4 The inverse problem

4.1 Formal divergence equations In this section we study the formal differential equations, closely related to the Euler-Lagrange equations of the calculus of variations on the jet prolongations of fibred manifolds. Essential parts of the proofs of our assertions are based on the trace decomposition theory, explained in Section 2.1.

Let $U \subset \mathbf{R}^n$ be an open set, let $W \subset \mathbf{R}^m$ be an open ball with centre at the origin, and denote $V = U \times W$. We consider V as a fibred manifold over U with the first Cartesian projection $\pi : V \to U$. As before, we denote by V^r the r-jet prolongation of V. V^r is explicitly expressed as

(1) $$V^r = U \times W \times L(\mathbf{R}^n, \mathbf{R}^m) \times L^2_{\text{sym}}(\mathbf{R}^n, \mathbf{R}^m) \times \ldots \times L^r_{\text{sym}}(\mathbf{R}^n, \mathbf{R}^m),$$

where $L^k_{\text{sym}}(\mathbf{R}^n, \mathbf{R}^m)$ is the vector space of k-linear, symmetric mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$. The Cartesian coordinates on V, and the associated jet coordinates on V^r, are denoted by x^i, y^σ, and $x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r}$, respectively.

Let $s \ge 1$ and let $f : V^s \to \mathbf{R}$ be a differentiable function. Our aim in this section is to find integrability conditions and solutions $g = g^i$ of the *formal divergence equation*

(2) $$d_i g^i = f,$$

whose components g^i are differentiable real functions on the set V^s. Since the *formal divergence* $d_i g^i$ is defined by

(3) $$d_i g^i = \frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \ldots + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} y^\sigma_{j_1 j_2 \ldots j_s i},$$

equation (2) is a first order partial differential equation. From this expression we immediately see that every solution $g = g^i$, defined on V^s, satisfies

(4) $$\frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} + \frac{\partial g^{j_1}}{\partial y^\sigma_{i j_2 j_3 \ldots j_s}} + \frac{\partial g^{j_2}}{\partial y^\sigma_{j_1 i j_3 j_4 \ldots j_s}} + \ldots + \frac{\partial g^{j_s}}{\partial y^\sigma_{j_1 j_2 \ldots j_{s-1} i}} = 0.$$

By a *solution of order r* of the formal divergence equation (2) we mean any system of functions $g = g^i$, defined on the set V^r for some r, satisfying condition (2). Clearly, a solution of order r is also a solution of order $r+1$.

Since in this case

(5) $$d_i g^i = \frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y_i^\sigma + \frac{\partial g^i}{\partial y_{j_1}^\sigma} y_{j_1 i}^\sigma + \frac{\partial g^i}{\partial y_{j_1 j_2}^\sigma} y_{j_1 j_2 i}^\sigma + \ldots + \frac{\partial g^i}{\partial y_{j_1 j_2 \ldots j_r}^\sigma} y_{j_1 j_2 \ldots j_r i}^\sigma,$$

we shall have several identities of the form (4).

Lemma 1 *Let* $f : V^s \to \mathbf{R}$ *be a differentiable function. Then if the formal divergence equation* (2) *has a solution, defined on the set* V^r, *where* $r \geq s$, *it also has a solution defined on* V^s. *The solution* $g = g^i$, *defined on* V^s *satisfies condition* (4) *and is polynomial in the variables* $y_{j_1 j_2 \ldots j_s}^\sigma$.

Proof 1. First we show that if a system of functions $g = g^i$ satisfies the condition

(6) $$\frac{\partial g^i}{\partial y_{j_1 j_2 \ldots j_r}^\sigma} + \frac{\partial g^{j_1}}{\partial y_{i j_2 j_3 \ldots j_r}^\sigma} + \frac{\partial g^{j_2}}{\partial y_{j_1 i j_2 j_3 \ldots j_r}^\sigma} + \ldots + \frac{\partial g^{j_r}}{\partial y_{j_1 j_2 \ldots j_{r-1} i}^\sigma} = 0,$$

then each component $g = g^i$ is a polynomial of degree $\leq n-1$ in the variables $y_{j_1 j_2 \ldots j_r}^\sigma$. To this purpose it will be convenient to work with multi-indices of the form $J = (j_1 j_2 \ldots j_r)$; we want to prove that

(7) $$\frac{\partial^n g^i}{\partial y_{J_1}^{\sigma_1} \partial y_{J_2}^{\sigma_2} \ldots \partial y_{J_n}^{\sigma_n}} = 0.$$

It is sufficient to show that all Young diagrams, defining the Young decomposition of the left-hand side of (7), vanish. Since this expression is already symmetric in the indices entering J_1, J_2, and J_n, only the diagrams, which contain any of the blocks J_1, J_2, and J_n in a row can define a nontrivial Young projector. A typical diagram is

(8)

J_1	J_2	$\ldots$	J_{k_2}	$\ldots$	J_{k_1}
J_{k_1+1}	J_{k_1+2}	$\ldots$	$J_{k_1+k_2}$		
$J_{k_1+k_2+1}$	$J_{k_1+k_2+2}$	$\ldots$			
$\ldots$					

(diagram with different position of indices in each row define analogous Young projectors). But the diagrams for the decomposition of (8) should also include the index i. If i stands in a row containing at least one of the blocks J_1, J_2, $\ldots$, J_n, we get necessarily the zero Young projector, by (6). Thus, nonzero projectors can possibly arise only from the diagrams, in which the index i is placed on the bottom:

(9)

J_1	J_2	$\dots$	J_{k_2}	$\dots$	J_{k_1}
J_{k_1+1}	J_{k_1+2}	$\dots$	$J_{k_1+k_2}$		
$J_{k_1+k_2+1}$	$J_{k_1+k_2+2}$	$\dots$			
$\dots$					
i					

However, we can use the skew-symmetry of the Young projector in the columns and interchange the indices in the first and last rows in the first column. We get the zero projector whenever $k_1 \geq 2$. Thus, we conclude that a non-trivial projector could only arise from the diagram

(10)

J_1
J_2
$\dots$
J_n
i

with i in the first column. But this diagram defines the zero projector, because it contains $n+1$ rows. This concludes the proof of identity (7).

2. Consider the formal divergence equation (2) with the right-hand side $f = f(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_s})$, and its solution $g = g^i$ of order $r \geq s+1$. Thus, we have

$$(11)\qquad \frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \dots + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \dots j_r}} y^\sigma_{j_1 j_2 \dots j_r i} = f,$$

and condition (6) is satisfied. Then by the first part of this proof,

$$(12)\qquad g^i = g^i_0 + g^i_1 + g^i_2 + \dots + g^i_{n-1},$$

where g^i_p is a homogeneous polynomial of degree p in the variables $y^\sigma_{j_1 j_2 \dots j_r}$. Substituting from (12) into (11) we get, because f does not depend on $y^\sigma_{j_1 j_2 \dots j_r}$,

$$(13)\qquad \frac{\partial g^i_0}{\partial x^i} + \frac{\partial g^i_0}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i_0}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i_0}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \dots + \frac{\partial g^i_0}{\partial y^\sigma_{j_1 j_2 \dots j_{r-1}}} y^\sigma_{j_1 j_2 \dots j_{r-1} i} = f.$$

Repeating this procedure, we get some functions $h = h^i$, defined on V^s, satisfying

$$\frac{\partial h^i}{\partial x^i} + \frac{\partial h^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial h^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial h^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \ldots + \frac{\partial h^i}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} y^\sigma_{j_1 j_2 \ldots j_s i} = 0, \tag{14}$$

as required.

Denote

$$\begin{aligned} &\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \\ &\omega_i = i_{\partial/\partial x^i}\omega_0 = \frac{1}{(n-1)!}\varepsilon_{i j_2 j_3 \ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \ldots \wedge dx^{j_n}. \end{aligned} \tag{15}$$

Consider a π^s- horizontal $(n-1)$-form η on V^s, expressed with respect to the bases ω_i and $dx^{j_2} \wedge dx^{j_3} \wedge \ldots \wedge dx^{j_n}$ as

$$\eta = g^i \omega_i = \frac{1}{(n-1)!} h_{j_2 j_3 \ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \ldots \wedge dx^{j_n}. \tag{16}$$

Note that from expressions (15), the components of the form η satisfy the transformation formulas

$$h_{j_2 j_3 \ldots j_n} = \varepsilon_{i j_2 j_3 \ldots j_n} g^i, \quad g^k = \frac{1}{(n-1)!}\varepsilon^{k j_2 j_3 \ldots j_n} h_{j_2 j_3 \ldots j_n}. \tag{17}$$

We want to find the transformation equations for the components g^i and $h_{j_1 j_2 \ldots j_{n-1}}$. Denote by Alt and Sym the *alternation* and *symmetrization* in the corresponding indices.

Lemma 2 *The functions* g^i *and* $h_{j_1 j_2 \ldots j_{n-1}}$ *satisfy*

$$\begin{aligned} &\frac{1}{r+1}\varepsilon_{i l_2 l_3 \ldots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1 k_2 \ldots k_s}} + \frac{\partial g^{k_1}}{\partial y^\sigma_{i k_2 k_3 \ldots k_s}} + \frac{\partial g^{k_2}}{\partial y^\sigma_{k_1 i k_3 k_4 \ldots k_s}} + \ldots + \frac{\partial g^{k_s}}{\partial y^\sigma_{k_1 k_2 \ldots k_{s-1} i}}\right) \\ &= \frac{\partial h_{l_2 l_3 \ldots l_n}}{\partial y^\sigma_{k_1 k_2 \ldots k_s}} - \frac{r(n-1)}{r+1}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{\partial y^\sigma_{i k_2 k_3 \ldots k_s}}\delta^{k_1}_{l_2} \quad \mathrm{Sym}(k_1 k_2 \ldots k_s) \quad \mathrm{Alt}(l_2 l_3 \ldots l_n) \end{aligned} \tag{18}$$

and are polynomial in the variables $y^\sigma_{k_1 k_2 \ldots k_s}$.

Proof 1. We have from (17)

(19) $$\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\dots k_s}} = \frac{1}{(n-1)!}\varepsilon^{ij_2j_3\dots j_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{k_1k_2\dots k_s}},$$

hence

(20) $$\begin{aligned}
&\frac{1}{s+1}\varepsilon_{il_2l_3\dots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\dots k_s}} + \frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\dots k_s}} + \frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\dots k_s}} + \dots + \frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\dots k_{s-1}i}}\right)\\
&= \frac{1}{s+1}\frac{1}{(n-1)!}\varepsilon_{il_2l_3\dots l_n}\varepsilon^{ij_2j_3\dots j_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{k_1k_2\dots k_s}}\\
&+ \frac{1}{s+1}\frac{1}{(n-1)!}\varepsilon_{il_2l_3\dots l_n}\varepsilon^{k_1j_2j_3\dots j_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\\
&+ \frac{1}{s+1}\frac{1}{(n-1)!}\varepsilon_{il_2l_3\dots l_n}\varepsilon^{k_2j_2j_3\dots j_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{k_1ik_3k_4\dots k_s}}\\
&+ \dots + \frac{1}{s+1}\frac{1}{(n-1)!}\varepsilon_{il_2l_3\dots l_n}\varepsilon^{k_sj_2j_3\dots j_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{k_1k_2\dots k_{s-1}i}}\\
&= \frac{1}{s+1}\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2\dots k_s}} + \frac{1}{s+1}\frac{n!}{(n-1)!}\delta^{k_1}_i\delta^{j_2}_{l_2}\delta^{j_3}_{l_3}\dots\delta^{j_n}_{l_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\quad \mathrm{Alt}(il_2l_3\dots l_n)\\
&+ \frac{1}{s+1}\frac{n!}{(n-1)!}\delta^{k_2}_i\delta^{j_2}_{l_2}\delta^{j_3}_{l_3}\dots\delta^{j_n}_{l_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{k_1ik_3k_4\dots k_s}}\quad \mathrm{Alt}(il_2l_3\dots l_n)\\
&+ \dots + \frac{1}{s+1}\frac{n!}{(n-1)!}\delta^{k_s}_i\delta^{j_2}_{l_2}\delta^{j_3}_{l_3}\dots\delta^{j_n}_{l_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{k_1k_2\dots k_{s-1}i}}\quad \mathrm{Alt}(il_2l_3\dots l_n).
\end{aligned}$$

To compute the alternations $\mathrm{Alt}(il_2l_3\dots l_n)$, we first alternate in $(l_2l_3\dots l_n)$ and then in $(il_2l_3\dots l_n)$. We get

(21) $$\begin{aligned}
&\delta^{k_1}_i\delta^{j_2}_{l_2}\delta^{j_3}_{l_3}\dots\delta^{j_n}_{l_n}\frac{\partial h_{j_2j_3\dots j_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\quad \mathrm{Alt}(il_2l_3\dots l_n)\\
&= \frac{1}{n}\left(\delta^{k_1}_i\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}} - \delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}} - \delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}} - \dots - \delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\right)\\
&= \frac{1}{n}\left(\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2k_3\dots k_s}} - \delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}} - \delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}} - \dots - \delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\right).
\end{aligned}$$

and similarly for the other terms. Altogether

$$\frac{1}{s+1}\varepsilon_{il_2l_3\dots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\dots k_s}}+\frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\dots k_s}}+\dots+\frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\dots k_{s-1}i}}\right)$$

$$=\frac{1}{s+1}\left(\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2\dots k_s}}+\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2k_3\dots k_s}}-\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}-\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\right.$$

$$-\dots-\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2k_3\dots k_s}}-\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}-\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}$$

$$-\dots-\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2k_3\dots k_s}}-\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_1k_3\dots k_s}}-\delta^{k_2}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_1k_3\dots k_s}}$$

$$-\dots-\delta^{k_2}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_1k_3\dots k_s}}+\dots+\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2k_3\dots k_s}}$$

$$\left.-\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}-\delta^{k_s}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}-\dots-\delta^{k_s}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}\right)$$

(22)
$$=\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2\dots k_s}}-\frac{1}{s+1}\left(\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\dots+\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\right.$$

$$+\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_1k_3\dots k_s}}+\delta^{k_2}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_1k_3\dots k_s}}+\dots+\delta^{k_2}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_1k_3\dots k_s}}$$

$$\left.+\dots+\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}+\delta^{k_s}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}+\dots+\delta^{k_s}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}\right)$$

$$=\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2\dots k_s}}$$

$$-\frac{n-1}{s+1}\frac{1}{n-1}\left(\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\dots+\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\right.$$

$$+\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_1k_3\dots k_s}}+\delta^{k_2}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_1k_3\dots k_s}}+\dots+\delta^{k_2}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_1k_3\dots k_s}}$$

$$\left.+\dots+\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}+\delta^{k_s}_{l_3}\frac{\partial h_{l_2il_4l_5\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}+\dots+\delta^{k_s}_{l_n}\frac{\partial h_{l_2l_3\dots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}\right)$$

and, with the help of alternations and symmetrizations,

$$\frac{1}{s+1}\varepsilon_{il_2l_3\dots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\dots k_s}}+\frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\dots k_s}}+\dots+\frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\dots k_{s-1}i}}\right)$$

$$=\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2\dots k_s}}-\frac{n-1}{s+1}\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}-\frac{n-1}{s+1}\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_1k_3\dots k_s}}$$

$$(23)\qquad -\dots-\frac{n-1}{s+1}\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}\quad \mathrm{Alt}(l_2l_3\dots l_n)$$

$$=\frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2\dots k_s}}-\frac{s(n-1)}{s+1}\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}-\frac{n-1}{s+1}\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_1k_3\dots k_s}}$$

$$-\dots-\frac{n-1}{s+1}\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_{s-1}k_1}}\quad \mathrm{Alt}(l_2l_3\dots l_n)\quad \mathrm{Sym}(k_1k_2\dots k_s).$$

2. Polynomiality of g^i has been verified in the proof of Lemma 1, and polynomiality of $h_{j_1j_2\dots j_{n-1}}$ follows from transformation formulas (17).

We say that a π^s-horizontal form η, defined on V^s, has a $\pi^{s,s-1}$-*projectable extension*, if there exists a form μ on V^{s-1} such that $\eta = h\mu$. Our objective now will be to find conditions for η ensuring that μ does exist. Let η be expressed in two bases of $(n-1)$-forms by (16),

$$(24)\qquad \eta = g^i\omega_i = \frac{1}{(n-1)!}h_{j_2j_3\dots j_n}dx^{j_2}\wedge dx^{j_3}\wedge\dots\wedge dx^{j_n}.$$

Theorem 1 *The following two conditions are equivalent:*
(a) *η has a $\pi^{s,s-1}$-projectable extension.*
(b) *The components g^i satisfy*

$$(25)\qquad \frac{\partial g^i}{\partial y^\sigma_{k_1k_2\dots k_s}}+\frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\dots k_s}}+\frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\dots k_s}}+\dots+\frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\dots k_{s-1}i}}=0.$$

(c) *The components $h_{i_1i_2\dots i_{n-1}}$ satisfy*

$$(26)\qquad \frac{\partial h_{l_2l_3\dots l_n}}{\partial y^\sigma_{k_1k_2\dots k_s}}-\frac{r(n-1)}{r+1}\frac{\partial h_{il_3l_4\dots l_n}}{\partial y^\sigma_{ik_2k_3\dots k_s}}\delta^{k_1}_{l_2}=0\quad \mathrm{Sym}(k_1k_2\dots k_s)\quad \mathrm{Alt}(l_2l_3\dots l_n).$$

Proof 1. To show that (a) implies (b), suppose that we have an $(n-1)$-form μ, defined on V^{s-1}, such that $\eta = h\mu$. Then $hd\eta = d_ig^i\cdot\omega_0$, which is a form on V^{s+1}. But $(\pi^{s,s-1})^*d\mu = d(\pi^{s,s-1})^*\mu$ thus $hd\mu + pd\mu = d(h\mu + p\mu)$, which implies that $hd\eta = hdh\mu = hd\mu$, so $hd\eta$ is $\pi^{s+1,s}$-projectable (with projection $hd\mu$). But

$$(27)\qquad \begin{aligned} hd\eta &= d_i g^i \cdot \omega_0 \\ &= \left(\frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y_i^\sigma + \frac{\partial g^i}{\partial y_{j_1}^\sigma} y_{j_1 i}^\sigma + \frac{\partial g^i}{\partial y_{j_1 j_2}^\sigma} y_{j_1 j_2 i}^\sigma + \ldots + \frac{\partial g^i}{\partial y_{j_1 j_2 \ldots j_s}^\sigma} y_{j_1 j_2 \ldots j_s i}^\sigma \right) \omega_0, \end{aligned}$$

so $\pi^{s+1,s}$-projectability implies (25).

2. (c) follows from (b) by Lemma 2.
3. Now we prove that (c) implies (a). Write η as in (24),

$$(28)\qquad \eta = \frac{1}{(n-1)!} h_{j_2 j_3 \ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \ldots \wedge dx^{j_n}.$$

By Lemma 2, the functions $h_{j_2 j_3 \ldots j_n}$ are polynomial in the variables y_{Jj}^σ, where J is a multi-index of length $s-1$. Thus,

$$(29)\qquad \begin{aligned} h_{i_1 i_2 \ldots i_{n-1}} &= B_{i_1 i_2 \ldots i_{n-1}} + B_{\sigma_1}^{J_1 k_1}{}_{i_1 i_2 \ldots i_{n-1}} y_{J_1 k_1}^{\sigma_1} + B_{\sigma_1}^{J_1 k_1}{}_{\sigma_2}^{J_2 k_2}{}_{i_1 i_2 \ldots i_{n-1}} y_{J_1 k_1}^{\sigma_1} y_{J_2 k_2}^{\sigma_2} \\ &\quad + \ldots + B_{\sigma_1}^{J_1 k_1}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{i_1 i_2 \ldots i_{n-1}} y_{J_1 k_1}^{\sigma_1} y_{J_2 k_2}^{\sigma_2} \ldots y_{J_{n-2} k_{n-2}}^{\sigma_{n-2}} \\ &\quad + B_{\sigma_1}^{J_1 k_1}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{i_1 i_2 \ldots i_{n-1}} y_{J_1 k_1}^{\sigma_1} y_{J_2 k_2}^{\sigma_2} \ldots y_{J_{n-2} k_{n-2}}^{\sigma_{n-2}} y_{J_{n-1} k_{n-1}}^{\sigma_{n-1}}. \end{aligned}$$

The coefficients in this expression are supposed to be symmetric in the multi-indices ${}_{\sigma}^{Jk}$, ${}_{\nu}^{Lj}$. By hypothesis the polynomials (29) satisfy condition (13)

$$(30)\qquad \begin{aligned} &\frac{\partial h_{i_2 i_3 \ldots i_n}}{\partial y_{Jk}^\sigma} - \frac{r(n-1)}{r+1} \frac{\partial h_{l i_3 i_4 \ldots i_n}}{\partial y_{Jl}^\sigma} \delta_{i_2}^k = 0 \\ &\quad \mathrm{Sym}(Jk) \quad \mathrm{Alt}(i_2 i_3 \ldots i_n), \end{aligned}$$

which reduces to some conditions for the coefficients. To find these conditions, we compute

$$(31)\qquad \begin{aligned} \frac{\partial h_{i_1 i_2 \ldots i_{n-1}}}{\partial y_{Jk}^\sigma} &= B_{\sigma}^{Jk}{}_{i_1 i_2 \ldots i_{n-1}} + 2 B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2}{}_{i_1 i_2 \ldots i_{n-1}} y_{J_2 k_2}^{\sigma_2} \\ &\quad + \ldots + (n-2) B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{i_1 i_2 \ldots i_{n-1}} y_{J_2 k_2}^{\sigma_2} y_{J_3 k_3}^{\sigma_3} \ldots y_{J_{n-2} k_{n-2}}^{\sigma_{n-2}} \\ &\quad + (n-1) B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{i_1 i_2 \ldots i_{n-1}} y_{J_2 k_2}^{\sigma_2} y_{J_3 k_3}^{\sigma_3} \ldots y_{J_{n-2} k_{n-2}}^{\sigma_{n-2}} y_{J_{n-1} k_{n-1}}^{\sigma_{n-1}}, \end{aligned}$$

and

$$
\begin{aligned}
(32)\qquad &\frac{\partial h_{li_2i_3\ldots i_{n-1}}}{\partial y^{\sigma}_{Jl}} = B^{Jl}_{\sigma\ \ li_2i_3\ldots i_{n-1}} + 2B^{Jl\ J_2k_2}_{\sigma\ \sigma_2\ \ li_2i_3\ldots i_{n-1}} y^{\sigma_2}_{J_2k_2} \\
&+\ldots+(n-2)B^{Jl\ J_2k_2\ \ldots\ J_{n-2}k_{n-2}}_{\sigma\ \sigma_2\ \ldots\ \sigma_{n-2}\ \ li_2i_3\ldots i_{n-1}} y^{\sigma_2}_{J_2k_2} y^{\sigma_3}_{J_3k_3}\ldots y^{\sigma_{n-2}}_{J_{n-2}k_{n-2}} \\
&+(n-1)B^{Jl\ J_2k_2\ \ldots\ J_{n-2}k_{n-2}\ J_{n-1}k_{n-1}}_{\sigma\ \sigma_2\ \ldots\ \sigma_{n-2}\ \ \sigma_{n-1}\ \ li_2i_3\ldots i_{n-1}} y^{\sigma_2}_{J_2k_2} y^{\sigma_3}_{J_3k_3}\ldots y^{\sigma_{n-2}}_{J_{n-2}k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1}k_{n-1}},
\end{aligned}
$$

from which we have, changing index notation,

$$
\begin{aligned}
(33)\qquad &\frac{\partial h_{li_3i_4\ldots i_n}}{\partial y^{\sigma}_{Jl}}\delta^k_{i_2} = B^{Jl}_{\sigma\ \ li_3i_4\ldots i_n}\delta^k_{i_2} + 2B^{Jl\ J_2k_2}_{\sigma\ \sigma_2\ \ li_3i_4\ldots i_n}\delta^k_{i_2} y^{\sigma_2}_{J_2k_2} \\
&+\ldots+(n-2)B^{Jl\ J_2k_2\ \ldots\ J_{n-2}k_{n-2}}_{\sigma\ \sigma_2\ \ldots\ \sigma_{n-2}\ \ li_3i_4\ldots i_n}\delta^k_{i_2} y^{\sigma_2}_{J_2k_2} y^{\sigma_3}_{J_3k_3}\ldots y^{\sigma_{n-2}}_{J_{n-2}k_{n-2}} \\
&+(n-1)B^{Jl\ J_2k_2\ \ldots\ J_{n-2}k_{n-2}\ J_{n-1}k_{n-1}}_{\sigma\ \sigma_2\ \ldots\ \sigma_{n-2}\ \ \sigma_{n-1}\ \ li_3i_4\ldots i_n}\delta^k_{i_2} y^{\sigma_2}_{J_2k_2} y^{\sigma_3}_{J_3k_3}\ldots y^{\sigma_{n-2}}_{J_{n-2}k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1}k_{n-1}} \\
&\mathrm{Sym}(Jk)\quad \mathrm{Alt}(i_2i_3\ldots i_n).
\end{aligned}
$$

Thus, comparing the coefficients in (33) and (31), condition (30) yields

$$
\begin{aligned}
(34)\qquad &B^{Jk}_{\sigma\ \ i_1i_2\ldots i_{n-1}} = \frac{s(n-1)}{s+1}B^{Jl}_{\sigma\ \ li_2i_3\ldots i_{n-1}}\delta^k_{i_1} \\
&B^{Jk\ J_2k_2}_{\sigma\ \sigma_2\ \ i_1i_2\ldots i_{n-1}} = \frac{s(n-1)}{s+1}B^{Jl\ J_2k_2}_{\sigma\ \sigma_2\ \ li_2i_3\ldots i_{n-1}}\delta^k_{i_1} \\
&\ldots \\
&B^{Jk\ J_2k_2\ J_3k_3\ \ldots\ J_{n-2}k_{n-2}}_{\sigma\ \sigma_2\ \sigma_3\ \ldots\ \sigma_{n-2}\ \ i_1i_2\ldots i_{n-1}} = \frac{s(n-1)}{s+1}B^{Jl\ J_2k_2\ J_3k_3\ \ldots\ J_{n-2}k_{n-2}}_{\sigma\ \sigma_2\ \sigma_3\ \ldots\ \sigma_{n-2}\ \ li_2i_3\ldots i_{n-1}}\delta^k_{i_1}, \\
&B^{Jk\ J_2k_2\ J_3k_3\ \ldots\ J_{n-1}k_{n-1}}_{\sigma\ \sigma_2\ \sigma_3\ \ldots\ \sigma_{n-1}\ \ i_1i_2\ldots i_{n-1}} = \frac{s(n-1)}{s+1}B^{Jl\ J_2k_2\ J_3k_3\ \ldots\ J_{n-1}k_{n-1}}_{\sigma\ \sigma_2\ \sigma_3\ \ldots\ \sigma_{n-1}\ \ li_2i_3\ldots i_{n-1}}\delta^k_{i_1} \\
&\mathrm{Sym}(Jk)\quad \mathrm{Alt}(i_1i_2\ldots i_{n-1}).
\end{aligned}
$$

On the other hand, any $(n-1)$-form μ on V^{s-1} can be expressed as

$$
(35)\qquad \mu = \mu_0 + \omega^{\nu}\wedge\Phi_{\nu} + d\omega^{\nu}\wedge\Psi_{\nu},
$$

where

$$
\begin{aligned}
(36)\qquad &\mu_0 = A_{i_1i_2\ldots i_{n-1}}dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_{n-1}} \\
&+ A^{J_1}_{\sigma_1\ \ i_2i_3\ldots i_{n-1}}dy^{\sigma_1}_{J_1}\wedge dx^{i_2}\wedge dx^{i_3}\wedge\ldots\wedge dx^{i_{n-1}} \\
&+ A^{J_1\ J_2}_{\sigma_1\ \sigma_2\ \ i_3i_3\ldots i_{n-1}}dy^{\sigma_1}_{J_1}\wedge dy^{\sigma_2}_{J_2}\wedge dx^{i_3}\wedge dx^{i_4}\wedge\ldots\wedge dx^{i_{n-1}} \\
&+\ldots+ A^{J_1\ J_2\ \ldots\ J_{n-1}}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_{n-1}}dy^{\sigma_1}_{J_1}\wedge dy^{\sigma_2}_{J_2}\wedge\ldots\wedge dy^{\sigma_{n-1}}_{J_{n-1}},
\end{aligned}
$$

and the coefficients are *traceless* (2.5, Theorem 11). Then $h\mu = h\mu_0$ because h is an exterior algebra homomorphism, annihilating the contact forms ω^ν, and

$$
\begin{aligned}
h\mu = (&A_{i_1 i_2 \dots i_{n-1}} + A_{\sigma_1}^{J_1}{}_{i_2 i_3 \dots i_{n-1}} y^{\sigma_1}_{J_1 i_1} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \dots i_{n-1}} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \\
&+ \dots + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-2}}^{J_{n-2}}{}_{i_{n-1}} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \dots y^{\sigma_{n-2}}_{J_{n-2} i_{n-2}} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-1}}^{j_{n-1}} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \dots y^{\sigma_{n-1}}_{J_{n-1} i_{n-1}}) \\
&\cdot dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}}.
\end{aligned} \tag{37}
$$

Now comparing the coefficients in (37) and (29) we see that the equation $h\mu = \eta$ for $\pi^{s,s-1}$-projectable extensions of the form η is equivalent with the system

$$
\begin{aligned}
&B_{i_1 i_2 \dots i_{n-1}} = A_{i_1 i_2 \dots i_{n-1}}, \\
&B_{\sigma_1}^{J_1 k_1}{}_{i_1 i_2 \dots i_{n-1}} = A_{\sigma_1}^{J_1}{}_{i_2 i_3 \dots i_{n-1}} \delta^{k_1}_{i_1} \quad \mathrm{Sym}(J_1 k_1) \quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1}), \\
&B_{\sigma_1}^{J_1 k_1}{}_{\sigma_2}^{j_2 k_2}{}_{i_1 i_2 \dots i_{n-1}} = A_{\sigma_1}^{J_1}{}_{\sigma_2}^{j_2}{}_{i_3 i_4 \dots i_{n-1}} \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} \quad \mathrm{Sym}(J_1 k_1) \quad \mathrm{Sym}(J_2 k_2) \\
&\quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1}), \\
&\dots \\
&B_{\sigma_1}^{J_1 k_1}{}_{\sigma_2}^{J_2 k_2} \dots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{i_1 i_2 \dots i_{n-1}} = A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-2}}^{J_{n-2}}{}_{i_{n-1}} \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} \dots \delta^{k_{n-2}}_{i_{n-2}} \\
&\quad \mathrm{Sym}(J_1 k_1) \quad \mathrm{Sym}(J_2 k_2) \quad \dots \mathrm{Sym}(J_{n-2} k_{n-2}) \quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1}), \\
&B_{\sigma_1}^{J_1 k_1}{}_{\sigma_2}^{J_2 k_2} \dots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{i_1 i_2 \dots i_{n-1}} = A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-2}}^{J_{n-2}}{}_{\sigma_{n-1}}^{j_{n-1}} \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} \dots \delta^{k_{n-2}}_{i_{n-2}} \delta^{k_{n-1}}_{i_{n-1}} \\
&\quad \mathrm{Sym}(j_1 k_1) \quad \mathrm{Sym}(j_2 k_2) \quad \dots \quad \mathrm{Sym}(j_{n-2} k_{n-2}) \\
&\quad \mathrm{Sym}(j_{n-1} k_{n-1}) \quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1})
\end{aligned} \tag{38}
$$

for unknown functions $A_{i_1 i_2 \dots i_{n-1}}$, $A_{\sigma_1}^{J_1}{}_{i_2 i_3 \dots i_{n-1}}$, $A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \dots i_{n-1}}$, $\dots$, $A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-2}}^{J_{n-2}}{}_{i_{n-1}}$, and $A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-2}}^{J_{n-2}}{}_{\sigma_{n-1}}^{J_{n-1}}$.

We can now solve this system with the help of the trace decomposition theory, namely with the trace decomposition formula of the symmetric-alternating tensors (2.1, Theorem 2). Consider each of equations (38) separately. The second equation is

$$B_{\sigma}^{Jk}{}_{i_1 i_2 \dots i_{n-1}} = A_{\sigma}^{J}{}_{i_2 i_3 \dots i_{n-1}} \delta^{k}_{i_1} \quad \mathrm{Sym}(Jk) \quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1}). \tag{39}$$

Denoting $B = B_{\sigma_1}^{J_1 k_1}{}_{i_1 i_2 \dots i_{n-1}}$ and $A = A_{\sigma}^{J}{}_{i_2 i_3 \dots i_{n-1}} \delta^{k}_{i_1}$, this equation can also be written as $B = \mathbf{q}\tilde{A}$ where $\tilde{A} = \tilde{A}_{\sigma}^{J}{}_{i_2 i_3 \dots i_{n-1}}$ is defined by

$$A_{\sigma}^{J}{}_{i_2 i_3 \dots i_{n-1}} = \frac{s(n-1)}{s+1} \tilde{A}_{\sigma}^{J}{}_{i_2 i_3 \dots i_{n-1}}. \tag{40}$$

But B satisfies the first condition (34), which can also be written as $B = \mathbf{q}\,\mathrm{tr}\,B$. Consequently, the trace decomposition formula yields $\tilde{A} = \mathrm{tr}\,\mathbf{q}\tilde{A} + \mathbf{q}\,\mathrm{tr}\,\tilde{A} = \mathrm{tr}\,B$ because $\tilde{A}$ is traceless; thus, we get a solution

$$(41) \qquad A = \frac{s(n-1)}{s+1}\tilde{A} = \frac{s(n-1)}{s+1}\mathrm{tr}\,B.$$

Next equation (38) is

$$(42) \qquad \begin{aligned} &B^{J_1k_1\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ i_1i_2\dots i_{n-1}} = A^{J_1\ J_2}_{\sigma_1\ \sigma_2\ i_3i_4\dots i_{n-1}}\delta^{k_1}_{i_1}\delta^{k_2}_{i_2} \quad \mathrm{Sym}(J_1k_1) \quad \mathrm{Sym}(J_2k_2) \\ &\mathrm{Alt}(i_1i_2\dots i_{n-1}). \end{aligned}$$

This equation can be understood as a condition for the trace decomposition of the tensor $B = B^{J_1k_1\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ i_1i_2\dots i_{n-1}}$, which according to (34) satisfies

$$(43) \qquad \begin{aligned} &B^{J_1k_1\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ i_1i_2\dots i_{n-1}} = \frac{s(n-1)}{s+1}B^{J_1l\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ li_2i_3\dots i_{n-1}}\delta^{k_1}_{i_1} \quad \mathrm{Sym}(J_1k_1) \\ &\mathrm{Alt}(i_1i_2\dots i_{n-1}). \end{aligned}$$

Analogously

$$(44) \qquad \begin{aligned} &B^{J_1k_1\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ i_1i_2\dots i_{n-1}} = \frac{s(n-1)}{s+1}B^{J_1k_1\ J_2l}_{\sigma_1\ \ \sigma_2\ \ li_2i_3\dots i_{n-1}}\delta^{k_2}_{i_1} \quad \mathrm{Sym}(J_2k_2) \\ &\mathrm{Alt}(i_1i_2\dots i_{n-1}). \end{aligned}$$

These conditions mean that B is a Kronecker tensor whose summands contain exactly one factor of the form δ^{α}_{i}, where α runs through J_1k_1 and i through the set $\{i_1, i_2, \dots, i_{n-1}\}$, and exactly one factor δ^{β}_{i}, where β runs through J_2k_2 and i through $\{i_1, i_2, \dots, i_{n-1}\}$; thus, B must be a linear combinations of the terms of the form $\delta^{j_1}_{i}\delta^{j_2}_{l}$, $\delta^{j_1}_{i}\delta^{k_2}_{l}$, $\delta^{k_1}_{i}\delta^{j_2}_{l}$, $\delta^{k_1}_{i}\delta^{k_2}_{l}$. From the complete trace decomposition theorem it now follows that the coefficients at these Kronecker tensors can be chosen traceless. This shows, however, that equation (42) has a solution $A^{J_1\ J_2}_{\sigma_1\ \sigma_2\ i_3i_4\dots i_{n-1}}$.

To complete the construction of the $\pi^{s,s-1}$-projectable extension μ of the form η, we proceed in the same way.

Now we can study integrability of the formal divergence equations. It is obvious that the formal divergence equation need not have a solution; for instance this is always the case when the right-hand side function f in (3) is not polynomial in the variables $y^{\sigma}_{j_1j_2\dots j_s}$. We introduce the concepts, related to f, which are responsible for the solvability.

We assign to any function $f : V^s \to \mathbf{R}$ the *Lagrangian* $\lambda_f = f\omega_0$ and the Euler-Lagrange form $E_f = E_{\sigma}(f)\omega^{\sigma} \wedge \omega_0$, where the components

$E_\sigma(f)$ are the *Euler-Lagrange expressions* associated with *f*,

(45) $$E_\sigma(f) = \frac{\partial f}{\partial y^\sigma} + \sum_{k=1}^{s} (-1)^k d_{p_1} d_{p_2} \dots d_{p_k} \frac{\partial f}{\partial y^\sigma_{p_1 p_2 \dots p_k}}.$$

Lemma 3 *For any function* $f : V^s \to \mathbf{R}$, *there exists an n-form* Θ_f, *defined on* V^{2s-1}, *such that* (a) $h\Theta_f = \lambda_f$, *and* (b) *the form* $p_1 d\Theta_f$ *is* ω^σ*-generated*.

Proof We take for Θ_f the the principal Lepage equivalent

(46) $$\Theta_f = f\omega_0 + \sum_{k=0}^{s} \left(\sum_{l=0}^{s-k-1} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \dots j_k p_1 p_2 \dots p_l i}} \right) \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i.$$

Now we can study solutions of the formal divergence equation (2).

Theorem 2 *Let* $f : V^s \to \mathbf{R}$ *be a function. The following two conditions are equivalent:*

(a) *The formal divergence equation* $d_i g^i = f$ *has a solution defined on the set* V^s.

(b) *The function f satisfies*

(47) $$E_\sigma(f) = 0.$$

Proof 1. Suppose that the formal divergence equation (2) has a solution $g = g^i$, defined on V^s. Differentiating $d_i g^i$, we get the formulas

(48) $$\frac{\partial d_i g^i}{\partial y^\sigma} = d_i \frac{\partial g^i}{\partial y^\sigma},$$

and for every $k = 1, 2, \dots, s$,

(49) $$\frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \dots i_k}} = d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \dots i_k}} + \frac{1}{k} \left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \dots i_k}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 \dots i_k}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 \dots i_k}} + \dots + \frac{\partial g^{i_k}}{\partial y^\sigma_{i_2 i_3 \dots i_{k-1}}} \right).$$

Using these formulas, we can compute the Euler-Lagrange expressions $E_\sigma(f) = E_\sigma(d_i g^i)$ in several steps. First, we have

$$E_\sigma(d_i g^i)$$

(50) $$= d_{i_1}\left(\frac{\partial g^{i_1}}{\partial y^\sigma} - \frac{\partial d_i g^i}{\partial y^\sigma_{i_1}} + d_{i_2}\frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2}} - \ldots + (-1)^s d_{i_2} d_{i_3} \ldots d_{i_s}\frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}\right)$$

$$= d_{i_1} d_{i_2}\left(-\frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} + \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2}} - d_{i_3}\frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 i_3}} + \ldots + (-1)^s d_{i_3} d_{i_4} \ldots d_{i_s}\frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}\right).$$

Second, using symmetrisation,

$$E_\sigma(d_i g^i) = d_{i_1} d_{i_2}\left(-\frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} + d_i\frac{\partial g^i}{\partial y^\sigma_{i_1 i_2}} + \frac{1}{2}\left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}}\right)\right.$$

(51) $$\left. - d_{i_3}\frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} + \ldots + (-1)^r d_{i_3} d_{i_4} \ldots d_{i_r}\frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 \ldots i_r}}\right)$$

$$= d_{i_1} d_{i_2} d_{i_3}\left(\frac{\partial g^{i_3}}{\partial y^\sigma_{i_1 i_2}} - \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} + \ldots + (-1)^r d_{i_4} d_{i_5} \ldots d_{i_r}\frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}\right).$$

Continuing this process we obtain after $s-1$ steps

(52) $$E_\sigma(d_i g^i) = (-1)^s d_{i_1} d_{i_2} \ldots d_{i_{s-1}} d_{i_s} d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}.$$

But since *f* is defined on V^s, the solution *g* necessarily satisfies

(53) $$\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s+1}}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 i_4 \ldots i_{s+1}}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 i_5 \ldots i_{s+1}}} + \ldots + \frac{\partial g^{i_{s+1}}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s-1} i_r i_1}} = 0.$$

proving that $E_\sigma(d_i g^i) = 0$.

2. Suppose that $E_\sigma(f) = 0$. We want to show that there exist functions $g^i : V^s \to \mathbf{R}$ such that $f = d_i g^i$. Let *I* be the fibred homotopy operator for differential forms on V^{2s}, associated with the projection $\pi : V \to U$ (Section 2.6, Theorem 12). We have

(54) $$\Theta_f = Id\Theta_f + dI\Theta_f + \Theta_0 = Ip_1 d\Theta_f + Ip_2 d\Theta_f + dI\Theta_f + \Theta_0,$$

where Θ_0 is an *n*-form, projectable on *U*. In this formula, $p_1 d\Theta_f = 0$ by hypothesis, and we have $\Theta_0 = d\vartheta_0$. Moreover $h\Theta_f = hd(I\Theta_f + \vartheta_0) = f\omega_0$. Defining functions g^i on V^{2s} by the condition

(55) $h(I\Theta_f + \vartheta_0) = g^i \omega_i$,

we see we have constructed a solution of the formal divergence equation. Explicitly, $hd(I\Theta_f + \vartheta_0) = hdh(I\Theta_f + \vartheta_0) = d_i g^i \cdot \omega_0 = f\omega_0$. Then, however, g^i is defined on V^s (Lemma 1).

Condition $E_\sigma(f) = 0$ (47) is called the *integrability condition* for the formal divergence equation (2).

A remarkable property of the solutions of the formal divergence equation is obtained when we combine Theorem 2 and Theorem 1; we see the solutions can be described as certain differential forms.

Theorem 3 *Let* $f: V^s \to \mathbf{R}$ *be a function, let* g^i *be a system of functions, defined on* V^s, *and let* $\eta = g^i \omega_i$. *Then the following conditions are equivalent:*

(a) *The system* g^i *is a solution of the formal divergence equation*

(56) $d_i g^i = f$.

(b) *There exists a projectable extension* μ *of the form* η *such that*

(57) $hd\mu = f\omega_0$.

Proof 1. If the functions g^i solve the formal divergence equation $d_i g^i = f$, then (3) is satisfied and η has a projectable extension μ by Theorem 1. Then $\eta = h\mu$, hence $(\pi^{s+1,s})^* hd\mu = hdh\mu = hd\eta = d_i g^i \cdot \omega_0 = f\omega_0$, proving (56).

2. If $g^i \omega_i = h\mu$ and $hd\mu = f\omega_0$, then $hd\mu = hdh\mu = d_1 g^i \cdot \omega_0$.

Remark 1 Theorem 3 says that equation (57) for an unknown $(n-1)$-form μ has a solution if and only if the formal divergence equation with right-hand side *f* has a solution.

4.2 Trivial Lagrangians Consider the Euler-Lagrange mapping, introduced in Section 3.5, (16), $\Omega^r_{n,X} W \ni \lambda \to E(\lambda) = E_\lambda \in \Omega^r_{n+1,Y} W$. The domain and the range of this mapping have the structure of Abelian groups (and real vector spaces), and the Euler-Lagrange mapping is a homomorphism of these Abelian groups. We say that a Lagrangian $\lambda \in \Omega^r_{n,X} W$ is *variationally trivial*, or *null*, if its Euler-Lagrange form vanishes, $E_\lambda = 0$. In this section we describe all variationally trivial Lagrangians, or, which is the same, the *kernel* of the Euler-Lagrange mapping $\lambda \to E(\lambda)$. To this purpose we use the formal divergence equations (Section 4.1), where the Euler-Lagrange expressions appear independently in the corresponding integrability conditions; the following result is merely a restatement of the theorems of Section 4.1.

Theorem 4 *Let* $\lambda \in \Omega_n^r W$ *be a Lagrangian. The following conditions are equivalent:*

(a) λ *is variationally trivial.*

(b) *For any fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *there exist functions* $g^i : V^r \to \mathbf{R}$, *such that on* V^r, $\lambda = \mathscr{L}\omega_0$, *where*

(1) $$\mathscr{L} = d_i g^i.$$

(c) *For every fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *there exists an* $(n-1)$*-form* $\mu \in \Omega_{n-1}^r V$ *such that on* V^r

(2) $$\lambda = hd\mu.$$

Proof 1. We show that (a) is equivalent with (b). Suppose that we have a variationally trivial form $\lambda \in \Omega_n^r W$. Write for any fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, $\lambda = \mathscr{L}\omega_0$. Since by hypothesis the Euler-Lagrange expressions $E_\sigma(\mathscr{L})$ vanish, consequently, by Theorem 2, $\mathscr{L} = d_i g^i$ for some functions g^i on V^2. The converse also follows from Theorem 2.

2. Equivalence of (a) and (c) follows from Theorem 3.

In general, Theorem 4 does not ensure global existence of the form μ or its exterior derivative $d\mu$. However, for first order Lagrangians we have a stronger result.

Corollary 1 *A first order Lagrange form* $\lambda \in \Omega_n^1 W$ *is variationally trivial if and only if there exists an n-form* $\eta \in \Omega_{n-1}^0 W$ *such that*

(3) $$\lambda = h\eta$$

and

(4) $$d\eta = 0.$$

Proof By Theorem 4, for any two points $y_1, y_2 \in Y$ there exist two $(n-1)$-forms $\mu_1, \mu_2 \in Y$, defined on a neighbourhood of y_1 and y_2, such that $hd\mu_1 = \lambda$ and $hd\mu_2 = \lambda$, respectively. Then $hd\mu_1 = hd\mu_2$, hence $hd(\mu_1 - \mu_2) = 0$ on the intersection of the neighbourhoods. But the horizontalization h, considered on forms on $J^0Y = Y$, is injective. Consequently, $d(\mu_1 - \mu_2) = 0$, and there exists an n-form $\eta \in \Omega_n^0 W$ whose restrictions agree with $d\mu_1$ and $d\mu_2$. Clearly, $d\eta = 0$.

4.3 Source forms and the Vainberg-Tonti Lagrangians For any positive integer s, a 1-contact form $\varepsilon \in \Omega^s_{n+1,Y}W$ is called a *source form* (Takens [21]). From this definition it follows that in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, ε has an expression

(1) $$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0,$$

where the components ε_σ depend on the jet coordinates $x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_s}$. Clearly, every Euler-Lagrange form E_λ is a source form, thus, the set of source forms contains the Euler-Lagrange forms as a subset.

We can assign to any source form a Lagrangian as follows. Let ε be a source form, defined on W^s, and let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fibred chart on Y, such that $V \subset W$, and the set $\psi(V)$ is star-shaped. Denote by I the fibred homotopy operator on V^s. Then $I\varepsilon$ is a π^s-horizontal form, that is, a *Lagrangian* for Y, defined on V^s. We denote

(2) $$\lambda_\varepsilon = I\varepsilon$$

and call λ_ε the *Vainberg-Tonti Lagrangian*, associated with the source form ε (and the fibred chart (V,ψ); cf. Tonti [22]).

Recall that $I\varepsilon$ is defined by the fibred homotopy $\chi_s : [0,1] \times V^s \to V^s$, where $\chi_s(t,(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_s})) = (x^i, ty^\sigma, ty^\sigma_{j_1}, ty^\sigma_{j_1 j_2}, \dots, ty^\sigma_{j_1 j_2 \dots j_s})$. Since χ_s satisfies $\chi_s^* \varepsilon = (\varepsilon_\sigma \circ \chi_s)(t\omega^\sigma + y^\sigma dt) \wedge \omega_0$, we have, integrating the coefficient in this expression at dt,

(3) $$\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0,$$

where

(4) $$\mathscr{L}_\varepsilon = y^\sigma \int_0^1 \varepsilon_\sigma \circ \chi_s \cdot dt,$$

or, which is the same,

(5) $$\begin{aligned}&\mathscr{L}_\varepsilon(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_s})\\ &\quad = y^\sigma \int_0^1 \varepsilon_\sigma(x^i, ty^\sigma, ty^\sigma_{j_1}, ty^\sigma_{j_1 j_2}, \dots, ty^\sigma_{j_1 j_2 \dots j_s}) dt.\end{aligned}$$

We can find the chart expression for the Euler-Lagrange form E_{λ_ε} of the Vainberg-Tonti Lagrangian λ_ε; recall that

(6) $$E_{\lambda_\varepsilon} = E_\sigma(\mathscr{L}_\varepsilon)\omega^\sigma \wedge \omega_0,$$

where

(7) $$E_\sigma(\mathscr{L}_\varepsilon) = \sum_{l=0}^{s}(-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \mathscr{L}_\varepsilon}{\partial y^\sigma_{p_1 p_2 \cdots p_l}}.$$

We need two formulas for the formal derivative operator d_i , stated in the following lemma. Note that in these formulas a specific summation convention is applied.

Lemma 4 (a) *For every function f on* V^p

(8) $$d_i(f \circ \chi_p) = d_i f \circ \chi_{p+1}.$$

(b) *For every function f on* V^s *and a collection of functions* $g^{p_1 p_2 \cdots p_k}$ *on* V^s *, symmetric in all superscripts,*

(9) $$\begin{aligned} & d_{p_1} d_{p_2} \dots d_{p_k} (f \cdot g^{p_1 p_2 \cdots p_k}) \\ & = \sum_{i=0}^{k} \binom{k}{i} d_{p_1} d_{p_2} \dots d_{p_i} f \cdot d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_k} g^{p_1 p_2 \cdots p_i p_{i+1} p_{i+2} \cdots p_k}. \end{aligned}$$

Proof (a) Formula (8) is an easy consequence of definitions.
(b) The proof is standard. We have

(10) $$\begin{aligned} d_{p_1}(f \cdot g^{p_1}) &= d_{p_1} f \cdot g^{p_1} + f \cdot d_{p_1} g^{p_1} \\ &= \binom{1}{0} d_{p_1} f \cdot g^{p_1} + \binom{1}{1} f \cdot d_{p_1} g^{p_1}, \end{aligned}$$

and

(10) $$\begin{aligned} & d_{p_1} d_{p_2} (f \cdot g^{p_1 p_2}) \\ & = d_{p_2}(d_{p_1} f \cdot g^{p_1 p_2} + f \cdot d_{p_1} g^{p_1 p_2}) \\ & = d_{p_2} d_{p_1} f \cdot g^{p_1 p_2} + d_{p_1} f \cdot d_{p_2} g^{p_1 p_2} + d_{p_2} f \cdot d_{p_1} g^{p_1 p_2} + f \cdot d_{p_1} d_{p_1} g^{p_1 p_2} \\ & = \binom{2}{0} d_{p_2} d_{p_1} f \cdot g^{p_1 p_2} + \binom{2}{1} d_{p_1} f \cdot d_{p_2} g^{p_1 p_2} + \binom{2}{2} f \cdot d_{p_1} d_{p_1} g^{p_1 p_2}. \end{aligned}$$

Then, supposing that

(11) $$\begin{aligned} & d_{p_1} d_{p_2} \dots d_{p_{k-1}} (f \cdot g^{p_1 p_2 \cdots p_{k-1}}) \\ & = \sum_{i=0}^{k-1} \binom{k-1}{i} d_{p_1} d_{p_2} \dots d_{p_i} f \cdot d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_{k-1}} g^{p_1 p_2 \cdots p_i p_{i+1} p_{i+2} \cdots p_{k-1}} \end{aligned}$$

$$
\begin{aligned}
&= \binom{k-1}{0} f \cdot d_{p_1} d_{p_2} \dots d_{p_{k-1}} g^{p_1 p_2 \cdots p_{k-1}} \\
&+ \binom{k-1}{1} d_{p_1} f \cdot d_{p_2} d_{p_3} \dots d_{p_{k-1}} g^{p_1 p_2 \cdots p_{k-1}} \\
&+ \binom{k-1}{2} d_{p_1} d_{p_2} f \cdot d_{p_3} d_{p_4} \dots d_{p_{k-1}} g^{p_1 p_2 \cdots p_{k-1}} \\
&+ \dots + \binom{k-1}{k-2} d_{p_1} d_{p_2} \dots d_{p_{k-2}} f \cdot d_{p_{k-1}} g^{p_1 p_2 \cdots p_{k-1}} \\
&+ \binom{k-1}{k-1} d_{p_1} d_{p_2} \dots d_{p_{k-1}} \cdot g^{p_1 p_2 \cdots p_{k-1}},
\end{aligned} \tag{11}
$$

we have

$$
\begin{aligned}
& d_{p_1} d_{p_2} \dots d_{p_{k-1}} d_{p_k} (f \cdot g^{p_1 p_2 \cdots p_{k-1} p_k}) \\
&= f \cdot d_{p_1} d_{p_2} \dots d_{p_{k-1}} d_{p_k} g^{p_1 p_2 \cdots p_{k-1} p_k} \\
&+ \left(\binom{k-1}{0} + \binom{k-1}{1} \right) d_{p_1} f \cdot d_{p_2} d_{p_3} \dots d_{p_k} g^{p_1 p_2 \cdots p_{k-1} p_k} \\
&+ \left(\binom{k-1}{1} + \binom{k-1}{2} \right) d_{p_1} d_{p_2} f \cdot d_{p_3} d_{p_4} \dots d_{p_k} g^{p_1 p_2 \cdots p_{k-1} p_k} \\
&+ \dots + \left(\binom{k-1}{k-2} + \binom{k-1}{k-1} \right) d_{p_1} d_{p_2} \dots d_{p_{k-1}} \cdot d_{p_k} g^{p_1 p_2 \cdots p_{k-1} p_k} \\
&+ \binom{k-1}{k-1} d_{p_k} d_{p_1} d_{p_2} \dots d_{p_{k-1}} \cdot g^{p_1 p_2 \cdots p_{k-1} p_k}
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
\binom{k-1}{p} + \binom{k-1}{p+1} &= \frac{(k-1)!}{p!(k-1-p)!} + \frac{(k-1)!}{(p+1)!(k-1-p-1)!} \\
&= \frac{(p+1)(k-1)!}{(p+1)!(k-p-1)!} + \frac{(k-p-1)(k-1)!}{(p+1)!(k-p-1)!} \\
&= \frac{k!}{(p+1)!(k-p-1)!} = \binom{k}{p+1},
\end{aligned} \tag{13}
$$

thus,

$$
\begin{aligned}
&d_{p_1}d_{p_2}\dots d_{p_{k-1}}d_{p_k}(f\cdot g^{p_1p_2\dots p_{k-1}p_k})\\
&\quad=\binom{k}{0}f\cdot d_{p_1}d_{p_2}\dots d_{p_{k-1}}d_{p_k}g^{p_1p_2\dots p_{k-1}p_k}\\
&\quad+\binom{k}{1}d_{p_1}f\cdot d_{p_2}d_{p_3}\dots d_{p_k}g^{p_1p_2\dots p_{k-1}p_k}\\
&\quad+\binom{k}{2}d_{p_1}d_{p_2}f\cdot d_{p_3}d_{p_4}\dots d_{p_k}g^{p_1p_2\dots p_{k-1}p_k}\\
&\quad+\dots+\binom{k}{k-1}d_{p_1}d_{p_2}\dots d_{p_{k-1}}\cdot d_{p_k}g^{p_1p_2\dots p_{k-1}p_k}\\
&\quad+\binom{k}{k}d_{p_k}d_{p_1}d_{p_2}\dots d_{p_{k-1}}\cdot g^{p_1p_2\dots p_{k-1}p_k}.
\end{aligned}
\tag{14}
$$

The Vainberg-Tonti Lagrangian allows us to assign to *any* source form, not only to an Euler-Lagrange form, a variational functional and the corresponding Euler-Lagrange equations for its extremals. Our aim now will be to find, with the help of Lemma 4, the corresponding Euler-Lagrange form of the Vainberg-Tonti lagrangian and compare it with the inicial source form. The following theorem describes the relationship of these two forms.

Theorem 5 *The Euler-Lagrange expressions of the Vainberg-Tonti Lagrangian* λ_ε *of a source form* $\varepsilon=\varepsilon_\sigma\omega^\sigma\wedge\omega_0$ *are*

$$
E_\sigma(\mathscr{L}_\varepsilon)=\varepsilon_\sigma-\sum_{k=0}^{s}y^\nu_{q_1q_2\dots q_k}\int H_{\sigma\ \nu}^{\ q_1q_2\dots q_k}(\varepsilon)\circ\chi_{2s}\cdot t\,dt,
\tag{15}
$$

where

$$
\begin{aligned}
H_{\sigma\ \nu}^{\ q_1q_2\dots q_k}(\varepsilon)&=\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\dots q_k}}-(-1)^k\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_k}}\\
&-\sum_{l=k+1}^{s}(-1)^l\binom{l}{k}d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}.
\end{aligned}
\tag{16}
$$

Proof We find a formula for the difference $\varepsilon_\sigma-E_\sigma(\mathscr{L}_\varepsilon)$. Consider the Euler-Lagrange form E_{λ_ε} (6) of the Vainberg-Tonti Lagrangian. Computing the derivatives we have

$$
\frac{\partial\mathscr{L}_\varepsilon}{\partial y^\sigma}=\int\varepsilon_\sigma\circ\chi_s\cdot dt+y^\nu\int\frac{\partial\varepsilon_\nu}{\partial y^\sigma}\circ\chi_s\cdot t\,dt,
\tag{17}
$$

and, by Lemma 4, (8) and (9), for every l, $1\le l\le s$,

$$d_{p_l}\dots d_{p_2}d_{p_1}\frac{\partial\mathscr{L}_\varepsilon}{\partial y^\sigma_{p_1p_2\dots p_l}}=d_{p_l}\dots d_{p_2}d_{p_1}\left(y^\nu\int\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2\dots p_l}}\circ\chi_s\cdot t\,dt\right)$$

(18)

$$=\sum_{i=0}^{l}\binom{l}{i}y^\nu_{p_1p_2\dots p_i}\cdot\int d_{p_{i+1}}d_{p_{i+2}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2\dots p_ip_{i+1}p_{i+2}\dots p_l}}\circ\chi_{s+l-i}\cdot t\,dt.$$

Then by (17) and (18),

$$E_\sigma(\mathscr{L}_\varepsilon)=\int\varepsilon_\sigma\circ\chi_s\cdot dt+y^\nu\int\frac{\partial\varepsilon_\nu}{\partial y^\sigma}\circ\chi_s\cdot t\,dt+\sum_{l=1}^{s}(-1)^l\sum_{i=0}^{l}\binom{l}{i}y^\nu_{p_1p_2\dots p_i}$$

(19)

$$\cdot\int d_{p_{i+1}}d_{p_{i+2}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2\dots p_ip_{i+1}p_{i+2}\dots p_l}}\circ\chi_{s+l-i}\cdot t\,dt.$$

On the other hand,

$$\varepsilon_\sigma=\int\frac{d}{dt}(\varepsilon_\sigma\circ\chi_s\cdot t)dt=\int\frac{d(\varepsilon_\sigma\circ\chi_s)}{dt}\cdot t\,dt+\int\varepsilon_\sigma\circ\chi_s\cdot dt$$

(20)

$$=\sum_{i=0}^{s}\int\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{p_1p_2\dots p_i}}\circ\chi_s\cdot y^\nu_{p_1p_2\dots p_i}\cdot t\,dt+\int\varepsilon_\sigma\circ\chi_s\cdot dt,$$

hence

$$\varepsilon_\sigma-E_\sigma(\mathscr{L}_\varepsilon)=\sum_{i=0}^{s}\int\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{p_1p_2\dots p_i}}\circ\chi_s\cdot y^\nu_{p_1p_2\dots p_i}\cdot t\,dt-y^\nu\int\frac{\partial\varepsilon_\nu}{\partial y^\sigma}\circ\chi_s\cdot t\,dt$$

$$-\sum_{l=1}^{s}(-1)^l\sum_{i=0}^{l}\binom{l}{i}y^\nu_{p_1p_2\dots p_i}\int d_{p_{i+1}}d_{p_{i+2}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2\dots p_ip_{i+1}p_{i+2}\dots p_l}}\circ\chi_{s+l-i}\cdot t\,dt$$

$$=\int\frac{\partial\varepsilon_\sigma}{\partial y^\nu}\circ\chi_s\cdot y^\nu\cdot t\,dt-y^\nu\int\frac{\partial\varepsilon_\nu}{\partial y^\sigma}\circ\chi_s\cdot t\,dt$$

(21)

$$-\sum_{l=1}^{s}(-1)^l\binom{l}{0}y^\nu\cdot\int d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2\dots p_l}}\circ\chi_{s+l}\cdot t\,dt$$

$$+\sum_{i=1}^{s}\int\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{p_1p_2\dots p_i}}\circ\chi_s\cdot y^\nu_{p_1p_2\dots p_i}\cdot t\,dt$$

$$-\sum_{l=1}^{s}(-1)^l\sum_{i=1}^{l}\binom{l}{i}y^\nu_{p_1p_2\dots p_i}\int d_{p_{i+1}}d_{p_{i+2}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2\dots p_ip_{i+1}p_{i+2}\dots p_l}}\circ\chi_{s+l-i}\cdot t\,dt.$$

We replace the summation through the pairs (l,i) in the double sum with the summation through (i,l), expressed by the scheme

$$
\begin{array}{ll}
 & (1,1) \\
 & (2,1),(2,2) \\
(22) & (3,1),(3,2),(3,3) \\
 & \dots \\
 & (s,1),(s,2),(s,3),\dots,(s-1,s),(s,s)
\end{array}
$$

Then it is easily seen that the same summation, but represented by the pairs (i,l), is expressed by the scheme

$$
\begin{array}{ll}
 & (1,1),(1,2),(1,3),\dots,(1,s-1),(1,s) \\
 & (2,2),(2,3),\dots,(2,s-1),(2,s) \\
(23) & \dots \\
 & (s-1,s-1),(s-1,s) \\
 & (s,s)
\end{array}
$$

Then the double sum in (21) becomes

$$
\begin{aligned}
&\sum_{l=1}^{s}(-1)^l \sum_{i=1}^{l}\binom{l}{i} y^{\nu}_{p_1 p_2\dots p_i} \int d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^{\sigma}_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi_{s+l-i} \cdot t\, dt \\
(24) \qquad &= \sum_{i=1}^{s}(-1)^i y^{\nu}_{p_1 p_2 \dots p_i} \int \frac{\partial \varepsilon_\nu}{\partial y^{\sigma}_{p_1 p_2 \dots p_l}} \circ \chi_s \cdot t\, dt \\
&+ \sum_{i=1}^{s}(-1)^l \sum_{l=i+1}^{s}\binom{l}{i} y^{\nu}_{p_1 p_2\dots p_i} \int d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^{\sigma}_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi_{s+l-i} \cdot t\, dt.
\end{aligned}
$$

Returning to (21) we get

$$
\begin{aligned}
&\varepsilon_\sigma - E_\sigma(\mathscr{L}_\varepsilon) \\
&= y^\nu \left(\int \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} - \sum_{l=1}^{s}(-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^{\sigma}_{p_1 p_2 \dots p_l}} \right) \circ \chi_{2s} \cdot tdt \\
(25) \qquad &+ y^{\nu}_{p_1 p_2 \dots p_i} \sum_{i=1}^{s} \left(\int \frac{\partial \varepsilon_\sigma}{\partial y^{\nu}_{p_1 p_2 \dots p_i}} - (-1)^i \frac{\partial \varepsilon_\nu}{\partial y^{\sigma}_{p_1 p_2 \dots p_l}} \right. \\
&\left. - \sum_{l=i+1}^{s}(-1)^l \binom{l}{i} \cdot d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^{\sigma}_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \right) \circ \chi_{2s-i} \cdot tdt.
\end{aligned}
$$

This formula proves Theorem 5.

We call the functions (16) the *Helmholtz expressions*, associated with the source form ε .

The following illustrative example describes the structure of the Helmholtz expressions for source forms of order $2r$.

Remark 2 If $s=2r$, we get

$$H_{\sigma}{}^{q_1q_2\ldots q_{2r}}_{\nu}(\varepsilon)=\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\ldots q_{2r}}}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\ldots q_{2r}}},$$

$$H_{\sigma}{}^{q_1q_2\ldots q_{2r-1}}_{\nu}(\varepsilon)=\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\ldots q_{2r-1}}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\ldots q_{2r-1}}}-\binom{2r}{2r-1}d_{p_{2r}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\ldots q_{2r-1}p_{2r}}},$$

$$H_{\sigma}{}^{q_1q_2\ldots q_{2r-2}}_{\nu}(\varepsilon)=\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\ldots q_{2r-2}}}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\ldots q_{2r-2}}}$$
$$+\binom{2r-1}{2r-2}d_{p_{2r-1}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\ldots q_{2r-2}p_{2r-1}}}-\binom{2r}{2r-2}d_{p_{2r-1}}d_{p_{2r}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\ldots q_{2r-2}p_{2r-1}p_{2r}}},$$

$\ldots$

$$H_{\sigma}{}^{q_1}_{\nu}(\varepsilon)=\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1}}-\binom{2}{1}d_{p_2}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1p_2}}$$
$$+\binom{3}{1}d_{p_2}d_{p_3}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1p_2p_3}}-\ldots+\binom{2r-1}{1}d_{p_2}d_{p_3}\ldots d_{p_{2r-1}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1p_2p_3\ldots p_{2r-1}}}$$
$$-\binom{2r}{1}d_{p_2}d_{p_3}\ldots d_{p_{2r}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1p_2p_3\ldots p_{2r}}},\tag{26}$$

$$H_{\sigma\nu}(\varepsilon)=\frac{\partial\varepsilon_\sigma}{\partial y^\nu}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma}+\binom{1}{0}d_{p_1}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1}}-\binom{2}{0}d_{p_1}d_{p_2}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2}}$$
$$+\ldots+\binom{s-1}{0}d_{p_1}d_{p_2}\ldots d_{p_{2r-1}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2\ldots p_{2r-1}}}$$
$$-\binom{2r}{0}d_{p_1}d_{p_2}\ldots d_{p_{2r}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{p_1p_2\ldots p_{2r}}}.$$

Remark 3 Theorem 5 describes the difference between the given source form ε and the Euler-Lagrange form of the Vainberg-Tonti Lagrangian; it states, in particular, that responsibility for the difference lies on the properties of the the Helmholtz expressions.

Now we specify this difference for *variational* source forms.

Lemma 5 *Let* $\lambda = \mathscr{L}\omega_0$ *be a Lagrangian, and let* Θ_λ *be its principal Lepage equivalent. Then the Vainberg-Tonti Lagrangian of the Euler-Lagrange form* $E_\lambda = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$ *,*

(27) $\lambda' = hId\Theta_\lambda,$

satisfies

(28) $\lambda' = \lambda - hd(I\Theta_\lambda + \mu_0),$

where μ_0 *is an* $(n-1)$ *-form on U.*

Proof Using the fibred homotopy operator, we can decompose the principal Lepage equivalent Θ_λ as $\Theta_\lambda = Id\Theta_\lambda + dI\Theta_\lambda + \Theta_0$, with Θ_0 defined on *X*. Then the horizontal component is

(29) $h\Theta_\lambda = hId\Theta_\lambda + hdI\Theta_\lambda + h\Theta_0 = hId\Theta_\lambda + hd(I\Theta_\lambda + \mu_0),$

where the Vainberg-Tonti Lagrangian is $hId\Theta_\lambda$.

Note that, in particular, formula (28) shows that the Vainberg-Tonti Lagrangian differs from the initial one by the term $hd(I\Theta_\lambda + \mu_0)$ that belongs to the kernel of the Euler-Lagrange mapping. This fact demonstrates that the Euler-Lagrange forms of these two Lagrangians coincide.

4.4 The inverse problem Our objective in this section is to study the *image* of the Euler-Lagrange mapping $\Omega^r_{n,X}W \ni \lambda \to E(\lambda) = E_\lambda \in \Omega^{2r}_{n+1,Y}W$ (Section 3.5), considered as a subset of the set of source forms (Section 4.3). We are interested in basic properties of this set, in particular, in a criterion when a source form belongs to the subset of the Euler-Lagrange forms.

The following theorem, completing Theorem 6, Section 3.5, shows that the image of the Euler-Lagrange mapping is closed under the Lie derivative with respect to projectable vector fields.

Theorem 6 *For any Lagrangian* $\lambda \in \Omega^r_{n,X}W$ *and any* π *-projectable vector field* Ξ *on* W *the Lie derivative* $\partial_{J^r\Xi}\lambda$ *belongs to the module* $\Omega^r_{n,X}W$ *and*

(1) $\partial_{J^r\Xi}E_\lambda = E_{\partial_{J^r\Xi}\lambda}.$

Proof Since $\lambda \in \Omega^r_{n,X}W$, then $\partial_{J^r\Xi}\lambda \in \Omega^r_{n,X}W$. If ρ_λ is a Lepage equivalent of λ , and $\rho_{\partial_{J^r\Xi}\lambda}$ is a Lepage equivalent of $\partial_{J^r\Xi}\lambda$, then, with the notation of 3.3, Theorem 3, $\rho_\lambda = \Theta_\lambda + d\eta + \mu$, and

(2) $$\rho_{\partial_{J^r\Xi}\lambda} = \Theta_{\partial_{J^r\Xi}\lambda} + d\eta' + \mu', \quad \partial_{J^r\Xi}\rho_\lambda = \partial_{J^r\Xi}\Theta_\lambda + d\partial_{J^r\Xi}\eta + \partial_{J^r\Xi}\mu.$$

The horizontal component is $h\partial_{J^r\Xi}\rho_\lambda = \partial_{J^{r+1}\Xi}h\rho_\lambda = \partial_{J^{r+1}\Xi}\lambda$, and $\partial_{J^r\Xi}\rho_\lambda$ is a Lepage form, because $p_1 d\partial_{J^r\Xi}\rho_\lambda = p_1 d\partial_{J^r\Xi}\Theta_\lambda = p_1\partial_{J^r\Xi}d\Theta_\lambda$ and the Lie derivative $\partial_{J^r\Xi}$ preserves contact forms (2.4, Theorem 10, (4)). Thus, the forms $\rho_{\partial_{J^r\Xi}\lambda}$ and $\partial_{J^r\Xi}\rho_\lambda$ are both Lepage forms, and have the same Lagrangians. Consequently, their Euler-Lagrange forms agree, $\partial_{J^r\Xi}E_\lambda = E_{\partial_{J^r\Xi}\lambda}$.

Consider a source form $\varepsilon \in \Omega^s_{n+1,Y}W$. We say that ε is *variational*, if

(3) $$\varepsilon = E_\lambda$$

for some *r* and some Lagrangian $\lambda \in \Omega^r_{n,X}W$. ε is said to be *locally variational*, if there exists an atlas on *Y*, consisting of fibred charts, such that for each chart (V,ψ), $\psi = (x^i, y^\sigma)$, belonging to this atlas, the restriction of ε to V^s is variational.

The *inverse problem* of the calculus of variations for source forms is the problem to find conditions ensuring the existence a Lagrangian λ, satisfying equation (3); then if these conditions are satisfied, to find all Lagrangian for ε. The *local inverse problem*, or *local variationality problem*, for a source form ε consists in finding existence (integrability) conditions and solutions $\mathscr{L}$ of the system of partial differential equations

(4) $$\varepsilon_\sigma = \frac{\partial\mathscr{L}}{\partial y^\sigma} + \sum_{l=1}^{r}(-1)^l d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial\mathscr{L}}{\partial y^\sigma_{p_1p_2\dots p_l}}$$

with given functions $\varepsilon_\sigma = \varepsilon_\sigma(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_s})$.

Remark 4 (Variationality of partial differential equations) The concept of local variationality can be applied to systems of partial differential equations. Having fixed the functions ε_σ, we sometimes say, without aspiration to rigour, that the *system of partial differential equations*

(5) $$\varepsilon_\sigma(x^i, y^\tau, y^\tau_{j_1}, y^\tau_{j_1j_2}, \dots, y^\tau_{j_1j_2\dots j_s}) = 0$$

is *variational*, it it coincides with the system of Euler-Lagrange equations of some Lagrangian. It is clear, however, that this concept of is not well defined; indeed, setting $\varepsilon'_\nu = \Phi^\nu_\sigma\varepsilon_\nu$ with any functions Φ^ν_σ such that $\det\Phi^\nu_\sigma \neq 0$, we get two equivalent systems $\varepsilon_\sigma = 0$ and $\varepsilon'_\nu = 0$, but it may happen that one of them is variational and the other is not. If (5) is *not* variational and there exists Φ^ν_σ such that the equivalent system

(6) $\Phi^{\nu}_{\sigma}\varepsilon_{\nu} = 0$

is variational, Φ^{ν}_{σ} are said to be the *variational integrating factors* for (5).

Let *r* be a fixed positive integer. In the following theorem we describe the subspace of the vector space of source forms, which is in general larger than the image of the Euler-Lagrange mapping, namely, the subspace of *locally variational forms*.

Theorem 7 *A source form* $\varepsilon \in \Omega^{s}_{n+1,Y}W$ *is locally variational if and only if there exists an integer q and a form* $F \in \Omega^{q}_{n+1}W$ *of order of contactness* ≥ 2 *such that* $d(\varepsilon + F) = 0$.

Proof 1. Suppose that ε is locally variational, and choose a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, such that ε is variational on *V*; then $\varepsilon = E_\lambda$ for some lagrangian $\lambda \in \Omega^{r}_{n,X}V$. Let Θ_λ denote the principal Lepage equivalent of λ, and set $F = p_2 d\Theta_\lambda$. Then $d(\varepsilon + F) = dd\Theta_\lambda = 0$.

2. Conversely, if for some fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, condition $d(\varepsilon + F) = 0$ holds on V^s, then $\varepsilon + F = d\rho$ for some ρ. ρ is obviously a Lepage form, thus, $\varepsilon = p_1 d\rho$, so ε is a locally variational form whose Lagrangian is $h\rho$.

The following lemma is needed in the proof of the theorem on the local inverse variational problem.

Lemma 6 *Let U be an open set in* $\mathbf{R}^n$ *such that for each point* $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)$ *the segment* $\{(tx_0^1, tx_0^2, \ldots, tx_0^n) \mid t \in [0,1]\}$ *belongs to U. Let* $f : U \to \mathbf{R}$ *be a function such that*

(7) $$\int_0^1 F(tx_0^1, tx_0^2, \ldots, tx_0^n)\,dt = 0$$

for all points $(x_0^1, x_0^2, \ldots, x_0^n) \in U$. *Then* $F = 0$.

Proof If (7) is true, then for any $s \in [0,1]$, $(sx_0^1, sx_0^2, \ldots, sx_0^n) \in U$, thus,

(8) $$\int_0^1 F(tsx_0^1, tsx_0^2, \ldots, tsx_0^n)\,dt = 0.$$

Differentiating with respect to *s* at $s = 1$

(10) $$\int_0^1 \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t\,dt = 0.$$

On the other hand,

$$\text{(11)} \qquad \frac{d}{dt}(tF(tx_0^1, tx_0^2, \ldots, tx_0^n)) = F(tx_0^1, tx_0^2, \ldots, tx_0^n) + \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t.$$

Integrating we have

$$\text{(12)} \qquad F(x_0^1, x_0^2, \ldots, x_0^n) = \int_0^1 F(tx_0^1, tx_0^2, \ldots, tx_0^n)\,dt + \int_0^1 \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t\,dt = 0.$$

Now we can study the local inverse problem of the calculus of variations. We wish to find integrability conditions for the system (4) and describe all solutions $\mathscr{L}$ of the system of partial differential equations (4) in an explicit form. To characterize locally variational forms, we need the *Helmholtz expressions* $H_{\sigma}{}^{q_1 q_2 \ldots q_k}_{\nu}(\varepsilon)$ (Section 4.3, (16)), where $k = 0,1,2,\ldots,s$, and s is the order or ε .

Theorem 8 *Let V be an open star-shaped set in $\mathbf{R}^m$, and let $\varepsilon_\sigma : V^s \to \mathbf{R}$ be differentiable functions. The following two conditions are equivalent:*

(a) *Equation*

$$\text{(15)} \qquad \varepsilon_\sigma = \frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{l=1}^{s} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}}$$

has a solution $\mathscr{L} : V^s \to \mathbf{R}$.

(b) For all $k = 0,1,2,\ldots,s$ *, the function* ε_σ *satisfy,*

$$\text{(16)} \qquad H_{\sigma}{}^{q_1 q_2 \ldots q_k}_{\nu}(\varepsilon) = 0$$

Proof 1. Suppose that the system (15) has a solution $\mathscr{L}$. Then ε is the Euler-Lagrange form of the Lagrangian $\lambda = \mathscr{L}\omega_0$, and $\varepsilon = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$. Since the Lagrangian λ and the Vainberg-Tonti Lagrangian have the same Euler-Lagrange form, the Helmholtz expressions satisfy

$$\text{(17)} \qquad \begin{aligned} &\sum_{k=0}^{s} y^\nu_{q_1 q_2 \ldots q_k} \int H_{\sigma}{}^{q_1 q_2 \ldots q_k}_{\nu}(\varepsilon) \circ \chi_{4r} \cdot t\,dt \\ &= \int_0^1 \sum_{k=0}^{s} (y^\nu_{q_1 q_2 \ldots q_k} H_{\sigma}{}^{q_1 q_2 \ldots q_k}_{\nu}(\varepsilon)) \circ \chi_{4r} \cdot dt = 0 \end{aligned}$$

(4.3, Theorem 5, (15)). Applying Lemma 6, we get

(18) $$\sum_{k=0}^{s} y^{\nu}_{q_1 q_2 \dots q_k} H_{\sigma\ \nu}^{\ q_1 q_2 \dots q_k}(\varepsilon) = 0.$$

Now suppose that the functions ε_σ do not satisfy conditions (16). Then there exists a point in V^s and some indices l and $\kappa, \lambda, p_1, p_2, \dots, p_l$ such that $H_{\kappa\ \lambda}^{\ p_1 p_2 \dots p_l}(\varepsilon) \neq 0$ at this point, and by continuity, $H_{\kappa\ \lambda}^{\ p_1 p_2 \dots p_l}(\varepsilon) \neq 0$ on a neighbourhood of this point. In particular, $H_{\kappa\ \lambda}^{\ p_1 p_2 \dots p_l}(\varepsilon) \neq 0$ on the intersection of this open set with the set defined by equations $y^{\nu}_{q_1 q_2 \dots q_k} = 0$, whenever the multi-index ${}^{\nu}_{q_1 q_2 \dots q_k}$ differs from ${}^{\lambda}_{p_1 p_2 \dots p_l}$. Then, however, the sum (18) is equal to $y^{\lambda}_{p_1 p_2 \dots p_l} H_{\kappa\ \lambda}^{\ p_1 p_2 \dots p_l}(\varepsilon)$ and is different from 0. This contradiction proves conditions (16).

2. Suppose that the system of functions ε_σ satisfies conditions (16) and denote by $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ the corresponding source form. Then the Euler-Lagrange expressions of the Vainberg-Tonti Lagrangian $\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0$,

(19) $$E_\sigma(\mathscr{L}_\varepsilon) = \varepsilon_\sigma - \sum_{k=0}^{s} y^{\nu}_{q_1 q_2 \dots q_k} \int H_{\sigma\ \nu}^{\ q_1 q_2 \dots q_k}(\varepsilon) \circ \chi_{2s} \cdot t\, dt,$$

reduce to $E_\sigma(\mathscr{L}_\varepsilon) = \varepsilon_\sigma$. In particular, ε has a Lagrangian of order s.

Remark 5 Integrability conditions (16) ensure existence of a Lagrangian of order s for a source form of order s. Existence of Lagrangians of order $r < s$ require additional properties of the source forms.

Remark 6 Condition (16) for $k = 0$ can also be easily proved by means of Theorem 6. If ε is a variational form, then for every π-vertical vector field Ξ

(20) $$\partial_{J^{2r}\Xi} \varepsilon = \partial_{J^{2r}\Xi} E_\lambda = E_{\partial_{J^r\Xi} \lambda},$$

therefore, the Lie derivative $\partial_{J^{2r}\Xi} \varepsilon$ is a variational form. Thus, $\partial_{J^{2r}\Xi} \varepsilon$ must satisfy condition (18), now written as

(21) $$\sum_{k=0}^{s} y^{\nu}_{q_1 q_2 \dots q_k} H_{\sigma\ \nu}^{\ q_1 q_2 \dots q_k}(\partial_{J^s\Xi} \varepsilon) = 0,$$

therefore,

(22) $$H_{\sigma\ \nu}^{\ q_1 q_2 \dots q_k}(\partial_{J^s\Xi} \varepsilon) = 0.$$

The functions $\varepsilon_\sigma(\Xi)$ in this formula are determined by the Lie derivative $\partial_{J^{2r}\Xi} \varepsilon$. Since

$$
\begin{aligned}
\partial_{J^{2r}\Xi}\varepsilon &= \varepsilon_\sigma(\Xi)\omega^\sigma \wedge \omega_0 \\
&= i_{J^{2r}\Xi}d\varepsilon + di_{J^{2r}\Xi}\varepsilon = i_{J^{2r}\Xi}(d\varepsilon_\sigma \wedge \omega^\sigma \wedge \omega_0) + d(\varepsilon_\sigma \Xi^\sigma \cdot \omega_0) \\
&= i_{J^{2r}\Xi}d\varepsilon_\sigma \cdot \omega^\sigma \wedge \omega_0 - \Xi^\sigma d\varepsilon_\sigma \wedge \omega_0 + d(\varepsilon_\sigma \Xi^\sigma) \wedge \omega_0 \\
&= \left(i_{J^{2r}\Xi}d\varepsilon_\sigma + \varepsilon_\nu \frac{\partial \Xi^\nu}{\partial y^\sigma} \right) \omega^\sigma \wedge \omega_0,
\end{aligned} \tag{22}
$$

we have

$$
\varepsilon_\sigma(\Xi) = i_{J^{2r}\Xi}d\varepsilon_\sigma + \varepsilon_\nu \frac{\partial \Xi^\nu}{\partial y^\sigma}. \tag{23}
$$

2. Fix an index τ and consider the vector field $\Xi = \partial / \partial y^\tau$ and its r-jet prolongation

$$
J^{2r}\Xi = \frac{\partial}{\partial y^\tau}. \tag{24}
$$

For this vector field expressions (23) reduce to

$$
\varepsilon_\sigma(\Xi) = \frac{\partial \varepsilon_\sigma}{\partial y^\tau}, \tag{25}
$$

Condition (20) implies

$$
\begin{aligned}
&\sum_{k=0}^{2r} y^\nu_{q_1 q_2 \dots q_k} \left(\frac{\partial \varepsilon_\sigma(\Xi)}{\partial y^\nu_{q_1 q_2 \dots q_k}} - (-1)^k \frac{\partial \varepsilon_\nu(\Xi)}{\partial y^\sigma_{q_1 q_2 \dots q_k}} \right. \\
&\quad \left. - \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu(\Xi)}{\partial y^\sigma_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \right) \\
&= \sum_{k=0}^{2r} y^\nu_{q_1 q_2 \dots q_k} \left(\frac{\partial^2 \varepsilon_\sigma}{\partial y^\tau \partial y^\nu_{q_1 q_2 \dots q_k}} - (-1)^k \frac{\partial^2 \varepsilon_\nu}{\partial y^\tau \partial y^\sigma_{q_1 q_2 \dots q_k}} \right. \\
&\quad \left. - \sum_{l=k+1}^{s} (-1)^l \binom{l}{k} \frac{\partial}{\partial y^\tau} d_{p_{k+1}} d_{p_{k+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \right) = 0
\end{aligned} \tag{26}
$$

because the partial derivative $\partial / \partial y^\tau$ commutes with the formal derivative d_p. Then

$$\sum_{k=0}^{2r} y^{\nu}_{q_1 q_2 \dots q_k} \frac{\partial}{\partial y^{\tau}} \Bigg(\frac{\partial \varepsilon_{\sigma}}{\partial y^{\nu}_{q_1 q_2 \dots q_k}} - (-1)^k \frac{\partial \varepsilon_{\nu}}{\partial y^{\sigma}_{q_1 q_2 \dots q_k}}$$

$$- \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \dots d_{p_l} \frac{\partial \varepsilon_{\nu}}{\partial y^{\sigma}_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \Bigg)$$

(27)
$$= \frac{\partial}{\partial y^{\tau}} \Bigg(\sum_{k=0}^{2r} y^{\nu}_{q_1 q_2 \dots q_k} \Bigg(\frac{\partial \varepsilon_{\sigma}}{\partial y^{\nu}_{q_1 q_2 \dots q_k}} - (-1)^k \frac{\partial \varepsilon_{\nu}}{\partial y^{\sigma}_{q_1 q_2 \dots q_k}}$$

$$- \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \dots d_{p_l} \frac{\partial \varepsilon_{\nu}}{\partial y^{\sigma}_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \Bigg) \Bigg)$$

$$- \frac{\partial \varepsilon_{\sigma}}{\partial y^{\tau}} + \frac{\partial \varepsilon_{\tau}}{\partial y^{\sigma}} + \sum_{l=1}^{2r} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \varepsilon_{\tau}}{\partial y^{\sigma}_{p_1 p_2 \dots p_l}} = 0.$$

But the first sum vanishes by (18), and the second sum gives

(28)
$$- \frac{\partial \varepsilon_{\sigma}}{\partial y^{\tau}} + \frac{\partial \varepsilon_{\tau}}{\partial y^{\sigma}} + \sum_{l=1}^{2r} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \varepsilon_{\tau}}{\partial y^{\sigma}_{p_1 p_2 \dots p_l}} = -H_{\sigma\,\nu}(\varepsilon) = 0,$$

which is the first one of equations (16).

Remark 7 One can also prove equation $H_{\sigma\,\tau}(\varepsilon) = 0$ by applying the integrability criterion for formal divergence equations. Consider the inverse problem equation

(29)
$$\varepsilon_{\sigma} = \frac{\partial \mathscr{L}}{\partial y^{\sigma}} - d_{p_1} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1}} + d_{p_1} d_{p_2} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1 p_2}}$$

$$- \dots + (-1)^{r-1} d_{p_1} d_{p_2} \dots d_{p_{r-1}} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1 p_2 \dots p_{r-1}}} + (-1)^r d_{p_1} d_{p_2} \dots d_{p_r} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1 p_2 \dots p_r}}$$

and suppose it has a solution $\mathscr{L}$. Denoting

(30)
$$\Phi^{p_1}_{\sigma} = \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1}} + d_{p_2} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1 p_2}} - \dots + (-1)^{r-1} d_{p_2} d_{p_3} \dots d_{p_{r-1}} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1 p_2 \dots p_{r-1}}}$$

$$+ (-1)^r d_{p_2} d_{p_3} \dots d_{p_r} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1 p_2 \dots p_r}},$$

we get the formal divergence equation

(31) $$\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma} = -d_{p_1}\Phi_\sigma^{p_1}.$$

Since by hypothesis there exists a solution, the integrability condition for this equation is satisfied, that is, by Section 4.1, Theorem 2,

(32) $$E_\tau\left(\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma}\right) = 0.$$

Explicitly, since the formal derivative d_i and the partial derivative $\partial/\partial y^\tau$ commute,

(33) $$\begin{aligned} E_\tau\left(\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma}\right) &= E_\tau(\varepsilon_\sigma) - \frac{\partial E_\tau(\mathscr{L})}{\partial y^\sigma} \\ &= \frac{\partial \varepsilon_\sigma}{\partial y^\tau} - d_{p_1}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1}} + d_{p_1}d_{p_2}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2}} \\ &\quad - \ldots + (-1)^{r-1} d_{p_1}d_{p_2}\ldots d_{p_{r-1}}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2\ldots p_{r-1}}} \\ &\quad + (-1)^r d_{p_1}d_{p_2}\ldots d_{p_r}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2\ldots p_r}} - \frac{\partial \varepsilon_\tau}{\partial y^\sigma} = 0. \end{aligned}$$

Comparing this formula with (14) we see we get $H_{\sigma\,\tau}(\varepsilon) = 0$.

Remark 8 (2nd order Helmholtz expressions) The Helmholtz conditions for a 2nd order source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ can also be written as

(34) $$\begin{aligned} H_{\sigma\,\nu}^{\ pq} &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}}, \\ H_{\sigma\,\nu}^{\ q} &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_q} - d_p\left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}}\right) + d_p H_{\sigma\,\nu}^{\ pq}, \\ H_{\sigma\,\nu} &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} - \frac{1}{2}d_p\left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_p} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_p}\right) + \frac{1}{2}d_p H_{\sigma\,\nu}^{\ p}. \end{aligned}$$

We can obtain these formulas by direct substitutions into

(35) $$H_{\sigma\,\nu}^{\ q} = \frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_q} - 2d_p\frac{\partial \varepsilon_\nu}{\partial y^\sigma_{qp}},$$

and

$$(36)\qquad \begin{aligned} H_{\sigma}{}^{q}{}_{\nu} &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_q} - 2 d_p \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{qp}}, \\ H_{\sigma\nu} &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + d_p \frac{\partial \varepsilon_\nu}{\partial y^\sigma_p} - d_p d_q \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}}. \end{aligned}$$

We have

$$(37)\qquad \begin{aligned} H_{\sigma}{}^{q}{}_{\nu} &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_q} - d_p \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} \right) - d_p \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} \right) \\ &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_q} - d_p \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} \right) + d_p H_{\sigma}{}^{pq}{}_{\nu}, \end{aligned}$$

and

$$(38)\qquad \begin{aligned} H_{\sigma\nu} &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + \frac{1}{2} d_p \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_p} - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_p} \right) + \frac{1}{2} d_p \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_p} + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_p} \right) \\ &\quad - \frac{1}{2} d_p d_q \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} \right) - \frac{1}{2} d_p d_q \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} \right) \\ &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + \frac{1}{2} d_p \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_p} - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_p} \right) \\ &\quad + \frac{1}{2} d_p \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_p} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_p} - d_q \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} \right) + d_q H_{\sigma}{}^{pq}{}_{\nu} \right) \\ &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + \frac{1}{2} d_p \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_p} - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_p} \right) + \frac{1}{2} d_p H_{\sigma}{}^{p}{}_{\nu}. \end{aligned}$$