3 Variational structures on fibred manifolds

3.1 Variational structures By a *variational structure* we mean in this work a pair (Y,ρ), where Y is a fibred manifold over an n-dimensional manifold X with projection π and ρ is an n-form on the r-jet prolongation J^rY.

Suppose that we have a variational structure (Y,ρ). Let $W\subset Y$ be an open set, and let $\Omega\subset\pi(W)$ be a compact, n-dimensional submanifold of X with boundary (a *piece* of X). Denote by $\Gamma_{\Omega,W}(\pi)$ the set of smooth sections of π over Ω, such that $\gamma(\Omega)\subset W$. Then for any section $\gamma\in\Gamma_{\Omega,W}(\pi)$ of Y, the pull-back $J^r\gamma*\rho$ is an n-form on a neighbourhood of Ω. Integrating $J^r\gamma*\rho$ on Ω, we get a real function $\Gamma_{\Omega,W}(\pi)\ni\gamma\rightarrow\rho_\Omega(\gamma)\in\mathbf{R}$, defined by

(1) $$\rho_\Omega(\gamma)=\int_\Omega J^r\gamma*\rho.$$

ρ_Ω is called the *variational functional*, associated with (Y,ρ) (over Ω).

The objective of the variational analysis on fibred manifolds is to study the behaviour of variational functionals ρ_Ω on the set of sections $\Gamma_\Omega(\pi)$, or on subsets of this set, defined by some additional conditions (*constraints*). In general, the set $\Gamma_\Omega(\pi)$ has no natural algebraic and topological structure; this fact prevents, in particular, to immediately apply to ρ_Ω the methods of the differentiation theory in topological vector spaces. Instead, the *variational method* is used, which consists of the study of the behaviour of each section $\gamma\in\Gamma_\Omega(\pi)$ on its 1-parameter deformations (variations) within $\Gamma_\Omega(\pi)$, and of the corresponding induced deformations (variations) of the value $\rho_\Omega(\gamma)$ of ρ_Ω. The *variational geometry* studies geometric, coordinate-independent properties of ρ_Ω.

For every r we denote by $\Omega^r_{n,X}W$ the submodule of the module Ω^r_nW, consisting of π^r-horizontal forms. Elements of the set $\Omega^r_{n,X}W$ are called *Lagrangians* (of order r) for the fibred manifold Y.

Let $\rho\in\Omega^r_nW$. There exists a unique Lagrangian $\lambda_\rho\in\Omega^{r+1}_{n,X}W$ such that

(2) $$J^{r+1}\gamma*\lambda_\rho=J^r\gamma*\rho$$

for all sections γ of Y. The n-form λ_ρ can alternatively be defined by the first canonical decomposition to the form ρ (Section 2.1)

(3) $$(\pi^{r+1,r})*\rho=h\rho+p_1\rho+p_2\rho+\ldots+p_n\rho$$

as the *horizontal component* of ρ ,

(4) $$\lambda_\rho = h\rho.$$

Property (2) says that the variational functional ρ_Ω can also be expressed as

(5) $$\rho_\Omega(\gamma) = \int_\Omega J^{r+1}\gamma * \lambda_\rho.$$

The $\pi^{r+1,r}$-horizontal *n*-form λ_ρ is called the *Lagrangian*, *associated with* the *n*-form ρ .

We give the chart expressions of ρ and $h\rho$ in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, on *Y* (or, more exactly, in the associated charts on J^rY and $J^{r+1}Y$). Recall that in multi-index notation the contact basis of 1-forms on V^r (and analogously on V^{r+1}) is defined to be the basis $(dx^i, \omega^\sigma_J, dy^\sigma_I)$, where the multi-indices satisfy $0 \le |J| \le r-1$, $|I| = r$, and

(6) $$\omega^\sigma_J = dy^\sigma_J - y^\sigma_{Jj}dx^j.$$

We also associate with the given chart the *n*-form (considered on $U = \pi(V) \subset X$, and also on V^r)

(7) $$\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n,$$

sometimes called the *local volume form*, associated with (V,ψ).

According to the second canonical decomposition theorem (2.3, Theorem 13), ρ has an expression

(8) $$\rho = \sum_{0\le|J|\le r-1} \omega^\sigma_J \wedge \Phi^J_\sigma + \sum_{|J|=r-1} d\omega^\sigma_J \wedge \Psi^J_\sigma + \rho_0,$$

where

(9) $$\begin{aligned} \rho_0 &= A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n} \\ &+ A^{J_1}_{\sigma_1}{}_{i_2 i_3 \ldots i_n} dy^{\sigma_1}_{J_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_n} \\ &+ A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2}{}_{i_3 i_4 \ldots i_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \ldots \wedge dx^{i_n} \\ &+ \ldots \\ &+ A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \ldots {}^{J_{n-1}}_{\sigma_{n-1}}{}_{i_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \ldots \wedge dy^{\sigma_{n-1}}_{J_{n-1}} \wedge dx^{i_n} \\ &+ A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \ldots {}^{J_n}_{\sigma_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \ldots \wedge dy^{\sigma_n}_{J_n}, \end{aligned}$$

and the coefficients $A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \ldots {}^{J_s}_{\sigma_s}{}_{i_{s+1} i_{s+2} \ldots i_n}$ are traceless. Then $h\rho = h\rho_0$ because *h* is an exterior algebra homomorphism, annihilating the contact forms ω^σ_J and $d\omega^\sigma_J$. Thus,

$$(10)\qquad \begin{aligned}\lambda_\rho = (&A_{i_1 i_2 \ldots i_n} + A^{J_1}_{\sigma_1\ i_2 i_3 \ldots i_n} y^{\sigma_1}_{J_1 i_1} + A^{J_1\ J_2}_{\sigma_1\ \sigma_2\ i_3 i_4 \ldots i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \\ &+ \ldots + A^{J_1\ J_2\ \ldots\ J_{n-1}}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_{n-1}\ i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \ldots y^{\sigma_{n-1}}_{J_{n-1} i_{n-1}} + A^{J_1\ J_2\ \ldots\ J_n\ J_n}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_n\ \sigma_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \ldots y^{\sigma_n}_{J_n i_n}) \\ &\cdot dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n}.\end{aligned}$$

Using the local volume form (7) we also write

$$(11)\qquad \lambda_\rho = \mathscr{L}\omega_0,$$

where

$$(12)\qquad \begin{aligned}\mathscr{L} = \varepsilon^{i_1 i_2 \ldots i_n} (&A_{i_1 i_2 \ldots i_n} + A^{J_1}_{\sigma_1\ i_2 i_3 \ldots i_n} y^{\sigma_1}_{J_1 i_1} + A^{J_1\ J_2}_{\sigma_1\ \sigma_2\ i_3 i_4 \ldots i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \\ &+ \ldots + A^{J_1\ J_2\ \ldots\ J_{n-1}}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_{n-1}\ i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \ldots y^{\sigma_{n-1}}_{J_{n-1} i_{n-1}} + A^{J_1\ J_2\ \ldots\ J_n}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \ldots y^{\sigma_n}_{J_n i_n}).\end{aligned}$$

$\mathscr{L}$ is a function on V^{r+1} called the *Lagrange function*, associated with ρ (or with the Lagrangian λ_ρ).

Remark 1 Sometimes the integration domain Ω in the variational functional ρ_Ω is not fixed, but is arbitrary. Then formula (1) defines a *family* of variational functionals labelled by Ω.

Remark 2 Orientability of the base *X* of the fibred manifold *Y* is not an essential assumption; replacing differential forms by *twisted base differential forms*, one can also develop the variational theory for *non-orientable* bases *X* (Krupka [10]). Variational functionals, defined on fibred manifolds over non-orientable bases, appear in the general relativity theory and field theory.

Remark 3 (The structure of Lagrange functions) Formulas (11) and (12) describe *general structure* of the Lagrangians, associated with the class of variational functionals (1). The Lagrange functions $\mathscr{L}$ that appear in chart descriptions of the Lagrangians are multi-linear, symmetric functions of the variables y^σ_I, where $|I| = r+1$.

Remark 4 (Lagrangians) The subset $\Omega^r_{n,X}W$ of forms, defining variational functionals (1), contains π^r*-horizontal forms* $\rho = A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n}$ (*Lagrangians of order r*). Then the associated Lagrangians λ_ρ coincide with ρ. Since $dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n} = \varepsilon^{i_1 i_2 \ldots i_n}\omega_0$, each Lagrangian can also be written in the form $\rho = \mathscr{L}\omega_0$. Concrete variational functionals are usually defined in this way.

The following lemma describes all *n*-forms $\rho \in \Omega_n^r W$, whose associated Lagrangians belong to the same module $\Omega_n^r W$, that is, are of order *r*.

Lemma 1 *For a form* $\rho \in \Omega_n^r W$ *the following two conditions are equivalent:*

(1) *The Lagrangian* λ_ρ *is defined on* $J^r W$.

(2) *In any fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, ρ *has an expression*

(13) $$\rho = \mathscr{L}\omega_0 + \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J| = r-1} d\omega_J^\sigma \wedge \Psi_\sigma^J$$

for some function $\mathscr{L}$ *and some forms* Φ_σ^J *and* Ψ_σ^J.

Proof This follows from (4) and (12).

3.2 Variational derivatives Let *U* be an open subset of *X*, let $\gamma : U \to Y$ be a section. Let Ξ be a π-projectable vector field on an open set $W \subset Y$ such that $\gamma(U) \subset W$. If α_t is the local 1-parameter group of Ξ, and $\alpha_{(0)t}$ its π-projection, then

(1) $$\gamma_t = \alpha_t \gamma \alpha_{(0)t}^{-1}$$

is a 1-parameter family of *sections* of *Y*, depending smoothly on the parameter *t*: Indeed, since $\pi\alpha_t = \alpha_{(0)t}\pi$, we have

(2) $$\pi\gamma_t(x) = \pi\alpha_t\gamma\alpha_{(0)t}^{-1}(x) = \alpha_{(0)t}\pi\gamma\alpha_{(0)t}^{-1}(x) = \alpha_{(0)t}\alpha_{(0)t}^{-1}(x) = x$$

on the domain of γ_t. The family γ_t is called the *variation*, or *deformation*, of the section γ, *induced* by the vector field Ξ.

Recall that a *vector field along* γ is a mapping $\Xi : U \to TY$ such that $\Xi(x) \in T_{\gamma(x)} Y$ for every point $x \in U$. Given Ξ, formula

(3) $$\xi = T\pi \cdot \Xi$$

then defines a vector field ξ on *U*, called the π*-projection* of Ξ.

The following theorem says that every vector field along a section γ can be extended to a π-projectable vector field, defined on an open set. Moreover, the *r*-jet prolongation of the extended field, considered along $J^r\gamma$, is independent of the prolongation.

Theorem 1 *Let* γ *be a section of Y defined on an open set* $U \subset X$, *let* Ξ *a vector field along* γ.

(a) *There exists a* π*-projectable vector field* $\tilde{\Xi}$, *defined on a neighbourhood of the set* $\gamma(U)$, *such that for each* $x \in U$

(4) $\tilde{\Xi}(\gamma(x)) = \Xi(x).$

(b) *Any two* π *-projectable vector fields* Ξ_1, Ξ_2, *defined on a neighbourhood of* $\gamma(U)$, *such that* $\Xi_1(\gamma(x)) = \Xi_2(\gamma(x)) = \Xi(x)$ *for all* $x \in U$, *satisfy*

(5) $J^r\Xi_1(J^r_x\gamma) = J^r\Xi_2(J^r_x\gamma).$

Proof (a) Choose a point $x_0 \in U$ and a fibred chart (V_0, ψ_0), $\psi_0 = (x_0^i, y_0^\sigma)$, at the point $\gamma(x_0) \in Y$ such that $\gamma(\pi(V_0)) \subset V_0$. In this chart

(6) $$\xi(\gamma(x)) = \xi^i(x)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(x)} + \Xi^\sigma(x)\left(\frac{\partial}{\partial y^\sigma}\right)_{\gamma(x)}.$$

We set for any $y \in V_0$, $\tilde{\xi}^i(y) = \xi^i(\pi(y))$, $\tilde{\Xi}^\sigma(y) = \Xi^\sigma(\pi(y))$, and define a vector field $\tilde{\Xi}$ on V_0 by

(7) $$\tilde{\Xi} = \tilde{\xi}^i \frac{\partial}{\partial x^i} + \tilde{\Xi}^\sigma \frac{\partial}{\partial y^\sigma}.$$

The vector field $\tilde{\Xi}$ satisfies $\tilde{\Xi}(\gamma(x)) = \Xi(\gamma(x))$ on $\pi(V_0)$.

Applying this construction to any point of the domain of definition U of Ξ we may suppose that we have families of fibred charts (V_ι, ψ_ι), $\psi_\iota = (x_\iota^i, y_\iota^\sigma)$, and vector fields $\tilde{\Xi}_\iota$, where ι runs through an index set I, such that $\gamma(\pi(V_\iota)) \subset V_\iota$ for every $\iota \in I$, $\tilde{\Xi}_\iota$ is defined on V_ι, and $\tilde{\Xi}_\iota(\gamma(x))) = \tilde{\Xi}(\gamma(x)))$ for all $\pi(V_\iota)$.

Let $(\chi_\iota)_{\iota \in I}$ be a partition of unity, subordinate to the covering $(V_\iota)_{\iota \in I}$ of the set $\gamma(U) \subset Y$. Setting

(8) $$\tilde{\Xi} = \sum_{\iota \in I} \chi_\iota \tilde{\Xi}_\iota,$$

we get a vector field defined on the open set $V = \bigcup V_\iota$. For any point $x \in U$ the point $\gamma(x)$ belongs to some of the sets V_ι, thus, $\gamma(U) \subset V$. The value of $\tilde{\Xi}$ at $\gamma(x)$ is

(9) $$\begin{aligned}\tilde{\Xi}(\gamma(x)) &= \sum_{\iota \in I} \chi_\iota(\gamma(x))\tilde{\Xi}_\iota(\gamma(x)) = \left(\sum_{\iota \in I} \chi_\iota(\gamma(x))\right)\Xi(\gamma(x)) \\ &= \Xi(\gamma(x))\end{aligned}$$

as required.

(b) This assertion follows from the formula for the *r*-jet prolongation of a π -projectable vector field (1.7, Lemma 12).

A π -projectable vector field $\tilde{\Xi}$, satisfying condition (a) of Theorem 1, is called a *π –projectable extension* of Ξ. Using (b) and any projectable extension $\tilde{\Xi}$, we may define, for the given section γ,

(10) $$J^r\Xi(J^r_x\gamma) = J^r\tilde{\Xi}(J^r_x\gamma).$$

Then $J^r\Xi$ is a vector field along the *r*-jet prolongation $J^r\gamma$ of γ; we call this vector field the *r-jet prolongation* of the vector field (along γ) Ξ.

Variations of sections induce the corresponding changes (variations) of the values of variational functionals. Let $\rho \in \Omega^r_n W$ be a form, $\Omega \subset \pi(W)$ a piece of *X*. Choose a section $\gamma \in \Gamma_{\Omega,W}(\pi)$ and a π -projectable vector field Ξ on *W*, and consider the variation of γ, induced by Ξ (formula (1)). Since the domain of γ_t contains Ω for all sufficiently small *t*, the value of the variational functional $\Gamma_{\Omega,W}(\pi) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$ at γ_t is defined, and we get a real-valued function, defined on a neighbourhood $(-\varepsilon,\varepsilon)$ of the point $0 \in \mathbf{R}$,

(11) $$(-\varepsilon,\varepsilon) \ni t \to \rho_{\alpha_{(0)t}(\Omega)}(\alpha_t\gamma\alpha^{-1}_{(0)t}) = \int_{\alpha_{(0)t}(\Omega)} J^r(\alpha_t\gamma\alpha^{-1}_{(0)t})^* \rho \in \mathbf{R}.$$

It is easily seen that this function is differentiable. Since

(12) $$J^r(\alpha_t\gamma\alpha^{-1}_{(0)t}))^* \rho = (\alpha^{-1}_{(0)t})^*(J^r\gamma)^*(J^r\alpha_t)^* \rho,$$

where $J^r\alpha_t$ is the local 1-paremeter group of the *r*-jet prolongation $J^r\Xi$ of the vector field Ξ, we have, using properties of the pull-back operation and the theorem on transformation of the integration domain,

(13) $$\int_{\alpha_{(0)t}(\Omega)} (J^r(\alpha_t\gamma\alpha^{-1}_{(0)t}))^* \rho = \int_\Omega J^r\gamma^*(J^r\alpha_t)^*\rho.$$

Thus, since Ω is compact, differentiability of (11) follows from the theorem on differentiation of an integral, depending upon a parameter.

Differentiating (11) at $t = 0$ one obtains, using (13) and the definition of the Lie derivative,

(14) $$\left(\frac{d}{dt}\rho_\Omega(\alpha_t\gamma\alpha^{-1}_{(0)t})\right)_0 = \int_\Omega J^r\gamma^* \partial_{J^r\Xi}\rho.$$

Note that this expression can be written, in the notation introduced by formula (1), as

(15) $$(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\rho.$$

The number (15) is called the *variation* of the integral variational functional ρ_Ω *at the point* γ *, induced by the vector field* Ξ *.*

This formula shows that the function $\Gamma_{\Omega,W}(\pi) \ni \gamma \to (\partial_{J^r\Xi}\lambda)_\Omega(\gamma) \in \mathbf{R}$ is the variational functional (over Ω), defined by the form $\partial_{J^r\Xi}\rho$. We call this functional the *variational derivative*, or the *first variation* of the variational functional ρ_Ω by the vector field Ξ.

Formula (15) admits a direct generalization. If Z is another π-projectable vector field on *W*, then the *second variational derivative*, or the *second variation*, of the variational function ρ_Ω by the vector fields Ξ and Z, is the mapping $\Gamma_{\Omega,W}(\pi) \ni \gamma \to (\partial_{J^r\mathrm{Z}}\partial_{J^r\Xi}\rho)_\Omega(\gamma) \in \mathbf{R}$, defined by the formula

(16) $$(\partial_{J^r\mathrm{Z}}\partial_{J^r\Xi}\rho)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\mathrm{Z}}\partial_{J^r\Xi}\rho.$$

It is now obvious how *higher-order variational derivatives* are defined: one should simply apply the Lie derivative (with respect to different vector fields) several times.

A section $\gamma \in \Gamma_{\Omega,W}(\pi)$ is called a *stable point* of the variational functional λ_Ω with respect to its variation Ξ, if

(17) $$(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = 0.$$

In practice, one usually requires that a section be a stable point with respect to a *family* of its variations, defined by the problem considered.

Formula (15) can also be expressed in terms of the Lagrangian $\lambda_\rho = h\rho$. Since for any π-projectable vector field Ξ the Lie derivatives commute with the horizontalisation,

(18) $$h\partial_{J^r\Xi}\rho = \partial_{J^r\Xi}h\rho,$$

(Section 1.7, Lemma 13), the first variation of the integral variational functional ρ_Ω at a point γ, induced by the vector field Ξ, can be written as

(19) $$(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = \int_\Omega J^{r+1}\gamma^*\partial_{J^{r+1}\Xi}\lambda_\rho.$$

3.3 Lepage forms We introduce in this subsection a class of *n*-forms ρ on J^rY by imposing certain conditions on the exterior derivative $d\rho$. In

Section 3.1 we considered integral variational functionals, defined by these forms. Deforming sections of Y by projectable vector fields Ξ, we came to the Lie derivative $\partial_{J^r\Xi}\rho$ under the integral sign, related with the induced deformations of the variational functionals. By the well-known (Cartan's) formula, $\partial_{J^r\Xi}\rho$ can be expressed as

(1) $$\partial_{J^r\Xi}\rho = i_{J^r\Xi}d\rho + di_{J^r\Xi}\rho.$$

we shall study the forms ρ for which, roughly speaking, the Cartan's formula defines the first variation formula known from the classical calculus of variations.

First we summarize some useful notation related with a chart (U,φ), $\varphi = (x^i)$, on an n-dimensional manifold X, and with a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y. We introduce with the help of the Levi-Civita symbol a new basis of forms on X, setting

(2) $$\omega_{k_1 k_2 \dots k_p} = \frac{1}{p!(n-p)!}\varepsilon_{k_1 k_2 \dots k_p i_{p+1} i_{p+2} \dots i_n} dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \dots \wedge dx^{i_n},$$

The inverse transformation formulas are

(3) $$\varepsilon^{k_1 k_2 \dots k_p l_{p+1} l_{p+2} \dots l_n}\omega_{k_1 k_2 \dots k_p} = dx^{l_{p+1}} \wedge dx^{l_{p+2}} \wedge \dots \wedge dx^{l_n}.$$

One can easily check that the forms $\omega_i = i_{\partial/\partial x^i}\omega_0$, introduced earlier, agree with the definition (2). Also note that

(4) $$\begin{aligned}\omega_{jk} &= i_{\partial/\partial x^j} i_{\partial/\partial x^k}\omega_0 \\ &= (-1)^{j+k} dx^1 \wedge dx^2 \wedge \dots \wedge dx^{j-1} \wedge dx^{j+1} \\ &\wedge \dots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \dots \wedge dx^n\end{aligned}$$

whenever $j < k$. Then it is immediate that

(5) $$dx^l \wedge \omega_{jk} = \delta^l_j \omega_k - \delta^l_k \omega_j,$$

We prove three lemmas characterizing the structure of n-forms on the r-jet prolongation J^rY, which are needed in computations.

Lemma 2 *Let ρ be an n-form. Suppose that ρ has in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, an expression*

(6) $$\rho = \rho_0 + \tilde{\rho} + d\eta$$

with the following properties:

(a) ρ_0 *is generated by the contact forms* ω^σ_J , $0 \le |J| \le r-1$, *that is,*

(7) $$\rho_0 = \sum_{0\le|J|\le r-1} \omega^\sigma_J \wedge \Phi^J_\sigma ,$$

where

(8) $$\Phi^J_\sigma = \Phi^J_{\sigma\,(1)} + \Phi^J_{\sigma\,(2)} + \tilde{\Phi}^J_\sigma ,$$

the forms $\Phi^J_{\sigma\,(1)}$ *are generated by the contact forms* ω^σ_J , $0 \le |J| \le r-1$, $\Phi^J_{\sigma\,(2)}$ *are generated by* $d\omega^\sigma_I$ *with* $|I| = r-1$, *and*

(9) $$\begin{aligned}\tilde{\Phi}^J_\sigma &= \tilde{\Phi}^J_{\sigma\; i_1 i_2 \ldots i_{n-1}} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}^{J\; I_1}_{\sigma\; \sigma_1\; i_2 i_3 \ldots i_{n-1}} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}^{J\; I_1\; I_2}_{\sigma\; \sigma_1\; \sigma_2\; i_3 i_4 \ldots i_{n-1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \ldots \wedge dx^{i_{n-1}} \\ &+ \ldots + \tilde{\Phi}^{J\; I_1\; I_2 \ldots I_{n-2}}_{\sigma\; \sigma_1\; \sigma_2 \ldots \sigma_{n-2}\; i_{n-1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_{n-2}}_{I_{n-2}} \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}^{J\; I_1\; I_2 \ldots I_{n-1}}_{\sigma\; \sigma_1\; \sigma_2 \ldots \sigma_{n-1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_{n-1}}_{I_{n-1}} ,\end{aligned}$$

where $|I_1|, |I_2|, \ldots, |I_{n-1}| = r$ *and all the coefficients* $\tilde{\Phi}^{J\; I_1}_{\sigma\; \sigma_1\; i_2 i_3 \ldots i_{n-1}}$, $\tilde{\Phi}^{J\; I_1\; I_2}_{\sigma\; \sigma_1\; \sigma_2\; i_3 i_4 \ldots i_{n-1}}$, $\ldots$, $\tilde{\Phi}^{J\; I_1\; I_2 \ldots I_{n-2}}_{\sigma\; \sigma_1\; \sigma_2 \ldots \sigma_{n-2}\; i_{n-1}}$ *are traceless.*

(b) η *is a contact form such that*

(10) $$\eta = \sum_{|I|=r-1} \omega^\sigma_I \wedge \Psi^I_\sigma ,$$

where the forms Ψ^I_σ *do not contain any exterior factor* ω^σ_J *with* $0 \le |J| \le r-1$.

(c) $\tilde{\rho}$ *has an expression*

(11) $$\begin{aligned}\tilde{\rho} &= A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n} \\ &+ A^{I_1}_{\sigma_1\; i_2 i_3 \ldots i_n} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_n} \\ &+ A^{I_1\; I_2}_{\sigma_1\; \sigma_2\; i_3 i_4 \ldots i_n} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \ldots \wedge dx^{i_n} \\ &+ \ldots + A^{I_1\; I_2 \ldots I_{n-1}}_{\sigma_1\; \sigma_2 \ldots \sigma_{n-1}\; i_n} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_{n-1}}_{I_{n-1}} \wedge dx^{i_n} \\ &+ A^{I_1\; I_2 \ldots I_n}_{\sigma_1\; \sigma_2 \ldots \sigma_n} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_n}_{I_n} ,\end{aligned}$$

where $|I_1|, |I_2|, \ldots, |I_n| = r$ *and all the coefficients* $A^{I_1}_{\sigma_1\; i_2 i_3 \ldots i_n}$, $A^{I_1\; I_2}_{\sigma_1\; \sigma_2\; i_3 i_4 \ldots i_n}$, $\ldots$, $A^{I_1\; I_2 \ldots I_{n-1}}_{\sigma_1\; \sigma_2 \ldots \sigma_{n-1}\; i_n}$ *are traceless.*

Proof Using the second canonical decomposition (2.5, Theorem 4), we can write ρ as

$$\rho = \rho_{(1)} + \rho_{(2)} + \tilde{\rho}, \tag{12}$$

where $\rho_{(1)}$ includes all ω_J^σ-generated terms, where $0 \le |J| \le r-1$, $\rho_{(2)}$ includes all $d\omega_I^\sigma$-generated terms with $|J| = r-1$, with traceless coefficients (and does not contain any exterior factor ω_J^σ), and $\tilde{\rho}$ is expressed by (11). Then

$$\begin{aligned} \rho_{(2)} &= \sum_{|I|=r-1} d\omega_I^\sigma \wedge \Psi_\sigma^I \\ &= d\left(\sum_{|I|=r-1} \omega_I^\sigma \wedge \Psi_\sigma^I\right) - \sum_{|I|=r-1} \omega_I^\sigma \wedge d\Psi_\sigma^I, \end{aligned} \tag{13}$$

so we get

$$\begin{aligned} \rho &= \rho_{(1)} - \sum_{|I|=r-1} \omega_I^\sigma \wedge d\Psi_\sigma^I + d\left(\sum_{|I|=r-1} \omega_I^\sigma \wedge \Psi_\sigma^I\right) + \tilde{\rho} \\ &= \rho_0 + d\left(\sum_{|I|=r-1} \omega_I^\sigma \wedge \Psi_\sigma^I\right) + \tilde{\rho}. \end{aligned} \tag{14}$$

Our next aim will be to find the chart expression for the horizontal and 1-contact components of the n-form

$$\tau = \rho_0 + \tilde{\rho} \tag{15}$$

from Lemma 2.

Lemma 3 *Suppose that* τ *has an expression* (6).

(a) *The horizontal component* $h\tau$ *is given by*

$$\begin{aligned} h\tau = (&A_{i_1 i_2 \dots i_n} + A_{\sigma_1}^{I_1}{}_{i_2 i_3 \dots i_n} y_{I_1 i_1}^{\sigma_1} + A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \\ &+ \dots + A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2}{}_{\dots}{}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &+ A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2}{}_{\dots}{}_{\sigma_n}^{I_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_n i_n}^{\sigma_n}) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}. \end{aligned} \tag{16}$$

(b) *The* 1*-contact component* $p_1\tau$ *is given by*

$$
\begin{aligned}
(17)\qquad p_1\tau = \sum_{0\le |J|\le r-1} (&\tilde{\Phi}_{\sigma}^{J}{}_{i_2 i_3 \dots i_n} + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_2 i_2}^{\sigma_2} + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \dots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\
&+ \dots + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3} \dots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\
&+ \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3} \dots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) \omega_J^{\sigma} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n} \\
&+ \sum_{|I|=r} (A_{\sigma}^{I}{}_{i_2 i_3 \dots i_n} + 2A_{\sigma_1}^{I}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_2 i_2}^{\sigma_2} + 3A_{\sigma}^{I}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \dots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\
&+ \dots + (n-1) A_{\sigma}^{I}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\
&+ n A_{\sigma}^{I}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) \omega_I^{\sigma} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}.
\end{aligned}
$$

Proof (a) Clearly, $h\tau = h\tilde{\rho}$ and (16) follows.
(b) The form $p_1\tau$ is given by

$$
(18)\qquad p_1\tau = \sum_{0\le |J|\le r-1} \omega_J^{\sigma} \wedge h\Phi_{\sigma}^{J} + p_1\tilde{\rho}.
$$

Then

$$
\begin{aligned}
(19)\qquad h\tilde{\Phi}_{\sigma}^{J} = (&\tilde{\Phi}_{\sigma}^{J}{}_{i_1 i_2 \dots i_{n-1}} + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_1}^{I_1}{}_{i_2 i_3 \dots i_{n-1}} y_{I_1 i_1}^{\sigma_1} + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_{n-1}} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \\
&+ \dots + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_{n-2}}^{I_{n-2}}{}_{i_{n-1}} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_{n-2} i_{n-2}}^{\sigma_{n-2}} \\
&+ \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_{n-1}}^{I_{n-1}} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}}) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}} \\
= (&\tilde{\Phi}_{\sigma}^{J}{}_{i_2 i_3 \dots i_n} + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_2 i_2}^{\sigma_2} + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \dots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\
&+ \dots + \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3} \dots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\
&+ \tilde{\Phi}_{\sigma}^{J}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3} \dots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n},
\end{aligned}
$$

and

$$
\begin{aligned}
(20)\qquad p_1\tilde{\rho} = (&A_{\sigma_1}^{I_1}{}_{i_2 i_3 \dots i_n} + 2A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_2 i_2}^{\sigma_2} + 3A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \dots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\
&+ \dots + (n-1) A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\
&+ n A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) \omega_{I_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n} \\
= \sum_{|I|=r} (&A_{\sigma}^{I}{}_{i_2 i_3 \dots i_n} + 2A_{\sigma_1}^{I}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_2 i_2}^{\sigma_2} + 3A_{\sigma}^{I}{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \dots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\
&+ \dots + (n-1) A_{\sigma}^{I}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\
&+ n A_{\sigma}^{I}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) \omega_I^{\sigma} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}.
\end{aligned}
$$

(17) now follows from (19) and (20).

We find the chart expression for the pull-back $(\pi^{r+1,r})^*\rho$. By Lemma 2

(21) $$(\pi^{r+1,r})^*\rho = h\tilde{\rho} + p_1(\rho_0 + \tilde{\rho}) + d\eta + \mu,$$

where $h\tilde{\rho} = h\tau$ and $p_1\rho_0 + p_1\tilde{\rho}$ are given by Lemma 3. We define f_0 and $f_\sigma^{J\ i}$ by the formulas

(22) $$h\tilde{\rho} = f_0\omega_0, \quad p_1(\rho_0 + \tilde{\rho}) = \sum_{0\le|J|\le r} f_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i.$$

Explicitly,

(23) $$\begin{aligned} f_0 = \varepsilon^{i_1i_2\dots i_n}(&A_{i_1i_2\dots i_n} + A^{I_1}_{\sigma_1\ i_2i_3\dots i_n} y^{\sigma_1}_{I_1i_1} + A^{I_1\ I_2}_{\sigma_1\ \sigma_2\ i_3i_4\dots i_n} y^{\sigma_1}_{I_1i_1} y^{\sigma_2}_{I_2i_2} \\ &+ \dots + A^{I_1\ I_2\ \dots\ I_{n-1}}_{\sigma_1\ \sigma_2\ \dots\ \sigma_{n-1}\ i_n} y^{\sigma_1}_{I_1i_1} y^{\sigma_2}_{I_2i_2} \dots y^{\sigma_{n-1}}_{I_{n-1}i_{n-1}} + A^{I_1\ I_2\ \dots\ I_n}_{\sigma_1\ \sigma_2\ \dots\ \sigma_n} y^{\sigma_1}_{I_1i_1} y^{\sigma_2}_{I_2i_2} \dots y^{\sigma_n}_{I_ni_n}), \end{aligned}$$

and, since $\varepsilon^{ii_2i_3\dots i_n}\omega_i = dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}$,

(24) $$\begin{aligned} f_\sigma^{J\ i} = \varepsilon^{ii_2i_3\dots i_n}(&\tilde{\Phi}^{J}_{\sigma\ i_2i_3\dots i_n} + \tilde{\Phi}^{J\ I_2}_{\sigma\ \sigma_2\ i_3i_4\dots i_n} y^{\sigma_2}_{I_2i_2} + \tilde{\Phi}^{J\ I_2\ I_3}_{\sigma\ \sigma_2\ \sigma_3\ i_4i_5\dots i_n} y^{\sigma_2}_{I_2i_2} y^{\sigma_3}_{I_3i_3} \\ &+ \dots + \tilde{\Phi}^{J\ I_2\ I_3\ \dots\ I_{n-1}}_{\sigma\ \sigma_2\ \sigma_3\ \dots\ \sigma_{n-1}\ i_n} y^{\sigma_2}_{I_2i_2} y^{\sigma_3}_{I_3i_3} \dots y^{\sigma_{n-1}}_{I_{n-1}i_{n-1}} + \tilde{\Phi}^{J\ I_2\ I_3\ \dots\ I_n}_{\sigma\ \sigma_2\ \sigma_3\ \dots\ \sigma_n} y^{\sigma_2}_{I_2i_2} y^{\sigma_3}_{I_3i_3} \dots y^{\sigma_n}_{I_ni_n}), \end{aligned}$$

and

(25) $$\begin{aligned} f_\sigma^{I\ i} = \varepsilon^{ii_2i_3\dots i_n}(&A^{I}_{\sigma\ i_2i_3\dots i_n} + 2A^{I\ I_2}_{\sigma\ \sigma_2\ i_3i_4\dots i_n} y^{\sigma_2}_{I_2i_2} + 3A^{I\ I_2\ I_3}_{\sigma\ \sigma_2\ \sigma_3\ i_4i_5\dots i_n} y^{\sigma_2}_{I_2i_2} y^{\sigma_3}_{I_3i_3} \\ &+ \dots + (n-1)A^{I\ I_2\ \dots\ I_{n-1}}_{\sigma\ \sigma_2\ \dots\ \sigma_{n-1}\ i_n} y^{\sigma_2}_{I_2i_2} y^{\sigma_3}_{I_3i_3} \dots y^{\sigma_{n-1}}_{I_{n-1}i_{n-1}} \\ &+ nA^{I\ I_2\ \dots\ I_n}_{\sigma\ \sigma_2\ \dots\ \sigma_n} y^{\sigma_2}_{I_2i_2} y^{\sigma_3}_{I_3i_3} \dots y^{\sigma_n}_{I_ni_n}), \end{aligned}$$

where $0 \le |J| \le r-1$ and $|I| = r$.

Lemma 4 *For $k \ge 1$ the forms $\omega^\sigma_{j_1j_2\dots j_k} \wedge \omega_i$ can be decomposed as*

(26) $$\begin{aligned} \omega^\sigma_{l_1l_2\dots l_k} \wedge \omega_i = &\frac{1}{k+1}(\omega^\sigma_{l_1l_2\dots l_k} \wedge \omega \\ &+ \omega^\sigma_{il_2l_3\dots l_k} \wedge \omega_{l_1} + \omega^\sigma_{l_1il_3l_4\dots l_k} \wedge \omega_{l_2} + \dots + \omega^\sigma_{l_1l_2\dots l_{k-1}i} \wedge \omega_{l_k}) \\ &+ \frac{1}{k+1}((\omega^\sigma_{l_1l_2\dots l_k} \wedge \omega_i - \omega^\sigma_{il_2l_3\dots l_k} \wedge \omega_{l_1}) + (\omega^\sigma_{l_1l_2\dots l_k} \wedge \omega_i - \omega^\sigma_{l_1il_3l_4\dots l_k} \wedge \omega_{l_2}) \\ &+ \dots + (\omega^\sigma_{l_1l_2\dots l_k} \wedge \omega_i - \omega^\sigma_{l_1l_2\dots l_{k-1}i} \wedge \omega_{l_k})). \end{aligned}$$

The forms $\omega^\sigma_{l_1l_2\dots l_k} \wedge \omega_i - \omega^\sigma_{l_1l_2\dots l_{p-1}il_{p+1}\dots l_{k-1}l_k} \wedge \omega_{l_p}$ are closed and can be expressed *as*

(27) $$\omega^\sigma_{l_1 l_2 \ldots l_k} \wedge \omega_i - \omega^\sigma_{l_1 l_2 \ldots l_{p-1} i l_{p+1} \ldots l_{k-1} l_k} \wedge \omega_{l_p} = d(\omega^\sigma_{l_1 l_2 \ldots l_{p-1} l_{p+1} \ldots l_{k-1} l_k} \wedge \omega_{i l_p}).$$

Proof Indeed, from (5)

(28) $$\begin{aligned} d\omega^\sigma_{l_1 l_2 \ldots l_{p-1} l_{p+1} \ldots l_{k-1} l_k} \wedge \omega_{l_p i} &= -\omega^\sigma_{l_1 l_2 \ldots l_{p-1} l_{p+1} \ldots l_{k-1} l_k j} \wedge dx^j \wedge \omega_{l_p i} \\ &= -\omega^\sigma_{l_1 l_2 \ldots l_{p-1} l_{p+1} \ldots l_{k-1} l_k j} \wedge dx^j \wedge \omega_{l_p i} = \omega^\sigma_{l_1 l_2 \ldots l_{p-1} l_{p+1} \ldots l_{k-1} l_k j} \wedge (\delta^j_i \omega_{l_p} - \delta^j_{l_p} \omega_i) \\ &= -\omega^\sigma_{l_1 l_2 \ldots l_{p-1} l_{p+1} \ldots l_{k-1} l_k l_p} \wedge \omega_i + \omega^\sigma_{l_1 l_2 \ldots l_{p-1} l_{p+1} \ldots l_{k-1} l_k i} \wedge \omega_{l_p}. \end{aligned}$$

Theorem 2 *For every fibred chart* (V, ψ), $\psi = (x^i, y^\sigma)$, *the pull-back* $(\pi^{r+1,r})^* \rho$ *has an expression*

(29) $$(\pi^{r+1,r})^* \rho = f_0 \omega_0 + \sum_{0 \le |J| \le r} P^{J\ i}_\sigma \omega^\sigma_J \wedge \omega_i + d\eta + \mu,$$

where the components $P^{J\ i}_\sigma$ *are symmetric in the superscripts,* η *is a contact form, and* μ *is a contact form whose order of contactness is* ≥ 2. *The functions* $P^{I\ i}_\sigma$ *such that* $|I| = r$ *satisfy*

(30) $$P^{I\ i}_\sigma = \frac{\partial f_0}{\partial y^\sigma_{Ii}}.$$

Proof We use (21) and (22) and apply Lemma 4 to the forms $f^{J\ i}_\sigma \omega^\sigma_J \wedge \omega_i$. Write with explicit index notation $f^{J\ i}_\sigma = P^{j_1 j_2 \cdots j_k\ i}_\sigma$. We have the decomposition

(31) $$f^{j_1 j_2 \cdots j_k\ i}_\sigma = P^{j_1 j_2 \cdots j_k\ i}_\sigma + Q^{j_1 j_2 \cdots j_k\ i}_\sigma,$$

where $P^{j_1 j_2 \cdots j_k\ i}_\sigma = f^{j_1 j_2 \cdots j_k\ i}_\sigma \operatorname{Sym}(j_1 j_2 \ldots j_k i)$ is the symmetric component and $Q^{j_1 j_2 \cdots j_k\ i}_\sigma$ is the complementary one. We have, for each k, $1 \le k \le r$,

(32) $$\begin{aligned} &f^{j_1 j_2 \cdots j_k\ i}_\sigma \omega^\sigma_{j_1 j_2 \cdots j_k} \wedge \omega_i \\ &= P^{j_1 j_2 \cdots j_k\ i}_\sigma \omega^\sigma_{j_1 j_2 \cdots j_k} \wedge \omega_i - \frac{1}{k+1} Q^{j_1 j_2 \cdots j_k\ i}_\sigma d(\omega^\sigma_{j_2 j_3 \cdots j_k} \wedge \omega_{j_1 i} \\ &+ \omega^\sigma_{j_1 j_3 j_4 \cdots j_k} \wedge \omega_{j_2 i} + \ldots + \omega^\sigma_{j_1 j_2 \cdots j_{k-1}} \wedge \omega_{j_k i}) \end{aligned}$$

$$= P_\sigma^{j_1 j_2 \dots j_k\ i} \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i - \frac{1}{k+1} d(Q_\sigma^{j_1 j_2 \dots j_k\ i} (\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1 i}$$
$$+ \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2 i} + \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k i}))$$
$$+ \frac{1}{k+1} dQ_\sigma^{j_1 j_2 \dots j_k\ i} \wedge (\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1 i} + \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2 i}$$
$$+ \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k i}).$$

The exterior derivative $dQ_\sigma^{j_1 j_2 \dots j_k\ i}$, when lifted to V^{r+2}, can be decomposed as

$$(\pi^{r+2,r+1})^* dQ_\sigma^{j_1 j_2 \dots j_k\ i} = hdQ_\sigma^{j_1 j_2 \dots j_k\ i} + pdQ_\sigma^{j_1 j_2 \dots j_k\ i}$$
$$= d_p Q_\sigma^{j_1 j_2 \dots j_k\ i} dx^p + pdQ_\sigma^{j_1 j_2 \dots j_k\ i}. \tag{33}$$

Substituting from (33) back to (32) we get 1-contact and a 2-contact summands. The 1-contact summands are equal to

$$hdQ_\sigma^{j_1 j_2 \dots j_k\ i} \wedge (\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1 i} + \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2 i} + \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k i})$$
$$= -(d_p Q_\sigma^{p j_2 j_3 \dots j_k\ i} \omega^\sigma_{j_2 j_3 \dots j_k} + d_p Q_\sigma^{j_1 p j_3 j_4 \dots j_k\ i} \omega^\sigma_{j_1 j_3 j_4 \dots j_k}$$
$$+ \dots + d_p Q_\sigma^{j_1 j_2 \dots j_{k-1} p\ i} \omega^\sigma_{j_1 j_2 \dots j_{k-1}}) \omega_i \tag{34}$$
$$+ d_p Q_\sigma^{j_1 j_2 \dots j_k\ p} (\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1} + \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2} + \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k})$$
$$= -k d_p (Q_\sigma^{p j_2 j_3 \dots j_k\ i} - Q_\sigma^{i j_2 j_3 \dots j_k\ p}) \omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_i.$$

Note that from the definition of the functions $Q_\sigma^{p j_2 j_3 \dots j_k\ i}$ and from formula (24) we easily see that this form is $\pi^{r+2,r+1}$-projectable. Thus, returning to (32), we have on V^{r+1}

$$f_\sigma^{j_1 j_2 \dots j_k\ i} \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i = P_\sigma^{j_1 j_2 \dots j_k\ i} \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i$$
$$- \frac{k}{k+1} d_p (Q_\sigma^{p j_2 j_3 \dots j_k\ i} - Q_\sigma^{i j_2 j_3 \dots j_k\ p}) \omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_i$$
$$- \frac{1}{k+1} d(Q_\sigma^{j_1 j_2 \dots j_k\ i} (\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1 i} + \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2 i} \tag{35}$$
$$+ \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k i}))$$
$$+ \frac{1}{k+1} pdQ_\sigma^{j_1 j_2 \dots j_k\ i} \wedge (\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1 i} + \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2 i}$$
$$+ \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k i}).$$

This sum replaces $f_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i$, where $|J| = k$, with the symmetrized term $P_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i$, a term $d_p(Q_\sigma^{pj_2j_3\dots j_k\ i} - Q_\sigma^{ij_2j_3\dots j_k\ p})\omega_{j_2j_3\dots j_k}^\sigma \wedge \omega_i$ containing $\omega_J^\sigma \wedge \omega_i$ with $|J| = k-1$, a closed form, and a 2-contact term.

Using these expressions in (21), written as

$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + \sum_{0\le|J|\le r} f_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i + d\eta + \mu, \tag{36}$$

we can redefine the coefficients and get

$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + \sum_{0\le|J|\le r-1} f_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i + \sum_{|J|\le r} P_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i + d\eta + \mu. \tag{37}$$

After r steps we get (29).

To prove (31), we differentiate (23) and compare the result with (25).

The following lemma concerns vector fields on any fibred manifold Y with base X and projection π.

Lemma 5 *Let ξ be a vector field on X. There exists a π-projectable vector field $\tilde{\xi}$ on Y whose π-projection is ξ.*

Proof We can construct $\tilde{\xi}$ by means of an atlas on Y, consisting of fibred charts, and a subordinate partition of unity; we proceed as in the proof of Theorem 1, Section 3.2.

Now we study properties of differential n-forms ρ, defined on $W^r \subset J^rY$, which play a key role in global variational geometry. To this purpose we write the decomposition formula (29) as

$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + P_\sigma^{\ i}\omega^\sigma \wedge \omega_i + \sum_{k=1}^{r} P_\sigma^{j_1j_2\dots j_k\ i}\omega_{j_1j_2\dots j_k}^\sigma \wedge \omega_i + d\eta + \mu, \tag{38}$$

where

$$P_\sigma^{j_1j_2\dots j_r\ i} = \frac{\partial f_0}{\partial y_{j_1j_2\dots j_r i}^\sigma}. \tag{39}$$

Lemma 6 *Let $\rho \in \Omega_n^r W$. The following three conditions are equivalent:*

(a) *$p_1 d\rho$ is a $\pi^{r+1,0}$-horizontal $(n+1)$-form.*

(b) *For each $\pi^{r,0}$-vertical vector field ξ on W^r,*

(40) $$hi_{\xi}d\rho = 0.$$

(c) *The pull-back* $(\pi^{r+1,r})^*\rho$ *has the chart expression* (38)*,such that the coefficients satisfy*

(41) $$\frac{\partial f_0}{\partial y^\sigma_{j_1j_2\dots j_k}} - d_iP_\sigma^{j_1j_2\dots j_k\ i} - P_\sigma^{j_1j_2\dots j_{k-1}\ j_k} = 0, \quad k = 1,2,\dots,r.$$

Proof 1. Let Ξ be a vector field on W^r, let $\tilde{\Xi}$ be a vector field on W^{r+1}, covering Ξ that is, such that $T\pi^{r+1,r}\cdot\tilde{\Xi} = \Xi\circ\pi^{r+1,r}$ (Lemma 5). Then $i_{\tilde{\Xi}}(\pi^{s+1,s})^*d\rho = (\pi^{s+1,s})^*i_{\Xi}d\rho$, and the forms on both sides can be canonically decomposed into their contact components. We have

(42) $$i_{\tilde{\Xi}}p_1d\rho + i_{\tilde{\Xi}}p_2d\rho + \dots + i_{\tilde{\Xi}}p_{n+1}d\rho = hi_{\Xi}d\rho + p_1i_{\Xi}d\rho + \dots + p_ni_{\Xi}d\rho.$$

Comparing the horizontal components on both sides we get

(43) $$hi_{\tilde{\Xi}}p_1d\rho = (\pi^{r+2,r+1})^*hi_{\Xi}d\rho.$$

Let $p_1d\rho$ be $\pi^{r+1,0}$-horizontal. Then if Ξ is $\pi^{r,0}$-vertical, $\tilde{\Xi}$ is $\pi^{r+1,0}$-vertical, and we get $hi_{\tilde{\Xi}}p_1d\rho = (\pi^{r+2,r+1})^*hi_{\Xi}d\rho = 0$, which implies, by injectivity of the mapping $(\pi^{r+2,r+1})^*$, that $hi_{\Xi}d\rho = 0$.

Conversely, let $hi_{\Xi}d\rho = 0$ for each $\pi^{r,0}$-vertical vector field ξ. Then by (43), $hi_{\tilde{\Xi}}p_1d\rho = i_{\tilde{\Xi}}p_1d\rho = 0$ for all $\pi^{r+1,r}$-projectable, $\pi^{r+1,0}$-vertical vector fields $\tilde{\Xi}$. If in a fibred chart,

(44) $$\tilde{\Xi} = \sum_{1\le k\le r}\Xi^\sigma_{j_1j_2\dots j_k}\frac{\partial}{\partial y^\sigma_{j_1j_2\dots j_k}}$$

and

(45) $$p_1d\rho = \sum_{1\le k\le r}A_\sigma^{j_1j_2\dots j_k}\omega^\sigma_{j_1j_2\dots j_k}\wedge\omega_0,$$

then we get

(46) $$A_\sigma^{j_1j_2\dots j_k} = 0, \quad 1\le k\le r,$$

proving $\pi^{r+1,0}$-horizontality of $p_1d\rho$. This proves that conditions (a) and (b) are equivalent.

2. Express $(\pi^{r+1,r})^*\rho$ in a fibred chart by (38). Then

$$p_1 d\rho = \left(\frac{\partial f_0}{\partial y^\sigma} - d_i P_\sigma{}^{i} \right) \omega^\sigma \wedge \omega_0 +$$

$$(47) \qquad + \sum_{k=1}^{r} \left(\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_i P_\sigma^{j_1 j_2 \dots j_k\ i} - P_\sigma^{j_1 j_2 \dots j_{k-1}\ j_k} \right) \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_0$$

$$+ \left(\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_{r+1}}} - P_\sigma^{j_1 j_2 \dots j_r\ j_{r+1}} \right) \omega^\sigma_{j_1 j_2 \dots j_r j_{r+1}} \wedge \omega_0$$

Formula (47) proves equivalence of conditions (a) and (c).

Any form $\rho \in \Omega^r_n W$ satisfying equivalent conditions of Lemma 6 is called a *Lepage form*.

Remark 5 (Existence of Lepage forms) Consider conditions (41). It is easily seen that this system has always a solution, and the solution is unique. Indeed, we have

$$P_\sigma^{j_1 j_2 \dots j_{k-1}\ j_k} = \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_{i_1} P_\sigma^{j_1 j_2 \dots j_k\ i_1}$$

$$= \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_{i_1} \left(\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k i_1}} - d_{i_2} P_\sigma^{j_1 j_2 \dots j_k i_1\ i_2} \right)$$

$$(48) \qquad = \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_{i_1} \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k i_1}} + d_{i_1} d_{i_2} P_\sigma^{j_1 j_2 \dots j_{k-1} i_1\ i_2}$$

$$= \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_{i_1} \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k i_1}} + d_{i_1} d_{i_2} \left(\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_{k-1} i_1 i_2}} - d_{i_3} P_\sigma^{j_1 j_2 \dots j_{k-1} i_1 i_2\ i_3} \right)$$

$$= \dots = \sum_{l=0}^{r+1-k} (-1)^l d_{i_1} d_{i_2} \dots d_{i_l} \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k i_1 i_2 \dots i_l}}.$$

so the coefficients $P_\sigma{}^{j_1}$, $P_\sigma^{j_1 j_2 \dots j_{k-1}\ j_k}$ are completely determined by f_0. In particular, Lepage forms always exist over fibred coordinate neighbourhoods. More precisely, one can also interpret this result in such a way that to any given form $\rho \in \Omega^r_n W$ and any fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, such that $V \subset W$, one can assign by the described construction a Lepage form, belonging to the module $\Omega^{r+1}_n V$.

Theorem 3 *A form* $\rho \in \Omega^r_n W$ *is a Lepage form if and only if for any fibred chart* (V, ψ), $\psi = (x^i, y^\sigma)$, *on Y such that* $V \subset W$, $(\pi^{r+1,r})^* \rho$ *has an expression*

(49) $(\pi^{r+1,r})^* \rho = \Theta + d\eta + \mu,$

where

(50) $$\Theta = f_0\omega_0 + \sum_{k=0}^{r}\left(\sum_{l=0}^{r-k}(-1)^l d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k p_1 p_2 \dots p_l i}}\right)\omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i,$$

f_0 *is a function, defined by the chart expression* $h\rho = f_0\omega_0$, *and the order of contactness of* η *is* ≥ 2.

Proof Suppose we have a Lepage form ρ expressed by (38) where conditions (41) are satisfied, and consider conditions (20). Then repeating (48) we get formula (50). The converse follows from (47) and (38).

The n-form Θ defined by (50), is sometimes called the *principal component* of the Lepage form ρ with respect to the fibred chart (V,ψ). Note that Θ depends only on the Lagrangian $h\rho = \lambda_\rho$ associated with ρ; the forms Θ constructed this way are defined only locally, but their horizontal components define a global form.

3.4 Euler-Lagrange forms We defined in Section 3.3 a Lepage form $\rho \in \Omega^r_n W$ by a condition on the exterior derivative $d\rho$; namely, we required that the 1-contact component $p_1 d\rho$ should belong to the ideal of forms, defined on W^{r+1}, generated in any fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, by the contact 1-forms ω^σ. Now we study the consequences of this definition for the exterior derivative $d\rho$. We express ρ as in formula (48), Section **3.3.**

Theorem 4 *If* $\rho \in \Omega^r_n W$ *is a Lepage form, then the form* $(\pi^{r+1,r})^* d\rho$ *has an expression*

(1) $(\pi^{r+1,r})^* d\rho = E + F,$

where E is a 1*-contact,* $(\pi^{r+1,0})$*-horizontal* $(n+1)$*-form, and F is a form whose order of contactness is* ≥ 2. *E is unique and has the chart expression*

(2) $$E = \left(\frac{\partial f_0}{\partial y^\sigma} - \sum_{l=1}^{r+1}(-1)^{l-1} d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \dots p_l}}\right)\omega^\sigma \wedge \omega_0.$$

Proof For any ρ, $E = p_1 d\rho$, and $F = p_2 d\rho + p_3 d\rho + \dots + p_{n+1} d\rho$. But for a Lepage form ρ, from 3.3, (48),

(3) $$E = p_1 d\rho = p_1 d\Theta = \left(\frac{\partial f_0}{\partial y^\sigma} - d_i P_\sigma{}^i \right) \omega^\sigma \wedge \omega_0,$$

where by 3.3, (47),

(4) $$P_\sigma{}^i = \sum_{l=0}^{s} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \dots p_l i}}.$$

The $(n+1)$-form E is called the *Euler-Lagrange form*, associated with the Lepage form ρ. Note that similarly as the form Θ, E depends only on the Lagrangian $\lambda_\rho = f_0 \omega_0$. The components of E

(5) $$E_\sigma(f_0) = \frac{\partial f_0}{\partial y^\sigma} - \sum_{l=1}^{r+1} (-1)^{l-1} d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \dots p_l}}$$

are called the *Euler-Lagrange expressions*, associated with f_0.

Sometimes we consider differential forms, defined on different order jet prolongations $J^r Y$ and $J^s Y$ of the fibred manifold *Y*, arising, however, by the pull-back by the corresponding canonical jet projection $\pi^{r,s}$. Then to avoid long notations, we usually omit the corresponding canonical pull-back mappings between two forms, defined on $J^r Y$ and $J^s Y$. Our aim will be to study Lepage forms with fixed (given) horizontal components.

As before, denote by $\Omega^r_{n,X} W$ the submodule of the module $\Omega^r_n W$, formed by π^r*-horizontal n-forms* (*Lagrangians of order r* for *Y*). Clearly, the set $\Omega^r_{n,X} W$ contains the Lagrangians λ_η, associated with *n*-forms $\eta \in \Omega^{r-1}_n W$.

The following is an existence theorem of Lepage forms whose horizontal component is a given Lagrangian.

Theorem 5 *To any Lagrangian* $\lambda \in \Omega^r_{n,X} W$ *there exists an integer s and a Lepage form* $\rho \in \Omega^s_n W$ *such that*

(51) $$h\rho = \lambda.$$

Proof We show that the theorem is true for $s = 2r - 1$. Choose an atlas $\{(V_\iota, \psi_\iota)\}$ on *Y*, consisted of fibred charts (V_ι, ψ_ι), $\psi_\iota = (x^i_\iota, y^\sigma_\iota)$, and a partition of unity $\{\chi_\iota\}$, subordinate to the covering $\{V_\iota\}$ of *Y*. The functions χ_ι define the (global) Lagrangians $\chi_\iota \lambda \in \Omega^r_n W$. We have, in the chart (V_ι, ψ_ι), with obvious notation,

(52) $$\lambda = \mathscr{L}_\iota \omega_{(\iota)0}.$$

Then we set for each ι

$$\Theta_\iota = \chi_\iota \mathscr{L}_\iota \omega_{(\iota)0} + \sum_{k=0}^{r-1} \left(\sum_{l=0}^{r-1-k} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial(\chi_\iota \mathscr{L}_\iota)}{\partial y_{(\iota)}{}^{\sigma}_{j_1 j_2 \ldots j_k p_1 p_2 \ldots p_l i}} \right) \omega_{(\iota)}{}^{\sigma}_{j_1 j_2 \ldots j_k} \wedge \omega_{(\iota)i} . \tag{53}$$

Thus, Θ_ι is the principal Lepage equivalent of the Lagrangian $\lambda = \mathscr{L}_\iota \omega_{(\iota)0}$. Since the family $\{\chi_\iota\}$ is locally finite, the family $\{\Theta_\iota\}$ is also locally finite, thus the sum $\rho = \sum \Theta_\iota$ is defined. Then we have $p_1 d\rho = \sum p_1 d\Theta_\iota$, thus, ρ is a Lepage form, because each of the forms Θ_ι is Lepage. It remains to show that $h\rho = \lambda$. We have $h\rho = \sum h\Theta_\iota = \sum \chi_\iota \mathscr{L}_\iota \omega_{(\iota)0}$. To compute this expression choose a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, such that the intersection $V \cap V_\iota$ is non-void for only finitely many indices ι. Using this chart, we have $\lambda = \mathscr{L}_\iota \omega_{(\iota)0} = \mathscr{L}_\iota \omega_0$ on $V \cap (\bigcup V_\iota)$ and, since

$$\omega_{(\iota)0} = \det\left(\frac{\partial x^i_{(\iota)}}{\partial x^j} \right) \omega_0 , \tag{54}$$

then

$$\mathscr{L}_\iota \det\left(\frac{\partial x^i_{(\iota)}}{\partial x^j} \right) = \mathscr{L}. \tag{55}$$

Consequently,

$$h\rho = \sum \chi_\iota \mathscr{L}_\iota \omega_{(\iota)0} = \sum \chi_\iota \mathscr{L}_\iota \det\left(\frac{\partial x^i_{(\iota)}}{\partial x^j} \right) \omega_0 = (\sum \chi_\iota) \mathscr{L} \omega_0 = \mathscr{L} \omega_0 \tag{56}$$

because $\sum \chi_\iota = 1$.

Let $\lambda \in \Omega^r_{n,X} W$ be a Lagrangian. A Lepage form $\rho \in \Omega^s_n W$ such that $h\rho = \lambda$ (possibly up to a canonical jet projection) is called a *Lepage equivalent* of λ.

If in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, λ is expressed as

$$\lambda = \mathscr{L} \omega_0 , \tag{57}$$

then the form

$$\Theta_{\mathscr{L}} = \mathscr{L}_\iota \omega_0 + \sum_{k=0}^{r-1} \left(\sum_{l=0}^{r-1-k} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \mathscr{L}}{\partial y_{(\iota)}{}^{\sigma}_{j_1 j_2 \ldots j_k p_1 p_2 \ldots p_l i}} \right) \omega^\sigma_{j_1 j_2 \ldots j_k} \wedge \omega_i \tag{58}$$

is called the *principal Lepage equivalent* of λ for the fibred chart (V,ψ).

Remark 6 The Lepage equivalent constructed in the proof of Theorem 5 is $\pi^{2r-1,r-1}$-horizontal, and its order of contactness is ≤ 1.

Remark 7 Theorem 5 says that the class of variational functionals, associated with the variational structures (π,ρ), introduced in Section **3.1**, remains the same when we restrict ourselves to *Lepage forms* ρ. Thus, from now on, we may suppose without loss of generality that the variational functionals

$$\Gamma_{\Omega,W}(\pi)\ni\gamma\to\rho_\Omega(\gamma)=\int_\Omega J^r\gamma*\rho\in\mathbf{R}, \tag{59}$$

are defined by Lepage forms.

Example 1 (Lepage forms of order 1) For Lagrangians $\lambda=\mathscr{L}\omega_0$ of order 1 we get the principal Lepage equivalent

$$\Theta=\mathscr{L}\omega_0+\frac{\partial\mathscr{L}}{\partial y_i^\sigma}\omega^\sigma\wedge\omega_i. \tag{60}$$

One can easily verify by a direct calculation that the form Θ is defined by (60) *globally*; it is called, due to P.L. Garcia [4], the *Poincare-Cartan form*.

Example 2 (Lepage forms of order 2) A Lagrangians $\lambda=\mathscr{L}\omega_0$ of order 2 has the principal Lepage equivalent

$$\Theta=\mathscr{L}\omega_0+\left(\frac{\partial\mathscr{L}}{\partial y_i^\sigma}-d_j\frac{\partial\mathscr{L}}{\partial y_{ij}^\sigma}\right)\omega^\sigma\wedge\omega_i+\frac{\partial\mathscr{L}}{\partial y_{ij}^\sigma}\omega_j^\sigma\wedge\omega_i. \tag{61}$$

The form (61) are *global*, although in general for higher order Lagrangians a similar assertion is not true. The Lepage form (61) was introduced by Krupka in [7]. The proof of invariance of Θ is routine. We shall verify the transformation properties of the forms $\omega_j^\sigma\wedge\omega_i+\omega_i^\sigma\wedge\omega_j$ with the help of explicit coordinate transformation (cf. 1.4, Example 5). We have

$$\bar\omega_i=i_{\partial/\partial\bar x^i}\bar\omega_0=\frac{\partial x^k}{\partial\bar x^i}\det\frac{\partial\bar x}{\partial x}i_{\partial/\partial x^k}\omega_0=\frac{\partial x^k}{\partial\bar x^i}\det\frac{\partial\bar x}{\partial x}\omega_k. \tag{62}$$

and

$$\bar{\omega}^\sigma_j = d\bar{y}^\sigma_j - \bar{y}^\sigma_{jl} d\bar{x}^l = \frac{\partial \bar{y}^\sigma_j}{\partial x^p} dx^p + \frac{\partial \bar{y}^\sigma_j}{\partial y^\nu} dy^\nu + \frac{\partial \bar{y}^\sigma_j}{\partial y^\nu_p} dy^\nu_p$$
$$- \left(\frac{\partial^2 \bar{y}^\sigma}{\partial x^s \partial x^m} + \frac{\partial^2 \bar{y}^\sigma}{\partial x^s \partial y^\mu} y^\mu_m + \frac{\partial^2 \bar{y}^\sigma}{\partial x^m \partial y^\nu} y^\nu_s + \frac{\partial^2 \bar{y}^\sigma}{\partial y^\mu \partial y^\nu} y^\nu_s y^\mu_m \right.$$
$$\left. + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{sm} \right) \frac{\partial x^m}{\partial \bar{x}^l} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^p} dx^p + \left(\frac{\partial \bar{y}^\sigma}{\partial x^s} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_s \right) \frac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^l} \frac{\partial \bar{x}^l}{\partial x^p} dx^p$$
$$= \left(\frac{\partial}{\partial x^p} \left(\frac{\partial \bar{y}^\sigma}{\partial x^l} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_l \right) \frac{\partial x^l}{\partial \bar{x}^j} + \left(\frac{\partial \bar{y}^\sigma}{\partial x^l} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_l \right) \frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^s} \frac{\partial \bar{x}^s}{\partial x^p} + \right.$$
$$(63) \qquad + \left(\frac{\partial \bar{y}^\sigma}{\partial x^l \partial y^\nu} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu \partial y^\tau} y^\tau_l \right) \frac{\partial x^l}{\partial \bar{x}^j} y^\nu_p + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} \frac{\partial x^q}{\partial \bar{x}^j} y^\nu_{pq}$$
$$- \left(\frac{\partial^2 \bar{y}^\sigma}{\partial x^s \partial x^m} + \frac{\partial^2 \bar{y}^\sigma}{\partial x^s \partial y^\mu} y^\mu_m + \frac{\partial^2 \bar{y}^\sigma}{\partial x^m \partial y^\nu} y^\nu_s + \frac{\partial^2 \bar{y}^\sigma}{\partial y^\mu \partial y^\nu} y^\nu_s y^\mu_m \right.$$
$$\left. \left. + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{sm} \right) \frac{\partial x^m}{\partial \bar{x}^l} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^p} + \left(\frac{\partial \bar{y}^\sigma}{\partial x^s} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_s \right) \frac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^l} \frac{\partial \bar{x}^l}{\partial x^p} \right) dx^p$$
$$+ \frac{\partial \bar{y}^\sigma_j}{\partial y^\nu} (dy^\nu - y^\nu_q dx^q) + \frac{\partial \bar{y}^\sigma_j}{\partial y^\nu_p} (dy^\nu_p - y^\nu_{pq} dx^q),$$

which result into the formula

$$(64) \qquad \bar{\omega}^\sigma_j = \frac{\partial \bar{y}^\sigma_j}{\partial y^\nu} \omega^\nu + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} \frac{\partial x^l}{\partial \bar{x}^j} \omega^\nu_p.$$

Then, however,

$$(65) \qquad \begin{aligned} \bar{\omega}^\sigma_j \wedge \bar{\omega}_i &= \frac{\partial x^k}{\partial \bar{x}^i} \det \frac{\partial \bar{x}}{\partial x} \left(\frac{\partial \bar{y}^\sigma_j}{\partial y^\nu} \omega^\nu + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} \frac{\partial x^p}{\partial \bar{x}^j} \omega^\nu_p \right) \wedge \omega_k \\ &= \det \frac{\partial \bar{x}}{\partial x} \left(\frac{\partial \bar{y}^\sigma_j}{\partial y^\nu} \frac{\partial x^k}{\partial \bar{x}^i} \omega^\nu + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^k}{\partial \bar{x}^i} \omega^\nu_p \right) \wedge \omega_k, \end{aligned}$$

which shows that the symmetrized expression $\bar{\omega}^\sigma_j \wedge \bar{\omega}_i + \bar{\omega}^\sigma_i \wedge \bar{\omega}_j$ transforms to the symmetrized expression $\omega^\nu_p \wedge \omega_k + \omega^\nu_k \wedge \omega_p$.

3.5 The Euler-Lagrange mapping Choosing for any Lagrangian $\lambda \in \Omega^r_{n,X} W$ a Lepage equivalent ρ of λ, we can construct the Euler-Lagrange form E associated to ρ (3.4, (3)); this $(n+1)$-form, depends on

λ only. We denote this form by E_λ and call it the *Euler-Lagrange form*, associated with λ. Denoting by $\Omega^r_{n+1,Y}W$ the module of $\pi^{2r-1,0}$ *-horizontal* $(n+1)$ *-forms* on W^{2r-1}, we get the mapping

$$\Omega^r_{n,X}W \ni \lambda \to E_\lambda \in \Omega^r_{n+1,Y}W \tag{16}$$

called the *Euler-Lagrange mapping*.

We can summarize basic properties of Lepage forms, namely their relations to the Euler-Lagrange forms, as follows. Denote by $\mathrm{Lep}^r_n W$ the real vector subspace of the vector space $\Omega^r_n W$, whose elements are Lepage forms. Taking into account properties of the exterior derivative of a Lepage form we see that the Euler-Lagrange mapping E makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Lep}^r_n W & \xrightarrow{\;h\;} & \Omega^{r+1}_{n,X}W \\ \downarrow d & & \downarrow E \\ \Omega^{r+1}_{n+1}W & \xrightarrow{\;p_1\;} & \Omega^{2(r+1)}_{n,Y}W \end{array} \tag{17}$$

The diagram demonstrates the relationship of the Euler-Lagrange mapping and the exterior derivative of differential forms in the spirit of the work or Th. Lepage.

The following theorem describes the behaviour of the Euler-Lagrange mapping under automorphisms of the underlying fibred manifold.

Theorem 6 *For each Lagrangian* λ *and each automorphism* α *of Y*

$$J^{2r}\alpha^* E_\lambda = E_{J^{2r}\alpha^*\lambda}. \tag{18}$$

Proof We apply Theorem 4 of Section 3.4 to Lepage equivalents. Let $\rho_\lambda \in \Omega^s_n W$ be any Lepage equivalent of λ. Then

$$(\pi^{s+1,s})^* d\rho = E_\lambda + F_\lambda. \tag{19}$$

It is easily seen that the pull-back $J^s\alpha^*\rho$ is a Lepage form whose Lagrangian is $hJ^s\alpha^*\rho = J^{s+1}\alpha^* h\rho = J^{s+1}\alpha^*\lambda$. Then from commutativity of the pull-back and the exterior derivative we have

$$(\pi^{s+1,s})^* dJ^s\alpha^*\rho = (\pi^{s+1,s})^* J^s\alpha^* d\rho = J^{s+1}\alpha^*(\pi^{s+1,s})^* d\rho, \tag{20}$$

from which we conclude that $J^{s+1}\alpha^* E_\lambda + J^{s+1}\alpha^* F_\lambda = E_{J^s\alpha^*\lambda} + F_{J^s\alpha^*\lambda}$. Theorem 6 now follows from the uniqueness of the 1-contact components.

3.6 The first variation formula Suppose we have a variational structure (Y,ρ), where Y is a fibred manifold with n-dimensional base X, and ρ is an n-form on the r-jet prolongation J^rY. Recall that for any piece Ω of X, and any open set $W \subset Y$, (Y,ρ) defines the variational functional $\Gamma_{\Omega,W}(\pi) \ni \gamma \to \rho_{\Omega,W}(\gamma) \in \mathbf{R}$ by

(1) $$\rho_\Omega(\gamma) = \int_\Omega J^r\gamma^*\rho$$

(Section 3.1). The *first variation* of this variational functional by a π-projectable vector field Ξ is the variational functional $\Gamma_{\Omega,W}(\pi) \ni \gamma \to (\partial_{J^r\Xi}\rho)_\Omega(\gamma) \in \mathbf{R}$, where

(2) $$(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\rho$$

(Section 3.2, (15)).

In this section we study a variational structure (Y,ρ) such that ρ is a *Lepage form*. Our main result of Section 3.5 (Theorem 5, Remark 7) shows that this assumption does not restrict the class of variational functionals. As before, denote by λ_ρ the *horizontal component* of an n-form ρ, that is, the *Lagrangian*, associated with ρ. For Lepage forms, the following theorem on the structure of the integrand in the first variation (2) is just a restatement of definitions.

Theorem 7 *Let $\rho \in \Omega_n^r W$ be a Lepage form, and let Ξ be a π-projectable vector field on W.*

(a) *The Lie derivative $\partial_{J^r\Xi}\rho$ can be expressed as*

(3) $$\partial_{J^r\Xi}\rho = i_{J^r\Xi}d\rho + di_{J^r\Xi}\rho.$$

(b) *If Ξ is π-vertical, then*

(4) $$\partial_{J^{r+1}\Xi}\lambda_\rho = i_{J^{r+1}\Xi}E_{\lambda_\rho} + hdi_{J^r\Xi}\rho.$$

(c) *For any section γ of Y with values in W,*

(5) $$J^r\gamma^*\partial_{J^r\Xi}\rho = J^{r+1}\gamma^*i_{J^{r+1}\Xi}E_{\lambda_\rho} + dJ^{r+1}\gamma^*i_{J^{r+1}\Xi}\rho.$$

(d) *For every piece Ω of X and every section γ of Y defined on Ω,*

(6) $$\int_\Omega J^r\gamma^*\partial_{J^r\Xi}\rho = \int_\Omega J^{r+1}\gamma^*i_{J^{r+1}\Xi}E_\lambda + \int_{\partial\Omega} J^{r+1}\gamma^*i_{J^{r+1}\Xi}\rho.$$

Proof (a) This is a standard Cartan's Lie derivative formula.

(b) If Ξ π-vertical, then from (3), $h\partial_{J^r\Xi}\rho = \partial_{J^r\Xi}h\rho = i_{J^r\Xi}p_1 d\rho + hdi_{J^r\Xi}\rho$, but $p_1 d\rho = E_{\lambda_\rho}$ because ρ is a Lepage form.

(c) Formula (4) can be proved by a straightforward calculation. We have

$$\begin{aligned} J^r\gamma^*\partial_{J^r\Xi}\rho &= J^{r+1}\gamma^* h\partial_{J^r\Xi}\rho = J^{r+1}\gamma^* h\partial_{J^r\Xi}\rho \\ &= J^{r+1}\gamma^* hi_{J^r\Xi}d\rho + J^{r+1}\gamma^* hdi_{J^r\Xi}\rho \\ &= J^{r+2}\gamma^* hi_{J^r\Xi}p_1 d\rho + J^{r+2}\gamma^* hi_{J^r\Xi}p_2 d\rho + J^r\gamma^* di_{J^r\Xi}\rho \\ &= J^{r+2}\gamma^* hi_{J^r\Xi}E_{\lambda_\rho} + J^r\gamma^* di_{J^r\Xi}\rho. \end{aligned} \tag{7}$$

(d) Integrating (5) and using the Stokes' theorem on integration of closed $(n-1)$-forms on pieces of n-dimensional manifolds we get (6).

Each of the formulas (3), (4) and (5) is called, in the context of the variational theory on fibred manifolds, the *infinitesimal first variation formula*; (6) is called the *integral first variation formula*.

Remark 8 Note that the infinitesimal first variation formula has no analogue in the *classical formulation* of the calculus of variations. The present formulation is based on the concepts of a Lepage form as well as of geometric concepts as the Lie derivative, exterior derivative and contraction of a form by a vector field.

Remark 9 Theorem 7 can be used to obtain the corresponding formulas for higher variational derivatives (cf. 3.2).

3.7 Extremals Let $U \subset X$ be an open set, $\gamma: U \to Y$ a section, and $\Xi: U \to TY$ a vector field along γ. The *support* of Ξ is the set $\operatorname{supp}\Xi = \operatorname{cl}\{x \in U \mid \Xi(x) \neq 0\}$ (here cl means *closure*). We know that each smooth vector field Ξ along γ can be smoothly prolonged to a π-projectable vector field $\tilde{\Xi}$ defined on a neighbourhood V of the set $\gamma(U) \subset Y$ (3.1, Theorem 1). $\tilde{\Xi}$ satisfies

$$\tilde{\Xi} \circ \gamma = \Xi. \tag{1}$$

Let $\Omega \subset X$ be a piece of X, $W \subset Y$ an open set, and let $\Gamma_{\Omega,W}(\pi)$ denote the set of sections $\gamma: U \to Y$ such that $\Omega \subset U$ and $\gamma(\Omega) \subset W$. Let $\rho \in \Omega_n^r W$ be a Lepage form. We say that a section $\gamma \in \Gamma_{\Omega,W}(\pi)$ is an *extremal* of the variational functional $\Gamma_{\Omega,W}(\pi) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$ on Ω, if for all π -projectable vector fields Ξ, such that $\mathrm{supp}(\Xi \circ \gamma) \subset \Omega$,

(2) $$\int_\Omega J^r\gamma^* \partial_{J^r\Xi}\rho = 0.$$

γ is called *extremal* of the variational functional $\Gamma_{\Omega,W}(\pi) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$, if it is an extremal on Ω for every Ω.

Thus, roughly speaking, the extremals are those sections γ for which the values $\rho_\Omega(\gamma)$ are not sensitive to small compact deformations of γ.

In the following necessary and sufficient conditions for a section to be an extremal, we use the *Euler-Lagrange form* $E_{h\rho}$, associated with the Lagrangian $\lambda_\rho = h\rho$, written in a fibred chart as

(3) $$E_{h\rho} = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0,$$

where the components $E_\sigma(\mathscr{L})$ are the *Euler-Lagrange expressions* (see 3.4, (5). Explicitly, if $h\rho = \mathscr{L}\omega_0$, then

(4) $$E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - \sum_{l=1}^{r+1} (-1)^{l-1} d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \dots p_l}}.$$

Theorem 8 *Let $\rho \in \Omega^r_n W$ be a Lepage form. Let $\gamma : U \to Y$ be a section, and $\Omega \subset U$ a piece of X. The following conditions are equivalent:*

(a) *γ is an extremal on Ω.*

(b) *For every π -vertical vector field Ξ defined on a neighbourhood of $\gamma(U)$, such that* $\mathrm{supp}(\Xi \circ \gamma) \subset \Omega$,

(5) $$J^r\gamma^* i_{J^r\Xi} d\rho = 0.$$

(c) *The Euler-Lagrange form associated with the Lagrangian $h\rho$ vanishes along $J^{r+1}\gamma$, i.e.,*

(6) $$E_{h\rho} \circ J^{r+1}\gamma = 0.$$

(d) *For every fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, such that $\pi(V) \subset U$ and $\gamma(\pi(V)) \subset V$, γ satisfies the system of partial differential equations*

(7) $$E_\sigma(\mathscr{L}) \circ J^{r+1}\gamma = 0, \quad 1 \le \sigma \le m.$$

Proof 1. We show that (a) implies (b). By Theorem 7, (d), for any piece Ω of *X* and any π -vertical vector field Ξ such that $\mathrm{supp}(\Xi \circ \gamma) \subset \Omega$,

(8) $$\int_\Omega J^r\gamma^* \partial_{J^1\Xi}\rho = \int_\Omega J^r\gamma^* i_{J^3\Xi}\, d\rho,$$

because the vector field $J^r\Xi$ vanishes along the boundary $\partial\Omega$. Then

(9) $$\int_\Omega J^r\gamma^* i_{J^r\Xi}\, d\rho = \int_\Omega J^{r+1}\gamma^* (\pi^{r+1,r})^* i_{J^r\Xi} d\rho = \int_\Omega J^{r+1}\gamma^* i_{J^{r+1}\Xi} p_1\, d\rho,$$

where $p_1 d\rho = E_{h\rho}$ is the Euler-Lagrange form.

If Ω is contained in a coordinate neighbourhood, the support $\operatorname{supp}(\Xi\circ\gamma)\subset\Omega$ lies in the same coordinate neighbourhood. Writing $\Xi = \Xi^\sigma \cdot \partial/\partial y^\sigma$ and $p_1 d\rho = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$, we get $i_{J^{r+1}\Xi} p_1 d\rho = E_\sigma(\mathscr{L})\Xi^\sigma \omega_0$ and

(10) $$J^r\gamma^* i_{J^r\Xi} d\rho = (E_\sigma(\mathscr{L})\circ J^{r+1}\gamma)\cdot(\Xi^\sigma \circ\gamma)\cdot\omega_0 .$$

Now supposing that $J^r\gamma^* i_{J^r\Xi} d\rho \neq 0$ for some π-vertical vector field Ξ, the first variation formula

(11) $$\int_\Omega J^r\gamma^* i_{J^r\Xi}\, d\rho = \int_\Omega (E_\sigma(\mathscr{L})\circ J^{r+1}\gamma)\cdot(\Xi^\sigma \circ\gamma)\cdot\omega_0$$

would give us a contradiction

(12) $$\int_\Omega J^3\gamma^* \partial_{J^r\Xi}\rho \neq 0.$$

Thus, (a) implies (b).

2. (c) is an immediate consequence of (b). Indeed, we can write with Ξ π-vertical

(13) $$\begin{aligned} J^r\gamma^* i_{J^r\Xi} d\rho &= (\pi^{r+1,r}\circ J^{r+1}\gamma)^* i_{J^r\Xi} d\rho = J^{r+1}\gamma^* (\pi^{r+1,r})^* i_{J^r\Xi} d\rho \\ &= J^{r+1}\gamma^* i_{J^{r+1}\Xi} (\pi^{r+1,r})^* d\rho = J^{r+1}\gamma^* i_{J^{r+1}\Xi} p_1 d\rho = J^{r+1}\gamma^* i_{J^{r+1}\Xi} E_{h\rho}. \end{aligned}$$

3. (d) is just a restatement of (b) for the components of the form $E_{h\rho}$.
4. We apply Theorem 7, (d).

Equations (7) are called the *Euler-Lagrange equations*; these equations are indeed related to the chosen fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$.

Remark 10 For a fixed fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, the Euler-Lagrange equations represent a system of partial differential equations of order $r+1$ for unknown functions $(x^i)\to\gamma^\sigma(x^i)$, where $1\le i\le n$ and

$1 \le \sigma \le m$. This fact is due to the origin of the Lagrange function $\mathcal{L}$ that comes from a Lepage form, which is of order *r*. If we start with a given Lagrangian of order *r*, then the Euler-Lagrange equations are of order $2r$. To get an extremal γ on a piece $\Omega \subset X$ we have to solve this system for every fibred chart (V_ι, ψ_ι), $\psi = (x_\iota^i, y_\iota^\sigma)$, from a collection of fibred charts, such that the sets $\pi(V_\iota)$ cover Ω; then the solutions $(x_\iota^i) \to \gamma_\iota^\sigma(x_\iota^i)$ should be used to find a section γ such that $\gamma_\iota^\sigma = y_\iota^\sigma \gamma \varphi_\iota^{-1}$ for all indices ι.

Remark 11 Properties of nonlinear equations (7) depend on the form ρ; their *global* structure is defined by condition (5). This condition says that a section γ is an extremal if and only if its *r*-jet prolongation is an *integral mapping* of an ideal of forms generated by the family of *n*-forms $i_{J^3\Xi} d\rho$. Using 3.3, Theorem 4, one can find explicit expressions for local generators of the ideal.