1 Jet prolongations of fibred manifolds

This chapter introduces fibred manifolds and their jet prolongations. First we recall properties of differentiable mappings of constant rank and introduce, with the help of rank, the notion of a fibred manifold. Then we define automorphisms of fibred manifolds as the mappings preserving their fibred structure. The r-jets of sections of a fibred manifold Y, with a fixed positive integer r, constitute a new fibred manifold, the *r-jet prolongation* J^rY of Y; we describe the structure of J^rY and a canonical construction of automorphisms of J^rY from automorphisms of the fibred manifold Y, the *r-jet prolongation*. The prolongation procedure immediately extends, via flows, to vector fields. For this background material we refer to Krupka [K17], Lee [L] and Saunders [S]).

These concepts are prerequisites for the geometric definition of *variations of sections* of a fibred manifold, extending the corresponding notion used in the classical multiple-integral variational theory on Euclidean spaces to smooth fibred manifolds.

1.1 The rank theorem

Recall that the *rank* of a linear mapping $u: E \to F$ of vector spaces is defined to be the dimension of its image space, $\operatorname{rank} u = \dim \operatorname{Im} u$. This definition applies to tangent mappings of differentiable mappings of smooth manifolds. Let $f: X \to Y$ be a C^r mapping of smooth manifolds, where $r \geq 1$. We define the *rank* of *f at a point* $x \in X$ to be the rank of the tangent mapping $T_x f: T_x X \to T_{f(x)} Y$. We denote

(1) $\qquad \operatorname{rank}_x f = \dim \operatorname{Im} T_x f.$

The function $x \to \operatorname{rank}_x f$, defined on X, is the *rank function*.

Elementary examples of real-valued functions f of one real variable show that the rank function is not, in general, locally constant. Our main objective in this section is to study differentiable mappings whose rank function *is* locally constant.

First we prove a manifold version of the constant rank theorem, a fundamental tool for a classification of differentiable mappings. The proof is based on the rank theorem in Euclidean spaces (see Appendix 3) and a standard use of charts on a smooth manifold.

Theorem 1 (Rank theorem) *Let X and Y be two manifolds,* $n = \dim X$, $m = \dim Y$, *and let q be a positive integer such that* $q \le \min(n,m)$. *Let* $W \subset X$ *be an open set, and let* $f : W \to Y$ *be a* C^r *mapping. The following conditions are equivalent:*

(1) *f has constant rank on W equal to q.*

(2) *To every point* $x_0 \in W$ *there exist a chart* (U,φ), $\varphi = (x^i)$ *at* x_0, *an open rectangle* $P \subset \mathbf{R}^n$ *with centre* 0 *such that* $\varphi(U) = P$, $\varphi(x_0) = 0$, *a chart* (V,ψ), $\psi = (y^\sigma)$, *at* $y_0 = f(x_0)$, *such that* $f(U) \subset V$, *and an open rectangle* $Q \subset \mathbf{R}^m$ *with centre* 0 *such that* $\psi(V) = Q$, $\psi(y_0) = 0$, *and*

$$y^\sigma \circ f = \begin{cases} x^\sigma, & \sigma = 1,2,\dots,q, \\ 0, & \sigma = q+1,q+2,\dots,m. \end{cases} \tag{2}$$

Proof 1. Suppose that *f* has constant rank on *W* equal to *q*. We choose a chart $(\bar{U},\bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, at x_0, and a chart $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, at y_0, and set $g = \bar{\psi} f \bar{\varphi}^{-1}$; g is a C^r mapping from $\bar{\varphi}(\bar{U}) \subset \mathbf{R}^n$ into $\bar{\psi}(\bar{V}) \subset \mathbf{R}^m$. Since for every tangent vector $\xi \in T_x X$ expressed as

$$\xi = \bar{\xi}^i \left(\frac{\partial}{\partial x^i} \right)_x, \tag{3}$$

we have

$$T_x f \cdot \xi = D_i(\bar{y}^\sigma f \bar{\varphi}^{-1})(\bar{\varphi}(x)) \bar{\xi}^i \left(\frac{\partial}{\partial y^\sigma} \right)_{f(x)}, \tag{4}$$

the rank of f at x is $\operatorname{rank} T_x f = \operatorname{rank} D_i(\bar{y}^\sigma f \bar{\varphi}^{-1})(\bar{\varphi}(x))$. Consequently, the rank of *f* is constant on the open set $\bar{\varphi}(\bar{U}) \subset \mathbf{R}^n$, and is equal to *q*. Shrinking $\bar{U}$ to a neighbourhood *U* of x_0 and $\bar{V}$ to a neighbourhood *V* of y_0 if necessary we may suppose that there exist an open rectangle $P \subset \mathbf{R}^n$ with centre 0, a diffeomorphism $\alpha : \bar{\varphi}(U) \to P$, an open rectangle $Q \subset \mathbf{R}^m$ with centre 0, and a diffeomorphism $\beta : \bar{\psi}(V) \to Q$, such that in the canonical coordinates z^i on *P* and w^σ on *Q*, $\beta g \alpha^{-1}(z^1,z^2,\dots,z^n) = (z^1,z^2,\dots,z^q,0,0,\dots,0)$. We set $\varphi = \alpha\bar{\varphi}$, $\varphi = (x^i)$, and $\psi = \beta\bar{\psi}$, $\psi = (y^\sigma)$. Then (U,φ) and (V,ψ) are charts on the manifolds *X* and *Y* respectively. In these charts, the mapping $\psi f \varphi^{-1}$ can be expressed as $\psi f \varphi^{-1} = \beta \bar{\psi} f \bar{\varphi}^{-1} \alpha^{-1} = \beta g \alpha^{-1}$; thus, for every point $x \in U$

$$\begin{aligned} \psi f(x) &= \psi f \varphi^{-1} \varphi(x) = \beta g \alpha^{-1} \varphi(x) \\ &= \beta g \alpha^{-1}(x^1(x),x^2(x),\dots,x^n(x)) \\ &= (x^1(x),x^2(x),\dots,x^q(x),0,0,\dots,0). \end{aligned} \tag{5}$$

In components,

(6) $$y^\sigma \circ f(x) = \begin{cases} x^\sigma(x), & \sigma = 1,2,\dots,q, \\ 0, & \sigma = q+1, q+2,\dots,m, \end{cases}$$

proving (2).

2. Conversely, suppose that on a neighbourhood of $x_0 \in W$ the mapping *f* is expressed by (2). Then $\operatorname{rank} T_{x_0} f = \operatorname{rank} D_i(y^\sigma f \varphi^{-1})(\varphi(x_0)) = q$.

Let $f: X \to Y$ be a C^r mapping, and let $x_0 \in X$ be a point. We say that *f* is a *constant rank mapping* at x_0, if there exists a neighbourhood *W* of x_0 such that the rank function $x \to \operatorname{rank}_x f$ is constant on *W*. Then the charts (U,φ) and (V,ψ) in which the mapping *f* has an expression (2), are said to be *adapted* to *f* at x_0, or just *f-adapted*. A C^r mapping *f* that is a constant rank mapping at every point is called a C^r mapping of *locally constant rank*.

A C^r mapping $f: W \to Y$ such that the tangent mapping $T_{x_0} f$ is *injective* is called an *immersion at* x_0. From the definition of the rank it is immediate that *f* is an immersion at x_0 if and only if $\operatorname{rank}_{x_0} f = n \le m$. If *f* is an immersion at every point of the set *W*, we say that *f* is an *immersion*.

From the rank theorem we get the following criterion.

Theorem 2 (Immersions) *Let X and Y be two manifolds,* $n = \dim X$, $m = \dim Y \ge n$. *Let* $f: X \to Y$ *be a* C^r *mapping,* $x_0 \in X$ *a point, and let* $y_0 = f(x_0)$. *The following two conditions are equivalent:*

(1) *f is an immersion at* x_0.

(2) *There exist a chart* (U,φ), $\varphi = (x^i)$ *at* x_0, *an open rectangle* $P \subset \mathbf{R}^n$ *with centre* 0 *such that* $\varphi(U) = P$ *and* $\varphi(x_0) = 0$, *a chart* (V,ψ), $\psi = (y^\sigma)$ *at* $y_0 = f(x_0)$, *and an open rectangle* $Q \subset \mathbf{R}^m$ *with centre* 0 *such that* $\psi(V) = Q$ *and* $\psi(y_0) = 0$, *such that in these charts f is expressed by*

(7) $$y^\sigma \circ f = \begin{cases} x^\sigma, & \sigma = 1,2,\dots,n, \\ 0, & \sigma = n+1, n+2,\dots,m. \end{cases}$$

Proof The matrix of the linear operator $T_{x_0} f$ in some charts (U,φ), $\varphi = (x^i)$, at x_0 and (V,ψ), $\psi = (y^\sigma)$, at y_0 is formed by partial derivatives $D_i(y^\sigma f \varphi^{-1})(\varphi(x_0))$, and is of dimension $n \times m$. If $\operatorname{rank} T_{x_0} f = n$ at x_0, then $\operatorname{rank} T_x f = n$ on a neighbourhood of x_0, by continuity of the determinant function. Equivalence of conditions (1) and (2) is now an immediate consequence of Theorem 1.

Let $f: X \to Y$ be an immersion, let $x_0 \in X$ be a point, and let (U,φ) and (V,ψ) be the charts from Theorem 2, (2). Shrinking *P* and *Q* if necessary we may suppose without loss of generality that the rectangle *Q* is of the form $Q = P \times R$, where *R* is an open rectangle in $\mathbf{R}^{m-n}$. Then the chart expression $\psi f \varphi^{-1}: P \to P \times R$ of the immersion *f* in these charts is the mapping $(x^1, x^2, \dots, x^n) \to (x^1, x^2, \dots, x^n, 0, 0, \dots, 0)$. The charts (U,φ), (V,ψ) with these properties are said to be *adapted* to the immersion *f* at x_0.

Example 1 (Sections) Let $s \geq r$, let $f : X \to Y$ be a surjective mapping of smooth manifolds. By a C^r *section*, or simply a *section* of *f* we mean a C^r mapping $\gamma : Y \to X$ such that

(8) $\qquad f \circ \gamma = \mathrm{id}_Y$.

Every section is an immersion. Indeed, $T_{\gamma(y)} f \circ T_y \gamma = \mathrm{id}_{T_y Y}$ at any point $y \in Y$. Thus, for any two tangent vectors $\xi_1, \xi_2 \in T_y Y$ satisfying the condition $T_y \gamma \cdot \xi_1 = T_y \gamma \cdot \xi_2$, we have $T_{\gamma(y)} f \circ T_y \gamma \cdot \xi_1 = T_{\gamma(y)} f \circ T_y \gamma \cdot \xi_2$. From this condition we conclude that $\xi_1 = \xi_2$.

A C^r mapping $f : W \to Y$ such that the tangent mapping $T_{x_0} f$ is surjective, is called a *submersion at* x_0. From the definition of the rank it is immediate that *f* is a submersion at x_0 if and only if $\mathrm{rank}_{x_0} = m \leq n$. A *submersion* $f : W \to Y$ is a C^r mapping that is a submersion at every point $x \in W$.

Theorem 3 (Submersions) *Let X and Y be manifolds, let* $n = \dim X$, $m = \dim Y$. *Let* $f : X \to Y$ *be a* C^r *mapping,* x_0 *a point of X,* $y_0 = f(x_0)$. *The following conditions are equivalent:*

(1) *f is a submersion at* x_0.

(2) *There exist a chart* (U, φ), $\varphi = (x^i)$, *at* x_0, *an open rectangle* $P \subset \mathbf{R}^n$ *with centre* 0 *such that* $\varphi(U) = P$, $\varphi(x_0) = 0$, *a chart* (V, ψ), $\psi = (y^\sigma)$, *at* $y_0 = f(x_0)$, *and an open rectangle* $Q \subset \mathbf{R}^m$ *with centre* 0 *such that* $\psi(V) = Q$, $\psi(y_0) = 0$, *such that*

(9) $\qquad y^\sigma \circ f = x^\sigma, \quad \sigma = 1, 2, \ldots, m.$

(3) *There exist a neighbourhood V of* y_0 *and a* C^r *section* $\gamma : V \to Y$ *such that* $\gamma(y_0) = x_0$.

Proof 1. Suppose that *f* is a submersion at x_0. Then $\mathrm{rank} T_x f = m$ on a neighbourhood of x_0, and equivalence of conditions (1) and (2) follows from Theorem 1.

2. Suppose that condition (2) is satisfied. Consider the chart expression $\psi f \varphi^{-1} : P \to Q$ of the submersion *f* that is equal to the Cartesian projection $(x^1, x^2, \ldots, x^m, x^{m+1}, x^{m+1}, \ldots, x^n) \to (x^1, x^2, \ldots, x^m)$. $\psi f \varphi^{-1}$ admits a C^r section δ. Since $\psi f \varphi^{-1} \circ \delta = \mathrm{id}_Q$ hence $f \varphi^{-1} \circ \delta = \psi^{-1}$. Setting $\gamma = \varphi^{-1} \delta \psi$ we have $f \gamma = f \varphi^{-1} \delta \psi = \psi^{-1} \psi = \mathrm{id}_V$ proving that γ is a section of *f*. This proves (3).

3. If *f* admits a C^r section γ defined on a neighbourhood *V* of a point *y*, then $f \circ \gamma = \mathrm{id}_V$ and $T_y (f \circ \gamma) = T_x f \circ T_y \gamma = T_y \,\mathrm{id}_V = \mathrm{id}_{T_y Y}$, where $x = \gamma(y)$. Thus $T_{x_0} f$ must be surjective, proving (1).

Let *f* be a C^r submersion, $x_0 \in X$ a point, and let (U, φ) and (V, ψ) be the charts from Theorem 3, (2). Shrinking *P* and *Q* if necessary we may suppose that the rectangle *P* is of the form $P = Q \times R$, where *R* is an open rectangle in $\mathbf{R}^{n-m}$. Then the chart expression (9) of the submersion *f* is the mapping $(x^1, x^2, \ldots, x^m, x^{m+1}, x^{m+1}, \ldots, x^n) \to (x^1, x^2, \ldots, x^m)$. The charts (U, φ),

(V,ψ) with these properties are said to be *adapted* to the submersion *f* at x_0.

Corollary 1 *A submersion is an open mapping*.

Proof In adapted charts, a submersion is expressed as a Cartesian projection that is an open mapping. Corollary 1 now follows from the definition of the manifold topology in which the charts are homeomorphisms.

Corollary 2 *Let $f:X\to Y$ be a submersion, (U,φ) a chart on X and (V,ψ) a chart on Y. If (U,φ) and (V,ψ) are adapted to f at a point $x_0\in X$, and $V=f(U)$, then the chart (V,ψ) is uniquely determined by (U,φ).*

Proof This is an immediate consequence of the definition of adapted charts and of Corollary 1.

Example 2 (Cartesian projections) Cartesian projections of the Cartesian product of C^∞ manifolds *X* and *Y*, $\mathrm{pr}_1:X\times Y\to X$ and $\mathrm{pr}_2:X\times Y\to Y$, are C^∞ submersions. Indeed, let us verify for instance the rank condition for the projection pr_1. If $(x,y)\in X\times Y$ is a point and (U,φ), $\varphi=(x^i)$ (resp. (V,ψ), $\psi=(y^\sigma)$) is a chart at *x* (resp. *y*), we have on the chart neighbourhood $U\times V\subset X\times Y$, $(x,y)=\psi^{-1}\psi(x,y)=\psi^{-1}(x^1,x^2,\dots,x^n,y^1,y^2,\dots,y^m)$ and $\mathrm{pr}_1(x,y)=x=\varphi^{-1}\varphi(x)=\varphi^{-1}(x^1,x^2,\dots,x^n)$. Then for all vectors $\xi\in T_xX$ and $\zeta\in T_yY$, expressed as

$$\text{(10)}\qquad \xi=\xi^i\left(\frac{\partial}{\partial x^i}\right)_x,\quad \zeta=\zeta^i\left(\frac{\partial}{\partial x^i}\right)_y,$$

equations of the projection pr_1 yield

$$\text{(11)}\qquad T_{(x,y)}\mathrm{pr}_1\cdot(\xi,\zeta)=\frac{\partial(x^i\circ\mathrm{pr}_1)}{\partial x^k}\xi^k\frac{\partial}{\partial x^i}+\frac{\partial(x^i\circ\mathrm{pr}_1)}{\partial y^\sigma}\zeta^\sigma\frac{\partial}{\partial y^\sigma}=\xi.$$

In particular, $T_{(x,y)}\mathrm{pr}_1$ is surjective so pr_1 is a surjective submersion.

Example 3 The tangent bundle projection is a surjective submersion. All tensor bundle projections are surjective submersions.

With the help of Corollary 1, submersions at a point can be characterized as follows.

Corollary 3 *Let X and Y be manifolds, $n=\dim X$, $m=\dim Y\le n$. A C^r mapping $f:X\to Y$ is a submersion at a point $x_0\in X$ if and only if there exist a neighbourhood U of x_0, an open rectangle $R\subset\mathbf{R}^{n-m}$, and a diffeomorphism $\chi:U\to f(U)\times\mathbf{R}^{n-m}$ such that $\mathrm{pr}_1\circ\chi=f$.*

Proof 1. Suppose *f* is a submersion at x_0, and choose some adapted charts (U,φ), $\varphi=(x^i)$, at x_0 and (V,ψ), $\psi=(y^\sigma)$ at y_0. Every point $x\in U$ has the coordinates $(x^1(x),x^2(x),\dots,x^m(x),x^{m+1}(x),x^{m+2}(x),\dots,x^n(x))$. We define a mapping $\chi:U\to Y\times\mathbf{R}^{n-m}$ by

(12) $\chi(x) = (f(x), x^{m+1}(x), x^{m+2}(x), \dots, x^n(x)).$

Then $\mathrm{pr}_1 \circ \chi = f$, and from Corollary 1, $f(U)$ is an open set in Y. It remains to show that χ is a diffeomorphism. We easily find the chart expression of the mapping χ with respect to the chart (U,φ) and the chart $(V \times \mathbf{R}^{n-m}, \eta)$, $\eta = (y^1, y^2, \dots y^m, t^1, t^2, \dots, t^{n-m})$, on $Y \times \mathbf{R}^{n-m}$, where t^k are the canonical coordinates on $\mathbf{R}^{n-m}$. We have for every $x \in U$, $y^\sigma \chi(x) = y^\sigma f(x) = x^\sigma(x)$, $1 \le \sigma \le m$, and $t^k \chi(x) = x^{m+k}(x)$, $1 \le k \le n-m$, that is,

(13) $$\begin{aligned} &y^i \chi = x^i, \quad i = 1,2,\dots,m, \\ &t^k \chi = x^{m+k}, \quad k = 1,2,\dots,n-m, \end{aligned}$$

that is, $\eta \circ \chi = \varphi$. Thus $\chi = \eta^{-1}\varphi$ is a diffeomorphism.

2. Conversely, if $\mathrm{pr}_1 \circ \chi = f$, we have $T_{x_0} f = T_{\chi(x_0)} \mathrm{pr}_1 \circ T_{x_0} \chi$, and since χ is by hypothesis a diffeomorphism, $\operatorname{rank} T_{x_0} f = \operatorname{rank} T_{\chi(x_0)} \mathrm{pr}_1$. But the rank of the projection pr_1 is m (Example 2).

1.2 Fibred manifolds

By a *fibred manifold structure* on C^∞ manifold Y we mean a C^∞ manifold X together with a surjective submersion $\pi : Y \to X$ of class C^∞. A manifold Y endowed with a fibred manifold structure is called a *fibred manifold* of class C^∞, or just a *fibred manifold*. X is the *base*, and π is the *projection* of the fibred manifold Y.

According to Section 1.1, Theorem 3 and Corollary 2, any manifold, endowed with a fibred manifold structure, admits the charts with some specific properties. Let Y be a fibred manifold with base X and projection π, $\dim X = n$, and $\dim Y = n + m$. By hypothesis, to every point $y \in Y$ there exists a chart at y, (V,ψ), $\psi = (u^i, y^\sigma)$, where $1 \le i \le n$, $1 \le \sigma \le m$, with the following properties:

(a) There exists a chart (U,φ), $\varphi = (x^i)$, at $x = \pi(y)$, where $1 \le i \le n$, in which the projection π is expressed by the equations $x^i \circ \pi = u^i$.

(b) $U = \pi(V)$.

The chart (V,ψ) with these properties is called a *fibred chart* on Y. The chart (U,φ) is defined uniquely, and is said to be *associated* with (V,ψ). Having in mind this correspondence, we usually write x^i instead of u^i, and denote a fibred chart as (V,ψ), $\psi = (x^i, y^\sigma)$.

Lemma 1 *Every fibred manifold has an atlas consisting of fibred charts.*

Proof An immediate consequence of the definition of a submersion.

A C^r *section* of the fibred manifold Y, defined on an open set $W \subset X$, is by definition a C^r section $\gamma : W \to Y$ of its projection π (cf. Section 1.1, Example 1). In terms of a fibred chart (V,ψ), $\psi=(x^i,y^\sigma)$, and the associated chart (U,φ), $\varphi=(x^i)$, such that $U\subset W$ and $\gamma(U)\subset V$, γ has equations of the form

(1) $$x^i\circ\gamma=x^i,\quad y^\sigma\circ\gamma=f^\sigma,$$

where f^σ are real C^r functions, defined on U.

Let Y_1 (resp. Y_2) be a fibred manifold with base X_1 (resp. X_2) and projection π_1 (resp. π_2). A C^r mapping $\alpha:W\to Y_2$, where W is an open set in Y_1, is called a C^r *morphism* of the fibred manifold Y_1 into Y_2, if there exists a C^r mapping $\alpha_0:W_0\to X_2$ where $W_0=\pi_1(W_1)$, such that

(2) $$\pi_2\circ\alpha=\alpha_0\circ\pi_1.$$

Note that W_0 is always an open set in X_1 (Section 1.1, Corollary 1). If α_0 exists it is unique, and is called the *projection* of α. We also say that α is a morphism *over* α_0. A morphism of fibred manifolds $\alpha:Y_1\to Y_2$ that is a diffeomorphism is called an *isomorphism*; the projection of an isomorphism of fibred manifolds is a diffeomorphism of their bases.

If the fibred manifolds Y_1 and Y_2 coincide, $Y_1=Y_2=Y$, then a morphism $\alpha:W\to Y$ is also called an *automorphism* of Y.

We find the expression of a morphism of fibred manifolds in fibred charts. Consider a fibred chart (V_1,ψ_1), $\psi_1=(x_1^i,y_1^\sigma)$, on Y_1 and a fibred chart (V_2,ψ_2), $\psi_2=(x_2^p,y_2^\tau)$, on Y_2 such that $\alpha(V_1)\subset V_2$. We have the commutative diagram

(3) $$\begin{array}{ccc} V_1 & \xrightarrow{\alpha} & V_2 \\ \downarrow & & \downarrow \\ \pi_1(V_1) & \xrightarrow{\alpha_0} & \pi_2(V_2) \end{array}$$

expressing condition (2). In terms of the charts we can write

(4) $$\begin{aligned} &\alpha_0\pi_1=\varphi_2^{-1}\circ\varphi_2\alpha_0\varphi_1^{-1}\circ\varphi_1\pi_1\psi_1^{-1}\circ\psi_1,\\ &\pi_2\alpha=\varphi_2^{-1}\circ\varphi_2\pi_2\psi_2^{-1}\circ\psi_2\alpha\psi_1^{-1}\circ\psi_1, \end{aligned}$$

so the commutativity yields

(5) $$\varphi_2\alpha_0\varphi_1^{-1}\circ\varphi_1\pi_1\psi_1^{-1}=\varphi_2\pi_2\psi_2^{-1}\circ\psi_2\alpha\psi_1^{-1}.$$

But in our fibred charts $\varphi_1\pi_1\psi_1^{-1}$ is the Cartesian projection $(x_1^i,y_1^\sigma)\to(x_1^i)$, and $\varphi_2\pi_2\psi_2^{-1}$ is the Cartesian projection $(x_2^p,y_2^\tau)\to(x_2^p)$. Consequently, writing in components

(6) $$\varphi_2\alpha_0\varphi_1^{-1}\circ\varphi_1\pi_1\psi_1^{-1}(x_1^i,y_1^\sigma)=\varphi_2\alpha_0\varphi_1^{-1}(x_1^i)=(x_2^p\alpha_0\varphi_1^{-1}(x_1^i)),$$

$$\begin{aligned}
&\varphi_2\pi_2\psi_2^{-1}\circ\psi_2\alpha\psi_1^{-1}(x_1^i,y_1^\sigma)\\
&\quad=\varphi_2\pi_2\psi_2^{-1}(x_2^p\alpha\psi_1^{-1}(x_1^i,y_1^\sigma),y_2^\tau\alpha\psi_1^{-1}(x_1^i,y_1^\sigma))\\
&\quad=(x_2^p\alpha\psi_1^{-1}(x_1^i,y_1^\sigma)),
\end{aligned}$$

we see that condition (5) implies $x_2^p\alpha_0\varphi_1^{-1}(x_1^i)=x_2^p\alpha\psi_1^{-1}(x_1^i,y_1^\sigma)$. This shows that the right-hand side expression is independent of the coordinates y_1^σ. Therefore, we conclude that the equations of the morphism α in fibred charts are always of the form

(7) $$x_2^p=f^p(x_1^i),\quad y_2^\tau=F^\tau(x_1^i,y_1^\sigma).$$

Let Y be a fibred manifold with base X and projection π. If Ξ is a tangent vector to Y at a point $y\in Y$, then the tangent vector ξ to X at $x=\pi(y)\in X$, defined by

(8) $$T_y\pi\cdot\Xi=\xi,$$

is called the π*-projection*, or simply the *projection* of Ξ. By definition of the submersion, the tangent mapping of the projection π at a point y, $T_y\pi:T_yY\to T_{\pi(x)}X$, is surjective.

A tangent vector $\Xi\in T_yY$ at a point $y\in Y$ is said to be π*-vertical,* if

(9) $$T_y\pi\cdot\Xi=0.$$

The vector subspace of T_yY consisted of π-vertical vectors, is denoted by VT_yY. If Ξ is expressed in a fibred chart (V,ψ), $\psi=(x^i,y^\sigma)$, by

(10) $$\Xi=\xi^i\left(\frac{\partial}{\partial x^i}\right)_y+\Xi^\sigma\left(\frac{\partial}{\partial y^\sigma}\right)_y,$$

then by (8)

(11) $$\xi=\xi^i\left(\frac{\partial}{\partial x^i}\right)_x=0.$$

Thus, Ξ is π-vertical if and only if

(12) $$\Xi=\Xi^\sigma\left(\frac{\partial}{\partial y^\sigma}\right)_y.$$

If in particular, $\dim Y=n+m$ and $\dim X=n$, then $\dim VT_yY=m$.

The subset VTY of the tangent bundle TY, defined by

(13) $$VTY=\bigcup_{y\in Y}VT_yY,$$

is a vector subbundle of TY .

The projection $\pi : Y \to X$ induces a vector bundle morphism $T\pi : TY \to TX$; from the definition of a fibred manifold it follows that the image is $\mathrm{Im} T\pi = TX$. The vector subbundle $VTY = \mathrm{Ker} T\pi$ of the vector bundle TY is called the *vertical subbundle* over *Y*.

Let ρ be a differential *k*-form, defined on an open set *W* in *Y*. We say that ρ is π *-horizontal*, or just *horizontal*, if it vanishes whenever one of its vector arguments is a π -vertical vector.

We describe the chart expressions of π -horizontal forms.

Lemma 2 *The form* ρ *is* π *-horizontal if and only if in any fibred chart* (V,ψ) *,* $\psi = (x^i, y^\sigma)$ *, it has an expression*

(14) $$\rho = \frac{1}{k!}\rho_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} .$$

Proof Choose a point $y \in V$ and express the form $\rho(y)$ as

(15) $$\rho(y) = \frac{1}{k!}\rho_{i_1 i_2 \dots i_k}(y) dx^{i_1}(y) \wedge dx^{i_2}(y) \wedge \dots \wedge dx^{i_k}(y) + dy^1(y) \wedge \rho_1(y) \\ + dy^2(y) \wedge \rho_2(y) + \dots + dy^m(y) \wedge \rho_m(y),$$

where the forms $\rho_1(y)$, $\rho_2(y)$, $\dots$, $\rho_m(y)$ do not contain $dy^1(y)$, the forms $\rho_2(y)$, $\rho_3(y)$, $\dots$, $\rho_m(y)$ do not contain $dy^1(y)$ and $dy^2(y)$, etc. Suppose that ρ is π -horizontal. Then contracting the form $\rho(y)$ by the vertical vector $(\partial / \partial y^1)_y$ we get $i_{(\partial/\partial y^1)_y} \rho(y) = \rho_1(y) = 0$. Contracting $\rho(y)$ by the vertical vector $(\partial / \partial y^2)_y$ we get $i_{(\partial/\partial y^2)_y} \rho(y) = \rho_2(y) = 0$, etc.. Clearly, which proves formula (14).

Example 4 The first Cartesian projection pr_1 of the product of Euclidean spaces $\mathbf{R}^n \times \mathbf{R}^m$ onto $\mathbf{R}^n$, restricted to the product of open sets $U \times V$, where $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$, is a fibred manifold over *U*. The restriction of pr_1 to *any* open set $W \subset \mathbf{R}^n \times \mathbf{R}^m$ is a fibred manifold over $\mathrm{pr}_1(W) \subset \mathbf{R}^n$.

Example 5 Moebius band is a fibred manifold over the circle.

A form ρ , defined on an open set *W* in *Y*, is said to be π *-projectable*, or just *projectable*, if there exists a form ρ_0 , defined on the set $\pi(W)$, such that

(16) $$\rho = \pi^* \rho_0 .$$

If the form ρ_0 exists, it is unique and is called the π *-projection*, of just the *projection* of ρ .

Convention Formula (16) shows that a π -projectable form can canonically be identified with its π -projection. Thus, to simplify the notation, we sometimes denote a π -projectable form $\pi^* \rho_0$ by its π -projection ρ_0 .

1.3 The contact of differentiable mappings

Let X and Y be two smooth manifolds, $n = \dim X$, and $m = \dim Y$. Let $x \in X$ be a point, $f_1 : W \to Y$ and $f_2 : W \to Y$ two mappings, defined on a neighbourhood W of x. We say that f_1, f_2 have the *contact of order* 0 at x, if

(1) $$f_1(x) = f_2(x).$$

Suppose that f_1 and f_2 are of class C^r, where r is a positive integer. We say that f_1, f_2 have the *contact of order r* at x, if they have the contact of order 0, and there exist a chart (U,φ), $\varphi = (x^i)$, at x and a chart (V,ψ), $\psi = (y^\sigma)$, at $f_1(x)$ such that $U \subset W$, $f_1(U), f_2(U) \subset V$, and

(2) $$D^k(\psi f_1 \varphi^{-1})(\varphi(x)) = D^k(\psi f_2 \varphi^{-1})(\varphi(x))$$

for all $k \le r$. These definitions immediately extend to C^∞ mappings f_1, f_2; in this case f_1, f_2 are said to have the *contact of order* ∞ at x, if they have the contact of order r for every r.

Writing in components $\psi f_1 \varphi^{-1} = y^\sigma f_1 \varphi^{-1}$, $\psi f_2 \varphi^{-1} = y^\sigma f_2 \varphi^{-1}$, we see at once that f_1 and f_2 have contact of order r if and only if $f_1(x) = f_2(x)$ and

(3) $$D_{i_1} D_{i_2} \ldots D_{i_k}(y^\sigma f_1 \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \ldots D_{i_k}(y^\sigma f_2 \varphi^{-1})(\varphi(x))$$

for all $k = 1,2,\ldots,r$, all σ and all $i_1, i_2, \ldots, i_k$ such that $1 \le \sigma \le m$ and $1 \le i_1 \le i_2 \le \ldots \le i_k \le n$.

We claim that if f_1, f_2 have contact of order r at a point x, then for any chart $(\bar{U},\bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, at x and any chart $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, at $f_1(x)$,

(4) $$D^k(\bar{\psi} f_1 \bar{\varphi}^{-1})(\bar{\varphi}(x)) = D^k(\bar{\psi} f_2 \bar{\varphi}^{-1})(\bar{\varphi}(x))$$

for all $k = 1,2,\ldots,r$. We can verify this formula by means of the chain rule for derivatives of mappings of Euclidean spaces. Using the charts (U,φ), (V,ψ) we express the derivative

(5) $$\begin{aligned} &D_{i_1} D_{i_2} \ldots D_{i_k}(\bar{y}^\sigma f_1 \bar{\varphi}^{-1})(\bar{\varphi}(x)) \\ &\quad = D_{i_1} D_{i_2} \ldots D_{i_k}(\bar{y}^\sigma \psi^{-1} \circ \psi f_1 \varphi^{-1} \circ \varphi \bar{\varphi}^{-1})(\bar{\varphi}(x)) \end{aligned}$$

as a polynomial in the variables $D_{j_1}(y^\nu f_1 \varphi^{-1})(\varphi(x))$, $D_{j_1} D_{j_2}(y^\nu f_1 \varphi^{-1})(\varphi(x))$, $\ldots$, $D_{j_1} D_{j_2} \ldots D_{j_k}(y^\nu f_1 \varphi^{-1})(\varphi(x))$. The derivative $D_{i_1} D_{i_2} \ldots D_{i_k}(\bar{y}^\sigma f_2 \bar{\varphi}^{-1})(\bar{\varphi}(x))$ is expressed by the same polynomial in the variables $D_{j_1}(y^\nu f_2 \varphi^{-1})(\varphi(x))$ $D_{j_1} D_{j_2}(y^\nu f_2 \varphi^{-1})(\varphi(x))$, $\ldots$, $D_{j_1} D_{j_2} \ldots D_{j_k}(y^\nu f_2 \varphi^{-1})(\varphi(x))$. Clearly, equality (4) now follows from (3).

Fix two points $x \in X$, $y \in Y$, and denote by $C^r_{(x,y)}(X,Y)$ the set of C^r mappings $f : W \to Y$, where W is a neighbourhood of x and $f(x) = y$.

The binary relation "*f, g have the contact of order r at x*" on $C^r_{(x,y)}(X,Y)$ is obviously *reflexive*, *transitive*, and *symmetric*, so is an equivalence relation. Equivalence classes of this equivalence relation are called *r-jets* with *source* x and *target* y. The r-jet whose representative is a mapping $f \in C^r_{(x,y)}(X,Y)$ is called the *r-jet of f at the point x*, and is denoted by $J^r_x f$. If there is no danger of misunderstanding, we call an r-jet with source x and target y an r-*jet*, or just a *jet*. The set of r-jets with source $x \in X$ and target $y \in Y$ is denoted by $J^r_{(x,y)}(X,Y)$.

Let $f \in C^r_{(x,y)}(X,Y)$ be a mapping, $f: W \to Y$, let U be a neighbourhood of x and V a neighbourhood of y. Assigning to f the restriction of f to the set $f^{-1}(V) \cap U \cap W$, we get a *bijection* $J^r_x f \to J^r_x(f|_{f^{-1}(V)\cap U\cap W})$ of the set $J^r_{(x,y)}(X,Y)$ onto $J^r_{(x,y)}(U,V)$.

Let X, Y, and Z be three smooth manifolds. Two r-jets $A \in J^r_{(x,u)}(X,Y)$, $A = J^r_x f$, and $B \in J^r_{(y,z)}(Y,Z)$, $B = J^r_y g$, are said to be *composable*, if they have representatives which are composable (as mappings), i.e., if $u = y$; this equality means that the target of A coincides with the source of B. In this case the composite $g \circ f$ of any representatives of A and B is a mapping of class C^r defined on a neighbourhood of x. It is easily seen that the r-jet $J^r_x(g \circ f)$ is independent of the representatives of the r-jets A and B. If $\bar{f}$ and $\bar{g}$ are such that $J^r_x f = J^r_x \bar{f}$ and $J^r_x g = J^r_x \bar{g}$, then for any charts (U,φ), $\varphi = (x^i)$ at x, (V,ψ), $\psi = (y^\sigma)$, at $y = f(x)$, and (W,η), $\eta = (z^p)$, at $z = g(y)$, the derivatives $D_{i_1} D_{i_2} \ldots D_{i_k}(z^p g f \varphi^{-1})(\varphi(x))$ are expressible in the form

$$(6) \qquad D_{i_1} D_{i_2} \ldots D_{i_k}(z^p g f \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \ldots D_{i_k}(z^p g \psi^{-1} \circ \psi f \varphi^{-1})(\varphi(x)).$$

for all $k = 1,2,\ldots,r$. By the chain rule for mappings of Euclidean spaces, expressions (6) are polynomial in the variables $D_{\nu_1} D_{\nu_2} \ldots D_{\nu_q}(z^p g \psi^{-1})(\psi(y))$ and $D_{i_1} D_{i_2} \ldots D_{i_m}(y^\nu f \varphi^{-1})(\varphi(x))$, where $m,q \le k$. The same polynomials in the derivatives $D_{\nu_1} D_{\nu_2} \ldots D_{\nu_q}(z^p \bar{g} \psi^{-1})(\psi(y))$, $D_{i_1} D_{i_2} \ldots D_{i_m}(y^\nu \bar{f} \varphi^{-1})(\varphi(x))$ are obtained when expressing $D_{i_1} D_{i_2} \ldots D_{i_k}(z^p \bar{g}\bar{f} \varphi^{-1})(\varphi(x))$ by means of the chain rule. Now since by definition

$$(7) \qquad \begin{aligned} D_{i_1} D_{i_2} \ldots D_{i_m}(y^\nu f \varphi^{-1})(\varphi(x)) &= D_{i_1} D_{i_2} \ldots D_{i_m}(y^\nu \bar{f} \varphi^{-1})(\varphi(x)), \\ D_{\nu_1} D_{\nu_2} \ldots D_{\nu_q}(z^p g \psi^{-1})(\psi(y)) &= D_{\nu_1} D_{\nu_2} \ldots D_{\nu_q}(z^p \bar{g} \psi^{-1})(\psi(y)), \end{aligned}$$

we have

$$(8) \qquad D_{i_1} D_{i_2} \ldots D_{i_k}(z^p g f \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \ldots D_{i_k}(z^p \bar{g}\bar{f} \varphi^{-1})(\varphi(x)).$$

This proves, that the r-jet $J^r_x(g \circ f)$ is independent of the choice of A and B.

If X, Y, and Z are three manifolds and $A \in J^r_{(x,y)}(X,Y)$, $A = J^r_x f$, and $B \in J^r_{(y,z)}(Y,Z)$, $B = J^r_y g$, are composable r-jets, we define

$$(9) \qquad B \circ A = J^r_x(g \circ f),$$

or, explicitly, $J^r_x g \circ J^r_x f = J^r_x(g \circ f)$. The r-jet $B \circ A$ is called the *composite* of A and B, and the mapping $(A,B) \to B \circ A$ of $J^r_{(x,f(x))}(X,Y) \times J^r_{(y,g(y))}(Y,Z)$ into $J^r_{(x,z)}(X,Z)$, where $z = g(y)$, is the *composition* of r-jets.

A chart on X at the point x and a chart on Y at the point y induce a chart on the set $J^r_{(x,y)}(X,Y)$. Let (U,φ), $\varphi = (x^i)$ (resp. (V,ψ), $\psi = (x^i, y^\sigma)$), be a chart on X (resp. Y). We assign to any r-jet $J^r_x f \in J^r_{(x,y)}(X,Y)$ the numbers

$$z^\sigma_{j_1 j_2 \dots j_k}(J^r_x \gamma) = D_{j_1} D_{j_2} \dots D_{j_k}(y^\sigma f \varphi^{-1})(\varphi(x)), \quad 1 \le k \le r. \tag{10}$$

Then the collection of functions $\chi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r})$, such that

$$1 \le j_1 \le j_2 \le \dots \le j_k \le n, \quad 1 \le \sigma \le m, \tag{11}$$

is a bijection of the set $J^r_{(x,y)}(X,Y)$ and the Euclidean space $\mathbf{R}^N$ of dimension

$$N = n + m\left(1 + n + \binom{n+1}{2} + \binom{n+2}{3} + \dots + \binom{n+r-1}{r}\right). \tag{12}$$

Thus, the pair $(J^r_{(x,y)}(X,Y), \chi^r)$ is a (global) chart on $J^r_{(x,y)}(X,Y)$. This chart is said to be *associated* with the charts (U,φ) and (V,ψ).

Lemma 3 (a) *The associated charts* $(J^r_{(x,y)}(X,Y), \chi^r)$*, such that the charts* (U,φ) *and* (V,ψ) *belong to smooth structures on X and Y, form a smooth atlas on* $J^r_{(x,y)}(X,Y)$*. With this atlas,* $J^r_{(x,y)}(X,Y)$ *is a smooth manifold of dimension N.*

(b) *The composition of jets*

$$J^r_{(x,y)}(X,Y) \times J^r_{(y,z)}(Y,Z) \ni (A,B) \to B \circ A \in J^r_{(x,z)}(X,Z) \tag{13}$$

is smooth.

Proof 1. It is enough to prove that the transformation equations between the associated charts are of class C^∞. However this follows from (5).

2. (b) is an immediate consequence of formula (6).

1.4 Jet prolongations of fibred manifolds

In this section we apply the concept of contact of differentiable mappings (Section 1.3) to C^r *sections* of fibred manifolds. We introduce the smooth manifold structure on the sets of jets of sections and establish the coordinate transformation formulas.

Let Y be a fibred manifold with base X and projection π, let $n = \dim X$ and $m = \dim Y - n$. We denote by $J^r Y$, where $r \ge 0$ is any integer, the set of r-jets $J^r_x \gamma$ of C^r *sections* γ of Y with source $x \in X$ and target $y = \gamma(x) \in Y$; if $r = 0$, then $J^0 Y = Y$. Note that the representatives of an r-jet $J^r_x \gamma$ are C^r sections $\gamma : W \to Y$, where W is an open set in X; the condition that γ is a section,

$$\pi \circ \gamma = \mathrm{id}_W \tag{1}$$

implies that the target $y = \gamma(x)$ of the r-jet $J^r_x\gamma$ belongs to the fibre $\pi^{-1}(x) \subset Y$ over the source point x. For any s such that $0 \le s \le r$ we have surjective mappings $\pi^{r,s} : J^rY \to J^sY$ and $\pi^r : J^rY \to X$, defined by the conditions

$$\pi^{r,s}(J^r_x\gamma) = J^s_x\gamma, \quad \pi^r(J^r_x\gamma) = x. \tag{2}$$

These mappings are called the *canonical jet projections*.

The smooth structure of the fibred manifold Y induces a smooth structure on the set J^rY. This is based on a canonical construction that assigns to any fibred chart on Y a chart on J^rY. Let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fibred chart on Y, and let (U,φ), $\varphi = (x^i)$, be the associated chart on X. We set $V^r = (\pi^{r,0})^{-1}(V)$, and introduce, for all values of the indices, a family of functions $x^i, y^\sigma, y^\sigma_{j_1 j_2 \dots j_k}$, defined on V^r, by

$$\begin{aligned} &x^i(J^r_x\gamma) = x^i(x),\\ &y^\sigma(J^r_x\gamma) = y^\sigma(\gamma(x)),\\ &y^\sigma_{j_1 j_2 \dots j_k}(J^r_x\gamma) = D_{j_1} D_{j_2} \dots D_{j_k}(y^\sigma \gamma \varphi^{-1})(\varphi(x)), \quad 1 \le k \le r. \end{aligned} \tag{3}$$

Then the collection of functions $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r})$, where the indices satisfy

$$1 \le i \le n, \quad 1 \le \sigma \le m, \quad 1 \le j_1 \le j_2 \le \dots \le j_k \le n, \quad k = 2,3,\dots,r, \tag{4}$$

is a bijection of the set V^r onto an open subset of the Euclidean space $\mathbf{R}^N$ of dimension

$$N = n + m\left(1 + n + \binom{n+1}{2} + \binom{n+2}{3} + \dots + \binom{n+r-1}{r}\right). \tag{5}$$

The pair (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r})$, is a chart on the set J^rY, which is said to be *associated* with the fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$.

Lemma 4 (Smooth structure on the set J^rY) *The set of associated charts (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r})$, such that the fibred charts (V,ψ) constitute an atlas on Y, is an atlas on J^rY.*

Proof Let $\mathscr{A}$ be an atlas on Y whose elements are fibred charts (Section 1.2, Lemma 1). One can easily check that $\mathscr{A}$ defines a topology on J^rY by requiring that for any fibred chart (V,ψ) from $\mathscr{A}$, the mapping $\psi^r : V^r \to \psi^r(V^r) \subset \mathbf{R}^N$ is a homeomorphism; we consider the set J^rY with this topology.

It is clear that the associated charts with fibred charts from $\mathscr{A}$ cover the set J^rY. Thus, to prove Lemma 4 it remains to check that the corresponding coordinate transformations are smooth.

Suppose we have two fibred chart on Y, (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\bar{V},\bar{\psi})$, $\bar{\psi}=(\bar{x}^i,\bar{y}^\sigma)$, such that $V\cap\bar{V}\neq\emptyset$. Consider the associated charts (V^r,ψ^r), $(\bar{V}^r,\bar{\psi}^r)$, and an element $J^r_x\gamma\in V^r\cap\bar{V}^r$. Let the coordinate transformation $\bar{\psi}\psi^{-1}$ be expressed by the equations

$$\bar{x}^i=f^i(x^j),\quad \bar{y}^\sigma=g^\sigma(x^j,y^\nu). \tag{6}$$

Note that the functions f^i and g^σ in formula (6) are defined by the formulas $\bar{x}^i(x)=\bar{x}^i\varphi^{-1}(\varphi(x))=f^i(\varphi(x))$ and $\bar{y}^\sigma(y)=\bar{y}^\sigma\psi^{-1}(\psi(y))=g^\sigma(\psi(y))$. We have

$$\begin{aligned}
&\bar{x}^i(J^r_x\gamma)=\bar{x}^i(x)=\bar{x}^i\varphi^{-1}(\varphi(x))=\bar{x}^i\varphi^{-1}(\varphi(J^r_x\gamma)),\\
&\bar{y}^\sigma(J^r_x\gamma)=\bar{y}^\sigma(\gamma(x)))=(\bar{y}^\sigma\psi^{-1}\circ\psi)(\gamma(x))=\bar{y}^\sigma\psi^{-1}(\psi(J^r_x\gamma)),\\
&\bar{y}^\sigma_{j_1j_2\dots j_k}(J^r_x\gamma)=D_{j_1}D_{j_2}\dots D_{j_k}(\bar{y}^\sigma\gamma\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad=D_{j_1}D_{j_2}\dots D_{j_k}(\bar{y}^\sigma\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x)).
\end{aligned} \tag{7}$$

From the chain rule it is now obvious that the left-hand sides, the coordinates of the r-jet $J^r_x\gamma$ in the chart $(\bar{V}^r,\bar{\psi}^r)$, depend smoothly on the coordinates of $J^r_x\gamma$ in the chart (V^r,ψ^r).

From now on, the set J^rY is always considered with the smooth structure, defined by Lemma 4, and is called the *r-jet prolongation* of the fibred manifold *Y*.

Lemma 5 *Each of the canonical jet projections* (2) *is smooth and defines a fibred manifold structure on the manifold* J^rY.

Proof Indeed, in the associated charts each of the canonical jet projections is expressed as a Cartesian projection, which is smooth.

Every C^r section $\gamma:W\to Y$, where *W* is an open set in *X*, defines a mapping

$$W\ni x\to J^r\gamma(x)=J^r_x\gamma\in J^rY, \tag{8}$$

called the *r-jet prolongation* of γ.

Example 6 (Coordinate transformations on J^2Y) Consider two fibred charts on a fibred manifold *Y*, (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\bar{V},\bar{\psi})$, $\bar{\psi}=(\bar{x}^i,\bar{y}^\sigma)$, such that $V\cap\bar{V}\neq\emptyset$. Suppose that the corresponding transformation equations are expressed as

$$\bar{x}^i=\bar{x}^i(x^j),\quad \bar{y}^\sigma=\bar{y}^\sigma(x^j,y^\nu). \tag{9}$$

Then the induced coordinate transformation on J^2Y is expressed by the equations

$$
\begin{aligned}
&\bar{x}^i = \bar{x}^i(x^j),\\
&\bar{y}^\sigma = \bar{y}^\sigma(x^j, y^\nu),\\
&\bar{y}^\sigma_{j_1} = \left(\frac{\partial \bar{y}^\sigma}{\partial x^l} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_l\right)\frac{\partial x^l}{\partial \bar{x}^{j_1}},\\
&\bar{y}^\sigma_{j_1 j_2} = \left(\frac{\partial^2 \bar{y}^\sigma}{\partial x^l \partial x^m} + \frac{\partial^2 \bar{y}^\sigma}{\partial x^l \partial y^\mu} y^\mu_m + \frac{\partial^2 \bar{y}^\sigma}{\partial x^m \partial y^\nu} y^\nu_l + \frac{\partial^2 \bar{y}^\sigma}{\partial y^\mu \partial y^\nu} y^\nu_l y^\mu_m\right.\\
&\qquad \left. + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{lm}\right)\frac{\partial x^m}{\partial \bar{x}^{j_2}}\frac{\partial x^l}{\partial \bar{x}^{j_1}} + \left(\frac{\partial \bar{y}^\sigma}{\partial x^l} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_l\right)\frac{\partial^2 x^l}{\partial \bar{x}^{j_1}\partial \bar{x}^{j_2}}.
\end{aligned}
\tag{10}
$$

To derive these equations, we use the chain rule for partial derivative operators. Let $J^2_x\gamma \in V^2 \cap \bar{V}^2$. The 2-jet $J^2_x\gamma$ has the coordinates

$$
\begin{aligned}
&x^i(J^2_x\gamma) = x^i(x),\\
&y^\sigma(J^2_x\gamma) = y^\sigma(\gamma(x)),\\
&y^\sigma_{j_1}(J^r_x\gamma) = D_{j_1}(y^\sigma\gamma\varphi^{-1})(\varphi(x)),\\
&y^\sigma_{j_1 j_2}(J^r_x\gamma) = D_{j_1}D_{j_2}(y^\sigma\gamma\varphi^{-1})(\varphi(x)),
\end{aligned}
\tag{11}
$$

and analogous formulas arise for the chart $(\bar{V}, \bar{\psi})$. Then by the chain rule

$$
\begin{aligned}
&D_{j_1}(\bar{y}^\sigma\gamma\bar{\varphi}^{-1})(\bar{\varphi}(x)) = D_{j_1}(\bar{y}^\sigma\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad = D_k(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(x^k\gamma\varphi^{-1})(\varphi\bar{\varphi}^{-1}(\bar{\varphi}(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad + D_\nu(\bar{y}^\sigma\psi^{-1}(\psi\gamma(x))D_l(y^\nu\gamma\varphi^{-1})(\varphi\bar{\varphi}^{-1}(\bar{\varphi}(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad = D_k(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))\delta^k_l D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad + D_\nu(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(y^\nu\gamma\bar{\varphi}^{-1})(\varphi(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad = (D_l(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x)) + D_\nu(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(y^\nu\gamma\bar{\varphi}^{-1})(\varphi(x)))\\
&\quad \cdot D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x)),
\end{aligned}
\tag{12}
$$

which proves the third one of equations (10). To prove the fourth equation, we differentiation (12) again and apply the chain rule. We can also derive the fourth equation by differentiating the third one.

Consider a morphism $\alpha: W \to Y$ of a fibred manifold Y with projection π. The projection $\alpha_0: \pi(W) \to X$ of the morphism α is a unique morphism of smooth manifolds such that

$$\pi \circ \alpha = \alpha_0 \circ \pi. \tag{13}$$

Suppose that α_0 is a diffeomorphism of the open subsets $\pi(W)$ and $U_0 = \alpha_0(\pi(W))$ in *X*. Then for any section γ of *Y*, defined on $\pi(W)$, formula $\gamma' = \alpha\gamma\alpha_0^{-1}$ defines a section of *Y* over U_0: indeed, since γ is a section, then $\pi\gamma' = \pi\alpha\gamma\alpha_0^{-1} = \alpha_0\pi\gamma\alpha_0^{-1} = \mathrm{id}_{U_0}$. In this sense α transforms sections γ of *Y* into sections $\alpha\gamma\alpha_0^{-1}$ of *Y*. In particular, setting for every *r*-jet $J_x^r\gamma \in W^r$

$$(14) \qquad J^r\alpha(J_x^r\gamma) = J_{\alpha_0}^r \alpha\gamma\alpha_0^{-1}$$

we get a mapping $J^r\alpha : W^r \to J^rY$. This mapping is differentiable, and satisfies, for all integers *s* such that $0 \le s \le r$,

$$(15) \qquad \pi^{r,s} \circ J^r\alpha = J^s\alpha \circ \pi^{r,s}, \quad \pi^r \circ J^r\alpha = \alpha_0 \circ \pi^r.$$

These formulas show that the *mapping* $J^r\alpha$ is a morphisms of the *r*-jet prolongation J^rY of the fibred manifold *Y* over J^sY for all *s* such that $0 \le s \le r$, and over *X*. $J^r\alpha$ is called the *r-jet prolongation* of the morphism $J^r\alpha$ of *Y*. Note that $J^r\alpha$ is *not* defined for morphisms α whose projections are *not* diffeomorphisms.

1.5 The horizontalization

Let *Y* be a fibred manifold with base *X* and projection π, $\dim X = n$ and $\dim Y = n + m$. For any open set $W \subset Y$ we denote by W^r the open set $(\pi^{r,0})^{-1}(W)$ in the *r*-jet prolongation J^rY of *Y*. We show that the fibred manifold structure on *Y* induces a vector bundle morphism between the tangent bundles $T^{r+1}Y$ and T^rY and study the decomposition of tangent vectors, associated with this mapping.

Let $J_x^{r+1}\gamma$ be a point of the manifold $J^{r+1}Y$. We assign to any tangent vector ξ of $J^{r+1}Y$ at the point $J_x^{r+1}\gamma$ a tangent vector of J^rY at the point $\pi^{r+1,r}(J_x^{r+1}\gamma) = J_x^r\gamma$ by

$$(1) \qquad h\xi = T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi.$$

We get a vector bundle morphism $h : TJ^{r+1}Y \to TJ^rY$ over the jet projection $\pi^{r+1,r}$, called the π*-horizontalization*, or simply the *horizontalization*. Sometimes we call $h\xi$ the *horizontal component* of ξ (note, however, that ξ and $h\xi$ do not belong to the same vector space). Using a complementary construction, one can also assign to every tangent vector $\xi \in TJ^{r+1}Y$ at the point $J_x^{r+1}\gamma \in J^{r+1}Y$ a tangent vector $p\xi \in TJ^rY$ at $J_x^r\gamma$ by the decomposition

$$(2) \qquad T\pi^{r+1,r} \cdot \xi = h\xi + p\xi.$$

$p\xi$ is called the *contact component* of the vector ξ.

Lemma 6 *The horizontal and contact components satisfy*

(3) $$T\pi^r \cdot h\xi = T\pi^{r+1} \cdot \xi, \quad T\pi^r \cdot p\xi = 0.$$

Proof The first property follows from (1). Then, however,

(4) $$\begin{aligned} T\pi^r \cdot p\xi &= T\pi^r \cdot T\pi^{r+1,r} \cdot \xi - T\pi^r \cdot h\xi = T\pi^{r+1} \cdot \xi - T\pi^r \cdot h\xi \\ &= T\pi^{r+1} \cdot \xi - T\pi^r \cdot T_x J^r\gamma \circ T\pi^{r+1} \cdot \xi = 0. \end{aligned}$$

Remark 1 If $h\xi = 0$, then necessarily $T\pi^{r+1} \cdot \xi = 0$ so ξ is π^{r+1}-vertical. This observation may explain why $h\xi$ is called the *horizontal component* of ξ.

One can easily find the chart expressions for the vectors $h\xi$ and $p\xi$. If in a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, ξ has an expression

(5) $$\xi = \xi^i \left(\frac{\partial}{\partial x^i} \right)_{J_x^{r+1}\gamma} + \sum_{k=0}^{r+1} \sum_{j_1 \le j_2 \le \ldots \le j_k} \Xi^\sigma_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \right)_{J_x^{r+1}\gamma},$$

then

(6) $$h\xi = \xi^i \left(\left(\frac{\partial}{\partial x^i} \right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \right)_{J_x^r\gamma} \right),$$

and

(7) $$p\xi = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} (\Xi^\sigma_{j_1 j_2 \ldots j_k} - y^\sigma_{j_1 j_2 \ldots j_k i} \xi^i) \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \right)_{J_x^r\gamma}.$$

Note that the conditions $h\xi = 0$ and $p\xi = 0$ do *not* imply $\xi = 0$; they are equivalent to the condition that ξ be $\pi^{r+1,r}$-vertical,

(8) $$\xi = \sum_{j_1 \le j_2 \le \ldots \le j_{r+1}} \Xi^\sigma_{j_1 j_2 \ldots j_{r+1}} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_{r+1}}} \right)_{J_x^{r+1}\gamma}.$$

The structure of the chart expression (6) can also be characterized by means of the vector fields d_i along the projection $\pi^{r+1,r}$, defined on V^{r+1} by

(9) $$d_i = \left(\frac{\partial}{\partial x^i} \right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \right)_{J_x^r\gamma}.$$

d_i is called the *i-th formal derivative operator* (relative to the fibred chart

(V,ψ)). Note that these vector fields are closely connected with the tangent mapping of the functions $f: J^rY \to \mathbf{R}$, composed with the prolongations $J^r\gamma$ of sections γ of *Y*. Namely, if (V,ψ), $\psi = (x^i, y^\sigma)$, is a fibred chart, $x \in \pi(U)$ a point and γ a section defined on *U*, then for every tangent vector $\xi_0 \in T_xX$, expressed as $\xi_0 = \xi_0^i (\partial / \partial x^i)_x$,

(10) $$T_x(f \circ J^r\gamma) \cdot \xi_0 = \left(\frac{\partial (f \circ J^r\gamma \circ \varphi^{-1})}{\partial x^k} \right)_x \xi^k.$$

For each *i* such that $1 \le i \le n$, the formula

(11) $$d_i f(J_x^{r+1}\gamma) = \left(\frac{\partial (f \circ J^r\gamma \circ \varphi^{-1})}{\partial x^k} \right)_x$$

defines a function $d_i f : V^{r+1} \to \mathbf{R}$, called the *i-th formal derivative* of the function *f* (relative to the given fibred chart). In the chart

(12) $$d_i f = \xi^i \frac{\partial f}{\partial x^i} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}.$$

Remark 2 Canonically extending the partial derivatives $\partial / \partial y^\sigma_{j_1 j_2 \ldots j_k}$ to *all* sequences $j_1, j_2, \ldots, j_k$, the formal derivative d_i can be expressed as

(13) $$d_i = \frac{\partial}{\partial x^i} + \sum_{k=0}^{r} y^\sigma_{j_1 j_2 \ldots j_k i} \frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}$$

(see Appendix 2).

Remark 3 In general, decomposition (2) of tangent vectors does *not* hold for vector fields. However, if ξ is a π^{r+1}*-vertical* vector field on W^{r+1}, then $h\xi$ is the zero vector field on W^r and condition (2) reduces to the $\pi^{r+1,r}$-projectability equation

(14) $$T\pi^{r+1,r} \cdot \xi = \xi_0 \circ \pi^{r+1,r}$$

for the $\pi^{r+1,r}$-projection ξ_0 of ξ. Thus $p\xi(J_x^{r+1}\gamma) = \xi_0(J_x^r\gamma)$.

1.6 Jet prolongations of automorphisms of fibred manifolds

Let (V,ψ), $\psi = (x^i, y^\sigma)$, be a chart, and let $f : V^r \to \mathbf{R}$ be a differentiable function. We set for every i, $1 \le i \le n$,

(1) $$d_i f = \frac{\partial f}{\partial x^i} + \sum_{0 \le k \le r} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i}.$$

In this formula the function $d_i f : V^{r+1} \to \mathbf{R}$ is the *i-th formal derivative* of the function *f* (Section 1.5). A notable formula

(2) $$d_i y^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k i}$$

says that d_i may be treated as a mapping, acting on jet coordinates of the given chart.

Let *r* be a positive integer. Consider an open set *W* in the fibred manifold *Y* and a C^r automorphism $\alpha : W \to Y$ with projection $\alpha_0 : W_0 \to X$, defined on an open set $W_0 = \pi(W)$. In this section we suppose that the projection α_0 is a C^r *diffeomorphism*.

Every section $\gamma : W_0 \to Y$ defines the mapping $\alpha\gamma\alpha_0^{-1} = \alpha \circ \gamma \circ \alpha_0^{-1}$; it is easily seen that this mapping is a section of *Y* over the open set $\alpha_0(W_0) \subset X$: indeed, using properties of morphisms and sections of fibred manifolds, we get $\pi \circ \alpha\gamma\alpha_0^{-1} = \alpha_0 \circ \pi \circ \gamma \circ \alpha_0^{-1} = \alpha_0 \circ \alpha_0^{-1} = \mathrm{id}_{W_0}$. Then, however, the *r*-jets of the section $x \to \alpha\gamma\alpha_0^{-1}(x)$ are defined and are elements of the set $J^r Y$. An *r*-jet $J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}$ can be decomposed as $J^r_{\gamma(x)}\alpha \circ J^r_x\gamma \circ J^r_{\alpha_0(x)}\alpha_0^{-1}$, so it is independent of the choice of the representative γ, and depends on the *r*-jet $J^r_x\gamma$ only. We set for every $J^r_x\gamma \in W^r = (\pi^{r,0})^{-1}(W)$

(3) $$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}.$$

This formula defines a mapping $J^r\alpha : W^r \to J^r Y$, called the *r-jet prolongation*, or just *prolongation* of the C^r automorphism α.

Note an immediate consequence of the definition (3). Given a C^r *section* $\gamma : W_0 \to Y$, then we have $J^r\alpha \circ J^r\gamma = J^r\alpha\gamma\alpha_0^{-1} \circ \alpha_0$ so the *r*-jet prolongation $J^r\alpha\gamma\alpha_0^{-1}$ of the section $\alpha\gamma\alpha_0^{-1}$ satisfies

(4) $$J^r\alpha\gamma\alpha_0^{-1} = J^r\alpha \circ J^r\gamma \circ \alpha_0^{-1}$$

on the set $\alpha_0(W_0)$. In particular, this formula shows that the *r*-jet prolongations of automorphisms carry sections of *Y* into sections of $J^r Y$ (over *X*).

We find the chart expression of the mapping $J^r\alpha$.

Lemma 7 *Suppose that in two fibred charts on Y,* (V,ψ), $\psi = (x^i, y^\sigma)$, *and* $(\bar V,\bar\psi)$, $\bar\psi = (\bar x^i, \bar y^\sigma)$, *on Y such that* $\alpha(V) \subset \bar V$, *the* C^r *automorphism* α *is expressed by equations*

(5) $$\bar x^i \circ \alpha(y) = f^i(x^j(x)), \quad \bar y^\sigma \circ \alpha(y) = F^\sigma(x^j(x), y^\nu(y)).$$

Then for every point $J^r_x\gamma \in V^r$, *the transformed point* $J^r\alpha(J^r_x\gamma)$ *has the coordinates*

(6) $$\begin{aligned} &\bar x^i \circ J^r\alpha(J^r_x\gamma) = f^i(x^j(x)), \\ &\bar y^\sigma \circ J^r\alpha(J^r_x\gamma) = F^\sigma(x^j(x), y^\nu(\gamma(x))), \end{aligned}$$

$$\begin{aligned}&\bar{y}^{\sigma}_{j_1 j_2 \ldots j_k} \circ J^r\alpha(J^r_x\gamma)\\&\quad = D_{j_1} D_{j_2} \ldots D_{j_k} (\bar{y}^{\sigma}\alpha\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))),\\&\quad 1 \le k \le r.\end{aligned}$$

Proof We have

$$\begin{aligned}&\bar{x}^i \circ J^r\alpha(J^r_x\gamma) = \bar{x}^i \circ \alpha_0(x)\\&\quad = \bar{x}^i\alpha_0\varphi^{-1}(\varphi(x)) = f^i(x^j(x)),\\&\bar{y}^{\sigma} \circ J^r\alpha(J^r_x\gamma) = \bar{y}^{\sigma} \circ \alpha(\gamma(x)) = \bar{y}^{\sigma}\alpha\psi^{-1}(\psi(\gamma(x)))\\&\quad = F^{\sigma}(x^j(x), y^{\nu}(\gamma(x))),\end{aligned} \tag{7}$$

and by definition

$$\begin{aligned}&\bar{y}^{\sigma}_{j_1 j_2 \ldots j_k} \circ J^r\alpha(J^s_x\gamma) = \bar{y}^{\sigma}_{j_1 j_2 \ldots j_k}(J^s_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1})\\&\quad = D_{j_1} D_{j_2} \ldots D_{j_k} (\bar{y}^{\sigma} \circ \alpha\gamma\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x)))\\&\quad = D_{j_1} D_{j_2} \ldots D_{j_k} (\bar{y}^{\sigma}\alpha\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))).\end{aligned} \tag{8}$$

Formula (6) contains partial derivatives of the functions f^i and F^{σ}, and also partial derivatives of the functions g^k, representing the chart expression $\varphi\alpha_0^{-1}\bar{\varphi}^{-1}$ of the inverse diffeomorphism α_0^{-1}. These functions are defined by

$$x^k \circ \alpha_0^{-1}(x') = g^k(\bar{x}^l(x')). \tag{9}$$

To obtain explicit dependence of the coordinates $\bar{y}^{\sigma}_{j_1 j_2 \ldots j_k}(J^r\alpha(J^r_x\gamma))$ on the coordinates of the *r*-jet $J^r_x\gamma$, we have to use the chain rule *k* times, which leads to polynomial dependence of the jet coordinates $\bar{y}^{\sigma}_{j_1 j_2 \ldots j_k}(J^r\alpha(J^r_x\gamma))$ on the jet coordinates $y^{\nu}_{i_1}(J^r_x\gamma)$, $y^{\nu}_{i_1 i_2}(J^r_x\gamma)$, $\ldots$, $y^{\nu}_{i_1 i_2 \ldots i_k}(J^r_x\gamma)$. This shows, in particular, that if α is of class C^r, then $J^r\alpha$ is of class C^0; if α is of class C^s, where $s \ge r$, then $J^r\alpha$ is of class C^{s-r}.

Equations (6) can be viewed as recurrence formulas for the chart expression of the mapping $J^r\alpha$. Writing

$$\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha(J^r_x\gamma) = (\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))), \tag{10}$$

we have

$$\begin{aligned}&\bar{y}^{\sigma}_{j_1 j_2 \ldots j_k} \circ J^r\alpha(J^r_x\gamma)\\&\quad = D_{j_k}(\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x)))\\&\quad = D_l(\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1})(\varphi(x)) D_{j_k}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))).\end{aligned} \tag{11}$$

Thus, if we already have the functions $\bar{y}^{\sigma}_{j_1 j_2 \dots j_{k-1}} \circ J^r\alpha$, then the functions $\bar{y}^{\sigma}_{j_1 j_2 \dots j_k} \circ J^r\alpha$ are determined by (6).

We derive explicit expressions for the second jet prolongation $J^2\alpha$.

Example 7 (2-jet prolongation of an automorphism) Let $r = 2$. We have from (5)

$$\begin{aligned}
&\bar{y}^{\sigma}_{j_1} \circ J^2\alpha(J^r_x\gamma) \\
&= D_{j_1}(\bar{y}^{\sigma}\alpha\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\
&= D_k(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) D_l(x^k\gamma\varphi^{-1})(\varphi(x)) D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\
&= D_k(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) \delta^k_l D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\
&+ D_{\lambda}(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) y^{\lambda}_l(J^r_x\gamma) D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\
&= (D_l(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) + D_{\lambda}(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) y^{\lambda}_l(J^s_x\gamma)) \\
&\cdot D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))),
\end{aligned} \tag{12}$$

or, in a different notation,

$$\bar{y}^{\sigma}_{j_1} \circ J^2\alpha(J^r_x\gamma) = d_l F^{\sigma}(J^r_x\gamma) \left(\frac{\partial g^l}{\partial \bar{x}^{j_1}} \right)_{\bar{\varphi}(\alpha_0(x))}, \tag{13}$$

where d_l denotes the formal derivative operator. Differentiating (12) or (13) again we get the following equations for the 2-jet prolongation $J^2\alpha$ of α:

$$\begin{aligned}
&\bar{x}^i = f^i(x^i), \quad \bar{y}^{\sigma} = F^{\sigma}(x^i, y^{\nu}), \quad \bar{y}^{\sigma}_{j_1} = d_{k_1} F^{\sigma} \cdot \frac{\partial g^{k_1}}{\partial \bar{x}^{j_1}}, \\
&\bar{y}^{\sigma}_{j_1 j_2} = d_{k_1} d_{k_2} F^{\sigma} \cdot \frac{\partial g^{k_1}}{\partial \bar{x}^{j_1}} \frac{\partial g^{k_2}}{\partial \bar{x}^{j_2}} + d_{k_1} F^{\sigma} \cdot \frac{\partial^2 g^{k_1}}{\partial \bar{x}^{j_1} \partial \bar{x}^{j_2}}.
\end{aligned} \tag{14}$$

We can easily prove the following statements.

Lemma 8 (a) *For any s such that* $0 \le s \le r$,

$$\pi^r \circ J^r\alpha = \alpha_0 \circ \pi^r, \quad \pi^{r,s} \circ J^r\alpha = J^s\alpha \circ \pi^{r,s}. \tag{15}$$

(b) *If two* C^r *automorphisms* α *and* β *of the fibred manifold Y are composable, then* $J^r\alpha$ *and* $J^r\beta$ *are composable and*

$$J^r\alpha \circ J^r\beta = J^r(\alpha \circ \beta). \tag{16}$$

Proof All these assertions are easy consequences of definitions.

Formulas (15) show that $J^r\alpha$ is an C^r automorphism of the *r*-jet prolongation J^rY of the fibred manifold Y, and also C^r automorphisms of J^rY over J^sY.

1.7 Jet prolongations of vector fields

Let Y be a fibred manifold with base X and projection π. Our aim in this section is to extend the theory of jet prolongations of automorphisms of a fibred manifold Y to local flows of vector fields, defined on Y.

Let Ξ be a C^r vector field on Y, let $y_0 \in Y$ be a point, and consider a local flow $\alpha^\Xi : (-\varepsilon,\varepsilon)\times V \to Y$ of Ξ at y_0 (see Appendix 4). As usual, define the mappings α_t^Ξ and α_y^Ξ by

(1) $$\alpha^\Xi(t,y) = \alpha_t^\Xi(y) = \alpha_y^\Xi(t).$$

Then for any point $y \in V$ the mapping $t \to \alpha_y^\Xi(t)$ is an integral curve of Ξ passing through y at $t = 0$, i.e.,

(2) $$T_t\alpha_y^\Xi = \Xi(\alpha_y^\Xi(t)), \quad \alpha_y^\Xi(0) = y.$$

Moreover, shrinking the domain of definition $(-\varepsilon,\varepsilon)\times V$ of α^Ξ to a subset $(-\kappa,\kappa)\times W \subset (-\varepsilon,\varepsilon)\times V$, where W is a neighbourhood of the point y_0, we have

(3) $$\alpha^\Xi(s+t,y) = \alpha^\Xi(s,\alpha^\Xi(t,y)), \quad \alpha^\Xi(-t,\alpha^\Xi(t,y)) = y$$

for all $(s,t) \in (-\kappa,\kappa)$ and $y \in W$ or, which is the same,

(4) $$\alpha_{s+t}^\Xi(y) = \alpha_s^\Xi(\alpha_t^\Xi(y)), \quad \alpha_{-t}^\Xi\alpha_t^\Xi(y) = y.$$

Note that the second formula implies

(5) $$(\alpha_t^\Xi)^{-1} = \alpha_{-t}^\Xi.$$

In the following lemma we study properties of flows of a π*-projectable* vector field.

Lemma 9 *Let* Ξ *be a* C^r *vector field on* Y*. The following two conditions are equivalent:*

(1) *The local* 1*-parameter groups of* Ξ *consist of* C^r *automorphisms of the fibred manifold* Y*.*

(2) Ξ *is* π*-projectable.*

Proof 1. Let Choose $y_0 \in Y$ be a point and let $x_0 = \pi(y_0)$. Choose a local flow $\alpha^\Xi : (-\varepsilon,\varepsilon)\times V \to Y$ at y_0, and suppose that the mappings $\alpha_t^\Xi : V \to Y$ are C^r automorphisms of Y. Then for each t there exists a unique C^r mapping $\alpha_t : U \to X$, where $U = \pi(V)$ is an open set, such that

(6) $\pi \circ \alpha_t^{\Xi} = \alpha_t \circ \pi$

on V. Setting $\alpha(t,x) = \alpha_t(x)$ we get a mapping $\alpha : (-\varepsilon,\varepsilon) \times U \to X$. It is easily seen that this mapping is of class C^r. Indeed, there exists a C^r section $\gamma : U \to Y$ such that $\gamma(x_0) = y_0$ (Section 1.1, Theorem 3); using this section we can write $\alpha(t,x) = \alpha_t(x) = \pi \circ \alpha_t^{\Xi} \circ \gamma(x) = \pi \circ \alpha^{\Xi}(t,\gamma(x))$, so α can be expressed as the composite of C^r-mappings. Since α satisfies $\alpha(0,x) = x$, setting

(7) $\xi(x) = T_0 \alpha_x \cdot 1$

we get a C^{r-1} vector field on U.

On the other hand, formula (6) implies $\pi \circ \alpha^{\Xi}(t,y) = \alpha(t,\pi(y))$, that is, $\pi \circ \alpha_y^{\Xi} = \alpha_{\pi(y)}$. Then from (2) $T_t(\pi \circ \alpha_y^{\Xi}) = T_{\alpha_y^{\Xi}(t)} \pi \cdot \Xi(\alpha_y^{\Xi}(t)) = T_t \alpha_{\pi(y)}$ and we have at the point $t = 0$

(8) $T_0 \alpha_{\pi(y)} = T_y \pi \cdot \Xi(y).$

Combining (7) and (8),

(9) $\xi(\pi(y)) = T_y \pi \cdot \Xi(y).$

This proves π-projectability of Ξ on V. π-projectability of Ξ (on Y) now follows form the uniqueness of the π-projection.

2. Suppose that Ξ is π-projectable and denote by ξ its π-projection. Then

(10) $T_y \pi \cdot \Xi(y) = \xi(\pi(y))$

at every point y of the fibred manifold Y. The local flow α^{Ξ} satisfies equation (2) $T_t \alpha_y^{\Xi} = \Xi(\alpha_y^{\Xi}(t))$. Applying the tangent mapping $T\pi$ to both sides we get

(11) $T_t(\pi \circ \alpha_y^{\Xi}) = T_{\alpha_y^{\Xi}(t)} \pi \cdot \Xi(\alpha_y^{\Xi}(t)) = \xi(\pi(\alpha_y^{\Xi}(t))).$

This equality means that the curve $t \to \pi(\alpha_y^{\Xi}(t)) = \alpha_{\pi(y)}^{\xi}(t)$ is an integral curve of the vector field ξ. Thus, denoting by α^{ξ} the local flow of ξ at the point $x_0 = \pi(y_0)$, we have

(12) $\pi(\alpha^{\Xi}(t,y)) = \alpha^{\xi}(t,\pi(y))$

as required.

Let Ξ be a π-projectable C^r vector field on Y, ξ its π-projection. Let α_t^{Ξ} (resp. α_t^{ξ}) be a local 1-parameter group of Ξ (resp. ξ). Since the mappings α_t^{ξ} are C^r diffeomorphisms, for each t the C^r automorphism α_t^{Ξ} can be prolonged to the jet prolongation J^sY of Y, for any s, $0 \le s \le r$. The prolonged mapping is an automorphism of the fibred manifold J^sY over X, defined by

$$(13)\qquad J^s\alpha_t^{\Xi}(J_x^r\gamma) = J^s_{\alpha_t^{\xi}(x)}\alpha_t^{\Xi}\gamma\alpha_{-t}^{\xi},$$

the *s-jet prolongation* of α_t^{Ξ}.

It is easily seen that there exists a unique C^s vector field on J^sY whose integral curves are exactly the curves $t \to J^s\alpha_t^{\Xi}(J_x^r\gamma)$. This vector field is defined by

$$(14)\qquad J^s\Xi(J_x^r\gamma) = \left(\frac{d}{dt}J^s\alpha_t^{\Xi}(J_x^r\gamma)\right)_0,$$

and is called the *r-jet prolongation* of the vector field Ξ. It follows from the definition that the vector field $J^s\Xi$ is π^s-projectable (resp. $\pi^{s,k}$-projectable for every k, $0 \le k \le s$) and its π^k-projection (resp. $\pi^{s,k}$-projection) is ξ (resp. $J^k\Xi$).

The following lemma explains the local structure of the jet prolongations of projectable vector fields (Krupka [13]); its proof is based on the chain rule.

Lemma 10 *Let Ξ be a π-projectable vector field on Y, expressed in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, by*

$$(15)\qquad \Xi = \xi^i\frac{\partial}{\partial x^i} + \Xi^\sigma\frac{\partial}{\partial y^\sigma}.$$

Then $J^s\Xi$ is expressed in the associated chart (V^s,ψ^s) by

$$(16)\qquad J^s\Xi = \xi^i\frac{\partial}{\partial x^i} + \Xi^\sigma\frac{\partial}{\partial y^\sigma} + \sum_{k=1}^{s}\sum_{j_1\le j_2\le\ldots\le j_k}\Xi^\sigma_{j_1j_2\ldots j_k}\frac{\partial}{\partial y^\sigma_{j_1j_2\ldots j_k}},$$

where the components $\Xi^\sigma_{j_1j_2\ldots j_k}$ are determined by the recurrence formula

$$(17)\qquad \Xi^\sigma_{j_1j_2\ldots j_k} = d_{j_k}\Xi^\sigma_{j_1j_2\ldots j_{k-1}} - y^\sigma_{j_1j_2\ldots j_{k-1}i}\frac{\partial\xi^i}{\partial x^{j_k}}.$$

Proof For all sufficiently small t we can express the local 1-parameter group of Ξ in one chart only. Equations of the C^r automorphism α_t^{Ξ} are expressed as

$$(18)\qquad x^i\circ\alpha_t^{\Xi}(y) = x^i\alpha_t^{\xi}(x),\quad y^\sigma\circ\alpha_t^{\Xi}(y) = y^\sigma\alpha_t^{\Xi}(y).$$

From these equations we obtain the components of the vector field Ξ in the form

(19) $$\xi^i(y) = \left(\frac{dx^i\alpha_t^\xi(x)}{dt}\right)_0, \quad \Xi^\sigma(y) = \left(\frac{dy^\sigma\alpha_t^\Xi(y)}{dt}\right)_0.$$

To determine the components of $J^s\Xi$ we use Lemma 9. The 1-parameter group of $J^s\Xi$ has the equations

(20) $$\begin{aligned} & x^i \circ J^r\alpha_t^\Xi(y) = x^i\alpha_t^\xi(x), \\ & y^\sigma \circ J^r\alpha_t^\Xi(y) = y^\sigma\alpha_t^\Xi(y), \\ & y^\sigma_{j_1 j_2 \dots j_k} \circ J^r\alpha_t^\Xi(J^r_x\gamma) \\ & \quad = D_{j_1}D_{j_2}\dots D_{j_k}(y^\sigma\alpha_t^\Xi\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha_{-t}^\xi\varphi^{-1})(\varphi(\alpha_t^\xi(x))), \quad 1 \le k \le s, \end{aligned}$$

so by (19) it is sufficient to determine $\Xi^\sigma_{j_1 j_2 \cdots j_k}$. By definition,

(21) $$\Xi^\sigma_{j_1 j_2 \cdots j_k}(J^r_x\gamma) = \left(\frac{d}{dt}(y^\sigma_{j_1 j_2 \cdots j_k} \circ J^r\alpha_t^\Xi)(J^r_x\gamma)\right)_0.$$

But

(22) $$\begin{aligned} & y^\sigma_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha_t^\Xi(J^r_x\gamma) \\ & \quad = D_{j_1}D_{j_2}\dots D_{j_{k-1}}(y^\sigma\alpha_t^\Xi\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha_{-t}^\xi\varphi^{-1})(\varphi(\alpha_t^\xi(x))) \\ & \quad = y^\sigma_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha_t^\Xi \circ J^r\gamma \circ \alpha_{-t}^\xi\varphi^{-1}(\varphi(\alpha_t^\xi(x))), \end{aligned}$$

thus,

(23) $$\begin{aligned} & y^\sigma_{j_1 j_2 \cdots j_k} \circ J^r\alpha_t^\Xi(J^r_x\gamma) \\ & \quad = D_{j_k}(y^\sigma_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha_t^\Xi \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\alpha_{-t}^\xi\varphi^{-1})(\varphi(\alpha_t^\xi(x))) \\ & \quad = D_l(y^\sigma_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha_t^\Xi \circ J^r\gamma \circ \varphi^{-1})(\varphi(x))D_{j_k}(x^l\alpha_{-t}^\xi\varphi^{-1})(\varphi(\alpha_t^\xi(x))). \end{aligned}$$

To obtain $\Xi^\sigma_{j_1 j_2 \cdots j_k}(J^r_x\gamma)$ (21) we differentiate the function

(24) $$(t, \varphi(x)) \to y^\sigma_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha_t^\Xi(J^r_x\gamma) = (y^\sigma_{j_1 j_2 \cdots j_{k-1}} \circ J^r\alpha_t^\Xi \circ J^r\gamma \circ \varphi^{-1})(\varphi(x))$$

with respect to *t* and x^l. Since the partial derivatives commute, we can first differentiate with respect to *t* at $t = 0$. We get the expression $\Xi^\sigma_{j_1 j_2 \cdots j_{k-1}}(J^r_x\gamma)$. Subsequent differentiation yields

(25) $$D_l(\Xi^\sigma_{j_1 j_2 \cdots j_{k-1}} \circ J^r\gamma \circ \varphi^{-1})(\varphi(x)) = d_l\Xi^\sigma_{j_1 j_2 \cdots j_{k-1}}(J^r_x\gamma),$$

where d_l is the formal derivative operator.

We should also differentiate expression $D_{j_k}(x^l\alpha_{-t}^\xi\varphi^{-1})(\varphi(\alpha_t^\xi(x)))$ with respect to *t*. We have the identity $D_l(x^k\alpha_{-t}^\xi\varphi^{-1} \circ \varphi\alpha_t^\xi\varphi^{-1})(\varphi(x)) = \delta^k_l$, that is,

(26) $$D_j(x^k\alpha_{-t}^{\xi}\varphi^{-1})(\varphi\alpha_t^{\xi}(x))D_l(x^j\alpha_t^{\xi}\varphi^{-1})(\varphi(x))=\delta_l^k.$$

From this formula

(27) $$\begin{aligned}&\frac{d}{dt}D_j(x^k\alpha_{-t}^{\xi}\varphi^{-1})(\varphi\alpha_t^{\xi}(x))\cdot D_l(x^j\alpha_t^{\xi}\varphi^{-1})(\varphi(x))\\&+D_j(x^k\alpha_{-t}^{\xi}\varphi^{-1})(\varphi\alpha_t^{\xi}(x))\cdot\frac{d}{dt}D_l(x^j\alpha_t^{\xi}\varphi^{-1})(\varphi(x))\\&=0\end{aligned}$$

thus, at $t=0$,

(28) $$\left(\frac{d}{dt}D_j(x^k\alpha_{-t}^{\xi}\varphi^{-1})(\varphi\alpha_t^{\xi}(x))\right)_0\cdot\delta_l^j+\delta_j^kD_l\xi^j(\varphi(x))=0,$$

hence

(29) $$\left(\frac{d}{dt}D_l(x^k\alpha_{-t}^{\xi}\varphi^{-1})(\varphi\alpha_t^{\xi}(x))\right)_0=-D_l\xi^k(\varphi(x)).$$

Now we can complete the differentiation of formula (23) at $t=0$. We have, using (25) and (29)

(30) $$\begin{aligned}\Xi^{\sigma}_{j_1j_2\dots j_k}(J_x^r\gamma)&=\left(\frac{d}{dt}(y^{\sigma}_{j_1j_2\dots j_k}\circ J^r\alpha_t^{\Xi})(J_x^r\gamma)\right)_0\\&=\left(\frac{d}{dt}D_l(y^{\sigma}_{j_1j_2\dots j_{k-1}}\circ J^r\alpha_t^{\Xi}\circ J^r\gamma\circ\varphi^{-1})(\varphi(x))\right)_0\delta^l_{j_k}\\&\quad+D_l(y^{\sigma}_{j_1j_2\dots j_{k-1}}\circ J^r\gamma\circ\varphi^{-1})(\varphi(x))\left(\frac{d}{dt}D_{j_k}(x^l\alpha_{-t}^{\xi}\varphi^{-1})(\varphi(\alpha_t^{\xi}(x)))\right)_0\\&=d_l\Xi^{\sigma}_{j_1j_2\dots j_{k-1}}(J_x^{r+1}\gamma)\delta^l_{j_k}\\&\quad-D_l(y^{\sigma}_{j_1j_2\dots j_{k-1}}\circ J^r\gamma\circ\varphi^{-1})(\varphi(x))D_{j_k}\xi^l(\varphi(x))\\&=d_{j_k}\Xi^{\sigma}_{j_1j_2\dots j_{k-1}}(J_x^r\gamma)-y^{\sigma}_{j_1j_2\dots j_{k-1}l}(J_x^r\gamma)D_{j_k}\xi^l(\varphi(x)),\end{aligned}$$

which coincides with (17).

Example 8 (2-jet prolongation of a vector field) Let a π-projectable vector field Ξ be expressed by

(31) $$\Xi=\xi^i\frac{\partial}{\partial x^i}+\Xi^{\sigma}\frac{\partial}{\partial y^{\sigma}}.$$

We can calculate the components of the second jet prolongation $J^2\Xi$ from Lemma 10. We get

(32) $$J^2\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \Xi^\sigma_j \frac{\partial}{\partial y^\sigma_j} + \sum_{j\le k} \Xi^\sigma_{jk} \frac{\partial}{\partial y^\sigma_{jk}},$$

we get

(33) $$\begin{aligned} \Xi^\sigma_j &= d_j\Xi^\sigma - y^\sigma_i \frac{\partial \xi^i}{\partial x^j}, \\ \Xi^\sigma_{jk} &= d_j d_k \Xi^\sigma - y^\sigma_{ij}\frac{\partial \xi^i}{\partial x^k} - y^\sigma_{ik}\frac{\partial \xi^i}{\partial x^j} - y^\sigma_i \frac{\partial^2 \xi^i}{\partial x^j \partial x^k}. \end{aligned}$$

In the following lemma we study the *Lie bracket* of r-jet prolongations of projectable vector fields.

Lemma 11 *For any two π-projectable vector fields Ξ and Z, the Lie bracket $[\Xi,\mathrm{Z}]$ is also π-projectable, and*

(34) $$J^r[\Xi,\mathrm{Z}] = [J^r\Xi, J^r\mathrm{Z}].$$

Proof 1. First we prove (34) for $r=1$. Suppose that in a fibred chart

(35) $$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}, \quad \mathrm{Z} = \zeta^k \frac{\partial}{\partial x^k} + \mathrm{Z}^\nu \frac{\partial}{\partial y^\nu}.$$

Then

(36) $$J^1\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \Xi^\sigma_j \frac{\partial}{\partial y^\sigma_j}, \quad J^1\mathrm{Z} = \zeta^i \frac{\partial}{\partial x^i} + \mathrm{Z}^\sigma \frac{\partial}{\partial y^\sigma} + \mathrm{Z}^\sigma_j \frac{\partial}{\partial y^\sigma_j},$$

where

(37) $$\Xi^\sigma_j = d_j\Xi^\sigma - y^\sigma_i \frac{\partial \xi^i}{\partial x^j}, \quad \mathrm{Z}^\sigma_j = d_j\mathrm{Z}^\sigma - y^\sigma_i \frac{\partial \zeta^i}{\partial x^j},$$

and

(38) $$\begin{aligned} [J^1\Xi, J^1\mathrm{Z}] &= \left(\frac{\partial \zeta^i}{\partial x^l}\xi^l - \frac{\partial \xi^i}{\partial x^l}\zeta^l\right)\frac{\partial}{\partial x^i} \\ &+ \left(\frac{\partial \mathrm{Z}^\sigma}{\partial x^l}\xi^l + \frac{\partial \mathrm{Z}^\sigma}{\partial y^\nu}\Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^l}\zeta^l - \frac{\partial \Xi^\sigma}{\partial y^\nu}\mathrm{Z}^\nu\right)\frac{\partial}{\partial y^\sigma} \\ &+ \left(\frac{\partial \mathrm{Z}^\sigma_j}{\partial x^l}\xi^l + \frac{\partial \mathrm{Z}^\sigma_j}{\partial y^\nu}\Xi^\nu + \frac{\partial \mathrm{Z}^\sigma_j}{\partial y^\nu_l}\Xi^\nu_l - \frac{\partial \Xi^\sigma_j}{\partial x^l}\zeta^l - \frac{\partial \Xi^\sigma_j}{\partial y^\nu}\mathrm{Z}^\nu - \frac{\partial \Xi^\sigma_j}{\partial y^\nu_l}\mathrm{Z}^\nu_l\right)\frac{\partial}{\partial y^\sigma_j}. \end{aligned}$$

On the other hand, denoting $\Theta = [\Xi,\mathrm{Z}]$ we have

$$\Theta = \vartheta^{ii} \frac{\partial}{\partial x^i} + \Theta^\sigma \frac{\partial}{\partial y^\sigma}, \tag{39}$$

where

$$\begin{aligned} &\vartheta^i = \frac{\partial \zeta^i}{\partial x^s} \xi^s - \frac{\partial \xi^i}{\partial x^s} \zeta^s, \\ &\Theta^\sigma = \frac{\partial Z^\sigma}{\partial x^s} \xi^s + \frac{\partial Z^\sigma}{\partial y^\nu} \Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^s} \zeta^s - \frac{\partial \Xi^\sigma}{\partial y^\nu} Z^\nu, \end{aligned} \tag{40}$$

and

$$J^1\Theta = \vartheta^i \frac{\partial}{\partial x^i} + \Theta^\sigma \frac{\partial}{\partial y^\sigma} + \Theta^\sigma_j \frac{\partial}{\partial y^\sigma_j}, \tag{41}$$

where

$$\Theta^\sigma_j = d_j \Theta^\sigma - y^\sigma_i \frac{\partial \vartheta^i}{\partial x^j}. \tag{42}$$

Comparing formulas (34) and (42) we see that to prove our assertion for $r = 1$ it is sufficient to show that

$$\begin{aligned} &d_j\left(\frac{\partial Z^\sigma}{\partial x^s} \xi^s + \frac{\partial Z^\sigma}{\partial y^\nu} \Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^s} \zeta^s - \frac{\partial \Xi^\sigma}{\partial y^\nu} Z^\nu \right) \\ &\quad - y^\sigma_i \frac{\partial}{\partial x^j} \left(\frac{\partial \zeta^i}{\partial x^s} \xi^s - \frac{\partial \xi^i}{\partial x^s} \zeta^s \right) \\ &\quad = \frac{\partial Z^\sigma_j}{\partial x^l} \xi^l + \frac{\partial Z^\sigma_j}{\partial y^\nu} \Xi^\nu + \frac{\partial Z^\sigma_j}{\partial y^\nu_l} \Xi^\nu_l - \frac{\partial \Xi^\sigma_j}{\partial x^l} \zeta^l - \frac{\partial \Xi^\sigma_j}{\partial y^\nu} Z^\nu - \frac{\partial \Xi^\sigma_j}{\partial y^\nu_l} Z^\nu_l. \end{aligned} \tag{43}$$

We shall consider the left- and right-hand sides of this formula separately. The left-hand side can be expressed as

$$\begin{aligned} &d_j \frac{\partial Z^\sigma}{\partial x^s} \xi^s + \frac{\partial Z^\sigma}{\partial x^s} \frac{\partial \xi^s}{\partial x^j} + d_j \frac{\partial Z^\sigma}{\partial y^\nu} \Xi^\nu + \frac{\partial Z^\sigma}{\partial y^\nu} d_j \Xi^\nu \\ &\quad - d_j \frac{\partial \Xi^\sigma}{\partial x^s} \zeta^s - \frac{\partial \Xi^\sigma}{\partial x^s} \frac{\partial \zeta^s}{\partial x^j} - d_j \frac{\partial \Xi^\sigma}{\partial y^\nu} Z^\nu - \frac{\partial \Xi^\sigma}{\partial y^\nu} d_j Z^\nu \\ &\quad - y^\sigma_i \left(\frac{\partial^2 \zeta^i}{\partial x^j \partial x^s} \xi^s + \frac{\partial \zeta^i}{\partial x^s} \frac{\partial \xi^s}{\partial x^j} - \frac{\partial^2 \xi^i}{\partial x^j \partial x^s} \zeta^s - \frac{\partial \xi^i}{\partial x^s} \frac{\partial \zeta^s}{\partial x^j} \right). \end{aligned} \tag{44}$$

The right-hand side of (43) is

$$
\begin{aligned}
&\left(d_j \frac{\partial Z^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \zeta^i}{\partial x^l \partial x^j}\right)\xi^l + d_j \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu + \frac{\partial}{\partial y_l^\nu}\left(d_j Z^\sigma - y_i^\sigma \frac{\partial \zeta^i}{\partial x^j}\right)\Xi_l^\nu \\
&-\left(d_j \frac{\partial \Xi^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \xi^i}{\partial x^l \partial x^j}\right)\zeta^l - d_j \frac{\partial \Xi^\sigma}{\partial y^\nu} Z^\nu - \frac{\partial}{\partial y_l^\nu}\left(d_j \Xi^\sigma - y_i^\sigma \frac{\partial \xi^i}{\partial x^j}\right) Z_l^\nu \\
&= \left(d_j \frac{\partial Z^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \zeta^i}{\partial x^l \partial x^j}\right)\xi^l + d_j \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu + \left(d_j \Xi^\nu - y_i^\nu \frac{\partial \xi^i}{\partial x^j}\right)\frac{\partial Z^\sigma}{\partial y^\nu} \\
(45)\qquad &-\left(d_l \Xi^\sigma - y_i^\sigma \frac{\partial \xi^i}{\partial x^l}\right)\frac{\partial \zeta^l}{\partial x^j} \\
&-\left(d_j \frac{\partial \Xi^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \xi^i}{\partial x^l \partial x^j}\right)\zeta^l - d_j \frac{\partial \Xi^\sigma}{\partial y^\nu} Z^\nu - \left(d_j Z^\nu - y_i^\nu \frac{\partial \zeta^i}{\partial x^j}\right)\frac{\partial \Xi^\sigma}{\partial y^\nu} \\
&+\left(d_l Z^\sigma - y_i^\sigma \frac{\partial \zeta^i}{\partial x^l}\right)\frac{\partial \xi^l}{\partial x^j}.
\end{aligned}
$$

In this formula

$$
\begin{aligned}
(46)\qquad & d_l Z^\sigma \frac{\partial \xi^l}{\partial x^j} - y_i^\nu \frac{\partial \xi^i}{\partial x^j}\frac{\partial Z^\sigma}{\partial y^\nu} = \frac{\partial Z^\sigma}{\partial x^l}\frac{\partial \xi^l}{\partial x^j} + \frac{\partial Z^\sigma}{\partial y^\nu} y_l^\nu \frac{\partial \xi^l}{\partial x^j} - y_i^\nu \frac{\partial \xi^i}{\partial x^j}\frac{\partial Z^\sigma}{\partial y^\nu} \\
&= \frac{\partial Z^\sigma}{\partial x^l}\frac{\partial \xi^l}{\partial x^j},
\end{aligned}
$$

and

$$
\begin{aligned}
(47)\qquad & -d_l \Xi^\sigma \frac{\partial \zeta^l}{\partial x^j} + y_i^\nu \frac{\partial \zeta^i}{\partial x^j}\frac{\partial \Xi^\sigma}{\partial y^\nu} = -\frac{\partial \Xi^\sigma}{\partial x^l}\frac{\partial \zeta^l}{\partial x^j} - \frac{\partial \Xi^\sigma}{\partial y^\nu} y_l^\nu \frac{\partial \zeta^l}{\partial x^j} + y_i^\nu \frac{\partial \zeta^i}{\partial x^j}\frac{\partial \Xi^\sigma}{\partial y^\nu} \\
&= -\frac{\partial \Xi^\sigma}{\partial x^l}\frac{\partial \zeta^l}{\partial x^j},
\end{aligned}
$$

thus,

$$
\begin{aligned}
&\frac{\partial Z_j^\sigma}{\partial x^l}\xi^l + \frac{\partial Z_j^\sigma}{\partial y^\nu}\Xi^\nu + \frac{\partial Z_j^\sigma}{\partial y_l^\nu}\Xi_l^\nu - \frac{\partial \Xi_j^\sigma}{\partial x^l}\zeta^l - \frac{\partial \Xi_j^\sigma}{\partial y^\nu} Z^\nu - \frac{\partial \Xi_j^\sigma}{\partial y_l^\nu} Z_l^\nu \\
&= \left(d_j \frac{\partial Z^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \zeta^i}{\partial x^l \partial x^j}\right)\xi^l + d_j \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu + d_j \Xi^\nu \frac{\partial Z^\sigma}{\partial y^\nu} + y_i^\sigma \frac{\partial \xi^i}{\partial x^l}\frac{\partial \zeta^l}{\partial x^j} \\
(48)\qquad &-\left(d_j \frac{\partial \Xi^\sigma}{\partial x^l} - y_i^\sigma \frac{\partial^2 \xi^i}{\partial x^l \partial x^j}\right)\zeta^l - d_j \frac{\partial \Xi^\sigma}{\partial y^\nu} Z^\nu - d_j Z^\nu \frac{\partial \Xi^\sigma}{\partial y^\nu} - y_i^\sigma \frac{\partial \zeta^i}{\partial x^l}\frac{\partial \xi^l}{\partial x^j}. \\
&+\frac{\partial Z^\sigma}{\partial x^l}\frac{\partial \xi^l}{\partial x^j} - \frac{\partial \Xi^\sigma}{\partial x^l}\frac{\partial \zeta^l}{\partial x^j}.
\end{aligned}
$$

This is, however, exactly expression (43), proving (34) for $r=1$.

2. In this part of the proof we consider the *r*-jet prolongation $J^{r-1}Y$ as a fibred manifold with base *X* and projection $\pi^{r-1}:J^{r-1}Y\to X$, and the 1-jet prolongation of this fibred manifold, $J^1J^{r-1}Y$. J^rY can be embedded in $J^1J^{r-1}Y$ by the *canonical injection*

(49) $$J^rY\ni J^r_x\gamma\to\iota(J^r_x\gamma)=J^1_xJ^{r-1}\gamma\in J^1J^{r-1}Y.$$

Obviously, ι is compatible with jet prolongations of automorphisms α of *Y* in the sense that

(50) $$\iota\circ J^r\alpha=(J^1J^{r-1}\alpha)\circ\iota.$$

Indeed, we have for any point $J^r_x\gamma$ from the domain of $J^r\alpha$

(51) $$\iota(J^r\alpha(J^r_x\gamma))=\iota(J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1})=J^1_{\alpha_0(x)}(J^{r-1}\alpha\gamma\alpha_0^{-1}),$$

and also

(52) $$J^1J^{r-1}\alpha(\iota(J^r_x\gamma))=J^1J^{r-1}\alpha(J^1_xJ^{r-1}\gamma)=J^1_{\alpha_0(x)}(J^{r-1}\alpha\circ J^{r-1}\gamma\circ\alpha_0^{-1}).$$

Thus (51) follows from the definition of the 1–jet prolongation of a fibred automorphism (Section 1.4, (14)).

Then, however, applying (52) to local 1-parameter groups of a π-projectable vector field Ξ, we get ι-compatibility of $J^1J^{r-1}\Xi$ and $J^r\Xi$,

(53) $$J^1J^{r-1}\Xi\circ\iota=T\iota\cdot J^r\Xi.$$

Since for any two π-projectable vector fields Ξ and Z the vector fields $J^1J^{r-1}\Xi$ $J^r\Xi$ and $J^1J^{r-1}\mathrm{Z}$ and $J^r\mathrm{Z}$ are ι-compatible, the corresponding Lie brackets are also ι-compatible and we have

(54) $$[J^1J^{r-1}\Xi,J^1J^{r-1}\mathrm{Z}]\circ\iota=T\iota\cdot[J^r\Xi,J^r\mathrm{Z}].$$

3. Using Part 1 of this proof, we now express the vector field on the left-hand side of (54) in a different way. First note that

(55) $$[J^1J^{r-1}\Xi,J^1J^{r-1}\mathrm{Z}]=J^1[J^{r-1}\Xi,J^{r-1}\mathrm{Z}].$$

But we may suppose that $[J^{r-1}\Xi,J^{r-1}\mathrm{Z}]=J^{r-1}[\Xi,\mathrm{Z}]$ (induction hypothesis), thus $[J^1J^{r-1}\Xi,J^1J^{r-1}\mathrm{Z}]=J^1J^{r-1}[\Xi,\mathrm{Z}]$. Restricting both sides by ι and applying (50),

(56) $$[J^1J^{r-1}\Xi,J^1J^{r-1}\mathrm{Z}]\circ\iota=J^1J^{r-1}[\Xi,\mathrm{Z}]\circ\iota=T\iota\cdot J^r[\Xi,\mathrm{Z}].$$

Now from (55) and (57) we conclude that $T\iota\cdot([J^r\Xi,J^r\mathrm{Z}]-J^r[\Xi,\mathrm{Z}])=0$. This implies, however, $[J^r\Xi,J^r\mathrm{Z}]-J^r[\Xi,\mathrm{Z}]=0$ because $T\iota$ is at every point injective.

This completes the proof of formula (34).

Remark 4 (Equations of the canonical injection) We find the chart expression of the canonical injection $\iota : J^rY \to J^1J^{r-1}Y$ (49) in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y and the induced fibred chart on J^rY. We also have a fibred chart on $J^1J^{r-1}Y$, induced by the fibred chart (V^{r-1},ψ^{r-1}), $\psi = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1j_2\ldots j_{r-1}})$, on $J^{r-1}Y$. We denote the fibred chart on $J^1J^{r-1}Y$ by (W,Ψ), where the coordinate functions are denoted as

(57) $$\Psi = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1j_2\ldots j_{r-1}}, y^\sigma_{\ ,k}, y^\sigma_{j_1,k}, y^\sigma_{j_1j_2,k}, \ldots, y^\sigma_{j_1j_2\ldots j_{r-1},k}).$$

Then by definition

(58) $$\begin{aligned} y^\sigma_{j_1j_2\ldots j_s,k} \circ \iota(J^r_x\gamma) &= D_k(y^\sigma_{j_1j_2\ldots j_s} \circ J^{r-1}\gamma \circ \varphi^{-1})(\varphi(x)) \\ &= D_kD_{j_1}D_{j_2}\ldots D_{j_s}(y^\sigma\gamma\varphi^{-1})(\varphi(x)) = y^\sigma_{j_1j_2\ldots j_sk}(J^r_x\gamma) \end{aligned}$$

for all $s = 1,2,\ldots,r-1$, so the canonical injection ι is expressed by the equations

(59) $$\begin{aligned} &x^i \circ \iota = x^i, \quad y^\sigma \circ \iota = y^\sigma, \quad y^\sigma_{j_1j_2\ldots j_s} \circ \iota = y^\sigma_{j_1j_2\ldots j_s}, \quad 1 \le s \le r-1, \\ &y^\sigma_{j_1j_2\ldots j_s,k} \circ \iota = y^\sigma_{j_1j_2\ldots j_sk}, \quad 1 \le s \le r-1. \end{aligned}$$

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