Appendix

Analysis on Euclidean spaces and smooth manifolds

In this appendix we summarize for the reference essential notions and theorems of differentiation and integration theory on Euclidean spaces as needed in this book. Main coordinate formulas of the calculus of vector fields and differential forms on smooth manifolds are also given. We have included elementary concepts from multi-linear algebra, and the trace decomposition theory over a real vector space.

1 Jets of mappings of Euclidean spaces

Let $L(\mathbf{R}^n,\mathbf{R}^m)$ be the vector space of *linear* mappings of $\mathbf{R}^n$ into $\mathbf{R}^m$, $L^k(\mathbf{R}^n,\mathbf{R}^m)$ the vector space of k-linear mappings of the Cartesian product $\mathbf{R}^n\times\mathbf{R}^n\times\ldots\times\mathbf{R}^n$ (k factors) into $\mathbf{R}^m$, and let $L^k_{\mathrm{sym}}(\mathbf{R}^n,\mathbf{R}^m)$ be the vector space of k-linear *symmetric* mappings from of $\mathbf{R}^n\times\mathbf{R}^n\times\ldots\times\mathbf{R}^n$ (k factors) into $\mathbf{R}^m$. Let $U\subset\mathbf{R}^n$ and $V\subset\mathbf{R}^m$ be open sets, and denote

(1) $$J^r(U,V)=U\times V\times L(\mathbf{R}^n,\mathbf{R}^m)\times L^2_{\mathrm{sym}}(\mathbf{R}^n,\mathbf{R}^m)\times\ldots\times L^r_{\mathrm{sym}}(\mathbf{R}^n,\mathbf{R}^m).$$

$J^r(U,V)$ is an open set in the Euclidean vector space

(2) $$\mathbf{R}^n\times\mathbf{R}^m\times L(\mathbf{R}^n,\mathbf{R}^m)\times L^2_{\mathrm{sym}}(\mathbf{R}^n,\mathbf{R}^m)\times\ldots\times L^r_{\mathrm{sym}}(\mathbf{R}^n,\mathbf{R}^m).$$

Using the canonical bases of the vector spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, this vector space can be identified with the Euclidean vector space $\mathbf{R}^N$ of dimension

(3) $$N=n+m\left(1+n+\binom{n+1}{2}+\binom{n+2}{2}+\ldots+\binom{n+r-1}{r}\right).$$

The set $J^r(U,V)$ can be identified with collections of real numbers $P=(x^i,y^\sigma,y^\sigma_{j_1},y^\sigma_{j_1j_2},\ldots,y^\sigma_{j_1j_2\ldots j_r})$, $1\le i,j_1,j_2,\ldots,j_r\le n$, $1\le\sigma\le m$, such that the systems $y^\sigma_{j_1j_2\cdots j_k}$ are *symmetric* in the subscripts. We call P an *r-jet*; the point $x\in U$, $x=x^i$ is called the *source* of P and the point $y\in V$, $y=y^\sigma$, *is called the* target of P.

We set for every point $P\in J^r(U,V)$, $P=(x^i,y^\sigma,y^\sigma_{j_1},y^\sigma_{j_1j_2},\ldots,y^\sigma_{j_1j_2\cdots j_r})$,

(4) $$x^i=x^i(P),\quad y^\sigma=y^\sigma(P),\quad y^\sigma_{j_1j_2\cdots j_k}=y^\sigma_{j_1j_2\cdots j_k}(P),\quad 1\le k\le r.$$

Then, by abuse of language, x^i, y^σ, and $y^\sigma_{j_1 j_2 \cdots j_k}$, denote both the components of P and also real-valued functions on $J^r(U,V)$. Restricting ourselves to independent functions, we get a global chart, the *canonical chart* $(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, $j_1 \le j_2 \le \ldots \le j_k$, defining the *canonical smooth manifold structure* on $J^r(U,V)$; elements of this chart are the *canonical coordinates* on $J^r(U,V)$. The set $J^r(U,V)$, endowed with its canonical smooth manifold structure, is called *the manifold of r-jets* (with *source in U* and *target in V*).

We sometimes express without notice an element $P \in J^r(U,V)$ as a collection of real numbers $P = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, subject to the condition $j_1 \le j_2 \le \ldots \le j_k$.

We show that the r-jets can be viewed as classes of mappings, transferring the source of an r-jet to its target. Given an r-jet $P = J^r(U,V)$, $P = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, one can always find a mapping $f = f^\sigma$, defined on a neighbourhood of the source $x \in U$, such that $f(x) = y$, whose derivatives satisfy

$$\begin{aligned} &D_{i_1} f^\sigma(x^i(P)) = y^\sigma_{i_1}(P), \quad D_{i_1} D_{i_2} f^\sigma(x^i(P)) = y^\sigma_{i_1 i_2}(P), \\ &\ldots, \quad D_{i_1} D_{i_2} \ldots D_{i_r} f^\sigma(x^i(P)) = y^\sigma_{j_1 j_2 \cdots j_r}(P). \end{aligned} \tag{5}$$

Indeed, one can choose for the components of f the polynomials

$$\begin{aligned} f^\sigma(t^j) = y^\sigma &+ \frac{1}{1!} y^\sigma_{j_1}(t^{j_1} - x^{j_1}) + \frac{1}{2!} y^\sigma_{j_1 j_2}(t^{j_1} - x^{j_1})(t^{j_2} - x^{j_2}) \\ &+ \ldots + \frac{1}{r!} y^\sigma_{j_1 j_2 \cdots j_r}(t^{j_1} - x^{j_1})(t^{j_2} - x^{j_2}) \cdots (t^{j_r} - x^{j_r}). \end{aligned} \tag{6}$$

Any mapping f, satisfying conditions (5), is called a *representative* of the r-jet P. Using representatives, we usually denote $P = J^r_x f$.

2 Summation conventions

This section contains some remarks to the summation conventions used in this book. We distinguish essentially three different cases:

(a) Summations through pairs of indices, one in contravariant and one in covariant position (the Einstein summation convention). In this case the summation symbol is not explicitly designated.

(b) Summations through more indices or multi-indices. In this case we usually omit the summation symbols for summations, which are evident.

(c) Summations of expressions through variables, labelled with non-decreasing sequences of integers. In this Appendix we discuss the corresponding conventions in more detail.

Let k be a positive integer, let $L^k\mathbf{R}^n$ be the vector space of collections of real numbers $u = u_{i_1 i_2 \ldots i_k}$, where $1 \le i_1, i_2, \ldots, i_k \le n$, and $J^k\mathbf{R}^n$ the vector space of collections of real numbers $v = v_{i_1 i_2 \ldots i_k}$, where $1 \le i_1 \le i_2 \le \ldots \le i_k \le n$. We introduce two mappings $\iota : J^k\mathbf{R}^n \to L^k\mathbf{R}^n$ and $\kappa : L^k\mathbf{R}^n \to J^k\mathbf{R}^n$ as follows. Choose a vector $v \in J^k\mathbf{R}^n$, $v = v_{i_1 i_2 \ldots i_k}$, where $1 \le i_1 \le i_2 \le \ldots \le i_k \le n$, and set for *any* sequence of the indices $j_1, j_2, \ldots, j_k$, not necessarily a non-decreasing one,

(1) $$v_{j_1 j_2 \ldots j_k} = v_{j_{\tau(1)} j_{\tau(2)} \ldots j_{\tau(k)}},$$

where τ is any permutation of the set $\{1,2,\ldots,k\}$, such that the subscripts satisfy $j_{\tau(1)} \le j_{\tau(2)} \le \ldots \le j_{\tau(k)}$. Then set

(2) $$\iota(v) = v_{j_1 j_2 \ldots j_k}.$$

The vector $\iota(v)$ is *symmetric* in all subscripts, and is called the *canonical extension* of v to $L^k\mathbf{R}^n$; the mapping ι is *the canonical extension* (by symmetry). If $u \in L^k\mathbf{R}^n$, $u = u_{i_1 i_2 \ldots i_k}$, set

(3) $$\kappa(u) = v_{j_1 j_2 \ldots j_k} = \frac{1}{k!}\sum_{\nu} u_{j_{\nu(1)} j_{\nu(2)} \ldots j_{\nu(k)}},$$

whenever $j_1 \le j_2 \le \ldots \le j_k$; κ is called the *symmetrization*. For any function $f : J^k\mathbf{R}^n \to \mathbf{R}$, the function $f \circ \kappa : L^k\mathbf{R}^n \to \mathbf{R}$ is called the *canonical extension* of *f*. When no misunderstanding may possibly arise we write just *f* instead of $f \circ \kappa$. Clearly definitions (2) and (3) imply

(4) $$\kappa \circ \iota = \mathrm{id}_{J^k\mathbf{R}^n}.$$

Note that in the finite-dimensional Euclidean vector space $\mathbf{R}^N$, the points of $\mathbf{R}^N$ are canonically identified with the canonical coordinates of these point. In what follows we shall consider the symbols $u_{i_1 i_2 \ldots i_k}$ and $v_{i_1 i_2 \ldots i_k}$ both as the points of $\mathbf{R}^N$ as well as the *canonical coordinates* on the vector spaces $L^k\mathbf{R}^n$ and $J^k\mathbf{R}^n$, respectively.

Denote

(5) $$N(j_1 j_2 \ldots j_k) = \frac{N_1! N_2! \ldots N_n!}{k!},$$

where N_l is the number of occurrences of the index $l = 1,2,\ldots,n$ in the k-tuple $(j_1, j_2, \ldots, j_k)$. The following lemma states two formulas how to express a linear form, whose variables are indexed with non-decreasing sequences; these formulas are based on simple algebraic relations.

Let

(6) $$\Phi = \sum_{i_1 \le i_2 \le \ldots \le i_k} A^{i_1 i_2 \ldots i_k} v_{i_1 i_2 \ldots i_k}$$

be a linear form on $J^k\mathbf{R}^n$.

Lemma 1 *A linear form* Φ (6) *on* $J^k\mathbf{R}^n$ *can be expressed as*

$$\Phi = B^{j_1 j_2 \dots j_k} v_{j_1 j_2 \dots j_k}, \tag{7}$$

where

$$B^{j_1 j_2 \dots j_k} = \frac{1}{N(j_1 j_2 \dots j_k)} A^{i_1 i_2 \dots i_k}. \tag{8}$$

Proof Supposing that $B^{j_1 j_2 \dots j_k}$ and $v_{j_1 j_2 \dots j_k}$ are symmetric, we have

$$\begin{aligned} B^{j_1 j_2 \dots j_k} v_{j_1 j_2 \dots j_k} &= \sum_{j_1, j_2, \dots, j_k} \sum_{\kappa} \frac{1}{k!} B^{j_{\kappa(1)} j_{\kappa(2)} \dots j_{\kappa(k)}} v_{j_{\kappa(1)} j_{\kappa(2)} \dots j_{\kappa(k)}} \\ &= \sum_{j_1 \le j_2 \le \dots \le j_k} \frac{1}{k!} N_1! N_2! \dots N_n! B^{j_1 j_2 \dots j_k} v_{j_1 j_2 \dots j_k} \\ &= \sum_{j_1 \le j_2 \le \dots \le j_k} N(i_1 i_2 \dots i_k) B^{j_1 j_2 \dots j_k} v_{j_1 j_2 \dots j_k}. \end{aligned} \tag{9}$$

If this expression equals Φ we get (8).

Lemma 1 can be applied to linear forms df, where $f : J^k\mathbf{R}^n \to \mathbf{R}$ is a function. df is defined by

$$df(v) \cdot \Xi = \sum_{i_1 \le i_2 \le \dots \le i_k} \left(\frac{\partial f}{\partial v_{i_1 i_2 \dots i_k}} \right)_v \Xi_{i_1 i_2 \dots i_k}, \tag{10}$$

where

$$\Xi = \sum_{i_1 \le i_2 \le \dots \le i_k} \Xi_{i_1 i_2 \dots i_k} \frac{\partial}{\partial v_{i_1 i_2 \dots i_k}} \tag{11}$$

is a tangent vector. But the chain rule yields $T_v f \cdot \Xi = T_{\iota(v)}(f \circ \kappa) \circ T_v \iota \cdot \Xi$, so we have the following assertion.

Lemma 2 *The linear form* df (10) *can be expressed as*

$$df(v) \cdot \Xi = \left(\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \dots j_k}} \right)_{\iota(v)} \Xi_{j_1 j_2 \dots j_k}. \tag{12}$$

Proof Using formula (4) we get from (10)

$$df(v) \cdot \Xi = \sum_{i_1 \le i_2 \le \dots \le i_k} \left(\frac{\partial (f \circ \kappa \circ \iota)}{\partial v_{i_1 i_2 \dots i_k}} \right)_v \Xi_{i_1 i_2 \dots i_k} \tag{13}$$

$$= \sum_{i_1 \le i_2 \le \ldots \le i_k} \sum_{j_1, j_2, \ldots, j_k} \left(\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} \right)_{\iota(v)} \left(\frac{\partial (u_{j_1 j_2 \ldots j_k} \circ \iota)}{\partial v_{i_1 i_2 \ldots i_k}} \right)_v \Xi_{i_1 i_2 \ldots i_k}$$

$$= \sum_{j_1, j_2, \ldots, j_k} \left(\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} \right)_{\iota(v)} \sum_{i_1 \le i_2 \le \ldots \le i_k} \left(\frac{\partial (u_{j_1 j_2 \ldots j_k} \circ \iota)}{\partial v_{i_1 i_2 \ldots i_k}} \right)_v \Xi_{i_1 i_2 \ldots i_k}.$$

But writing

(14) $$T_v \iota \cdot \Xi = \Xi_{j_1 j_2 \ldots j_k} \frac{\partial}{\partial u_{j_1 j_2 \ldots j_k}},$$

we see that $T_v \iota$ extends the components $\Xi_{i_1 i_2 \ldots i_k}$, $i_1 \le i_2 \le \ldots \le i_k$ by the index symmetry,

(15) $$\Xi_{j_1 j_2 \ldots j_k} = \sum_{i_1 \le i_2 \le \ldots \le i_k} \left(\frac{\partial (u_{j_1 j_2 \ldots j_k} \circ \iota)}{\partial v_{i_1 i_2 \ldots i_k}} \right)_v \Xi_{i_1 i_2 \ldots i_k}.$$

Thus, using the symmetric components (15), one can also express the exterior derivative df (13) as in (12).

Corollary 1 *Let* $f : J^k\mathbf{R}^n \to \mathbf{R}$ *be a function,* $v \in J^k\mathbf{R}^n$ *a point, and let* $\Xi = \Xi_{j_1 j_2 \ldots j_k}$*, where* $1 \le j_1 \le j_2 \le \ldots \le j_k \le n$*, be the components of a tangent vector of* $J^k\mathbf{R}^n$ *at the point v. Then the derivatives of the functions f and* $f \circ \kappa$ *satisfy*

(16) $$\sum_{j_1 \le j_2 \le \ldots \le j_k} \left(\frac{\partial f}{\partial v_{j_1 j_2 \ldots j_k}} \right)_v \Xi_{j_1 j_2 \ldots j_k} = \left(\frac{\partial (f \circ \kappa)}{\partial u_{i_1 i_2 \ldots i_k}} \right)_{\iota(v)} \Xi_{i_1 i_2 \ldots i_k}.$$

Proof (16) follows from (10) and (12).

Corollary 2 (a) *Partial derivatives of the functions f and* $f \circ \kappa$ *satisfy the condition*

(17) $$\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} = N(j_1 j_2 \ldots j_k) \frac{\partial f}{\partial v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}}} \circ \kappa,$$

where λ *is any permutation of the index set* $\{1,2,\ldots,k\}$*, such that* $j_{\lambda(1)} \le j_{\lambda(2)} \le \ldots \le j_{\lambda(k)}$*, and*

(18) $$\frac{\partial f}{\partial v_{i_1 i_2 \ldots i_k}} = \frac{1}{N(i_1 i_2 \ldots i_k)} \frac{\partial (f \circ \kappa)}{\partial u_{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(k)}}} \circ \iota$$

for any permutation τ.

(b) *For any permutation* $l_{\tau(1)}, l_{\tau(2)}, \ldots, l_{\tau(k)}$ *of the indices* $l_1, l_2, \ldots, l_k$ *, the derivatives of the function* $f \circ \kappa$ *satisfy*

(19) $$\frac{\partial (f \circ \kappa)}{\partial u_{l_{\tau(1)} l_{\tau(2)} \ldots l_{\tau(k)}}} = \frac{\partial (f \circ \kappa)}{\partial u_{l_1 l_2 \ldots l_k}}.$$

Proof (a) From the chain rule we have for any $(j_1, j_2, \ldots, j_k)$

(20) $$\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} = \sum_{i_1 \le i_2 \le \ldots \le i_k} \left(\frac{\partial f}{\partial v_{i_1 i_2 \ldots i_k}} \circ \kappa \right) \frac{\partial (v_{i_1 i_2 \ldots i_k} \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}},$$

But from equation (3), there is exactly one non-zero term on the right-hand side, namely the term in which $(i_1 i_2 \ldots i_k) = (j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)})$, such that $j_{\lambda(1)} \le j_{\lambda(2)} \le \ldots \le j_{\lambda(k)}$ for some permutation λ. Then

(21) $$\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} = \frac{\partial f}{\partial v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}}} \circ \kappa \cdot \frac{\partial (v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}} \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}},$$

where by (3)

(22) $$v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}} \circ \kappa = \frac{1}{k!} \sum_{\tau} u_{j_{\tau(1)} j_{\tau(2)} \ldots j_{\tau(k)}}.$$

Differentiating (18) we get

(23) $$\begin{aligned} \frac{\partial (v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}} \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} &= \frac{1}{k!} \sum_{\tau} \frac{\partial u_{j_{\tau(1)} j_{\tau(2)} \ldots j_{\tau(k)}}}{\partial u_{j_1 j_2 \ldots j_k}} \\ &= \frac{N_1! N_2! \ldots N_n!}{k!}. \end{aligned}$$

Substituting from (23) back to (21) we have

(24) $$\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} = N(j_1 j_2 \ldots j_k) \frac{\partial f}{\partial v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}}} \circ \kappa.$$

Conversely, given a k-tuple of indices $(i_1, i_2, \ldots, i_k)$ such that $1 \le i_1 \le i_2 \le \ldots \le i_k \le n$, we get from (24) and (4)

(25) $$\frac{\partial f}{\partial v_{i_1 i_2 \ldots i_k}} = \frac{1}{N(i_1 i_2 \ldots i_k)} \frac{\partial (f \circ \kappa)}{\partial u_{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(k)}}} \circ \iota$$

for any permutation τ. Formulas (24) and (25) prove Corollary 2.

(b) Formula (19) follows from (17).

Remark Formula (16) can also be used, with obvious simplification, in the form

$$\text{(26)} \qquad \sum_{j_1 \le j_2 \le \ldots \le j_k} \frac{\partial f}{\partial u_{j_1 j_2 \ldots j_k}} \Xi_{j_1 j_2 \ldots j_k} = \frac{\partial f}{\partial u_{i_1 i_2 \ldots i_k}} \Xi_{i_1 i_2 \ldots i_k}.$$

3 The rank theorem

In the following two basic theorems of analysis of real-valued functions on finite-dimensional Euclidean spaces we denote by x^i and y^σ the canonical coordinates on the Euclidean spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively.

Theorem 1 (The Rank theorem) *Let W be an open set in* $\mathbf{R}^n$, *and let* $f: W \to \mathbf{R}^m$ *be a* C^r *mapping. Let* $q \le \min(m,n)$ *be a positive integer. The following conditions are equivalent:*

(1) *The mapping f has constant rank* $\operatorname{rank} Df(x) = q$ *on W.*

(2) *For every point* $x_0 \in W$ *there exist a neighbourhood U of* x_0 *in W, an open rectangle* $P \subset \mathbf{R}^n$ *with centre* 0*, a* C^r *diffeomorphism* $\varphi: U \to P$ *such that* $\varphi(x_0) = 0$*, a neighbourhood V of* $f(x_0)$ *such that* $f(U) \subset V$*, an open rectangle* $Q \subset \mathbf{R}^m$ *with centre* 0*, and a* C^r *diffeomorphism* $\psi: V \to Q$ *such that* $\psi(f(x_0)) = 0$*, and on P,*

$$\text{(1)} \qquad \psi f \varphi^{-1}(x^1, x^2, \ldots, x^q, x^{q+1}, x^{q+2}, \ldots, x^n) = (x^1, x^2, \ldots, x^q, 0, 0, \ldots, 0).$$

Formula (1) can be expressed in terms of *equations* of the mapping $\psi f \varphi^{-1}$, which are of the form

$$\text{(2)} \qquad y^\sigma \circ f = \begin{cases} x^\sigma, & 1 \le \sigma \le q, \\ 0, & q+1 \le \sigma \le m. \end{cases}$$

In particular, if $q = n \le m$, then $\psi f \varphi^{-1}$ is the restriction of the canonical injection $(x^1, x^2, \ldots, x^n) \to (x^1, x^2, \ldots, x^n, 0, 0, \ldots, 0)$ of Euclidean spaces; if $q = n = m$, $\psi f \varphi^{-1}$ is the restriction of the identity mapping of $\mathbf{R}^n$; if the dimensions *n* and *m* satisfy $n > m$, then $\psi f \varphi^{-1}$ is the restriction of the Cartesian projection $(x^1, x^2, \ldots, x^m, x^{m+1}, x^{m+2}, \ldots, x^n) \to (x^1, x^2, \ldots, x^m)$ of Euclidean spaces.

The following is an immediate consequence of Theorem 1.

Theorem 2 (The Inverse function theorem) *Let* $W \subset \mathbf{R}^n$ *be an open set, and let* $f: W \to \mathbf{R}^n$ *be a* C^r *mapping. Suppose that* $\det Df(x_0) \ne 0$ *at a point* $x_0 \in W$*. Then there exists a neighbourhood U of* x_0 *in W and a neighbourhood V of* $f(x_0)$ *in* $\mathbf{R}^n$ *such that* $f(U) = V$ *and the restriction* $f|_U: U \to V$ *is a* C^r *diffeomorphism.*

4 Local flows of vector fields

In this book, the symbol $T_x f$ denotes the tangent mapping of a mapping f at a point x. Sometimes we also use another notation, which may simplify calculations and resulting formulas. If $t \to \zeta(t)$ is a *curve* in a manifold, then its tangent vector at a point t_0 is denoted by either of the symbols

$$(1) \qquad T_{t_0}\zeta \cdot 1, \quad \left(\frac{d\zeta}{dt}\right)_{t_0}.$$

The tangent vector field is denoted by

$$(2) \qquad T_t\zeta \cdot 1 = \frac{d\zeta}{dt}.$$

Note, however, that sometimes the symbol $d\zeta / dt$ may cause notational problems when using the chain rule.

The following is a well-known result of the theory of integral curves of vector fields on smooth manifolds.

Theorem (The local flow theorem) *Let $r \geq 1$ and let ξ be a C^r vector field on a smooth manifold X.*

(a) *For every point $x_0 \in X$ there exists an open interval J containing the point $0 \in \mathbf{R}$, a neighbourhood V of x_0, and a unique C^r mapping $\alpha : J \times V \to X$ such that for every point $x \in V$, $\alpha(0,x) = x$ and the mapping $J \ni t \to \alpha_x(t) = \alpha(t,x) \in X$ satisfies*

$$(3) \qquad T_t\alpha_x = \xi(\alpha_x(t)).$$

(b) *There exist a subinterval K of J with centre* 0 *and a neighbourhood W of x_0 in V such that*

$$(4) \qquad \alpha(s+t,x) = \alpha(s,\alpha(t,x)), \quad \alpha(-t,\alpha(t,x)) = x$$

for all points $(s,t) \in K$ and $x \in W$. For every $t \in K$, the mapping $W \ni x \to \alpha(t,x) \in X$ is a C^k diffeomorphism.

Condition (3) means that $t \to \alpha_x(t)$ is an *integral curve* of the vector field ξ, and the mapping $(t,x) \to \alpha_x(t) = \alpha(t,x)$ is a *local flow* of ξ at the point x_0; we also say that α is a *local flow* of ξ *on the set V*. Equation (3) can also be written as

$$(5) \qquad \frac{d\alpha_x}{dt} = \xi(\alpha_x(t)).$$

5 Calculus on manifolds

In this Appendix we give a list of basic rules and coordinate formulas of the calculus of differential forms and vector fields on smooth manifolds.

We use the following notation:

Tf	tangent mapping of a differentiable mapping f
$f^*\eta$	pull-back of a differential form η by f
$[\xi,\zeta]$	Lie bracket of vector fields ξ and ζ
d	exterior derivative of a differential form
$i_\xi\eta$	contraction of a differential form η by a vector field ξ
$\partial_\xi\eta$	Lie derivative of a differential form η by a vector field ξ

Theorem 1 (The pull-back of a differential form) *Let X, Y and Z be smooth manifolds.*

(a) *For any differentiable mapping $f : X \to Y$, any p-form η and any q-form ρ on Y*

(1) $$f^*(\eta \wedge \rho) = f^*\eta \wedge f^*\rho.$$

(b) *Let $f : X \to Y$ and $g : Y \to Z$ be differentiable mappings. Then for any p-form μ on Z*

(2) $$f^*g^*\mu = (g \circ f)^*\mu.$$

Theorem 2 (Exterior derivative) *Let X and Y be smooth manifolds.*

(a) *For any p-form η and q-form ρ on X*

(3) $$d(\eta \wedge \rho) = d\eta \wedge \rho + (-1)^p \eta \wedge d\rho.$$

(b) *For every p-form η on X*

(4) $$d(d\eta) = 0.$$

(c) *For any differentiable mapping $f : X \to Y$ and any p-form η on Y*

(5) $$df^*\eta = f^*d\eta.$$

Theorem 3 (Contraction of forms by a vector field) *Let X and Y be smooth manifolds.*

(a) *Let η be a p-form on X, and let ξ and ζ be two vector fields on X. Then*

(6) $$i_\zeta i_\xi \eta = -i_\xi i_\zeta \eta.$$

(b) *Let* η *be a p-form,* ρ *a q-form, and let* ζ *be a vector field on X. Then*

(7) $$i_\zeta(\eta \wedge \rho) = i_\zeta \eta \wedge \rho + (-1)^p \eta \wedge i_\zeta \rho.$$

(c) *Let* $f : X \to Y$ *be a differentiable mapping,* η *a p-form on Y, and let* ξ *be a vector field on X,* ζ *a vector field on Y. Suppose that* ξ *and* ζ *are f-related. Then*

(8) $$f^* i_\zeta \eta = i_\xi f^* \eta.$$

Theorem 4 (Lie derivative) (a) *Let X be a smooth manifold,* η *a p-form,* ρ *a q-form, and let* ξ *and* ζ *be vector fields on X. Then*

(9) $$\partial_\xi \eta = i_\xi d\eta + d i_\xi \eta,$$

(10) $$\partial_\xi d\eta = d\partial_\xi \eta,$$

(11) $$\partial_\xi(\eta \wedge \rho) = \partial_\xi \eta \wedge \rho + \eta \wedge \partial_\xi \rho,$$

(12) $$i_{[\xi,\zeta]}\eta = \partial_\xi i_\zeta \eta - i_\zeta \partial_\xi \eta,$$

(13) $$\partial_{[\xi,\zeta]}\eta = \partial_\xi \partial_\zeta \eta - \partial_\zeta \partial_\xi \eta.$$

(b) *Let* $f : X \to Y$ *be a differentiable mapping of smooth manifolds, let* ξ *be a vector field on X, and* ζ *be a vector field on Y. Suppose that* ξ *and* ζ *are f-compatible. Then for any p-form* η *on Y*

(14) $$f^* \partial_\zeta \eta = \partial_\xi f^* \eta.$$

Theorem 5 *Let X and Y be smooth manifolds,* $f : X \to Y$ *a* C^1 *mapping. Let* (U,φ), $\varphi = (x^i)$, *be a chart on X, and* (V,ψ), $\psi = (y^\sigma)$, *a chart on Y, such that* $f(U) \subset V$.

(a) *For any point* $x \in U$ *and any tangent vector* $\xi \in T_x X$ *at the point x, expressed as*

(15) $$\xi = \xi^k \left(\frac{\partial}{\partial x^k} \right)_x,$$

the image $Tf \cdot \xi$ *is*

(16) $$Tf \cdot \xi = \left(\frac{\partial (y^\sigma f \varphi^{-1})}{\partial x^i} \right)_{\varphi(x)} \xi^i \left(\frac{\partial}{\partial y^\sigma} \right)_{f(x)}.$$

(b) *The pull-back* $f^* \eta$ *of a differential p-form* η *on Y, expressed as*

(17) $$\eta = \frac{1}{p!}\eta_{i_1 i_2 \ldots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p},$$

is given by

(18) $$f^*\eta = \frac{1}{p!}\frac{\partial(y^{\sigma_1} f\varphi^{-1})}{\partial x^{i_1}}\frac{\partial(y^{\sigma_2} f\varphi^{-1})}{\partial x^{i_2}}\ldots\frac{\partial(y^{\sigma_p} f\varphi^{-1})}{\partial x^{i_p}} \cdot(\eta_{\sigma_1\sigma_2\ldots\sigma_p}\circ f)\cdot dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}.$$

Theorem 6 *Let* (U,φ), $\varphi = (x^i)$*, be a chart on X.*

(a) *For any two vector fields* ξ *and* ζ *on X, expressed by*

(19) $$\xi = \xi^i \frac{\partial}{\partial x^i}, \quad \zeta = \zeta^i \frac{\partial}{\partial x^i},$$

the Lie bracket $[\xi,\zeta]$ *is expressed by*

(20) $$[\xi,\zeta] = \left(\frac{\partial \zeta^i}{\partial x^l}\xi^l - \frac{\partial \xi^i}{\partial x^l}\zeta^l\right)\frac{\partial}{\partial x^i}.$$

(b) *The exterior derivative* df *of a function* $f : X \to \mathbf{R}$ *is expressed by*

(21) $$df = \frac{\partial f}{\partial x^k}dx^k.$$

The exterior derivative $d\eta$ *of a p-form* η (17) *has the chart expression*

(22) $$d\eta = \frac{1}{p!}d\eta_{i_1 i_2 \ldots i_p} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p},$$

where the exterior derivative $d\eta_{i_1 i_2 \ldots i_p}$ *is of is determined by formula* (21).

(c) *The contraction* $i_\xi \eta$ *of the form* η (17) *by a vector field* ξ (19) *has the chart expression*

(23) $$i_\xi \eta = \frac{1}{(k-1)!}\eta_{s i_1 i_2 \ldots i_{k-1}}\xi^s dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{k-1}}.$$

(d) *The Lie derivative* $\partial_\xi \eta$ *of the form* η (17) *by a vector field* ξ (19) *has the chart expression*

(24) $$\partial_\xi \eta = \frac{1}{p!}\left(\frac{\partial \xi^s}{\partial x^{i_1}}\eta_{s i_2 i_3 \ldots i_p} - \frac{\partial \xi^s}{\partial x^{i_2}}\eta_{s i_1 i_3 i_4 \ldots i_p} + \frac{\partial \xi^s}{\partial x^{i_3}}\eta_{s i_1 i_2 i_4 i_5 \ldots i_p} - \ldots + (-1)^{p-1}\frac{\partial \xi^s}{\partial x^{i_p}}\eta_{s i_1 i_2 \ldots i_{p-1}} + \frac{\partial \eta_{i_1 i_2 \ldots i_p}}{\partial x^k}\xi^k\right)dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}.$$

6 Fibred homotopy operators

In this section we study differential forms, defined on open star-shaped sets U in an Euclidean space $\mathbf{R}^n$ and on trivial fibred manifolds $U \times V$, where V is an open star-shaped set in $\mathbf{R}^m$. Our aim will be to investigate properties of the exterior derivative operator d on U and on $U \times V$.

First we consider a differential k-form ρ, where $k \geq 1$, defined on an open star-shaped set $U \subset \mathbf{R}^n$ with centre at the origin $0 \in \mathbf{R}^n$. We shall study the equation

(1) $$d\eta = \rho$$

for an unknown $(k-1)$-form η on V. Denote by x^i the canonical coordinates on U. Define a mapping $\chi : [0,1] \times V \to V$ as the restriction of the image of the mapping $(s, x^1, x^2, \dots, x^n) = (sx^1, sx^2, \dots, sx^n)$ from $\mathbf{R} \times \mathbf{R}^n$ to $\mathbf{R}^n$ to the open set V; thus in short

(2) $$\chi(s, x^i) = (sx^i).$$

Then

(3) $$\chi^* dx^i = x^i ds + s dx^i.$$

Consider the pull-back $\chi^* \rho$ which is a k-form on a neighbourhood of the set $[0,1] \times V$. Obviously, there exists a unique decomposition

(4) $$\chi^* \rho = ds \wedge \rho^{(0)}(s) + \rho'(s),$$

such that the k-forms $\rho^{(0)}(s)$ and $\rho'(s)$ do not contain ds. Note that by formula (3), $\rho'(s)$ arises from ρ by replacing each factor dx^i with sdx^i, and by replacing each coefficient f with $f \circ \chi$. Thus, $\rho'(s)$ obeys

(5) $$\rho'(1) = \rho, \quad \rho'(0) = 0.$$

We set

(6) $$I\rho = \int_0^1 \rho^{(0)}(s),$$

where the expression on the right-hand side means integration of the coefficients in the form $\rho^{(0)}(s)$ over s from 0 to 1.

Lemma 1 *Let U be an open ball in $\mathbf{R}^n$ with centre* 0.

(a) *For every differentiable function $f : U \to \mathbf{R}$,*

(7) $$f = Idf + f(0).$$

(b) *Suppose that* $k \geq 1$. *Then for any differential k-form* ρ *on U,*

$$\rho = Id\rho + dI\rho. \tag{8}$$

Proof 1. If f is a function, then $df = (\partial f / \partial x^i)dx^i$, and we have by (3) $\chi^* df = ((\partial f / \partial x^i) \circ \chi) \cdot (x^i ds + s dx^i)$. Consequently,

$$Idf = x^i \int_0^1 \left(\frac{\partial f}{\partial x^i} \circ \chi \right) ds. \tag{9}$$

Now (7) follows from the identity

$$\begin{aligned} f - f(0) &= (f \circ \chi)|_{s=1} - (f \circ \chi)|_{s=0} = \int_0^1 \frac{d(f \circ \chi)}{ds} ds \\ &= x^i \int_0^1 \left(\frac{\partial f}{\partial x^i} \circ \chi \right) ds. \end{aligned} \tag{10}$$

2. Let $k = 1$. Then ρ has an expression $\rho = B_i dx^i$, and the pull-back $\chi^* \rho$ is given by $\chi^* \rho = x^i (B_i \circ \chi) ds + (B_i \circ \chi) s dx^i$. Differentiating we get

$$\begin{aligned} \chi^* d\rho = d\chi^* \rho &= ds \wedge \left(-d(x^i (B_i \circ \chi) + \frac{\partial((B_i \circ \chi)s)}{\partial s} dy^i \right) \\ &+ s \frac{\partial(B_i \circ \chi)}{\partial x^j} dx^j \wedge dx^i, \end{aligned} \tag{11}$$

hence

$$I\rho = x^i \int_0^1 B_i \circ \chi \cdot ds. \tag{12}$$

Thus,

$$Id\rho = \int_0^1 \left(\frac{\partial((B_i \circ \chi)s)}{\partial s} - \frac{\partial(x^j \cdot B_j \circ \chi)}{\partial x^i} \right) ds \cdot dx^i, \tag{13}$$

and

$$dI\rho = \int_0^1 \frac{\partial(x^j \cdot B_j \circ \chi)}{\partial x^i} ds \cdot dx^i. \tag{14}$$

Consequently,

$$\begin{aligned} Id\rho + dI\rho &= \int_0^1 \left(\frac{\partial((B_i \circ \chi)s)}{\partial s} \right) ds \cdot dx^i \\ &= ((B_i \circ \chi \cdot s)|_{s=1} - (B_i \circ \chi \cdot s)|_{s=0}) dx^i = \rho. \end{aligned} \tag{15}$$

3. Let $k \geq 2$. Write ρ in the form

$$\rho = dx^i \wedge \Psi_i, \tag{16}$$

and define differential forms $\Psi_i^{(0)}(s)$ and $\Psi_i'(s)$ by

$$\chi^* \Psi_i = ds \wedge \Psi_i^{(0)}(s) + \Psi_i'(s). \tag{17}$$

Then

$$\begin{aligned} \chi^* \rho &= (s dx^i + x^i ds) \wedge (ds \wedge \Psi_i^{(0)}(s) + \Psi_i'(s)) \\ &= ds \wedge (-s dx^i \wedge \Psi_i^{(0)}(s) + y^\sigma \Psi_i'(s)) + s dy^i \wedge \Psi_i'(s). \end{aligned} \tag{18}$$

Thus,

$$I\rho = \int_0^1 (-s\, dx^i \wedge \Psi_i^{(0)}(s) + x^i \Psi_i'(s)). \tag{19}$$

To determine $Id\rho$, we compute $\chi^* d\rho$. Property $\chi^* d\rho = d\chi^* \rho$ of the pull-back yields

$$\begin{aligned} \chi^* d\rho &= -ds \wedge (s dx^i \wedge d\Psi_i^{(0)}(s) + dx^i \wedge \Psi_i'(s) \\ &\quad + x^i d\Psi_i'(s)) - dx^i \wedge d(s\Psi_i'(s))) \\ &= ds \wedge \Bigg(-s dx^i \wedge d\Psi_i^{(0)}(s) - dx^i \wedge \Psi_i'(s) \\ &\quad - x^i d\Psi_i'(s) + dx^i \wedge \frac{\partial(s\Psi_i'(s))}{\partial s} \Bigg) - dx^i \wedge dx^j \wedge \frac{\partial(s\Psi_i'(s))}{\partial x^j}, \end{aligned} \tag{20}$$

where $\partial\eta(s)/\partial s$ denotes the form, arising from $\eta(s)$ by differentiation with respect to s, followed by multiplication by ds. Now by (20) and (6),

$$\begin{aligned} Id\rho &= -dx^i \wedge \int_0^1 s\, d\Psi_i^{(0)}(s) - dx^i \wedge \int_0^1 \Psi_i'(s) \\ &\quad - x^i \int_0^1 d\Psi_i'(s) + dx^i \wedge \int_0^1 \frac{\partial(s\Psi_i'(s))}{\partial s}. \end{aligned} \tag{21}$$

It is important to notice that the exterior derivatives $d\Psi_\sigma^{(0)}(s)$, and $d\Psi_\sigma'(s)$ have the meaning of the derivatives with respect to x^i (the terms containing ds are cancelled; see the definition of I (4), (6)).

Now we easily get

$$Id\rho + dI\rho = dx^i \wedge \int_0^1 \frac{\partial(s\Psi_i'(s))}{\partial s}. \tag{22}$$

Remembering that the integral symbol denotes integration of *coefficients* in the corresponding forms with respect to the parameter s from 0 to 1, and using (5), one obtains

$$Id\rho + dI\rho = dx^i \wedge (1 \cdot \Psi_i'(1) - 0 \cdot \Psi_i'(0)) \tag{23}$$
$$= dx^i \wedge \Psi_i'(1) = dx^i \wedge \Psi_i = \rho,$$

as desired.

As an immediate consequence, we get the following statement.

Lemma 2 (The Volterra-Poincare lemma) *Let U be an open ball in* $\mathbf{R}^n$ *with centre* 0*,* ρ *a differential k-form on U, where* $k \geq 1$. *The following two conditions are equivalent:*

(a) *There exists a form* η *on U such that*

$$d\eta = \rho. \tag{24}$$

(b) ρ *satisfies*

$$d\rho = 0. \tag{25}$$

Proof If $d\eta = \rho$ for some η, we have $d\rho = dd\eta = 0$. Conversely, if $d\rho = 0$, we take $\eta = I\rho$ in Lemma 1.

Condition (25) is sometimes called *integrability condition* for the differential equation (24).

Now we consider a different kind of differential equations, reducing to (1) for differential forms of sufficiently high degree. Let *U* be an open set *U* in $\mathbf{R}^n$, and *V* an open ball *V* in $\mathbf{R}^m$ with centre at the origin. Denote by π the first Cartesian projection of $U \times V$ onto *U*. Suppose we are given ρ on $U \times V$, where ρ is a positive integer. Our objective will be to study the equation

$$d\eta + \pi^* \eta_0 = \rho \tag{26}$$

for the unknowns a $(k-1)$-form η on $U \times V$, and a *k*-form η_0 on *U*.

Let (x^i, y^σ), where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, be the canonical coordinates on $U \times V$, and $\zeta : U \to U \times V$ be the *zero section* of $U \times V$. Consider the mapping $(s,(x^1,x^2,\ldots,x^n,y^1,y^2,\ldots,y^m)) \to (x^1,x^2,\ldots,x^n,sy^1,sy^2,\ldots,sy^m)$ of $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^m$ with values in $\mathbf{R}^m \times \mathbf{R}^m$. Restricting the range of this mapping to $U \times V$, we define a mapping $\chi : [0,1] \times U \times V \to U \times V$ by

$$\chi(s,(x^i, y^\sigma)) = (x^i, sy^\sigma). \tag{27}$$

Then

$$\chi^* dx^i = dx^i, \quad \chi^* dy^\sigma = y^\sigma ds + s dy^\sigma. \tag{28}$$

Consider the pull-back $\chi^* \rho$, which is a *k*-form on a neighbourhood of the set $[0,1]\times U\times V$. There exists a unique decomposition

$$\chi^*\rho = ds\wedge\rho^{(0)}(s)+\rho'(s) \tag{29}$$

such that the k-forms $\rho^{(0)}(s)$ and $\rho'(s)$ do not contain ds. Note that by (28), $\rho'(s)$ arises from ρ by replacing each factor dy^σ with sdy^σ, and by replacing each coefficient f with $f\circ\chi$; the factors dx^i remain unchanged. Thus, $\rho'(s)$ obeys

$$\rho'(1)=\rho,\quad \rho'(0)=\pi^*\zeta^*\rho. \tag{30}$$

Let $k\geq 1$. We define

$$I\rho=\int_0^1\rho^{(0)}(s), \tag{31}$$

where the expression on the right-hand side means integration of the coefficients in the form $\rho^{(0)}(s)$ over s from 0 to 1.

Theorem 1 *Let $U\subset\mathbf{R}^n$ be an open set, and let $V\subset\mathbf{R}^m$ be an open ball with centre* 0.

(a) *For every differentiable function $f:U\times V\to\mathbf{R}$,*

$$f=Idf+\pi^*\zeta^*f. \tag{32}$$

(b) *Let $k\geq 1$. Then for every differential k-form ρ on the Cartesian product $U\times V$,*

$$\rho=Id\rho+dI\rho+\pi^*\zeta^*\rho. \tag{33}$$

Proof 1. We have

$$df=\frac{\partial f}{\partial x^i}dx^i+\frac{\partial f}{\partial y^\sigma}dy^\sigma, \tag{34}$$

and by (28)

$$\chi^*f=\left(\frac{\partial f}{\partial x^i}\circ\chi\right)dx^i+\left(\frac{\partial f}{\partial y^\sigma}\circ\chi\right)(y^\sigma ds+sdy^\sigma). \tag{35}$$

Now the identity

$$\begin{aligned} f-\pi^*\zeta^*f&=f\circ\chi|_{s=1}-f\circ\chi|_{s=0}\\ &=\int_0^1\frac{d(f\circ\chi)}{ds}ds=y^\sigma\int_0^1\left(\frac{\partial f}{\partial y^\sigma}\circ\chi\right)ds=Idf, \end{aligned} \tag{36}$$

which follows from (31), gives the result.

2. Let $k=1$. Then ρ has an expression $\rho = A_i dx^i + B_\sigma dy^\sigma$, thus

$$\begin{aligned}\chi^*\rho &= (A_i\circ\chi)dx^i + (B_\sigma\circ\chi)(s dy^\sigma + y^\sigma ds)\\ &= y^\sigma(B_\sigma\circ\chi)ds + (A_i\circ\chi)dx^i + (B_\sigma\circ\chi)s dy^\sigma,\end{aligned} \tag{37}$$

and

$$\begin{aligned}\chi^* d\rho &= d\chi^*\rho\\ &= ds\wedge\left(-d(y^\sigma(B_\sigma\circ\chi)) + \frac{\partial(A_i\circ\chi)}{\partial s}dx^i + \frac{\partial((B_\sigma\circ\chi)s)}{\partial s}dy^\sigma\right)\\ &+\left(\frac{\partial(A_i\circ\chi)}{\partial x^j}dx^j + \frac{\partial(A_i\circ\chi)}{\partial y^\nu}dy^\nu\right)\wedge dx^i\\ &+ s\left(\frac{\partial(B_\sigma\circ\chi)}{\partial x^j}dx^j + \frac{\partial(B_\sigma\circ\chi)}{\partial y^\nu}dy^\nu\right)\wedge dy^\sigma,\end{aligned} \tag{38}$$

hence

$$I\rho = y^\sigma\int_0^1 B_\sigma\circ\chi\cdot ds, \tag{39}$$

and

$$\begin{aligned}Id\rho &= \int_0^1\left(\frac{\partial(A_i\circ\chi)}{\partial s} - \frac{\partial(y^\nu\cdot B_\nu\circ\chi)}{\partial x^i}\right)ds\cdot dx^i\\ &+\int_0^1\left(\frac{\partial((B_\sigma\circ\chi)s)}{\partial s} - \frac{\partial(y^\nu\cdot B_\nu\circ\chi)}{\partial y^\sigma}\right)ds\cdot dy^\sigma.\end{aligned} \tag{40}$$

We also get

$$dI\rho = y^\sigma\int_0^1\frac{\partial(B_\sigma\circ\chi)}{\partial x^i}ds\cdot dx^i + \int_0^1\frac{\partial(y^\nu\cdot B_\nu\circ\chi)}{\partial y^\sigma}ds\cdot dy^\sigma, \tag{41}$$

consequently,

$$\begin{aligned}Id\rho + dI\rho &= A_i\circ\chi\,|_{s=1} - A_i\circ\chi\,|_{s=0} + (B_\sigma\circ\chi\cdot s)|_{s=1} - (B_\sigma\circ\chi\cdot s)|_{s=0}\\ &= \rho - \pi^*\zeta^*\rho.\end{aligned} \tag{42}$$

Let $k\geq 2$. Write ρ in the form $\rho = dx^i\wedge\Phi_i + dy^\sigma\wedge\Psi_\sigma$, and define differential forms $\Phi_i^{(0)}(s)$, $\Phi_i'(s)$, $\Psi_\sigma^{(0)}(s)$ by

$$\begin{aligned}\chi^*\Phi_i &= ds\wedge\Phi_i^{(0)}(s) + \Phi_i'(s),\\ \chi^*\Psi_\sigma &= ds\wedge\Psi_\sigma^{(0)}(s) + \Psi_\sigma'(s).\end{aligned} \tag{43}$$

Then

$$\begin{aligned}\chi^*\rho &= ds\wedge(-dx^i\wedge\Phi_i^{(0)}(s)-sdy^\sigma\Psi_\sigma^{(0)}(s)+y^\sigma\Psi'_\sigma(s))\\ &\quad +dx^i\wedge\Phi'_i(s)+sdy^\sigma+sy^\sigma\Psi'_\sigma(s)).\end{aligned}\tag{44}$$

Thus,

$$I\rho=-dx^i\wedge\int_0^1\Phi_i^{(0)}(s)-dy^\sigma\wedge\int_0^1(s\Psi_\sigma^{(0)}(s)+y^\sigma\Psi'_\sigma(s))ds.\tag{45}$$

To determine $Id\rho$, we compute $\chi^*d\rho$. We get

$$\begin{aligned}\chi^*d\rho &= d\chi^*\rho\\ &=-ds\wedge(dx^i\wedge d\Phi_i^{(0)}(s))+sdy^\sigma\wedge d\Psi_\sigma^{(0)}(s)+dy^\sigma\wedge\Psi'_\sigma(s)\\ &\quad +y^\sigma d\Psi'_\sigma(s))-dx^i\wedge d\Phi'_i(s)-dy^\sigma\wedge d(s\Psi'_\sigma(s)))\\ &=ds\wedge\left(-dx^i\wedge d\Phi_i^{(0)}(s))+dx^i\wedge\frac{\partial\Phi'_i(s)}{\partial s}-sdy^\sigma\wedge d\Psi_\sigma^{(0)}(s)\right.\\ &\quad\left.-dy^\sigma\wedge\Psi'_\sigma(s)-y^\sigma d\Psi'_\sigma(s)+dy^\sigma\wedge\frac{\partial(s\Psi'_\sigma(s))}{\partial s}\right)\\ &\quad -dx^i\wedge\left(dx^j\wedge\frac{\partial\Phi'_i(s)}{\partial x^j}+dy^\nu\wedge\frac{\partial\Phi'_i(s)}{\partial y^\nu}\right)\\ &\quad -dy^\sigma\wedge\left(dx^j\wedge\frac{\partial(s\Psi'_\sigma(s))}{\partial x^j}+dy^\nu\wedge\frac{\partial(s\Psi'_\sigma(s))}{\partial y^\nu}\right),\end{aligned}\tag{46}$$

where $\partial\eta(s)/\partial s$ denotes the form, arising by differentiation of $\eta(s)$ with respect to s, followed by multiplication by ds. Now by (45) and (30),

$$\begin{aligned}Id\rho &=-dx^i\wedge\int_0^1 d\Phi_i^{(0)}(s)-dy^\sigma\wedge\int_0^1 s\,d\Psi_\sigma^{(0)}(s)-dy^\sigma\wedge\int_0^1\Psi'_\sigma(s)\\ &\quad -y^\sigma\int_0^1 d\Psi'_\sigma(s)+dx^i\wedge\int_0^1\frac{\partial\Phi'_i(s)}{\partial s}+dy^\sigma\wedge\int_0^1\frac{\partial(s\Psi'_\sigma(s))}{\partial s}.\end{aligned}\tag{47}$$

Note that the expressions $d\Phi_i^{(0)}(s)$, $d\Psi_\sigma^{(0)}(s)$, and $d\Psi'_\sigma(s)$ have the meaning of the exterior derivatives with respect to x^i, y^σ (the terms containing ds are cancelled; see the definition of I (30), (31)).

Now

$$Id\rho+dI\rho=dx^i\wedge\int_0^1\frac{\partial\Phi'_i(s)}{\partial s}+dy^\sigma\wedge\int_0^1\frac{\partial(s\Psi'_\sigma(s))}{\partial s},\tag{48}$$

and using formula (30),

$$Id\rho+dI\rho=dx^i\wedge(\Phi'_i(1)-\Phi'_i(0))+dy^\sigma\wedge(1\cdot\Psi'_\sigma(1)-0\cdot\Psi'_\sigma(0))\tag{49}$$

$$= dx^i \wedge \Phi_i'(1) + dy^\sigma \wedge \Psi_\sigma'(1) - dx^i \wedge \Phi_i'(0)$$
$$= dx^i \wedge \Phi_i + dy^\sigma \wedge \Psi_\sigma - dx^i \wedge \pi^*\zeta^*\Phi_i$$
$$= \rho - \pi^*\zeta^*\rho.$$

As a consequence, we have the following statement.

Theorem 2 (The fibred Volterra-Poincare lemma) *Let $U \subset \mathbf{R}^n$ be an open set, $V \subset \mathbf{R}^m$ an open ball with centre 0. Let $k \geq 1$ and let ρ be a differential k-form on $U \times V$. The following two conditions are equivalent:*

(a) *There exist a $(k-1)$-form η on $U \times V$ and a k-form η_0 on U such that*

(50) $$d\eta + \pi^*\eta_0 = \rho.$$

(b) *The form $d\rho$ is π-projectable and its π-projection is $d\eta_0$.*

Proof Suppose we have some forms η and η_0 satisfying condition (a). Then $d\rho = d\pi^*\eta_0 = \pi^* d\eta_0$ proving (b).

Conversely, if $d\rho$ is π-projectable, then by the definition of *I*, $Id\rho = 0$, and then by Theorem 1, $\rho = Id\rho + dI\rho + \pi^*\zeta^*\rho = d\eta + \pi^*\eta_0$ proving (a).

We also get two assertions on *projectability* of forms, and non-uniqueness of solutions of equation (26).

Corollary 1 *Let $U \subset \mathbf{R}^n$ be an open set, $V \subset \mathbf{R}^m$ an open ball with centre the origin 0, ρ a differential form on $U \times V$. The following two conditions are equivalent:*

(1) *There exists a form η on U such that $\rho = \pi^*\eta$.*

(2) $Id\rho + dI\rho = 0$.

Proof This follows from Theorem 1.

Corollary 2 *Suppose that the form $d\rho$ is π-projectable. Let (η, η_0) and $(\tilde{\eta}, \tilde{\eta}_0)$ be two solutions of equation (26). Then there exist a $(p-1)$-form τ on $U \times V$ and a $(p-1)$-form χ on U such that*

(51) $$\tilde{\eta} = \eta + \pi^*\chi + d\tau, \quad \tilde{\eta}_0 = \eta_0 - d\chi.$$

Proof By hypothesis,

(52) $$d\eta + \pi^*\eta_0 = \rho, \quad d\tilde{\eta} + \pi^*\tilde{\eta}_0 = \rho.$$

These equations imply $d\eta + \pi^*\eta_0 = d\tilde{\eta} + \pi^*\tilde{\eta}_0$ hence $\pi^* d\eta_0 = \pi^* d\tilde{\eta}_0$. But for any section δ of the projection π,

(53) $$\delta^*\pi^* d\eta_0 = d\eta_0 = \delta^*\pi^* d\tilde{\eta}_0 = d\tilde{\eta}_0.$$

Thus, by the Volterra-Poincaré lemma, $\tilde{\eta}_0 - \eta_0 = d\chi$ for some χ. Then, however, $d\eta + \pi^*\eta_0 = d\tilde{\eta} + \pi^*(\eta_0 + d\chi)$, and

(54) $$d(\eta - \tilde{\eta} - \pi^*\chi) = 0.$$

Applying the Volterra-Poincaré lemma again we get (51).

Remark 1 (The Volterra-Poincare lemma on manifolds) Let X be an n-dimensional manifold. Every point $x \in X$ has a neighbourhood U such that the decomposition of forms, given in Theorem 1, is defined on U. Indeed, if (U,φ) is a chart at x such that $\varphi(U)$ is an open ball with centre $0 \in \mathbf{R}^n$, then formulas $\rho = \varphi^*\mu$ and $(\varphi^{-1})^*\rho = \mu$ establish a bijective correspondence between forms on U and $\varphi(U)$, commuting with the exterior derivative d. In general, this correspondence does not provide a construction of solutions of differential equations (1) and (26), defined *globally* on X.

Remark 2 For k-forms ρ such that $k = n$, always $d\eta_0 = 0$ hence $\eta_0 = d\tau$ and equation $d\eta + \pi^*\eta_0 = \rho$ (26) reduces to $d\eta = \rho$ (1). The same is true for $k > n$ because in this case $\eta_0 = 0$.

Turning back to the definition of the fibred homotopy operator I (31) we have the following explicit assertion.

Lemma 3 *Let ρ be a differential k-form on the product of open sets $U \times V$, considered as a fibered manifold over U, expressed in the canonical coordinates (x^i, y^σ) on $U \times V$ as*

(55) $$\rho = \frac{1}{p!} A_{\sigma_1\sigma_2\ldots\sigma_p\ i_1 i_2 \ldots i_q} dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \ldots \wedge dy^{\sigma_p} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p},$$

where $k = p + q$. Then the fibred homotopy operator I is given by

(56) $$I\rho = y^\sigma \int_0^1 (A_{\sigma\sigma_1\sigma_2\ldots\sigma_{p-1}\ i_1 i_2 \ldots i_q}(x^j, sy^\nu) s^{p-1}\, ds \cdot dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \ldots \wedge dy^{\sigma_{p-1}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}.$$

I satisfies

(57) $$I^2\rho = 0.$$

Proof The homotopy $(x^i, y^\sigma) \to \chi(s,(x^i, y^\sigma)) = (x^i, sy^\sigma)$ yields

(58) $$\begin{aligned}\chi^*\rho = \frac{1}{p!}(p y^\sigma (A_{\sigma\sigma_1\sigma_2\ldots\sigma_{p-1}\ i_1 i_2 \ldots i_q} \circ \chi) s^{p-1} ds \wedge dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \ldots \wedge dy^{\sigma_{p-1}} \\ + (A_{\sigma_1\sigma_2\ldots\sigma_p\ i_1 i_2 \ldots i_q} \circ \chi) s^p dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \ldots \wedge dy^{\sigma_p}) \\ \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}\end{aligned}$$

which implies that

$$\begin{aligned} I\rho &= (A_{\sigma_1\sigma_2\ldots\sigma_p\ i_1i_2\ldots i_q}\circ\chi)\chi^*(dy^{\sigma_1}\wedge dy^{\sigma_2}\wedge\ldots\wedge dy^{\sigma_p}) \\ &= y^\sigma\int_0^1(A_{\sigma\sigma_1\sigma_2\ldots\sigma_{p-1}\ i_1i_2\ldots i_q}\circ\chi)s^{p-1}\,ds\cdot dy^{\sigma_1}\wedge dy^{\sigma_2}\wedge\ldots\wedge dy^{\sigma_{p-1}} \\ &\quad\wedge dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_p}. \end{aligned} \tag{59}$$

Identity (57) is now an immediate consequence of formula (56).

7 Differential ideals

For basic concepts of the theory of differential ideals and related topics we refer to Bryant, Chern, Gardner, Goldschmidt and Griffiths [Br].

Let X be an n-dimensional smooth manifold. We denote by $\Lambda^p TX$ the bundle of alternating p-forms over X; in this notation, $\Lambda^1 TX = T^*X$ is the *cotangent bundle* of X. Sections of the bundle $\Lambda^p TX$, *differential p-forms* on X, form a *module* over the ring of functions, denoted by $\Omega_p X$. The direct sum

$$\Omega X = \Omega_0 X\oplus\Omega_1 X\oplus\Omega_2 X\oplus\ldots\oplus\Omega_n X \tag{1}$$

together with the exterior multiplication of forms is the *exterior algebra* of X. We usually consider elements of $\Omega_p X$ as elements of ΩX. The multiplication $\wedge$ in ΩX is *associative* and *distributive*, but *not* commutative; instead we have for any $\eta\in\Omega_p X$ and $\rho\in\Omega_q X$,

$$\eta\wedge\rho = (-1)^{pq}\rho\wedge\eta. \tag{2}$$

A subset $\Theta\subset\Omega X$ is called an *ideal*, if the following two conditions are satisfied:

(a) Θ is a subgroup of the additive group of ΩX.

(b) If $\eta\in\Theta$ and $\rho\in\Omega X$ then $\eta\wedge\rho\in\Theta$.

An ideal $\Theta\subset\Omega X$ is called a *differential ideal*, if for any $\eta\in\Theta$ also $d\eta\in\Theta$; thus, a differential ideal is an ideal *closed* under exterior derivative operation.

Any non-empty set $\theta\subset\Omega X$ generates a subgroup Θ_θ of the additive group of ΩX, formed by (finite) sums

$$\mu = \sum\eta_k\wedge\rho_k, \tag{3}$$

where $\eta_k\in\theta$ and $\rho_k\in\Omega X$. Θ_θ is an ideal, which is a subset of *any* ideal containing θ; it is said to be *generated* by the set θ (or by the *generators* $\eta\in\theta$). If the set θ is *finite*, we say that Θ_θ is *finitely generated*.

Let $\mathcal{V}X$ denote the module of vector fields on X. We denote

(4) $\mathscr{A}(\Theta) = \{\xi \in \mathscr{V}X \mid i_\xi \eta \subset \Theta, \eta \in \Theta\}.$

This set, the *Cauchy characteristic space* of Θ, has the structure of a subgroup of the additive group of $\mathscr{V}X$. The annihilator

(5) $\mathscr{C}(\Theta) = \{\mu \in \Omega X \mid i_\xi \mu = 0, \xi \in \mathscr{A}(\Theta)\}$

is the *retracting subspace* of Θ.

8 The Levi-Civita symbol

We introduce in this appendix a real-valued function, defined on the symmetric group $\tau \in S_n$, the *Levi-Civita symbol*, playing an essential role in algebraic computations with skew-symmetric expressions. We also derive basic computation formulas for the Levi-Civita symbol, needed in this book.

Any permutation $\tau \in S_n$ can be written as the composition of transpositions τ_k, i.e., $\tau = \tau_M \circ \tau_{M-1} \circ \ldots \circ \tau_2 \circ \tau_1$. This decomposition of τ is not unique, but the number $\operatorname{sgn}\tau = (-1)^M$, the *sign* of the permutation τ, is independent of the choice of the decomposition. If $\operatorname{sgn}\tau = 1$ (resp. $\operatorname{sgn}\tau = -1$), the permutation τ is called *even* (resp. *odd*). The function $S_k \ni \tau \to \operatorname{sgn}\tau \in \{1, -1\}$ is sometimes called the *sign function*. As an immediate consequence of the definition, we have

(1) $\operatorname{sgn}(\nu \cdot \tau) = \operatorname{sgn}\nu \cdot \operatorname{sgn}\tau$

for all permutations $\nu, \tau \in S_r$.

The *sign function* $\tau \to \operatorname{sgn}\tau$ can be considered as a function on the set of *distinct* n-tuples $(i_1, i_2, \ldots, i_n)$ of integers, such that $1 \le i_1, i_2, \ldots, i_n \le n$. We define the *Levi-Civita,* or *permutation symbol* $\varepsilon_{i_1 i_2 \ldots i_n}$ setting $\varepsilon_{i_1 i_2 \ldots i_n} = 1$ if the n-tuple $(i_1, i_2, \ldots, i_n)$ is an even permutation of $(1, 2, \ldots, n)$, $\varepsilon_{i_1 i_2 \ldots i_n} = -1$ if $(i_1, i_2, \ldots, i_n)$ is an odd permutation of $(1, 2, \ldots, n)$, and $\varepsilon_{i_1 i_2 \ldots i_n} = 0$ whenever at least two of the indices coincide. Clearly,

(2) $$\varepsilon_{i_1 i_2 \ldots i_n} = \sum_{\tau \in S_n} \operatorname{sgn}\tau \cdot \delta^1_{i_{\tau(1)}} \delta^2_{i_{\tau(2)}} \ldots \delta^n_{i_{\tau(n)}}.$$

Sometimes it is convenient to express this formula in a different form, without explicit mentioning the permutations τ. To this purpose we introduce the *alternation operation* in the indices $(i_1, i_2, \ldots, i_n)$, denoted $\operatorname{Alt}(i_1 i_2 \ldots i_n)$, by the formula

(3) $$\frac{1}{n!} \sum_{\tau \in S_n} \operatorname{sgn}\tau \cdot \delta^1_{i_{\tau(1)}} \delta^2_{i_{\tau(2)}} \ldots \delta^n_{i_{\tau(n)}} = \delta^1_{i_1} \delta^2_{i_2} \ldots \delta^n_{i_n} \quad \operatorname{Alt}(i_1 i_2 \ldots i_n).$$

It is understood in this formula that the operator $\operatorname{Alt}(i_1 i_2 \ldots i_n)$ is applied to the right-hand side expression, and replaces explicit expression on the left-hand side. From (3) we get, in particular,

(4) $$\varepsilon_{i_1 i_2 \ldots i_n} = n!\delta^1_{i_{\tau(1)}}\delta^2_{i_{\tau(2)}}\ldots\delta^n_{i_{\tau(n)}} \quad \mathrm{Alt}(i_1 i_2 \ldots i_n).$$

Formula (4) indicates that the Levi-Civita symbols $\varepsilon_{i_1 i_2 \ldots i_n}$ and $\varepsilon^{i_1 i_2 \ldots i_n}$ can be expressed by means of determinants. We have

(5) $$\varepsilon_{i_1 i_2 \ldots i_n} = \begin{vmatrix} \delta^1_{i_1} & \delta^2_{i_1} & \ldots & \delta^n_{i_1} \\ \delta^1_{i_2} & \delta^2_{i_2} & \ldots & \delta^n_{i_2} \\ \ldots & & & \\ \delta^1_{i_n} & \delta^2_{i_n} & \ldots & \delta^n_{i_n} \end{vmatrix}, \quad \varepsilon^{i_1 i_2 \ldots i_n} = \begin{vmatrix} \delta^{i_1}_1 & \delta^{i_2}_1 & \ldots & \delta^{i_n}_1 \\ \delta^{i_1}_2 & \delta^{i_2}_2 & \ldots & \delta^{i_n}_2 \\ \ldots & & & \\ \delta^{i_1}_n & \delta^{i_2}_n & \ldots & \delta^{i_n}_n \end{vmatrix}.$$

Clearly, multiplying these determinants, we get

(6) $$\varepsilon_{i_1 i_2 \ldots i_n}\varepsilon^{j_1 j_2 \ldots j_n} = \begin{vmatrix} \delta^1_{i_1} & \delta^2_{i_1} & \ldots & \delta^n_{i_1} \\ \delta^1_{i_2} & \delta^2_{i_2} & \ldots & \delta^n_{i_2} \\ \ldots & & & \\ \delta^1_{i_n} & \delta^2_{i_n} & \ldots & \delta^n_{i_n} \end{vmatrix} \begin{vmatrix} \delta^{j_1}_1 & \delta^{j_2}_1 & \ldots & \delta^{j_n}_1 \\ \delta^{j_1}_2 & \delta^{j_2}_2 & \ldots & \delta^{j_n}_2 \\ \ldots & & & \\ \delta^{j_1}_n & \delta^{j_2}_n & \ldots & \delta^{j_n}_n \end{vmatrix}$$
$$= \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \ldots & \delta^{j_n}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \ldots & \delta^{j_n}_{i_2} \\ \ldots & & & \\ \delta^{j_1}_{i_n} & \delta^{j_2}_{i_n} & \ldots & \delta^{j_n}_{i_n} \end{vmatrix} = n!\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\ldots\delta^{j_n}_{i_n} \quad \mathrm{Alt}(i_1 i_2 \ldots i_n).$$

Lemma 1 (a) *For every* k *such that* $1 \le k \le n$,

(7) $$\varepsilon_{i_1 i_2 \ldots i_k s_{k+1} s_{k+2} \ldots s_n}\varepsilon^{j_1 j_2 \ldots j_k s_{k+1} s_{k+2} \ldots s_n} = k!(n-k)!\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\ldots\delta^{j_k}_{i_k} \quad \mathrm{Alt}(i_1 i_2 \ldots i_k).$$

(b) *For every* k *such that* $0 \le k \le s \le n$,

(8) $$\frac{1}{s!}\binom{n-k}{n-s}\varepsilon_{j_1 j_2 \ldots j_k j_{k+1} j_{k+2} \ldots j_s i_{s+1} i_{s+2} \ldots i_n}\delta^{l_{k+1}}_{i_{k+1}}\delta^{l_{k+2}}_{i_{k+2}}\ldots\delta^{l_s}_{i_s} \quad \mathrm{Alt}(i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n)$$
$$= \frac{1}{k!(s-k)!}\delta^{l_{k+1}}_{j_{k+1}}\delta^{l_{k+2}}_{j_{k+2}}\ldots\delta^{l_s}_{j_s}\varepsilon_{j_1 j_2 \ldots j_k i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n}$$
$$\mathrm{Alt}(j_1 j_2 \ldots j_k j_{k+1} j_{k+2} \ldots j_s).$$

Proof 1. Setting

$$\Delta^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots i_l} = \delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\ldots\delta^{j_l}_{i_l} \quad \mathrm{Alt}(i_1 i_2 \ldots i_l), \tag{9}$$

we have

$$\begin{aligned}
\Delta^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots i_l} &= \delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\ldots\delta^{j_{l-1}}_{i_{l-1}}\delta^{j_l}_{i_l} \quad \mathrm{Alt}(i_1 i_2 \ldots i_{l-1}) \quad \mathrm{Alt}(i_1 i_2 \ldots i_l)\\
&= \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_2 \ldots i_{l-1}}\delta^{j_l}_{i_l} \quad \mathrm{Alt}(i_1 i_2 \ldots i_l)\\
&= \frac{1}{l}(\Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_2 \ldots i_{l-1}}\delta^{j_l}_{i_l} - \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_l i_2 i_3 \ldots i_{l-1}}\delta^{j_l}_{i_1} - \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_l i_3 i_4 \ldots i_{l-1}}\delta^{j_l}_{i_2}\\
&\quad - \ldots - \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_2 \ldots i_{l-3} i_l i_{l-1}}\delta^{j_l}_{i_{l-2}} - \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_2 \ldots i_{l-3} i_{l-2} i_l}\delta^{j_l}_{i_{l-1}}).
\end{aligned} \tag{10}$$

Note that contracting this expression we obtain

$$\Delta^{i_1 i_2 \ldots i_{l-1} s}_{i_1 i_2 \ldots i_{l-1} s} = \frac{n-l+1}{l}\Delta^{i_1 i_2 \ldots i_{l-1}}_{i_1 i_2 \ldots i_{l-1}}. \tag{11}$$

Now formula (6) can be written in the form

$$\varepsilon_{i_1 i_2 \ldots i_n}\varepsilon^{j_1 j_2 \ldots j_n} = n!\Delta^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}. \tag{12}$$

Contracting (12) in one pair of indices, we get

$$\varepsilon_{i_1 i_2 \ldots i_{n-1} s}\varepsilon^{j_1 j_2 \ldots j_{n-1} s} = n!\Delta^{j_1 j_2 \ldots j_{n-1} s}_{i_1 i_2 \ldots i_{n-1} s} = (n-1)!1!\Delta^{j_1 j_2 \ldots j_{n-1}}_{i_1 i_2 \ldots i_{n-1}}, \tag{13}$$

proving (7) for $k=1$. After $n-k$ contractions we obtain

$$\Delta^{j_1 j_2 \ldots j_k s_{k+1} s_{k+2} \ldots s_n}_{i_1 i_2 \ldots i_k s_{k+1} s_{k+2} \ldots s_n} = \frac{1}{(n-n+k)!}\Delta^{j_1 j_2 \ldots j_k}_{i_1 i_2 \ldots i_k} = \frac{1}{k!}\Delta^{j_1 j_2 \ldots j_k}_{i_1 i_2 \ldots i_k}, \tag{14}$$

which leads to (7).

2. To prove formula (8), consider the tensors

$$\begin{aligned}
&\binom{n-k}{n-s}\frac{1}{s!}\varepsilon_{j_1 j_2 \ldots j_k j_{k+1} j_{k+2} \ldots j_s i_{s+1} i_{s+2} \ldots i_n}\delta^{l_{k+1}}_{i_{k+1}}\delta^{l_{k+2}}_{i_{k+2}}\ldots\delta^{l_s}_{i_s}\\
&\quad \mathrm{Alt}(i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\frac{1}{k!(s-k)!}\varepsilon_{j_1 j_2 \ldots j_k i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n}\delta^{l_{k+1}}_{j_{k+1}}\delta^{l_{k+2}}_{j_{k+2}}\ldots\delta^{l_s}_{j_s}\\
&\quad \mathrm{Alt}(j_1 j_2 \ldots j_k j_{k+1} j_{k+2} \ldots j_s).
\end{aligned} \tag{16}$$

Suppose that the component (15) is different from 0. Then

(a) the set $\{i_{k+1}, i_{k+2}, \ldots, i_s, i_{s+1}, i_{s+2}, \ldots, i_n\}$ consists of distinct elements,
(b) the set $\{j_1, j_2, \ldots, j_k, j_{k+1}, j_{k+2}, \ldots, j_s\}$ consists of distinct elements,
(c) the set $\{l_{k+1}, l_{k+2}, \ldots, l_s\}$ satisfies

$$(17)\qquad \begin{aligned}&\{i_{k+1},i_{k+2},\dots,i_s,i_{s+1},i_{s+2},\dots,i_n\}\cap\{j_1,j_2,\dots,j_k,j_{k+1},j_{k+2},\dots,j_s\}\\ &\quad=\{l_{k+1},l_{k+2},\dots,l_s\}.\end{aligned}$$

Take $j_{k+1}=l_{k+1}$, $j_{k+2}=l_{k+2}$, $\dots$, $j_s=l_s$. Then (15) reduces to

$$(18)\qquad \begin{aligned}&\binom{n-k}{n-s}\frac{1}{s!}\varepsilon_{j_1j_2\dots j_kl_{k+1}l_{k+2}\dots _s i_{s+1}i_{s+2}\dots i_n}\delta^{l_{k+1}}_{i_{k+1}}\delta^{l_{k+2}}_{i_{k+2}}\dots\delta^{l_s}_{i_s}\\ &\quad\mathrm{Alt}(i_{k+1}i_{k+2}\dots i_si_{s+1}i_{s+2}\dots i_n).\end{aligned}$$

There exist exactly one $(s-k)$-tuple in the set $i_{k+1},i_{k+2},\dots,i_s,i_{s+1},i_{s+2},\dots,i_n$, say $i_{k+1},i_{k+2},\dots,i_s$ such that $\delta^{l_{k+1}}_{i_{k+1}}\delta^{l_{k+2}}_{i_{k+2}}\dots\delta^{l_s}_{i_s}=1$. Then

$$(19)\qquad i_{k+1}=j_{k+1}=l_{k+1},\quad i_{k+2}=j_{k+2}=l_{k+2},\quad \dots,\quad i_s=j_s=l_s,$$

and (19) gives the expression

$$(20)\qquad \frac{(n-s)!}{(n-k)!}\binom{n-k}{n-s}\frac{1}{s!}\varepsilon_{j_1j_2\dots j_kl_{k+1}l_{k+2}\dots _s i_{s+1}i_{s+2}\dots i_n}=\frac{1}{s!(s-k)!}\varepsilon_{j_1j_2\dots j_kl_{k+1}l_{k+2}\dots _s i_{s+1}i_{s+2}\dots i_n}.$$

Compute now (16) for the same indices, satisfying conditions (20). We get

$$(21)\qquad \begin{aligned}&\frac{1}{k!(s-k)!}\frac{k!}{s!}\delta^{l_{k+1}}_{j_{k+1}}\delta^{l_{k+2}}_{j_{k+2}}\dots\delta^{l_s}_{j_s}\varepsilon_{j_1j_2\dots j_kl_{k+1}l_{k+2}\dots l_si_{s+1}i_{s+2}\dots i_n}\\ &\quad=\frac{1}{s!(s-k)!}\varepsilon_{j_1j_2\dots j_kl_{k+1}l_{k+2}\dots l_si_{s+1}i_{s+2}\dots i_n}.\end{aligned}$$

This shows that if the component (15) is different from 0, then also the component (16) is different from 0, and is equal to (15).

Conversely, if (16) is different from 0, then

$$(22)\qquad \begin{aligned}&\frac{1}{k!(s-k)!}\delta^{l_{k+1}}_{j_{k+1}}\delta^{l_{k+2}}_{j_{k+2}}\dots\delta^{l_s}_{j_s}\varepsilon_{j_1j_2\dots j_ki_{k+1}i_{k+2}\dots i_si_{s+1}i_{s+2}\dots i_n}\\ &\quad\mathrm{Alt}(j_1j_2\dots j_kj_{k+1}j_{k+2}\dots j_s),\end{aligned}$$

we obtain again conditions (a), (b), and (c).

Corollary 1 *If* $k=n$, (7) *coincides with* (6). *If* $k=0$, *we have*

$$(23)\qquad \varepsilon_{s_1s_2\dots s_n}\varepsilon^{s_1s_2\dots s_n}=n!.$$

Corollary 2 (Bases of forms) *Let X be an n-dimensional smooth manifold, and let* (U,φ), $\varphi=(x^i)$, *be a chart on X. Then the forms*

$$(24)\qquad \omega_0=\frac{1}{n!}\varepsilon_{i_1i_2\dots i_n}dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_n},$$

$$\omega_{k_1k_2\dots k_p} = \frac{1}{(n-p)!}\varepsilon_{k_1k_2\dots k_{p-1}k_pi_{p+1}i_{p+2}\dots i_n} dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \dots \wedge dx^{i_n}, \tag{24}$$
$$1 \le p \le n-1,$$

define bases of n-forms, $(n-1)$ *-forms,* $\dots$ *,* 2*-forms, and* 1*-forms, respectively. The inverse transformation formulas are*

$$\begin{aligned} &\varepsilon^{l_1l_2\dots l_n}\omega_0 = dx^{l_1} \wedge dx^{l_2} \wedge \dots \wedge dx^{l_n}, \\ &\varepsilon^{k_1k_2\dots k_pl_{p+1}l_{p+2}\dots l_n}\omega_{k_1k_2\dots k_p} = dx^{l_{p+1}} \wedge dx^{l_{p+2}} \wedge \dots \wedge dx^{l_n}, \\ &1 \le p \le n \end{aligned} \tag{25}$$

Proof Immediate: The forms (24) are defined by

$$\begin{aligned} &\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n, \quad \omega_{k_1} = i_{\partial/\partial x^{k_1}}\omega_0, \quad \omega_{k_1k_2} = i_{\partial/\partial x^{k_2}}\omega_{k_1}, \\ &\dots, \quad \omega_{k_1k_2\dots k_p} = i_{\partial/\partial x^{k_p}}\omega_{k_1k_2\dots k_{p-1}}, \quad \dots, \quad \omega_{k_1k_2\dots k_{n-1}} = i_{\partial/\partial x^{k_{n-1}}}\omega_{k_1k_2\dots k_{n-2}}, \end{aligned} \tag{26}$$

and are linearly independent.

9 The trace decomposition

This appendix is devoted to specific algebraic methods, used in the decomposition theory of differential forms on jet manifolds. We present elementary trace decomposition formulas and and their proofs (Krupka [K15]).

Beside the usual index notation we also use multi-indices of the form $I = (i_1i_2\dots i_k)$, where r and n are positive integers, $k = 0,1,2,\dots,r$, and $1 \le i_1, i_2, \dots, i_k \le n$. The number k is called the *length* of I and is denoted by $|I|$. For any index j, such that $1 \le j \le n$ we denote by Ij the multi-index $(i_1i_2\dots i_kj)$. The symbol $\mathrm{Alt}(i_1i_2\dots i_k)$ (resp. $\mathrm{Sym}(i_1i_2\dots i_k)$) denotes *alternation* (resp. *symmetrisation*) in the indices i_1, i_2, $\dots$, i_k.

Let E be an n-dimensional vector space, E^* its dual vector space, and let r and s be two non-negative integers; suppose that at least one of these integers is non-zero. Then by a *tensor of type* (r,s) over E we mean a multilinear mapping $U: E^* \times E^* \times \dots \times E^* \times E \times E \times \dots \times E \to \mathbf{R}$ (r factors E^*, s factors E); r (resp. s) is called the *contravariant* (resp. *covariant*) *degree* of U. A tensor of type $(r,0)$ (resp. $(0,s)$) is called *contravariant* (*covariant*) of degree r (resp. s). The set of tensors of type (r,s) considered with its natural real vector space structure, is called the *tensor space of type* (r,s) over E, and is denoted by T^r_sE.

Let e_i be a basis of the vector space E, e^i the dual basis of E^*. The tensors $\mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \dots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \dots \otimes \mathrm{e}^{i_s}$, $1 \le j_1, j_2, \dots, j_r, i_1, i_2, \dots, i_s \le n$, form a *basis* of the vector space T^r_sE. Each tensor $U \in T^r_sE$ has a unique expression

(1) $$U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}\, \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \ldots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \ldots \otimes \mathrm{e}^{i_s},$$

where the numbers $U^{i_1 i_2 \ldots i_r}{}_{i_1 i_2 \ldots i_s}$ are the *components* of *U* in the basis e_i.

Remark 1 If a basis of the vector space *E* is fixed, it is sometimes convenient to denote the tensors simply by their components; in this case a tensor *U* of type (r,s) over *E* is usually written as

(2) $$U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}.$$

Remark 2 The *canonical basis* of the vector space $E = \mathbf{R}^n$ consists of the vectors $\mathrm{e}_1 = (1,0,0,\ldots,0)$, $\mathrm{e}_2 = (0,1,0,0,\ldots,0)$, $\ldots$, $\mathrm{e}_n = (0,0,\ldots,0,1)$. The basis of the tensor space $T_s^r \mathbf{R}^n$, associated with $(\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n)$ is also called *canonical*. A tensor $U \in T_s^r \mathbf{R}^n$ can be expressed either by formula (1) or by (2); these formulas define the *canonical identification* of the vector space $T_s^r \mathbf{R}^n$ with the vector space $\mathbf{R}^N$ of the collections $U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}$, where $N = \dim T_s^r \mathbf{R}^n = n^{rs}$.

Remark 3 The *transformation equations* for the associated bases in $T_s^r E$ are easily derived from the transformation equations for bases of the vector space *E*. Suppose we have two bases e_i and $\overline{\mathrm{e}}_i$ of *E*. Let $\overline{\mathrm{e}}_i = A_i^p \mathrm{e}_p$ and $\overline{\mathrm{e}}^i = B_p^i \mathrm{e}^p$ be the corresponding transformation equations. Then

(3) $$A_i^q B_p^i = \delta_p^q,$$

where δ_q^p is the *Kronecker symbol*, $\delta_p^p = 1$ and $\delta_q^p = 0$ if $p \neq q$, and

(4) $$\begin{aligned} &\overline{\mathrm{e}}_{j_1} \otimes \overline{\mathrm{e}}_{j_2} \otimes \ldots \otimes \overline{\mathrm{e}}_{j_r} \otimes \overline{\mathrm{e}}^{i_1} \otimes \overline{\mathrm{e}}^{i_2} \otimes \ldots \otimes \overline{\mathrm{e}}^{i_s} \\ &\quad = A_{j_1}^{p_1} A_{j_2}^{p_2} \ldots A_{j_r}^{p_r} B_{q_1}^{i_1} B_{q_2}^{i_2} \ldots B_{q_s}^{i_s}\, \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \otimes \ldots \otimes \mathrm{e}_{p_r} \otimes \mathrm{e}^{q_1} \otimes \mathrm{e}^{q_2} \otimes \ldots \otimes \mathrm{e}^{q_s}. \end{aligned}$$

Expressing a tensor $U \in T_s^r E$ as in (1), we have

(5) $$\begin{aligned} U &= \overline{U}^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}\, \overline{\mathrm{e}}_{j_1} \otimes \overline{\mathrm{e}}_{j_2} \otimes \ldots \otimes \overline{\mathrm{e}}_{j_r} \otimes \overline{\mathrm{e}}^{i_1} \otimes \overline{\mathrm{e}}^{i_2} \otimes \ldots \otimes \overline{\mathrm{e}}^{i_s} \\ &= U^{p_1 p_2 \ldots p_r}{}_{q_1 q_2 \ldots q_s}\, \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \otimes \ldots \otimes \mathrm{e}_{p_r} \otimes \mathrm{e}^{q_1} \otimes \mathrm{e}^{q_2} \otimes \ldots \otimes \mathrm{e}^{q_s}. \end{aligned}$$

Clearly, then

(6) $$U^{p_1 p_2 \ldots p_r}{}_{q_1 q_2 \ldots q_s} = A_{j_1}^{p_1} A_{j_2}^{p_2} \ldots A_{j_r}^{p_r} B_{q_1}^{i_1} B_{q_2}^{i_2} \ldots B_{q_s}^{i_s} \overline{U}^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}.$$

The *Kronecker tensor* over *E* is a $(1,1)$-tensor δ, defined in any basis of *E* as

(7) $$\delta = \mathrm{e}_i \otimes \mathrm{e}^i.$$

It is immediately seen that the tensor δ does not depend on the choice of the basis e_i. We can also write $\delta = \delta^i_j \mathrm{e}_i \otimes \mathrm{e}^j$, where δ^i_j is the *Kronecker symbol* (Remark 3).

This definition can be extended to tensors of type (r,s) for any positive integers r and s. Let α and β be integers such that $1 \le \alpha \le r$, $1 \le \beta \le s$, and let e_i be a basis of E. We introduce a linear mapping $\iota^\alpha_\beta : T^{r-1}_{s-1}E \to T^r_s E$ as follows. For every $V \in T^{r-1}_{s-1}E$,

(8) $$V = V^{j_1 j_2 \ldots j_{r-1}}{}_{i_1 i_2 \ldots i_{s-1}} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \ldots \otimes \mathrm{e}_{j_{r-1}} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \ldots \otimes \mathrm{e}^{i_{s-1}},$$

define a tensor $\iota^\alpha_\beta V \in T^r_s E$ by

(9) $$\iota^\alpha_\beta V = W^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \ldots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \ldots \otimes \mathrm{e}^{i_s},$$

where

(10) $$W^{j_1 j_2 \ldots j_{\alpha-1} j_\alpha j_{\alpha+1} \ldots j_r}{}_{i_1 i_2 \ldots i_{\beta-1} i_\beta i_{\beta+1} \ldots i_s} = \delta^{j_\alpha}_{i_\beta} V^{j_1 j_2 \ldots j_{\alpha-1} j_{\alpha+1} \ldots j_r}{}_{i_1 i_2 \ldots i_{\beta-1} i_{\beta+1} \ldots i_s}.$$

Thus,

(11) $$\begin{aligned} \iota^\alpha_\beta V = V^{j_1 j_2 \ldots j_{r-1}}{}_{i_1 i_2 \ldots i_{s-1}} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \ldots \otimes \mathrm{e}_{j_{\alpha-1}} \otimes \mathrm{e}_s \otimes \mathrm{e}_{j_{\alpha+1}} \otimes \ldots \otimes \mathrm{e}_{j_r} \\ \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \ldots \otimes \mathrm{e}^{i_{\beta-1}} \otimes \mathrm{e}^s \otimes \mathrm{e}^{i_{\beta+1}} \otimes \ldots \otimes \mathrm{e}^{i_s} \end{aligned}$$

(summation through s on the right-hand side). It is easily verified that this tensor is independent of the choice of e_i.

The mapping ι^α_β defined by formulas (9), (10) is the (α,β)*-canonical injection*. A tensor $U \in T^r_s E$, belonging to the vector subspace generated by the subspaces $\iota^\alpha_\beta(T^{r-1}_{s-1}E) \subset T^r_s E$, where $1 \le \alpha \le r$ and $1 \le \beta \le s$, is called a *Kronecker tensor*, or a tensor of *Kronecker type*.

A tensor $V \in T^r_s E$, $V = V^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s}$, is a Kronecker tensor if and only if there exist some tensors $V^{(p)}_{(q)} \in T^{r-1}_{s-1}E$, $V^{(p)}_{(q)} = V^{(p) k_1 k_2 \ldots k_{r-1}}_{(q) \; l_1 l_2 \ldots l_{s-1}}$, where the indices satisfy $1 \le p \le r$, $1 \le q \le s$, such that $V^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s}$ can be expressed in the form

(12) $$\begin{aligned} & V^{j_1 j_2 \ldots j_r}{}_{l_1 l_2 \ldots l_s} \\ & = \delta^{j_1}_{l_1} V^{(1) j_2 j_3 \ldots j_r}_{(1) \; l_2 l_3 \ldots l_s} + \delta^{j_1}_{l_2} V^{(1) j_2 j_3 \ldots j_r}_{(2) \; l_1 l_3 \ldots l_s} + \ldots + \delta^{j_1}_{l_s} V^{(1) j_2 j_3 \ldots j_r}_{(s) \; l_1 l_2 \ldots l_{s-1}} \\ & + \delta^{j_2}_{l_1} V^{(2) j_1 j_3 \ldots j_r}_{(1) \; l_2 l_3 \ldots l_s} + \delta^{j_2}_{l_2} V^{(2) j_1 j_3 \ldots j_r}_{(2) \; l_1 l_3 \ldots l_s} + \ldots + \delta^{j_2}_{l_s} V^{(2) j_1 j_3 \ldots j_r}_{(s) \; l_1 l_2 \ldots l_{s-1}} \\ & + \ldots \\ & + \delta^{j_r}_{l_1} V^{(r) j_1 j_2 \ldots j_{r-1}}_{(1) \; l_2 l_3 \ldots l_s} + \delta^{j_r}_{l_2} V^{(r) j_1 j_2 \ldots j_{r-1}}_{(2) \; l_1 l_3 \ldots l_s} + \ldots + \delta^{j_r}_{l_s} V^{(r) j_1 j_2 \ldots j_{r-1}}_{(s) \; l_1 l_2 \ldots l_{s-1}}. \end{aligned}$$

A tensor $U \in T^r_s E$ expressed as in (1), is said to be *traceless*, if its traces are all zero,

$$
(13)\qquad
\begin{aligned}
&U^{sl_1l_2\ldots l_{r-1}}{}_{sj_1j_2\ldots j_{s-1}}=0,\quad U^{l_1sl_2\ldots l_{r-1}}{}_{sj_1j_2\ldots j_{s-1}}=0,\quad \ldots,\quad U^{l_1l_2\ldots l_{r-1}s}{}_{sj_1j_2\ldots j_{s-1}}=0,\\
&U^{sl_1l_2\ldots l_{r-1}}{}_{j_1sj_2\ldots j_{s-1}}=0,\quad U^{l_1sl_2\ldots l_{r-1}}{}_{j_1sj_2\ldots j_{s-1}}=0,\quad \ldots,\quad U^{l_1l_2\ldots l_{r-1}s}{}_{j_1sj_2\ldots j_{s-1}}=0,\\
&\ldots\\
&U^{sl_1l_2\ldots l_{r-1}}{}_{j_1j_2\ldots j_{s-1}s}=0,\quad U^{l_1sl_2\ldots l_{r-1}}{}_{j_1j_2\ldots j_{s-1}s}=0,\quad \ldots,\quad U^{l_1l_2\ldots l_{r-1}s}{}_{j_1j_2\ldots j_{s-1}s}=0.
\end{aligned}
$$

To prove a theorem of the decomposition of the tensor space T_s^rE by the trace operation, recall that every scalar product g on the vector space E induces a scalar product on T_s^rE as follows. Let g be expressed in a basis as

$$(14)\qquad g(\xi,\zeta)=g_{ij}\xi^i\zeta^j,$$

where $\xi=\xi^i$, $\zeta=\zeta^i$ are any vectors from E. Let $U,V\in T_s^rE$ be any tensors, $U=U^{j_1j_2\ldots j_r}{}_{i_1i_2\ldots i_s}$, $V=V^{i_1i_2\ldots i_r}{}_{j_1j_2\ldots j_s}$. We define a bilinear form on T_s^rE, denoted by the same letter, g, by

$$(15)\qquad \begin{aligned} &g(U,V)\\ &\quad=g_{j_1k_1}g_{j_2k_2}\ldots g_{j_rk_r}g^{i_1l_1}g^{i_2l_2}\ldots g^{i_sl_s}U^{j_1j_2\ldots j_r}{}_{i_1i_2\ldots i_s}V^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_s l_1l_2\ldots l_s}.\end{aligned}$$

Lemma 1 *Formula* (15) *defines a scalar product on the tensor space* T_s^rE.

Proof Only positive definiteness of the bilinear form (15) needs proof. If we choose a basis of E such that $g_{jk}=\delta_{jk}$, then $g(U,V)$ (15) has an expression

$$(16)\qquad g(U,V)=\sum_{k_1,k_2,\ldots,k_r}\sum_{l_1,l_2,\ldots,l_s}U^{j_1j_2\cdots j_r}{}_{l_1l_2\ldots l_s}V^{j_1j_2\cdots j_r}{}_{l_1l_2\ldots l_s}.$$

Obviously, this is the Euclidean scalar product, which is positive definite.

Theorem 1 (The trace decomposition theorem) *The vector space T_s^rE is the direct sum of its vector subspaces of traceless and Kronecker tensors.*

Proof We want to show that any tensor $W\in T_s^rE$, has a unique decomposition of the form $W=U+V$, where U is traceless and V is of Kronecker type. To prove existence of the decomposition, consider a scalar product g (16) on T_s^rE. It is immediately seen that the orthogonal complement of the subspace of Kronecker tensors coincides with the subspace of traceless tensors. Indeed, if $U\in T_s^rE$, $U=U^{i_1i_2\ldots i_r}{}_{j_1j_2\ldots j_s}$, then calculating the scalar product $g(U,V)$ for any tensor $V\in T_s^rE$, $V=V^{k_1k_2\ldots k_r}{}_{l_1l_2\ldots l_s}$, satisfying condition (12), the condition

$$(17)\qquad g(U,V)=0$$

implies that U must be traceless. The uniqueness of the direct sum follows from the orthogonality of subspaces of traceless and Kronecker tensors in $T^r_s E$ with respect to the scalar product g.

Theorem 1 states that every tensor $W \in T^r_s E$, $W = W^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s}$ is expressible in the form

$$
\begin{aligned}
W^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s} &= U^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s} \\
&+ \delta^{i_1}_{l_1} V^{(1)\, i_2 i_3 \dots i_r}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_1}_{l_2} V^{(1)\, i_2 i_3 \dots i_r}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{i_1}_{l_s} V^{(1)\, i_2 i_3 \dots i_r}_{(s)}{}_{l_1 l_2 \dots l_{s-1}} \\
&+ \delta^{i_2}_{l_1} V^{(2)\, i_1 i_3 \dots i_r}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_2}_{l_2} V^{(2)\, i_1 i_3 \dots i_r}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{i_2}_{l_s} V^{(2)\, i_1 i_3 \dots i_r}_{(s)}{}_{l_1 l_2 \dots l_{s-1}} \\
&+ \dots \\
&+ \delta^{i_r}_{l_1} V^{(r)\, i_1 i_2 \dots i_{r-1}}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_r}_{l_2} V^{(r)\, i_1 i_2 \dots i_{r-1}}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{k_r}_{l_s} V^{(r)\, i_1 i_2 \dots i_{r-1}}_{(s)}{}_{l_1 l_2 \dots l_{s-1}},
\end{aligned}
\tag{18}
$$

where $U = U^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s}$ is a uniquely defined traceless tensor, and for every p and q such that $1 \le p \le r$, $1 \le q \le s$, the tensor $V^{(p)}_{(q)} = V^{(p)\, i_1 i_2 \dots i_{r-1}}_{(q)}{}_{l_1 l_2 \dots l_{s-1}}$ belongs to the tensor space $T^{r-1}_{s-1} E$.

Remark 4 The traceless component $U^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s}$ and the complementary Kronecker component of the tensor W in (18) are determined uniquely. However, this does not imply, in general, that the tensors $V^{(p)}_{(q)}$ are unique. If the contravariant and covariant degrees satisfy $r + s \le n + 1$, then the tensors $V^{(p)}_{(q)}$ may not be unique.

Formula (18) is called the *trace decomposition formula*.

Denote by E^r_s the vector subspace of tensors $U = U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s}$ in the tensor space $T^r_s E$, *symmetric* in the superscripts and *skew-symmetric* in the subscripts; sometimes these tensors are *symmetric-skew-symmetric*. We wish to find the trace decomposition formula for the tensors, belonging to the tensor space E^r_s. Set

$$\operatorname{tr} U = U^{k j_1 j_2 \dots j_{r-1}}{}_{k i_1 i_2 \dots i_{s-1}}, \tag{19}$$

and

$$
\begin{aligned}
\mathbf{q} U = \frac{(r+1)(s+1)}{n+r-s} \delta^{j_1}_{i_1} U^{j_2 j_3 \dots j_{r+1}}{}_{i_2 i_3 \dots i_{s+1}} \quad & \operatorname{Alt}(i_1 i_2 \dots i_{s+1}) \\
\operatorname{Sym}(j_1 j_2 \dots j_{r+1}).&
\end{aligned}
\tag{20}
$$

These formulas define two linear mappings $\operatorname{tr} : E^r_s \to E^{r-1}_{s-1}$ and $\mathbf{q} : E^r_s \to E^{r+1}_{s+1}$.

Theorem 2 (a) *Any tensor* $U \in E^r_s$ *has a decomposition*

$$U = \operatorname{tr} \mathbf{q} U + \mathbf{q} \operatorname{tr} U. \tag{21}$$

(b) *The mappings* tr *and* $\mathbf{q}$ *satisfy*

$$\operatorname{tr}\operatorname{tr} U = 0, \quad \mathbf{q}\mathbf{q} U = 0. \tag{22}$$

Proof (a) Using (20) we have, with obvious notation,

$$\mathbf{q}U = \frac{r+1}{n+r-s}(\delta_{i_1}^{j_1} U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - \delta_{i_2}^{j_1} U^{j_2 j_3 \cdots j_{r+1}}{}_{i_1 i_3 i_4 \ldots i_{s+1}} \tag{23}$$
$$- \delta_{i_3}^{j_1} U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_1 i_4 i_5 \ldots i_{s+1}} - \ldots - \delta_{i_{s+1}}^{j_1} U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_s i_1}) \quad \mathrm{Sym}(j_1 j_2 \ldots j_{r+1}).$$

Thus,

$$\operatorname{tr}\mathbf{q}U = \frac{1}{n+r-s}(\delta_k^k U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - \delta_{i_2}^k U^{j_2 j_3 \cdots j_{r+1}}{}_{k i_3 i_4 \ldots i_{s+1}} \tag{24}$$
$$- \delta_{i_3}^k U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 k i_4 i_5 \ldots i_{s+1}} - \ldots - \delta_{i_{s+1}}^k U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_s k}$$
$$+ \delta_k^{j_2} U^{k j_3 j_4 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - \delta_{i_2}^{j_2} U^{k j_3 j_4 \cdots j_{r+1}}{}_{k i_3 i_4 \ldots i_{s+1}}$$
$$- \delta_{i_3}^{j_2} U^{k j_3 j_4 \cdots j_{r+1}}{}_{i_2 k i_4 i_5 \ldots i_{s+1}} - \ldots - \delta_{i_{s+1}}^{j_2} U^{k j_3 j_4 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_s k}$$
$$+ \delta_k^{j_3} U^{j_2 k j_4 j_5 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - \delta_{i_2}^{j_3} U^{j_2 k j_4 j_5 \cdots j_{r+1}}{}_{k i_3 i_4 \ldots i_{s+1}}$$
$$- \delta_{i_3}^{j_3} U^{j_2 k j_4 j_5 \cdots j_{r+1}}{}_{i_2 k i_4 i_5 \ldots i_{s+1}} - \ldots - \delta_{i_{s+1}}^{j_3} U^{j_2 k j_4 j_5 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_s k}$$
$$+ \ldots + \delta_k^{j_{r+1}} U^{j_2 j_3 \cdots j_r k}{}_{i_2 i_3 \ldots i_{s+1}} - \delta_{i_2}^{j_{r+1}} U^{j_2 j_3 \cdots j_r k}{}_{k i_3 i_4 \ldots i_{s+1}}$$
$$- \delta_{i_3}^{j_{r+1}} U^{j_2 j_3 \cdots j_r k}{}_{i_2 k i_4 i_5 \ldots i_{s+1}} - \ldots - \delta_{i_{s+1}}^{j_{r+1}} U^{j_2 j_3 \cdots j_r k}{}_{i_2 i_3 \ldots i_s k}).$$

Computing the traces we get

$$\operatorname{tr}\mathbf{q}U = \frac{1}{n+r-s}(n U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 i_4 \ldots i_{s+1}} - U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 i_4 i_5 \ldots i_{s+1}} \tag{25}$$
$$- \ldots - U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_s i_{s+1}} + U^{j_2 j_3 j_4 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - \delta_{i_2}^{j_2} U^{k j_3 j_4 \cdots j_{r+1}}{}_{k i_3 i_4 \ldots i_{s+1}}$$
$$- \delta_{i_3}^{j_2} U^{k j_3 j_4 \cdots j_{r+1}}{}_{i_2 k i_4 i_5 \ldots i_{s+1}} - \ldots - \delta_{i_{s+1}}^{j_2} U^{k j_3 j_4 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_s k}$$
$$+ U^{j_2 j_3 j_4 j_5 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - \delta_{i_2}^{j_3} U^{j_2 k j_4 j_5 \cdots j_{r+1}}{}_{k i_3 i_4 \ldots i_{s+1}}$$
$$- \delta_{i_3}^{j_3} U^{j_2 k j_4 j_5 \cdots j_{r+1}}{}_{i_2 k i_4 i_5 \ldots i_{s+1}} - \ldots - \delta_{i_{s+1}}^{j_3} U^{j_2 k j_4 j_5 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_s k}$$
$$+ \ldots + U^{j_2 j_3 \cdots j_r j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - \delta_{i_2}^{j_{r+1}} U^{j_2 j_3 \cdots j_r k}{}_{k i_3 i_4 \ldots i_{s+1}}$$
$$- \delta_{i_3}^{j_{r+1}} U^{j_2 j_3 \cdots j_r k}{}_{i_2 k i_4 i_5 \ldots i_{s+1}} - \ldots - \delta_{i_{s+1}}^{j_{r+1}} U^{j_2 j_3 \cdots j_r k}{}_{i_2 i_3 \ldots i_s k}).$$

Further straightforward calculations yield

$$\operatorname{tr}\mathbf{q}U = U^{j_2 j_3 \cdots j_{r+1}}{}_{i_2 i_3 \ldots i_{s+1}} - \frac{rs}{n+r-s}\delta_{i_2}^{j_2} U^{k j_3 j_4 \cdots j_{r+1}}{}_{k i_3 i_4 \ldots i_{s+1}} \tag{26}$$
$$\mathrm{Sym}(j_2 j_3 \ldots j_{r+1}) \quad \mathrm{Alt}(i_2 i_3 \ldots i_{s+1}).$$

But by (19), the second term is exactly $\mathbf{q}\operatorname{tr}u$, proving (21).

(b) Formulas (22) are immediate.

Formula (21) is the *trace decomposition formula* for tensors $U \in E_s^r$.

The following assertion is a consequence of Theorem 2. It states, in particular, that the decomposition (21) of a tensor $U \in E^r_s$ is unique.

Theorem 3 *Let* $U \in E^r_s$.

(a) *Equation* $\mathbf{q}V + \mathrm{tr}\,W = U$ *for unknown tensors* $V \in E^{r-1}_{s-1}$ *and* $W \in E^{r+1}_{s+1}$ *has a unique solution such that* $\mathrm{tr}\,V = 0$, $\mathbf{q}W = 0$. *This solution is given by* $V = \mathrm{tr}\,U$, $W = \mathbf{q}U$.

(b) *Equation* $\mathbf{q}X = U$ *has a solution* $X \in E^{r-1}_{s-1}$ *if and only if* $\mathbf{q}U = 0$. *If this condition is satisfied, then* $X = \mathrm{tr}\,U$ *is a solution. Any other solution is of the form* $X' = X + \mathbf{q}Y$ *for some tensor* $Y \in E^{r-1}_{s-2}$.

Proof (a) If $\mathbf{q}V + \mathrm{tr}\,W = U$, $\mathrm{tr}\,V = 0$ then $V = \mathrm{tr}\,\mathbf{q}V = \mathrm{tr}\,U$ because $\mathrm{tr}\,\mathrm{tr}\,W = 0$; if $\mathbf{q}W = 0$, then $W = \mathbf{q}\,\mathrm{tr}\,W = \mathbf{q}(U - \mathbf{q}V) = \mathbf{q}U$.

(b) If equation $\mathbf{q}X = U$ has a solution U, then necessarily $\mathbf{q}U = 0$. Conversely, if $\mathbf{q}U = 0$, then $U = \mathbf{q}\,\mathrm{tr}\,U$ and $X = \mathrm{tr}\,U$ solves equation $\mathbf{q}X = U$. Clearly, the tensors $X' = X + \mathbf{q}Y$, where $Y \in E^{r-1}_{s-2}$ also solve this equation.

Example 1 We find the trace decomposition formula (21) for $r = 1$. Writing $U = U^{j_1}{}_{i_1 i_2 \ldots i_s}$, we have $\mathrm{tr}\,U = U^k{}_{k i_1 i_2 \ldots i_{s-1}}$ and

(27) $$\mathbf{q}\,\mathrm{tr}\,U = \frac{1}{n+1-s}(\delta^{j_1}_{i_1} U^k{}_{k i_2 i_3 \ldots i_s} + \delta^{j_1}_{i_2} U^k{}_{i_1 k i_3 i_4 \ldots i_s} + \ldots + \delta^{j_1}_{i_s} U^k{}_{i_1 i_2 \ldots i_{s-1} k}).$$

Analogously

(28) $$\begin{aligned} \mathbf{q}U &= \frac{2(s+1)}{n+1-s}\delta^{j_1}_{i_1} U^{j_2}{}_{i_2 i_3 \ldots i_{s+1}} \quad \mathrm{Alt}(i_1 i_2 \ldots i_{s+1}) \quad \mathrm{Sym}(j_1 j_2) \\ &= \frac{1}{n+1-s}(\delta^{j_1}_{i_1} U^{j_2}{}_{i_2 i_3 \ldots i_{s+1}} - \delta^{j_1}_{i_2} U^{j_2}{}_{i_1 i_3 i_4 \ldots i_{s+1}} - \ldots - \delta^{j_1}_{i_{s+1}} U^{j_2}{}_{i_2 i_3 \ldots i_s i_1} \\ &\quad + \delta^{j_2}_{i_1} U^{j_1}{}_{i_2 i_3 \ldots i_{s+1}} - \delta^{j_2}_{i_2} U^{j_1}{}_{i_1 i_3 i_4 \ldots i_{s+1}} - \ldots - \delta^{j_2}_{i_{s+1}} U^{j_1}{}_{i_2 i_3 \ldots i_s i_1}) \end{aligned}$$

hence

(29) $$\begin{aligned} \mathrm{tr}\,\mathbf{q}U &= \frac{1}{n+1-s}(nU^{j_2}{}_{i_2 i_3 \ldots i_{s+1}} - (s-1)U^{j_2}{}_{i_2 i_3 i_4 \ldots i_{s+1}} \\ &\quad - \frac{1}{n+1-s}(\delta^{j_2}_{i_2} U^k{}_{k i_3 i_4 \ldots i_{s+1}} + \delta^{j_2}_{i_3} U^k{}_{i_2 k i_4 i_5 \ldots i_{s+1}} + \ldots + \delta^{j_2}_{i_{s+1}} U^k{}_{i_2 i_3 \ldots i_s k}) \\ &= U^{j_2}{}_{i_2 i_3 \ldots i_{s+1}} - \mathbf{q}\,\mathrm{tr}\,U. \end{aligned}$$

Formulas (28) and (30) yield $U = \mathrm{tr}\,\mathbf{q}U + \mathbf{q}\,\mathrm{tr}\,U$. In particular, if $r = 1$ and $s = n$, then $U = U^j{}_{i_1 i_2 \ldots i_n}$, $\mathrm{tr}\,U = U^s{}_{s i_1 i_2 \ldots i_{n-1}}$ and $\mathbf{q}U = 0$. Thus,

(30) $$\begin{aligned} U &= n\delta^j_{i_1} U^s{}_{s i_2 i_3 \ldots i_n} \quad \mathrm{Alt}(i_1 i_2 \ldots i_n) \\ &= \delta^j_{i_1} U^s{}_{s i_2 i_3 \ldots i_n} + \delta^j_{i_2} U^s{}_{i_1 s i_3 i_4 \ldots i_n} + \ldots + \delta^j_{i_n} U^s{}_{i_1 i_2 \ldots i_{n-1} s}. \end{aligned}$$

Example 2 We determine decomposition (21) for $r = 2$ and $s = n-1$, and find explicit expressions for the traceless and Kronecker components $\mathrm{tr}\,\mathbf{q}U$ and $\mathbf{q}\,\mathrm{tr}U$ of the tensor U. Writing $U = U^{j_1 j_2}{}_{i_1 i_2 \ldots i_{n-1}}$ and using the proof of Theorem 2 we have

$$
\begin{aligned}
\mathrm{tr}\,\mathbf{q}U = {} & U^{j_2 j_3}{}_{i_2 i_3 \ldots i_n} \\
& -\frac{1}{3}(\delta^{j_2}_{i_2} U^{kj_3}{}_{k i_3 i_4 \ldots i_n} + \delta^{j_2}_{i_3} U^{kj_3}{}_{i_2 k i_4 i_5 \ldots i_n} + \ldots + \delta^{j_2}_{i_n} U^{kj_3}{}_{i_2 i_3 \ldots i_{n-1} k} \\
& + \delta^{j_3}_{i_2} U^{j_2 k}{}_{k i_3 i_4 i_5 \ldots i_n} + \delta^{j_3}_{i_3} U^{j_2 k}{}_{i_2 k i_4 i_5 \ldots i_n} + \ldots + \delta^{j_3}_{i_n} U^{j_2 k}{}_{i_2 i_3 \ldots i_{n-1} k})
\end{aligned} \tag{31}
$$

and

$$
\begin{aligned}
\mathbf{q}\,\mathrm{tr}U & = \frac{1}{3}(\delta^{j_2}_{i_2} U^{kj_3}{}_{k i_3 i_4 \ldots i_n} + \delta^{j_2}_{i_3} U^{kj_3}{}_{i_2 k i_4 i_5 \ldots i_n} + \ldots + \delta^{j_2}_{i_n} U^{kj_3}{}_{i_2 i_3 \ldots i_{n-1} k} \\
& + \delta^{j_3}_{i_2} U^{j_2 k}{}_{k i_3 i_4 i_5 \ldots i_n} + \delta^{j_3}_{i_3} U^{j_2 k}{}_{i_2 k i_4 i_5 \ldots i_n} + \ldots + \delta^{j_3}_{i_n} U^{j_2 k}{}_{i_2 i_3 \ldots i_{n-1} k}) \\
& = \frac{1}{3}(\delta^{j_2}_{i_2} U^{kj_3}{}_{k i_3 i_4 \ldots i_n} - \delta^{j_2}_{i_3} U^{kj_3}{}_{k i_2 i_4 i_5 \ldots i_n} - \ldots - \delta^{j_2}_{i_n} U^{kj_3}{}_{k i_3 \ldots i_{n-1} i_2} \\
& + \delta^{j_3}_{i_2} U^{kj_2}{}_{k i_3 i_4 i_5 \ldots i_n} - \delta^{j_3}_{i_3} U^{kj_2}{}_{k i_2 i_4 i_5 \ldots i_n} - \ldots - \delta^{j_3}_{i_n} U^{kj_2}{}_{k i_3 \ldots i_{n-1} i_2}) \\
& = \frac{2(n-1)}{3} \delta^{j_2}_{i_2} U^{kj_3}{}_{k i_3 i_4 \ldots i_n} \quad \mathrm{Sym}(j_2 j_3) \quad \mathrm{Alt}(i_2 i_3 \ldots i_n).
\end{aligned} \tag{32}
$$

Let s and j be positive integers such that $j \le s \le n$. Consider the vector space of tensors $X = X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s}$, indexed with multi-indices I_1, I_2, $\ldots$, I_j of length r and indices i_{j+1}, i_{j+2}, $\ldots$, i_s, such that $1 \le i_{j+1}, i_{j+2}, \ldots i_s \le n$, *symmetric* in the superscripts entering each of the multi-indices, and skew-*symmetric* in the subscripts. Our objective will be to solve the system of homogeneous equations

$$
\begin{aligned}
& \delta^{p_1}_{p_1} \delta^{p_2}_{p_2} \ldots \delta^{p_j}_{p_j} X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s} = 0 \quad \mathrm{Alt}(p_1 p_2 \ldots p_j i_{j+1} i_{j+2} \ldots i_s) \\
& \mathrm{Sym}(I_1 p_1) \quad \mathrm{Sym}(I_2 p_2) \quad \ldots \quad \mathrm{Sym}(I_j p_j)
\end{aligned} \tag{33}
$$

for an unknown tensor X. In this formula, the alternation operation is applied to the subscripts, and the symmetrizations to the superscripts, and then the summations through double indices are provided.

In the proof of the following theorem we want to distinguish between two groups of indices in the expression $\delta^{p_1}_{i_1} \delta^{p_2}_{i_2} \ldots \delta^{p_j}_{i_j} X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s}$; the indices labelling the tensor $X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s}$ will be called *interior* (the complementary indices, labelling the Kronecker tensors, are called *exterior*).

Theorem 4 *Let q and j be positive integers such that $1 \le j \le s \le n$. Let $X = X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_q}$ be a tensor, indexed with multi-indices I_1, I_2, $\ldots$, I_j of length r and indices i_{j+1}, i_{j+2}, $\ldots$, i_s, such that $1 \le i_{j+1}, i_{j+2}, \ldots, i_s \le n$, symmetric in the superscripts entering each of the multi-indices, and skew-*

symmetric in the subscripts. Then X *satisfies equation* (33) *if and only if it is a Kronecker tensor.*

Proof 1. Suppose we have a tensor $X = X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_s}$, satisfying equations (34). We want to show that X is a Kronecker tensor. Consider a fixed component $X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_s}$. Choose p_1, p_2, $\dots$, p_j and i_1, i_2, $\dots$, i_j such that the s-tuples $(p_1, p_2, \dots, p_j, i_{j+1}, i_{j+2}, \dots, i_s)$ and $(i_1, i_2, \dots, i_j, i_{j+1}, i_{j+2}, \dots, i_s)$ consist of mutually different indices, and consider expression

$$\begin{gathered}\delta^{p_1}_{i_1} \delta^{p_2}_{i_2} \dots \delta^{p_j}_{i_j} X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_s} \quad \mathrm{Alt}(i_1 i_2 \dots i_j i_{j+1} \dots i_s) \\ \mathrm{Sym}(I_1 p_1) \quad \mathrm{Sym}(I_2 p_2) \quad \dots \quad \mathrm{Sym}(I_j p_j).\end{gathered} \tag{34}$$

The summations in (34) are defined by the alternation $\mathrm{Alt}(i_1 i_2 \dots i_j i_{j+1} \dots i_s)$ and the symmetrizations $\mathrm{Sym}(I_1 p_1)$, $\mathrm{Sym}(I_2 p_2)$, $\dots$, $\mathrm{Sym}(I_j p_j)$. We divide the summands in four groups according to the positions of the indices p_1, p_2, $\dots$, p_j and i_1, i_2, $\dots$, i_j.

(a) None of the indices p_1, p_2, $\dots$, p_j and i_1, i_2, $\dots$, i_j is interior.

(b) None of the indices p_1, p_2, $\dots$, p_j is interior, at least one of the indices i_1, i_2, $\dots$, i_j is interior.

(c) At least one of the indices p_1, p_2, $\dots$, p_j is interior, none of the indices i_1, i_2, $\dots$, i_j is interior.

(d) At least one of the indices p_1, p_2, $\dots$, p_j is interior, and at least one of the indices i_1, i_2, $\dots$, i_j is interior.

Equation (34) involves expressions (34) such that $i_1 = p_1$, $i_2 = p_2$, $\dots$, $i_q = p_q$. For this choice of indices the terms (a) become

$$\begin{gathered}\delta^{p_1}_{p_1} \delta^{p_2}_{p_2} \dots \delta^{p_q}_{p_q} X^{I_1 I_2 \dots I_q}{}_{i_{q+1} i_{q+2} \dots i_s} \quad \mathrm{Alt}(p_1 p_2 \dots p_q i_{q+1} i_{q+2} \dots i_s) \\ \mathrm{Sym}(I_1 p_1) \quad \mathrm{Sym}(I_2 p_2) \quad \dots \quad \mathrm{Sym}(I_q p_q)\end{gathered} \tag{35}$$

(no summation through p_1, p_2, $\dots$, p_q). Expressions (b) and (c) vanish identically because the indices $(i_1, i_2, \dots, i_q, i_{q+1}, i_{q+2}, \dots, i_s)$ are mutually different and $X^{I_1 I_2 \dots I_q}{}_{i_{q+1} i_{q+2} \dots i_s}$ is skew-symmetric in the subscripts. The terms in (d) are of Kronecker type, each summand is a multiple of the Kronecker symbol δ^{α}_{β}, where $\alpha \notin \{p_1, p_2, \dots, p_q\}$ and $\beta \in \{i_{q+1} i_{q+2} \dots i_s\}$.

Thus, (34) is the sum of the terms (a) and (d). But the left-hand side of equation (33) is determined from (34) by the trace operation in $i_1 = p_1$, $i_2 = p_2$, $\dots$, $i_q = p_q$. The terms entering (a) lead to an expression of the form cX, where c is a non-zero constant, namely to the expression

$$\begin{aligned}&\frac{j!}{s!((r+1)!)^q} \delta^{p_1}_{p_1} \delta^{p_2}_{p_2} \dots \delta^{p_j}_{p_j} X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_s} \quad \mathrm{Alt}(p_1 p_2 \dots p_j) \\ &\quad = \frac{1}{s!((r+1)!)^j} \det \delta^{p_i}_{p_l} \cdot X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_s}.\end{aligned} \tag{36}$$

Since the contraction of the terms (d) in $i_1 = p_1$, $i_2 = p_2$, $\dots$, $i_q = p_q$ does not influence the factors δ^{α}_{β}, (d) leads to a Kronecker tensor.

Corollary 1 *Assume that in addition to the assumptions of* Theorem 4, *the tensor* $X = X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_s}$ *is traceless. Then*

$$X^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2} \dots i_s} = 0. \tag{37}$$

Proof This follows from Theorem 4, and from the orthogonality of traceless and Kronecker tensors.

Example 3 For tensors of lower degrees equations (33) can be solved directly by means of the decomposition of the unknown tensor X. Consider for example the system

$$\delta^{p_1}_{p_1} \delta^{p_2}_{p_2} X^{i_1 i_2}{}_{i_3} = 0 \quad \mathrm{Alt}(p_1 p_2 i_3) \quad \mathrm{Sym}(i_1 p_1) \quad \mathrm{Sym}(i_2 p_2) \tag{38}$$

for a *traceless* tensor $X = X^{i_1 i_2}{}_k$. The decomposition of the left-hand side is

$$\begin{aligned}
&\delta^{p_1}_{p_1}\delta^{p_2}_{p_2}X^{i_1i_2}{}_{i_3} + \delta^{i_1}_{p_1}\delta^{p_2}_{p_2}X^{p_1i_2}{}_{i_3} + \delta^{p_1}_{p_1}\delta^{i_2}_{p_2}X^{i_1p_2}{}_{i_3} + \delta^{i_1}_{p_1}\delta^{i_2}_{p_2}X^{p_1p_2}{}_{i_3} \\
&- \delta^{p_1}_{p_2}\delta^{p_2}_{p_1}X^{i_1i_2}{}_{i_3} - \delta^{i_1}_{p_2}\delta^{p_2}_{p_1}X^{p_1i_2}{}_{i_3} - \delta^{p_1}_{p_2}\delta^{i_2}_{p_1}X^{i_1p_2}{}_{i_3} - \delta^{i_1}_{p_2}\delta^{i_2}_{p_1}X^{p_1p_2}{}_{i_3} \\
&- \delta^{p_1}_{i_3}\delta^{p_2}_{p_2}X^{i_1i_2}{}_{p_1} - \delta^{i_1}_{i_3}\delta^{p_2}_{p_2}X^{p_1i_2}{}_{p_1} - \delta^{p_1}_{i_3}\delta^{i_2}_{p_2}X^{i_1p_2}{}_{p_1} - \delta^{i_1}_{i_3}\delta^{i_2}_{p_2}X^{p_1p_2}{}_{p_1} \\
&+ \delta^{p_1}_{p_2}\delta^{p_2}_{i_3}X^{i_1i_2}{}_{p_1} + \delta^{i_1}_{p_2}\delta^{p_2}_{i_3}X^{p_1i_2}{}_{p_1} + \delta^{p_1}_{p_2}\delta^{i_2}_{i_3}X^{i_1p_2}{}_{p_1} + \delta^{i_1}_{p_2}\delta^{i_2}_{i_3}X^{p_1p_2}{}_{p_1} \\
&- \delta^{p_1}_{p_1}\delta^{p_2}_{i_3}X^{i_1i_2}{}_{p_2} - \delta^{i_1}_{p_1}\delta^{p_2}_{i_3}X^{p_1i_2}{}_{p_2} - \delta^{p_1}_{p_1}\delta^{i_2}_{i_3}X^{i_1p_2}{}_{p_2} - \delta^{i_1}_{p_1}\delta^{i_2}_{i_3}X^{p_1p_2}{}_{p_2} \\
&+ \delta^{p_1}_{i_3}\delta^{p_2}_{p_1}X^{i_1i_2}{}_{p_2} + \delta^{i_1}_{i_3}\delta^{p_2}_{p_1}X^{p_1i_2}{}_{p_2} + \delta^{p_1}_{i_3}\delta^{i_2}_{p_1}X^{i_1p_2}{}_{p_2} + \delta^{i_1}_{i_3}\delta^{i_2}_{p_1}X^{p_1p_2}{}_{p_2}.
\end{aligned} \tag{39}$$

Contraction in p_1 and p_2 gives the expression

$$\begin{aligned}
&n^2 X^{i_1i_2}{}_{i_3} + nX^{i_1i_2}{}_{i_3} + nX^{i_1i_2}{}_{i_3} + X^{i_1i_2}{}_{i_3} - nX^{i_1i_2}{}_{i_3} \\
&- X^{i_1i_2}{}_{i_3} - X^{i_1i_2}{}_{i_3} - X^{i_2i_1}{}_{i_3} - X^{i_1i_2}{}_{i_3} - X^{i_1i_2}{}_{i_3} \\
&+ X^{i_1i_2}{}_{i_3} - nX^{i_1i_2}{}_{i_3} - X^{i_1i_2}{}_{i_3} + X^{i_1i_2}{}_{i_3} \\
&= (n^2 - 2)X^{i_1i_2}{}_{i_3} - X^{i_2i_1}{}_{i_3}.
\end{aligned} \tag{40}$$

Since this expression should vanish, we get $(n^2 - 2)X^{i_1i_2}{}_{i_3} - X^{i_2i_1}{}_{i_3} = 0$ which is only possible when $X^{i_1i_2}{}_{i_3} = 0$.

10 Bases of forms

We summarize for reference some useful formulas for the bases of differential forms on an n-dimensional manifold X.

Lemma 1 (Bases of forms) *Let X be an n-dimensional smooth manifold, and let* (U,φ), $\varphi=(x^i)$, *be a chart on X. Then the forms*

$$\omega_0=\frac{1}{n!}\varepsilon_{i_1i_2\dots i_n}dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_n} \tag{1}$$

and

$$\omega_{k_1k_2\dots k_p}=\frac{1}{(n-p)!}\varepsilon_{k_1k_2\dots k_pi_{p+1}i_{p+2}\dots i_n}dx^{i_{p+1}}\wedge dx^{i_{p+2}}\wedge\dots\wedge dx^{i_n}, \quad 1\le p\le n-1, \tag{2}$$

define bases of n-forms and $(n-p)$*-forms on* U. *The transformation formulas to the canonical bases are*

$$\varepsilon^{k_1k_2\dots k_pl_{p+1}l_{p+2}\dots l_n}\omega_{k_1k_2\dots k_p}=dx^{l_{p+1}}\wedge dx^{l_{p+2}}\wedge\dots\wedge dx^{l_n}. \tag{3}$$

Proof See Appendix 8.

The *Jacobian determinant* of a transformation $\bar{x}^p=\bar{x}^p(x^1,x^2,\dots,x^n)$, $\det(\partial\bar{x}^p/\partial x^p)$, has the following basic properties:

Lemma 2 (Jacobians) (a) *The local volume forms on X are on intersections of the charts are related by the formula*

$$\bar{\omega}_0=\det\left(\frac{\partial\bar{x}^p}{\partial x^p}\right)\omega_0. \tag{4}$$

(b) *The derivative of the Jacobian satisfies*

$$\frac{\partial}{\partial\bar{x}^m}\det\left(\frac{\partial x^r}{\partial\bar{x}^s}\right)=\det\left(\frac{\partial x^r}{\partial\bar{x}^s}\right)\cdot\frac{\partial^2x^p}{\partial\bar{x}^m\partial\bar{x}^q}\frac{\partial\bar{x}^q}{\partial x^p}. \tag{5}$$

(c) *The* $(n-1)$*-forms* ω_k *and* $\bar{\omega}_i$ obey the transformation formulas

$$\bar{\omega}_i=\frac{\partial x^k}{\partial\bar{x}^i}\det\frac{\partial\bar{x}^r}{\partial x^s}\cdot\omega_k. \tag{6}$$

Proof (b) To verify formula (5), consider any regular matrix a and its inverse a^{-1},

$$(7)\qquad a=\begin{pmatrix} a_1^1 & a_2^1 & a_n^1 \\ a_1^2 & a_2^2 & a_n^2 \\ & & \\ a_1^n & a_2^n & a_n^n \end{pmatrix},\quad a^{-1}=\begin{pmatrix} b_1^1 & b_2^1 & b_n^1 \\ b_1^2 & b_2^2 & b_n^2 \\ & & \\ b_1^n & b_2^n & b_2^n \end{pmatrix},$$

and compute the derivative $\partial \det a / \partial a_q^p$. Multilinearity and the Laplace decomposition with respect to the *s-th* row of the determinant of a yields $\det a = a_1^s A_1^s + a_2^s A_2^s + \ldots + a_n^s A_n^s$, where with algebraic complements A_k^s. Thus

$$(8)\qquad \frac{\partial \det a}{\partial a_q^p} = A_q^p .$$

But a is regular, so the inverse matrix satisfies

$$(9)\qquad \begin{pmatrix} b_1^1 & b_2^1 & \ldots & b_n^1 \\ b_1^2 & b_2^2 & \ldots & b_n^2 \\ \ldots & & & \\ b_1^n & b_2^n & \ldots & b_n^n \end{pmatrix} = \begin{pmatrix} \frac{A_1^1}{\det a} & \frac{A_1^2}{\det a} & \ldots & \frac{A_1^n}{\det a} \\ \frac{A_2^1}{\det a} & \frac{A_2^2}{\det a} & \ldots & \frac{A_2^n}{\det a} \\ \ldots & & & \\ \frac{A_n^1}{\det a} & \frac{A_n^2}{\det a} & \ldots & \frac{A_n^n}{\det a} \end{pmatrix},$$

hence $A_q^p = \det a \cdot b_p^q$ and we conclude that

$$(10)\qquad \frac{\partial \det a}{\partial a_q^p} = \det a \cdot b_p^q .$$

Now substituting

$$(11)\qquad a_s^r = \frac{\partial x^r}{\partial \overline{x}^s},\quad b_s^r = \frac{\partial \overline{x}^r}{\partial x^s},$$

we get

$$(12)\qquad \frac{\partial}{\partial \overline{x}^m}\det\left(\frac{\partial x^r}{\partial \overline{x}^s}\right) = \sum_{p,q}\frac{\partial \det a}{\partial a_q^p}\frac{\partial a_q^p}{\partial \overline{x}^m} = \det\left(\frac{\partial x^r}{\partial \overline{x}^s}\right)\cdot\frac{\partial^2 x^p}{\partial \overline{x}^m\,\partial \overline{x}^q}\frac{\partial \overline{x}^q}{\partial x^p}.$$

(c) Using the transformation properties of the forms ω_0 and $\bar{\omega}_0$ (formula (4),

$$(13)\qquad \bar{\omega}_i = i_{\partial/\partial \overline{x}^i}\bar{\omega}_0 = \frac{\partial x^k}{\partial \overline{x}^i}\det\frac{\partial \overline{x}}{\partial x}\cdot i_{\partial/\partial x^k}\omega_0 = \frac{\partial x^k}{\partial \overline{x}^i}\det\frac{\partial \overline{x}}{\partial x}\cdot\omega_k .$$

Remark (Different bases) Sometimes it is convenient to consider bases of forms, differening from the forms (2) by a constant factor. If we set

$$\omega_{k_1k_2\dots k_p} = \frac{1}{p!(n-p)!}\varepsilon_{k_1k_2\dots k_pi_{p+1}i_{p+2}\dots i_n}dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \dots \wedge dx^{i_n}. \tag{14}$$

then for example

$$\begin{aligned} dx^l \wedge \omega_{k_1k_2} &= \frac{1}{2!(n-2)!}\varepsilon_{k_1k_2i_3i_4\dots i_n}dx^l \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_n}, \\ &= \frac{1}{2!(n-2)!}\varepsilon_{k_1k_2i_3i_4\dots i_n}\varepsilon^{pli_3i_4\dots i_n}\omega_p = \frac{2!(n-2)!}{2!(n-2)!}\frac{1}{2}(\delta^{p_1}_{k_1}\delta^l_{k_2} - \delta^{p_1}_{k_2}\delta^l_{k_1})\omega_{p_1} \\ &= \frac{1}{2}(\delta^l_{k_2}\omega_{k_1} - \delta^l_{k_1}\omega_{k_2}), \end{aligned} \tag{15}$$

etc. (cf. Appendix 8).