontal component* of ρ. Then

(10) $$h\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q) = \rho(J_x^r\gamma)(h\xi_1,h\xi_2,\ldots,h\xi_q).$$

We also introduce the *contact component* of ρ by

(11) $$p\rho = p_1\rho + p_2\rho + \ldots + p_q\rho.$$

We shall say that ρ is *k-contact*, if

(12) $$(\pi^{r+1,r})^*\rho = p_k\rho.$$

Summarizing, any q-form $\rho \in \Omega_q^r W$, where $q \geq 1$, can be expressed as

(13) $$(\pi^{r+1,r})^*\rho = h\rho + p\rho = h\rho + \sum_{i=1}^{q} p_i\rho = \sum_{i=0}^{q} p_i\rho.$$

This formula will be referred to as the *first canonical decomposition* of the form ρ (note however, the decomposition concerns rather the pull-back $(\pi^{r+1,r})^*\rho$ than ρ itself).

We extend these definitions to 0-forms (functions). We define the *horizontal* and *contact components* of a function $f : W^r \to \mathbf{R}$ as

(14) $$hf = (\pi^{r+1,r})^* f, \quad pf = 0.$$

Clearly, then the first canonical decomposition (13) remains valid.

The following observation is immediate.

Lemma 2 *If* $q - k > n$, *then*

(15) $$h\rho = 0, \quad p_1\rho = 0, \quad p_2\rho = 0, \quad \ldots, \quad p_{q-n-1} = 0.$$

Proof Indeed, expression $\rho(J_x^r\gamma)(p\xi_{j_1}, p\xi_{j_2},\ldots,p\xi_{j_k}, h\xi_{j_{k+1}},\ldots,h\xi_{j_q})$ in (8) is a $(q-k)$-linear function of vectors $\zeta_{j_{k+1}} = T\pi^{r+1}\cdot\xi_{j_{k+1}}$, $\zeta_{j_{k+2}} = T\pi^{r+1}\cdot\xi_{j_{k+2}}$, $\ldots$, $\zeta_{j_q} = T\pi^{r+1}\cdot\xi_{j_q}$, belonging to the tangent space T_xX. Consequently, if

$q-k>n=\dim X$, then $p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\dots,\xi_q)=0$.

One can determine the chart expressions of contact components of a form by means of the formula $(\pi^{r+1,r})^* dy^\sigma_{i_1i_2\dots i_k}=\omega^\sigma_{i_1i_2\dots i_k}+y^\sigma_{i_1i_2\dots i_kl}dx^l$. To this purpose it will be convenient to use the *multi-index notation*; the results can immediately be restated in the standard index notation.

We introduce multi-indices $I=(i_1i_2\dots i_k)$, where $k=0,1,2,\dots,r$ and the entries are indices such that $1\le i_1,i_2,\dots,i_k\le n$. The number *k* is the *length* of *I* and is denoted by $|I|$. If *i* is any integer such that $1\le i\le n$, we denote by Ii the multi-index $Ii=(i_1i_2\dots i_ki)$; the length of Ii is $|Ii|=k+1$.

We also introduce the symbol $\mathrm{Alt}(i_1i_2\dots i_k)$ to denote *alternation* in the indices $i_1,i_2,\dots,i_k$; writing $U^{i_1i_2\dots i_k}$ $\mathrm{Alt}(i_1i_2\dots i_k)$ we mean the skew-symmetric component of $U^{i_1i_2\dots i_k}$.

The following is a chart expression formula for the *k*-contact components of some special *q*-forms.

Lemma 3 *Let W be an open set in Y, an integer,* $\rho\in\Omega^r_qW$ *a form, and let* (V,ψ) *,* $\psi=(x^i,y^\sigma)$ *, be a fibred chart on Y such that* $V\subset W$ *. Assume that* ρ *has on* V^r *the chart expression*

$$\rho=\sum_{s=0}^{q}\frac{1}{s!(q-s)!}A^{I_1\ I_2\ \dots\ I_s}_{\sigma_1\sigma_2\dots\sigma_s\ i_{s+1}i_{s+2}\dots i_q}dy^{\sigma_1}_{I_1}\wedge dy^{\sigma_2}_{I_2}\wedge\dots\wedge dy^{\sigma_s}_{I_s}\wedge dx^{i_{s+1}}\wedge dx^{i_{s+2}}\wedge\dots\wedge dx^{i_q},\tag{16}$$

with multi-indices of length r. Then the k-contact component $p_k\rho$ *of* ρ *has on* V^{r+1} *the chart expression*

$$p_k\rho=\frac{1}{k!(q-k)!}B^{I_1\ I_2\ \dots\ I_k}_{\sigma_1\sigma_2\dots\sigma_k\ i_{k+1}i_{k+2}\dots i_q}\omega^{\sigma_1}_{I_1}\wedge\omega^{\sigma_2}_{I_2}\wedge\dots\wedge\omega^{\sigma_k}_{I_k}\wedge dx^{i_{k+1}}\wedge dx^{i_{k+2}}\wedge\dots\wedge dx^{i_q},\tag{17}$$

where

$$B^{I_1\ I_2\ \dots\ I_k}_{\sigma_1\sigma_2\dots\sigma_k\ i_{k+1}i_{k+2}\dots i_q}=\sum_{s=k}^{q}\binom{q-k}{q-s}\tilde{A}^{I_1\ I_2\ \dots\ I_k\ I_{k+1}\ I_{k+2}\ \dots\ I_s}_{\sigma_1\sigma_2\dots\sigma_k\sigma_{k+1}\sigma_{k+2}\dots\sigma_s\ i_{s+1}i_{s+2}\dots i_q}y^{\sigma_{k+1}}_{I_{k+1}i_{k+1}}y^{\sigma_{k+2}}_{I_{k+2}i_{k+2}}\dots y^{\sigma_s}_{I_si_s}\,\mathrm{Alt}(i_{k+1}i_{k+2}\dots i_si_{s+1}\dots i_q),\tag{18}$$

and $\tilde{A}^{I_1\ I_2\ \dots\ I_k\ I_{k+1}\ I_{k+2}\ \dots\ I_s}_{\sigma_1\sigma_2\dots\sigma_k\sigma_{k+1}\sigma_{k+2}\dots\sigma_s\ i_{s+1}i_{s+2}\dots i_q}$ *are traceless components of the coefficients* $A^{I_1\ I_2\ \dots\ I_k\ I_{k+1}\ I_{k+2}\ \dots\ I_s}_{\sigma_1\sigma_2\dots\sigma_k\sigma_{k+1}\sigma_{k+2}\dots\sigma_s\ i_{s+1}i_{s+2}\dots i_q}$.

Proof To derive formula (17), we pull-back ρ to V^{r+1} and express the form $(\pi^{r+1,r})^*\rho$ in terms of the contact basis; in the multi-index notation the transformation equations are

$$dx^i = dx^i, \quad dy_I^\sigma = \omega_I^\sigma + y_{Ii}^\sigma dx^i, \quad 0 \le |I| \le r \tag{19}$$

(Theorem 5). Thus, we set in (16)

$$dy_{I_l}^{\sigma_l} = \omega_{I_l}^{\sigma_l} + y_{I_l i_l}^{\sigma_l} dx^{i_l}, \tag{20}$$

and consider the terms such that $s \ge 1$. Then the form $dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge \ldots \wedge dy_{I_s}^{\sigma_s}$ is equal to

$$(\omega_{I_1}^{\sigma_1} + y_{I_1 i_1}^{\sigma_1} dx^{i_1}) \wedge (\omega_{I_2}^{\sigma_2} + y_{I_2 i_2}^{\sigma_2} dx^{i_2}) \wedge \ldots \wedge (\omega_{I_s}^{\sigma_s} + y_{I_s i_s}^{\sigma_s} dx^{i_s}). \tag{21}$$

Collecting together all terms homogeneous of degree k in the 1-forms $\omega_{I_l}^{\sigma_l}$ we get $\binom{s}{k}$ summands with exactly k entries the contact 1-forms $\omega_{I_l}^{\sigma_l}$. Thus, using symmetry properties of the components $A_{\sigma_1\ \sigma_1 \ldots \sigma_s\ i_{s+1} i_{s+2} \ldots i_q}^{I_1\ I_1\ \ I_s}$ in (16) and interchanging multi-indices, we see the terms containing k entries $\omega_{I_l}^{\sigma_l}$ are, for fixed s and each $k = 1,2,\ldots,s$, given by

$$\begin{aligned}\frac{1}{s!(q-s)!}\binom{s}{k} A_{\sigma_1\ \sigma_2 \ldots \sigma_s\ i_{s+1} i_{s+2} \ldots i_q}^{I_1\ I_2\ \ I_s} y_{I_{k+1} i_{k+1}}^{\sigma_{k+1}} y_{I_{k+2} i_{k+2}}^{\sigma_{k+2}} \ldots y_{I_s i_s}^{\sigma_s} \omega_{I_1}^{\sigma_1} \wedge \omega_{I_2}^{\sigma_2} \wedge \ldots \wedge \omega_{I_k}^{\sigma_k} \\ \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \ldots \wedge dx^{i_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \ldots \wedge dx^{i_q}.\end{aligned} \tag{22}$$

Expressing the factor as

$$\frac{1}{s!(q-s)!}\binom{s}{k} = \frac{1}{k!(q-k)!} \frac{k!(q-k)!}{s!(q-s)!} \frac{s!}{k!(s-k)!} = \frac{1}{k!(q-k)!}\binom{q-k}{q-s}, \tag{23}$$

we can write expression (21) as

$$\begin{aligned}\frac{1}{k!(q-k)!}\binom{q-k}{q-s} A_{\sigma_1\ \sigma_2 \ldots \sigma_s\ i_{s+1} i_{s+2} \ldots i_q}^{I_1\ I_2\ \ I_s} y_{I_{k+1} i_{k+1}}^{\sigma_{k+1}} y_{I_{k+2} i_{k+2}}^{\sigma_{k+2}} \ldots y_{I_s i_s}^{\sigma_s} \omega_{I_1}^{\sigma_1} \wedge \omega_{I_2}^{\sigma_2} \\ \wedge \ldots \wedge \omega_{I_k}^{\sigma_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \ldots \wedge dx^{i_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \ldots \wedge dx^{i_q}.\end{aligned} \tag{24}$$

Formula (24) is valid for each $s = 1,2,\ldots,q$ and each $k = 1,2,\ldots,s$, and we need the sum of all these terms to get expression (16). To this purpose we shall use the summation defined as follows. Instead of the summation through the pairs (s,k), given by the table

(25)

s	1	2	3	$\dots$	$q-1$	q
k	1	1,2	1,2,3	$\dots$	$1,2,3,\dots,q-1$	$1,2,3,\dots,q$

we pass to the summation over the pairs (k,s) given by

(26)

k	1	2	3	$\dots$	$q-1$	q
s	$1,2,3,\dots,q$	$2,3,\dots,q$	$3,4,\dots,q$	$\dots$	$q-1,q$	q

Now we can substitute from (24) back to (16). We have, with multi-indices of length r,

$$\begin{aligned}\rho = {} & \frac{1}{q!}A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} \\ & + \sum_{s=1}^{q}\sum_{k=1}^{s} \frac{1}{k!(q-k)!}\binom{q-k}{q-s} A^{I_1\, I_2\, \dots\, I_s}_{\sigma_1\, \sigma_2 \dots \sigma_s\, i_{s+1} i_{s+2} \dots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \dots y^{\sigma_s}_{I_s i_s} \\ & \cdot \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_s} \wedge dx^{i_{s+1}} \wedge \dots \wedge dx^{i_q}\end{aligned} \tag{27}$$

hence

$$\begin{aligned}\rho = {} & \frac{1}{q!}A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} \\ & + \sum_{k=1}^{q} \frac{1}{k!(q-k)!}\left(\sum_{s=k}^{q}\binom{q-k}{q-s} A^{I_1\, I_2\, \dots\, I_s}_{\sigma_1\, \sigma_2 \dots \sigma_s\, i_{s+1} i_{s+2} \dots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \dots y^{\sigma_s}_{I_s i_s}\right) \\ & \cdot \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_q}.\end{aligned} \tag{28}$$

Note that in (18) the coefficients $A^{I_1\, I_2\, \dots\, I_k\, I_{k+1}\, I_{k+2}\, \dots\, I_s}_{\sigma_1\, \sigma_2 \dots \sigma_k\, \sigma_{k+1}\, \sigma_{k+2} \dots \sigma_s\, i_{s+1} i_{s+2} \dots i_q}$ can be replaced with their *traceless* components $\tilde{A}^{I_1\, I_2\, \dots\, I_k\, I_{k+1}\, I_{k+2}\, \dots\, I_s}_{\sigma_1\, \sigma_2 \dots \sigma_k\, \sigma_{k+1}\, \sigma_{k+2} \dots \sigma_s\, i_{s+1} i_{s+2} \dots i_q}$. Indeed, applying to these coefficients the first trace decomposition theorem, we easily see that the Kronecker components vanish identically. Moreover, if one of the traces vanishes, then using index symmetries we see that all the traces must also vanish.

This proves formulas (17) and (18).

It remains to prove invariance of the forms (17); their independence on fibred charts is, however, an immediate consequence of the properties of contact basis (Theorem 5).

The operators p_0, p_1, p_2, $\dots$, p_q defined by formula (17), behave like projectors:

Corollary 1 *We have for any k and l*

$$p_k p_l \rho = \begin{cases} (\pi^{r+2,r+1})^* p_k \rho, & k = l, \\ 0, & k \neq l. \end{cases} \tag{29}$$

The following observation is an application of Lemma 3 to the exterior derivative operator.

Corollary 2 *We have for any k*

$$(\pi^{r+2,r+1})^* p_k d\rho = p_k dp_{k-1}\rho + p_k dp_k \rho. \tag{30}$$

Proof The first canonical decomposition applied to both sides of the identity $d(\pi^{r+1,r})^* \rho = (\pi^{r+1,r})^* d\rho$ gives

$$\begin{aligned} &p_0 d\rho + p_1 d\rho + p_2 d\rho + \ldots + p_q d\rho + p_{q+1} d\rho \\ &\quad = dp_0\rho + dp_1\rho + dp_2\rho + \ldots + dp_{q-1}\rho + dp_q\rho. \end{aligned} \tag{31}$$

But from Lemma 3, the decomposition of $p_k d\rho$ depends only on $p_k dp_{k-1}\rho$ and $p_k dp_k \rho$, and $p_k d\rho = p_k d_{k-1} p\rho + p_k dp_p \rho$. Decomposing both sides of (31) and applying Corollary 1 we get $(\pi^{r+2,r+1})^* p_k d\rho = p_k dp_{k-1}\rho + p_k dp_k \rho$.

The following theorem describes the local structure of the horizontalization.

Theorem 9 *Let W be an open set in the fibred manifold Y. Then the horizontalization* $\Omega^r W \ni \rho \to h\rho \in \Omega^{r+1} W$ *is a unique* $\mathbf{R}$*-linear, exterior-product-preserving mapping such that for any function* $f : W^r \to \mathbf{R}$, *and any fibred chart* (V, ψ), $\psi = (x^i, y^\sigma)$, *with* $V \subset W$,

$$hf = f \circ \pi^{r+1,r}, \quad hdf = d_i f \cdot dx^i, \tag{32}$$

where

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{j_1 \le j_2 \le \ldots \le j_k} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i}. \tag{33}$$

Proof The proof that *h*, defined by (10) and (14), has the desired properties, is standard. To prove uniqueness, note that (32) and (33) imply

$$hdx^i = dx^i, \quad hdy^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k i} dx^i. \tag{34}$$

Now it is easy to check that any two mappings h_1, h_2 satisfying the assumptions of Theorem 9, which agree on functions and their exterior derivatives, coincide.

Remark 4 The mapping $p_0 : \Omega^r W \to \Omega^{r+1} W$ is a homomorphism of exterior algebras. For any positive integer k, the mapping $p_k : \Omega^r W \to \Omega^{r+1} W$ satisfies

(35) $$p_k(\rho + \eta) = p_k \rho + p_k \eta, \quad p_k(f\rho) = (f \circ \pi^{r+1,r}) p_k \rho$$

for any function $f \in \Omega_0^r W$, but is *not* a homomorphism of exterior algebras.

2.4 Contact components and geometric operations The following theorem summarizes basic properties of contact components of a form with respect to differential-geometric operations, such as the wedge product $\wedge$, the contraction i_ζ of a form by a vector ζ, and the Lie derivative ∂_ξ by a vector field ξ.

Theorem 10 *Let* ρ *and* η *be two differential forms on* $W^r \subset J^r Y$, $J_x^r \gamma \in J^r Y$ *a point,* Z *a* π^{r+1}*-vertical,* $\pi^{r+1,r}$*-projectable vector field on* W^{r+1} *with* $\pi^{r+1,r}$*-projection* Z_0*, and* Ξ *a* π*-projectable vector field on* Y*. Let* α *be an automorphism of* Y*. Then for every* k *such that the corresponding expressions are defined,*

(1) $$p_k(\rho \wedge \eta) = \sum_{i+j=k} p_i \rho \wedge p_j \eta,$$

(2) $$i_{\mathrm{Z}} p_k \rho = p_{k-1} i_{\mathrm{Z}_0} \rho,$$

(3) $$p_k(J^r \alpha^* \rho) = J^{r+1} \alpha^* p_k \rho,$$

(4) $$p_k(\partial_{J^r \Xi} \rho) = \partial_{J^{r+1} \Xi} p_k \rho,$$

(5) $$i_{J^{r+1}\Xi} p_k \rho = p_k(i_{J^r \Xi} \rho).$$

Proof Formulas (1) – (5) are immediate consequences of definitions: To get (1) we express the pull-back $(\pi^{r+1,r})^*(\rho \wedge \eta) = (\pi^{r+1,r})^* \rho \wedge (\pi^{r+1,r})^* \eta$ and apply 2.3, Corollary 1. (2) follows from the definition of the horizontalization of vectors. Formulas (3) and (4) follow immediately from the commutativity property $(\pi^{r+1,r})^* J^r \alpha^* \rho = J^{r-1} \alpha^* (\pi^{r+1,r})^* \rho$. Finally, (5) follows from (2).

2.5 The second canonical decomposition The following assertion, describing a decomposition of differential forms on the r-jet prolongations J^rY, relative to a given fibred chart, plays a basic role in the proofs.

Theorem 11 (Second canonical decomposition) *Let q be arbitrary, and let $\rho \in \Omega_q^r W$ be a q-form. Let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y such that $V \subset W$. Then ρ has on V^r a unique expression*

(1) $$\rho = \rho_1 + \rho_2 + \tilde{\rho},$$

with the following properties:

(a) *ρ_1 is generated by contact 1-forms ω_J^σ with $0 \le |J| \le r-1$.*

(b) *ρ_2 is generated by contact 2-forms $d\omega_I^\sigma$ with $|I| = r-1$ and does not contain any factor ω_J^σ, that is,*

(2) $$\begin{aligned}\rho_2 = \sum B^{J_1}_{\nu_1}{}^{J_2}_{\nu_2}\cdots{}^{J_p}_{\nu_p}{}^{I_{p+1}}_{\sigma_{p+1}}{}^{I_{p+2}}_{\sigma_{p+2}}\cdots{}^{I_s}_{\sigma_s}{}_{i_{s+1}i_{s+2}\dots i_Q} d\omega^{\nu_1}_{J_1} \wedge d\omega^{\nu_2}_{J_2} \wedge \dots \wedge d\omega^{\nu_p}_{L_p} \\ \wedge dy^{\sigma_{p+1}}_{I_{p+1}} \wedge dy^{\sigma_{p+2}}_{I_{p+2}} \wedge \dots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \dots \wedge dx^{i_Q},\end{aligned}$$

where the summation is taking place through $p \ge 1$, $q = p + Q$, the multi-indices satisfy

(3) $$0 \le |J_1|, |J_2|, \dots, |J_p| \le r-1, \quad |I_{p+1}|, |I_{p+2}|, \dots, |I_{p+s}| = r,$$

and the components $B^{J_1}_{\nu_1}{}^{J_2}_{\nu_2}\cdots{}^{J_p}_{\nu_p}{}^{I_{p+1}}_{\sigma_{p+1}}{}^{I_{p+2}}_{\sigma_{p+2}}\cdots{}^{I_p}_{\sigma_p}{}_{i_{l+p+1}i_{l+p+2}\dots i_Q}$ are completely traceless.

(c) *$\tilde{\rho}$ has an expression*

(4) $$\begin{aligned}\tilde{\rho} &= C_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} \\ &+ C^{I_1}_{\sigma_1}{}_{i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_q} \\ &+ C^{I_1}_{\sigma_1}{}^{I_2}_{\sigma_2}{}_{i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_q} \\ &+ \dots + C^{I_1}_{\sigma_1}{}^{I_2}_{\sigma_2}\cdots{}^{I_{q-1}}_{\sigma_{q-1}}{}_{i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ &+ C^{I_1}_{\sigma_1}{}^{I_2}_{\sigma_2}\cdots{}^{I_q}_{\sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_q}_{I_q},\end{aligned}$$

where $|I_1|, |I_2|, \dots, |I_{q-1}| = r$, and all the components $C^{I_1}_{\sigma_1}{}_{i_2 i_3 \dots i_q}$, $C^{I_1}_{\sigma_1}{}^{I_2}_{\sigma_2}{}_{i_2 i_3 \dots i_q}$, $\dots$, $C^{I_1}_{\sigma_1}{}^{I_2}_{\sigma_2}\cdots{}^{I_{q-1}}_{\sigma_{q-1}}{}_{i_2 i_3 \dots i_q}$ are completely traceless.

Proof Express ρ in the contact basis. Then $\rho = \mu_0 + \mu'$, where μ_0 is generated by contact 1-forms ω_J^σ with $0 \le |J| \le r-1$ and μ' is of the form

$$\begin{aligned}\mu' &= A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} \\ &+ A_{\sigma_1 \; i_2 i_3 \dots i_q}^{I_1} dy_{I_1}^{\sigma_1} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} \\ &+ A_{\sigma_1 \; \sigma_2 \; i_2 i_3 \dots i_q}^{I_1 \; I_2} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_q} \\ &+ \dots + A_{\sigma_1 \; \sigma_2 \dots \sigma_{q-1} \; i_q}^{I_1 \; I_2 \dots I_{q-1}} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge \dots \wedge dy_{I_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q} \\ &+ A_{\sigma_1 \; \sigma_2 \dots \sigma_q}^{I_1 \; I_2 \dots I_q} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge \dots \wedge dy_{I_q}^{\sigma_q},\end{aligned} \tag{5}$$

where $|I_1|,|I_2|,\dots,|I_{q-1}| = r$. Applying to the coefficients $A_{\sigma_1 \; i_2 i_3 \dots i_q}^{I_1}$, $A_{\sigma_1 \; \sigma_2 \; i_2 i_3 \dots i_q}^{I_1 \; I_2}$, $\dots$, $A_{\sigma_1 \; \sigma_2 \dots \sigma_{q-1} \; i_q}^{I_1 \; I_2 \dots I_{q-1}}$ the complete trace decomposition theorem, we get formula (1) satisfying conditions (a), (b), and (c).

Formula (1) is the *second canonical decomposition* of the form ρ. The form $\tilde{\rho}$ is defined in a given fibred chart uniquely, and is sometimes called the *traceless component* of ρ.

2.6 Fibred homotopy operators In this section we study differential forms, defined on the trivial fibred manifold $U \times V$, whose base U is an open set in $\mathbf{R}^n$, and V is an open ball in $\mathbf{R}^m$ with centre at the origin. Our aim will be to investigate properties of the exterior derivative operator and differential equations, related with this operator. As a particular case we discuss the fibred homotopy operator on the s-jet prolongation $W^s = J^s(U \times V)$ of the Cartesian product $W = U \times V$.

First we consider a differential form ρ on an open ball V in the Euclidean space $\mathbf{R}^n$ with centre at the origin. We shall study the equation

$$d\eta = \rho \tag{1}$$

for an unknown $(k-1)$-form η on V; if η exists, it is called a *solution* of equation (1).

Consider an open ball $V \subset \mathbf{R}^m$ with centre 0, and denote by y^σ the canonical coordinates on V. Define a mapping $\chi : [0,1] \times V \to V$ by

$$\chi(s, y^\sigma) = (s y^\sigma). \tag{2}$$

Then

$$\chi^* dy^\sigma = y^\sigma ds + s dy^\sigma. \tag{3}$$

For any k-form ρ on V, where $k \ge 1$, consider the pull-back $\chi^* \rho$ which is a k-form on the set $[0,1] \times V$. Obviously, there exists a unique decomposition

(4) $$\chi^* \rho = ds \wedge \rho^{(0)}(s) + \rho'(s),$$

such that the k-forms $\rho^{(0)}(s)$ and $\rho'(s)$ do not contain ds. Note that by (3), $\rho'(s)$ arises from ρ by replacing each factor dy^σ with sdy^σ, and by replacing each coefficient f with $f \circ \chi$. Thus, $\rho'(s)$ obeys

(5) $$\rho'(1) = \rho, \quad \rho'(0) = 0.$$

Let k be a positive integer. Define for every k-form ρ

(6) $$I\rho = \int_0^1 \rho^{(0)}(s),$$

where the expression on the right-hand side means integration of the coefficients in the form $\rho^{(0)}(s)$ over s from 0 to 1.

Lemma 4 *Let V be an open ball in* $\mathbf{R}^m$ *with centre* 0.
(a) *For every differentiable function* $f : V \to \mathbf{R}$,

(7) $$f = Idf + f(0).$$

(b) *Suppose that* $k \geq 1$. *Then for any differential k-form* ρ *on V,*

(8) $$\rho = Id\rho + dI\rho.$$

Proof 1. If f is a function, then $df = (\partial f / \partial y^\sigma) dy^\sigma$, and we have by (3) $\chi^* df = ((\partial f / \partial y^\sigma) \circ \chi) \cdot (y^\sigma ds + sdy^\sigma)$. Consequently,

(9) $$Idf = y^\sigma \int_0^1 \left(\frac{\partial f}{\partial y^\sigma} \circ \chi \right) ds.$$

Now (7) follows from the identity

(10) $$\begin{aligned} f - f(0) &= (f \circ \chi)|_{s=1} - (f \circ \chi)|_{s=0} = \int_0^1 \frac{d(f \circ \chi)}{ds} ds \\ &= y^\sigma \int_0^1 \left(\frac{\partial f}{\partial y^\sigma} \circ \chi \right) ds. \end{aligned}$$

2. Let $k = 1$. Then ρ has an expression $\rho = B_\sigma dy^\sigma$, and the pull-back $\chi^* \rho$ is given by $\chi^* \rho = y^\sigma (B_\sigma \circ \chi) ds + (B_\sigma \circ \chi) sdy^\sigma$. Differentiating we get from this formula

$$\begin{aligned}\chi^* d\rho = d\chi^*\rho &= ds \wedge \left(-d(y^\sigma (B_\sigma \circ \chi) + \frac{((B_\sigma \circ \chi)s)}{s} dy^\sigma \right) \\ &+ s \frac{(B_\sigma \circ \chi)}{y^\nu} dy^\nu \wedge dy^\sigma,\end{aligned} \tag{11}$$

hence

$$I\rho = y^\sigma \int_0^1 B_\sigma \circ \chi \cdot ds. \tag{12}$$

Thus,

$$Id\rho = \int_0^1 \left(\frac{\partial((B_\sigma \circ \chi)s)}{\partial s} - \frac{\partial(y^\nu \cdot B_\nu \circ \chi)}{\partial y^\sigma} \right) ds \cdot dy^\sigma, \tag{13}$$

and

$$dI\rho = \int_0^1 \frac{\partial(y^\nu \cdot B_\nu \circ \chi)}{\partial y^\sigma} ds \cdot dy^\sigma. \tag{14}$$

Consequently,

$$\begin{aligned}Id\rho + dI\rho &= \int_0^1 \left(\frac{\partial((B_\sigma \circ \chi)s)}{\partial s} \right) ds \cdot dy^\sigma \\ &= ((B_\sigma \circ \chi \cdot s)|_{s=1} - (B_\sigma \circ \chi \cdot s)|_{s=0}) dy^\sigma \\ &= \rho.\end{aligned} \tag{15}$$

3. Let $k \geq 2$. Write ρ in the form

$$\rho = dy^\sigma \wedge \Psi_\sigma, \tag{16}$$

and define differential forms $\Psi_\sigma^{(0)}(s)$ and $\Psi'_\sigma(s)$ by

$$\chi^* \Psi_\sigma = ds \wedge \Psi_\sigma^{(0)}(s) + \Psi'_\sigma(s). \tag{17}$$

Then

$$\begin{aligned}\chi^*\rho &= (sdy^\sigma + y^\sigma ds) \wedge (ds \wedge \Psi_\sigma^{(0)}(s) + \Psi'_\sigma(s)) \\ &= sdy^\sigma \wedge (ds \wedge \Psi_\sigma^{(0)}(s) + \Psi'_\sigma(s)) + y^\sigma ds \wedge \Psi'_\sigma(s) \\ &= ds \wedge (-sdy^\sigma \wedge \Psi_\sigma^{(0)}(s) + y^\sigma \Psi'_\sigma(s)) + sdy^\sigma \wedge \Psi'_\sigma(s).\end{aligned} \tag{18}$$

Thus,

$$I\rho = \int_0^1 (-s\,dy^\sigma \wedge \Psi_\sigma^{(0)}(s) + y^\sigma \Psi'_\sigma(s)). \tag{19}$$

To determine $Id\rho$, we compute $\chi^* d\rho$. We get

$$\begin{aligned}
\chi^* d\rho &= d\chi^* \rho \\
&= -ds \wedge (s dy^\sigma \wedge d\Psi_\sigma^{(0)}(s) + dy^\sigma \wedge \Psi'_\sigma(s) + y^\sigma d\Psi'_\sigma(s)) \\
&\quad - dy^\sigma \wedge d(s\Psi'_\sigma(s))) \\
&= ds \wedge \big(-s dy^\sigma \wedge d\Psi_\sigma^{(0)}(s) - dy^\sigma \wedge \Psi'_\sigma(s) \\
&\quad - y^\sigma d\Psi'_\sigma(s) + dy^\sigma \wedge \frac{\partial(s\Psi'_\sigma(s))}{\partial s}\Big) - dy^\sigma \wedge dy^\nu \wedge \frac{\partial(s\Psi'_\sigma(s))}{\partial y^\nu},
\end{aligned} \tag{20}$$

where $\partial\eta(s)/\partial s$ denotes the form, arising from $\eta(s)$ by differentiation with respect to s, followed by multiplication by ds. Now by (20) and (6),

$$\begin{aligned}
Id\rho &= -dy^\sigma \wedge \int_0^1 s\,d\Psi_\sigma^{(0)}(s) - dy^\sigma \wedge \int_0^1 \Psi'_\sigma(s) \\
&\quad - y^\sigma \int_0^1 d\Psi'_\sigma(s) + dy^\sigma \wedge \int_0^1 \frac{\partial(s\Psi'_\sigma(s))}{\partial s}
\end{aligned} \tag{21}$$

It is important to notice that the exterior derivatives $d\Psi_\sigma^{(0)}(s)$, and $d\Psi'_\sigma(s)$ have the meaning of the derivatives with respect to y^σ (the terms containing ds are cancelled; see the definition of I (4), (6)).

Now we easily get

$$Id\rho + dI\rho = dy^\sigma \wedge \int_0^1 \frac{\partial(s\Psi'_\sigma(s))}{\partial s}. \tag{22}$$

Remembering that the integral symbol denotes integration of *coefficients* in the corresponding forms with respect to the parameter s from 0 to 1, and using (5), one obtains

$$\begin{aligned}
Id\rho + dI\rho &= dy^\sigma \wedge (1 \cdot \Psi'_\sigma(1) - 0 \cdot \Psi'_\sigma(0)) \\
&= dy^\sigma \wedge \Psi'_\sigma(1) = dy^\sigma \wedge \Psi_\sigma = \rho,
\end{aligned} \tag{23}$$

as desired.

As an immediate consequence, we get the following statement.

Lemma 5 (The Volterra-Poincare lemma) *Let V be an open ball in* $\mathbf{R}^m$ *with centre* 0, ρ *a differential k-form on V, where* $k \geq 1$. *The following two conditions are equivalent:*

(a) *There exists a form* η *on V such that*

(24) $$d\eta = \rho$$

(b) ρ *satisfies*

(25) $$d\rho = 0.$$

Proof If $d\eta = \rho$ for some η, we have $d\rho = dd\eta = 0$. Conversely, if $d\rho = 0$, we take $\eta = I\rho$ in Lemma 4.

Condition (25) is sometimes called *integrability condition* for the differential equation (24).

We now consider differential equations for differential forms, defined on the Cartesian product of open sets in Euclidean spaces, more general than equations (1). We suppose we are given an open set *U* in $\mathbf{R}^n$, and an open ball *V* in $\mathbf{R}^m$ with centre at the origin. We denote by π the first Cartesian projection of $U \times V$ onto *U*. Let *k* be a positive integer, and let ρ be a *k*-form on $U \times V$. We study the equation

(26) $$d\eta + \pi^* \eta_0 = \rho$$

for the unknowns a $(k-1)$-form η on $U \times V$, and a *k*-form η_0 on *U*. Any pair (η, η_0) satisfying (26) is called a *solution* of equation (26).

We denote by (x^i, y^σ) the canonical coordinates on $U \times V$, and by $\zeta : U \to U \times V$ the constant *zero section* of $U \times V$. We define a mapping $\chi : [0,1] \times U \times V \to U \times V$ by

(27) $$\chi(s, (x^i, y^\sigma)) = (x^i, sy^\sigma).$$

Then

(28) $$\chi^* dx^i = dx^i, \quad \chi^* dy^\sigma = y^\sigma ds + s dy^\sigma.$$

For any *k*-form ρ on $U \times V$, where $k \geq 1$, consider the pull-back $\chi^* \rho$ which is a *k*-form on the set $[0,1] \times U \times V$. Obviously, there exists a unique decomposition

(29) $$\chi^* \rho = ds \wedge \rho^{(0)}(s) + \rho'(s)$$

such that the *k*-forms $\rho^{(0)}(s)$ and $\rho'(s)$ do not contain ds. Note that by

(28), $\rho'(s)$ arises from ρ by replacing each factor dy^σ by sdy^σ, and by replacing each coefficient *f* by $f \circ \chi$; the factors dx^i remain unchanged. Thus, $\rho'(s)$ obeys

(30) $$\rho'(1) = \rho, \quad \rho'(0) = \pi^* \zeta^* \rho.$$

Let $k \geq 1$. We define

(31) $$I\rho = \int_0^1 \rho^{(0)}(s),$$

where the expression on the right means integration of the coefficients in the form $\rho^{(0)}(s)$ over *s* from 0 to 1.

Theorem 12 *Let* $U \subset \mathbf{R}^n$ *be an open set, and let* $V \subset \mathbf{R}^m$ *be an open ball with centre* 0.

(a) *For every differentiable function* $f : U \times V \to \mathbf{R}$,

(32) $$f = Idf + \pi^* \zeta^* f.$$

(b) *Let* $k \geq 1$. *Then for every differential k-form* ρ *on* $U \times V$,

(33) $$\rho = Id\rho + dI\rho + \pi^* \zeta^* \rho.$$

Proof 1. We have

(34) $$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^\sigma} dy^\sigma,$$

and by (28)

(35) $$\chi^* f = \left(\frac{\partial f}{\partial x^i} \circ \chi \right) dx^i + \left(\frac{\partial f}{\partial y^\sigma} \circ \chi \right) (y^\sigma ds + sdy^\sigma).$$

Now the identity

(36) $$\begin{aligned} f - \pi^* \zeta^* f &= f \circ \chi|_{s=1} - f \circ \chi|_{s=0} \\ &= \int_0^1 \frac{d(f \circ \chi)}{ds} ds = y^\sigma \int_0^1 \left(\frac{\partial f}{\partial y^\sigma} \circ \chi \right) ds = Idf, \end{aligned}$$

which follows from (31), gives the result.

2. Consider the case $k = 1$. Then the form ρ has an expression $\rho = A_i dx^i + B_\sigma dy^\sigma$, thus

(37)
$$\begin{aligned}\chi^*\rho &= (A_i\circ\chi)dx^i + (B_\sigma\circ\chi)(s\,dy^\sigma + y^\sigma ds)\\ &= y^\sigma(B_\sigma\circ\chi)ds + (A_i\circ\chi)dx^i + (B_\sigma\circ\chi)s\,dy^\sigma,\end{aligned}$$

and

(38)
$$\begin{aligned}\chi^*d\rho &= d\chi^*\rho\\ &= ds\wedge\left(-d(y^\sigma(B_\sigma\circ\chi)) + \frac{\partial(A_i\circ\chi)}{\partial s}dx^i + \frac{\partial((B_\sigma\circ\chi)s)}{\partial s}dy^\sigma\right)\\ &+\left(\frac{\partial(A_i\circ\chi)}{\partial x^j}dx^j + \frac{\partial(A_i\circ\chi)}{\partial y^\nu}dy^\nu\right)\wedge dx^i\\ &+ s\left(\frac{\partial(B_\sigma\circ\chi)}{\partial x^j}dx^j + \frac{\partial(B_\sigma\circ\chi)}{\partial y^\nu}dy^\nu\right)\wedge dy^\sigma,\end{aligned}$$

hence

(39)
$$I\rho = y^\sigma\int_0^1 B_\sigma\circ\chi\cdot ds,$$

and

(40)
$$\begin{aligned}Id\rho &= \int_0^1\left(\frac{\partial(A_i\circ\chi)}{\partial s} - \frac{\partial(y^\nu\cdot B_\nu\circ\chi)}{\partial x^i}\right)ds\cdot dx^i\\ &+\int_0^1\left(\frac{\partial((B_\sigma\circ\chi)s)}{\partial s} - \frac{\partial(y^\nu\cdot B_\nu\circ\chi)}{\partial y^\sigma}\right)ds\cdot dy^\sigma,\end{aligned}$$

and

(41)
$$dI\rho = y^\sigma\int_0^1\frac{\partial(B_\sigma\circ\chi)}{\partial x^i}ds\cdot dx^i + \int_0^1\frac{\partial(y^\nu\cdot B_\nu\circ\chi)}{\partial y^\sigma}ds\cdot dy^\sigma.$$

Consequently,

(42)
$$\begin{aligned}Id\rho + dI\rho &= A_i\circ\chi\,|_{s=1} - A_i\circ\chi\,|_{s=0} + (B_\sigma\circ\chi\cdot s)\,|_{s=1} - (B_\sigma\circ\chi\cdot s)\,|_{s=0}\\ &= \rho - \pi^*\zeta^*\rho.\end{aligned}$$

Let $k\ge 2$. Write ρ in the form $\rho = dx^i\wedge\Phi_i + dy^\sigma\wedge\Psi_\sigma$, and define differential forms $\Phi_i^{(0)}(s)$, $\Phi_i'(s)$, $\Psi_\sigma^{(0)}(s)$ by

(43)
$$\chi^*\Phi_i = ds\wedge\Phi_i^{(0)}(s) + \Phi_i'(s),\quad \chi^*\Psi_\sigma = ds\wedge\Psi_\sigma^{(0)}(s) + \Psi_\sigma'(s).$$

Then

$$\begin{aligned}\chi^*\rho &= dx^i \wedge (ds \wedge \Phi_i^{(0)}(s) + \Phi'_i(s)) \\ &\quad + (s dy^\sigma + y^\sigma ds) \wedge (ds \wedge \Psi_\sigma^{(0)}(s) + \Psi'_\sigma(s)) \\ &= ds \wedge (-dx^i \wedge \Phi_i^{(0)}(s) - s dy^\sigma \Psi_\sigma^{(0)}(s) + y^\sigma \Psi'_\sigma(s)) \\ &\quad + dx^i \wedge \Phi'_i(s) + s dy^\sigma + s y^\sigma \Psi'_\sigma(s)).\end{aligned} \tag{44}$$

Thus,

$$I\rho = -dx^i \wedge \int_0^1 \Phi_i^{(0)}(s) - dy^\sigma \wedge \int_0^1 (s\Psi_\sigma^{(0)}(s) + y^\sigma \Psi'_\sigma(s)) ds. \tag{45}$$

To determine $Id\rho$, we compute $\chi^* d\rho$. We get

$$\begin{aligned}\chi^* d\rho &= d\chi^*\rho \\ &= -ds \wedge (dx^i \wedge d\Phi_i^{(0)}(s)) + s dy^\sigma \wedge d\Psi_\sigma^{(0)}(s) + dy^\sigma \wedge \Psi'_\sigma(s) \\ &\quad + y^\sigma d\Psi'_\sigma(s)) - dx^i \wedge d\Phi'_i(s) - dy^\sigma \wedge d(s\Psi'_\sigma(s))) \\ &= ds \wedge \left(-dx^i \wedge d\Phi_i^{(0)}(s)) + dx^i \wedge \frac{\partial \Phi'_i(s)}{\partial s} - s dy^\sigma \wedge d\Psi_\sigma^{(0)}(s)\right. \\ &\quad \left. - dy^\sigma \wedge \Psi'_\sigma(s) - y^\sigma d\Psi'_\sigma(s) + dy^\sigma \wedge \frac{\partial (s\Psi'_\sigma(s))}{\partial s}\right) \\ &\quad - dx^i \wedge \left(dx^j \wedge \frac{\partial \Phi'_i(s)}{\partial x^j} + dy^\nu \wedge \frac{\partial \Phi'_i(s)}{\partial y^\nu}\right) \\ &\quad - dy^\sigma \wedge \left(dx^j \wedge \frac{\partial (s\Psi'_\sigma(s))}{\partial x^j} + dy^\nu \wedge \frac{\partial (s\Psi'_\sigma(s))}{\partial y^\nu}\right),\end{aligned} \tag{46}$$

where $\partial\eta(s)/\partial s$ denotes the form, arising by differentiation of $\eta(s)$ with respect to s, followed by multiplication by ds. Now by (45) and (30),

$$\begin{aligned}Id\rho &= -dx^i \wedge \int_0^1 d\Phi_i^{(0)}(s) - dy^\sigma \wedge \int_0^1 s\, d\Psi_\sigma^{(0)}(s) - dy^\sigma \wedge \int_0^1 \Psi'_\sigma(s) \\ &\quad - y^\sigma \int_0^1 d\Psi'_\sigma(s) + dx^i \wedge \int_0^1 \frac{\partial \Phi'_i(s)}{\partial s} + dy^\sigma \wedge \int_0^1 \frac{\partial (s\Psi'_\sigma(s))}{\partial s}.\end{aligned} \tag{47}$$

Note that the expressions $d\Phi_i^{(0)}(s)$, $d\Psi_\sigma^{(0)}(s)$, and $d\Psi'_\sigma(s)$ have the meaning of the exterior derivatives with respect to x^i, y^σ (the terms containing ds are cancelled; see the definition of I (30), (31)).

Now we easily get

(48) $$Id\rho + dI\rho = dx^i \wedge \int_0^1 \frac{\partial \Phi_i'(s)}{\partial s} + dy^\sigma \wedge \int_0^1 \frac{\partial (s\Psi_\sigma'(s))}{\partial s},$$

and using formula (30), one obtains

(49) $$\begin{aligned} Id\rho + dI\rho &= dx^i \wedge (\Phi_i'(1) - \Phi_i'(0)) + dy^\sigma \wedge (1 \cdot \Psi_\sigma'(1) - 0 \cdot \Psi_\sigma'(0)) \\ &= dx^i \wedge \Phi_i'(1) + dy^\sigma \wedge \Psi_\sigma'(1) - dx^i \wedge \Phi_i'(0) \\ &= dx^i \wedge \Phi_i + dy^\sigma \wedge \Psi_\sigma - dx^i \wedge \pi * \zeta * \Phi_i \\ &= \rho - \pi * \zeta * \rho. \end{aligned}$$

As a consequence, we have the following statement.

Theorem 13 (The fibred Volterra-Poincare lemma) *Let $U \subset \mathbf{R}^n$ be an open set, $V \subset \mathbf{R}^m$ an open ball with centre 0. Let k be a positive integer, and let ρ be a differential k-form on $U \times V$. The following two conditions are equivalent:*

(a) *There exist a $(k-1)$-form η on $U \times V$ and a k-form η_0 on U such that*

(50) $$d\eta + \pi * \eta_0 = \rho.$$

(b) *The form $d\rho$ is π-projectable.*

Proof Suppose we have some forms η and η_0 satisfying condition (a). Then $d\rho = d\pi * \eta_0 = \pi * d\eta_0$ proving (b). Conversely, if $d\rho$ is π-projectable, then by the definition of *I*, $Id\rho = 0$, and then (a) follows from the identity $\rho = Id\rho + dI\rho + \pi * \zeta * \rho = d\eta + \pi * \eta_0$ (Theorem 12).

We also get two assertions that concern *projectability* of forms, and non-uniqueness of solutions of equation (26).

Corollary 1 *Let $U \subset \mathbf{R}^n$ be an open set, $V \subset \mathbf{R}^m$ an open ball with centre the origin 0, ρ a differential form on $U \times V$. The following two conditions are equivalent:*

(1) *There exists a form η on U such that $\rho = \pi^* \eta$.*

(2) $Id\rho + dI\rho = 0$.

Proof This follows from Theorem 12.

Corollary 2 *Suppose that the form $d\rho$ is π-projectable. Let (η, η_0) and $(\tilde{\eta}, \tilde{\eta}_0)$ be two solutions of equation (26). Then there exist a $(p-1)$-form τ on $U \times V$ and a $(p-1)$-form χ on U such that*

(51) $\quad \tilde{\eta} = \eta + \pi^* \chi + d\tau, \quad \tilde{\eta}_0 = \eta_0 - d\chi.$

Proof By hypothesis,

(52) $\quad d\eta + \pi^* \eta_0 = \rho, \quad d\tilde{\eta} + \pi^* \tilde{\eta}_0 = \rho.$

These equations imply $d\eta + \pi^* \eta_0 = d\tilde{\eta} + \pi^* \tilde{\eta}_0$ hence $\pi^* d\eta_0 = \pi^* d\tilde{\eta}_0$. But for any section δ of the projection π,

(53) $\quad \delta^* \pi^* d\eta_0 = d\eta_0 = \delta^* \pi^* d\tilde{\eta}_0 = d\tilde{\eta}_0.$

Thus, by the Volterra-Poincaré lemma, $\tilde{\eta}_0 - \eta_0 = d\chi$ for some χ. Then, however, $d\eta + \pi^* \eta_0 = d\tilde{\eta} + \pi^*(\eta_0 + d\chi)$, and

(54) $\quad d(\eta - \tilde{\eta} - \pi^* \chi) = 0$

Applying the Volterra-Poincaré lemma again we get (51).

Remark 5 Let X be an n-dimensional manifold. Every point $x \in X$ has a neighbourhood U such that the decomposition of forms, given in Lemma 4, is defined on U. Indeed, if (U, φ) is a chart at x such that $\varphi(U)$ is an open ball with centre $0 \in \mathbf{R}^n$, then formulas $\rho = \varphi^* \mu$ and $(\varphi^{-1})^* \rho = \mu$ establish a bijective correspondence between forms on U and $\varphi(U)$, that commutes with the exterior derivative d. Note, however, that in general, this correspondence does not allow us to construct solutions of exterior differential equations (1) and (26), defined *globally* on X.

In our subsequent constructions we need the fibred homotopy operator on the *s-jet prolongation* $W^s = J^s(U \times V)$ of the Cartesian product $W = U \times V$; explicitly,

(55) $\quad W^s = U \times V \times L(\mathbf{R}^n, \mathbf{R}^m) \times L^2_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m) \times \ldots \times L^s_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m),$

where $L^k_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m)$ is the vector space of k-linear, symmetric mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$. The Cartesian coordinates on V and the associated jet coordinates on W^s are denoted by x^i, y^σ and $x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_s}$, respectively. We have a mapping χ_s from the set $[0,1] \times W^s$ to W^s, given by

(56) $\quad \chi_s(t, (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_s})) = (x^i, ty^\sigma, ty^\sigma_{j_1}, ty^\sigma_{j_1 j_2}, \ldots, ty^\sigma_{j_1 j_2 \ldots j_s}).$

χ_s defines the *fibered homotopy operator* I_s, assigning to a k-form ρ on V^s, where $k \geq 1$, a $(k-1)$-form $I_s \rho$ on W^s. To recall the definition of I_s, it is convenient to use a multi-index notation. Suppose we have a form $\rho \in \Omega^s_q W$, expressed by

$$\rho = \sum_{0\le j\le q} \frac{1}{j!(q-j)!} A^{I_1\,I_2\,\ldots\,I_j}_{\sigma_1\,\sigma_2\,\ldots\,\sigma_j\;i_{j+1}i_{j+2}\ldots i_k} dy^{\sigma_1}_{I_1}\wedge dy^{\sigma_2}_{I_2}\wedge\ldots\wedge dy^{\sigma_j}_{I_j} \wedge dx^{i_{j+1}}\wedge dx^{i_{j+2}}\wedge\ldots\wedge dx^{i_q}, \tag{57}$$

where the multi-indices are of the form $I = (p_1 p_2 \ldots p_k)$, and $|I| = k$ is the *length* of *I*; the summation in (57) is taking place through multi-indices of length $\le s$. Then

$$I_s\rho = \int_0^1 I_{s,0}\rho \tag{58}$$

(integration through *t* from 0 to 1), where $I_{s,0}\rho$ is defined by the decomposition

$$\chi^*_s\rho = dt\wedge I_{s,0}\rho + I'_s\rho \tag{59}$$

such that the forms $I_{s,0}\rho$ and $I'_s\rho$ do not contain dt. The mapping χ_s satisfies

$$\chi^*_s dx^i = dx^i,\quad \chi^*_s dy^\sigma_J = y^\sigma_J dt + t dy^\sigma_J,\quad 0\le |J|\le s, \qquad \chi^*_s\omega^\sigma_J = y^\sigma_J dt + t\omega^\sigma_J,\quad 0\le |J|\le s-1. \tag{60}$$

We have from (60)

$$\chi^*_s\rho = \sum_{j=1}^{q} \frac{1}{j!(q-j)!} A^{I_1\,I_2\,\ldots\,I_j}_{\sigma_1\,\sigma_2\,\ldots\,\sigma_j\;i_{j+1}i_{j+2}\ldots i_q}\circ\chi_s\cdot(y^{\sigma_1}_{I_1}dt + t dy^{\sigma_1}_{I_1}) \wedge(y^{\sigma_2}_{I_2}dt + t dy^{\sigma_2}_{I_2})\wedge\ldots\wedge(y^{\sigma_j}_{I_j}dt + t dy^{\sigma_j}_{I_j})\wedge dx^{i_{j+1}}\wedge dx^{i_{j+2}}\wedge\ldots\wedge dx^{i_q}, \tag{61}$$

hence

$$I_{s,0}\rho = \sum_{j=1}^{q} \frac{1}{(j-1)!(q-j)!} y^{\sigma_1}_{I_1} A^{I_1\,I_2\,\ldots\,I_j}_{\sigma_1\,\sigma_2\,\ldots\,\sigma_j\;i_{j+1}i_{j+2}\ldots i_q}\circ\chi_s\cdot t^{j-1} \cdot dy^{\sigma_2}_{I_2}\wedge dy^{\sigma_3}_{I_3}\wedge\ldots\wedge dy^{\sigma_j}_{I_j}\wedge dx^{i_{j+1}}\wedge dx^{i_{j+2}}\wedge\ldots\wedge dx^{i_q}, \tag{62}$$

and

$$I_s\rho = y^{\sigma_1}_{I_1}\sum_{j=1}^{q}\frac{1}{(j-1)!(q-j)!}\int_0^1 A^{I_1\ I_2\ \dots\ I_j}_{\sigma_1\ \sigma_2\ \dots\ \sigma_j\ i_{j+1}i_{j+2}\dots i_q}\circ\chi_s\cdot t^{j-1}\,dt$$
$$\cdot\, dy^{\sigma_2}_{I_2}\wedge dy^{\sigma_3}_{I_3}\wedge\dots\wedge dy^{\sigma_j}_{I_j}\wedge dx^{i_{j+1}}\wedge dx^{i_{j+2}}\wedge\dots\wedge dx^{i_q}. \tag{63}$$

In the following theorem ζ is the zero section of W^s over U.

Theorem 14 (a) *The mapping* χ_s *satisfies*

$$\rho = I_s d\rho + dI_s\rho + (\pi^s)^*\zeta^*\rho. \tag{64}$$

(b) *If* $d\rho = 0$ *, then there exists a* $(q-1)$ *-form* η *such that*

$$\rho = d\eta. \tag{65}$$

Proof For the proof see Theorem 13. Clearly, condition $d\rho = 0$ implies $d\zeta^*\rho = 0$; thus, to prove (65) we can integrate the form $\zeta^*\rho$ by means of the standard Volterra-Poincaré lemma, and then apply formula (64).