# 3 Formal divergence equations

In this chapter we introduce formal divergence equations on Euclidean spaces and study their basic properties. These partial differential equations naturally appear in the variational geometry on fibred manifolds, but also have a broader meaning related to differential equations, conservation laws, and integration of forms on manifolds with boundary. A formal divergence equation is not always integrable; we show that the obstructions are connected with the *Euler-Lagrange expressions* known from the higher-order variational theory of multiple integrals. If a solution exists, then it defines a solution of the associated "ordinary" divergence equation along any section of the underlying fibred manifold. The notable fact is that the solutions of formal divergence equations of order *r* are in one-one correspondence with a class of differential forms on the $(r-1)$-st jet prolongation of the underlying fibred manifold, defined by the exterior derivative operator.

The chapter extends the theory introduced in Krupka [K14].

## 3.1 Formal divergence equations

Let $U\subset\mathbf{R}^n$ be an open set, let $V\subset\mathbf{R}^m$ be an open ball with centre $0\in\mathbf{R}^m$, and denote $W=U\times V$. We consider *W* as a fibred manifold over *U* with the first Cartesian projection $\pi:W\to U$. As before, we denote by $W^r$ the *r*-jet prolongation of *W*. The set $W^r$ can explicitly be expressed as the Cartesian product

(1) $$W^r=U\times V\times L(\mathbf{R}^n,\mathbf{R}^m)\times L^2_{\mathrm{sym}}(\mathbf{R}^n,\mathbf{R}^m)\times\ldots\times L^r_{\mathrm{sym}}(\mathbf{R}^n,\mathbf{R}^m),$$

where $L^k_{\mathrm{sym}}(\mathbf{R}^n,\mathbf{R}^m)$ is the vector space of *k*-linear, symmetric mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$. The Cartesian coordinates on *W*, and the associated jet coordinates on $W^r$, are denoted by $(x^i,y^\sigma)$, and $(x^i,y^\sigma,y^\sigma_{j_1},y^\sigma_{j_1j_2},\ldots,y^\sigma_{j_1j_2\ldots j_r})$, respectively.

Let $s\geq 1$ and let $f:W^s\to\mathbf{R}$ be a function. In this section we study the differential equation

(2) $$d_ig^i=f$$

for a collection $g=g^i$ of differentiable functions $g^i:W^r\to\mathbf{R}$, where $r\geq s$, and

(3) $$d_ig^i=\frac{\partial g^i}{\partial x^i}+\sum_{0\leq k\leq s}\sum_{1\leq i\leq n}\sum_{j_1\leq j_2\leq\ldots\leq j_k}\frac{\partial g^i}{\partial y^\sigma_{j_1j_2\ldots j_k}}y^\sigma_{j_1j_2\ldots j_ki}$$

is the *formal divergence* of the collection $g^i$. Equation (2) is the *formal divergence equation*, and $g^i$ is its *solution of order* $r$. Clearly a solution of order $r$ is also a solution of order $r+1$. Our aim will be to find all solutions of order $s$, defined on the same domain as the function $f$.

In expression (3) we differentiate with respect to independent variables $y^\sigma_{j_1 j_2 \ldots j_k}$, where $j_1 \le j_2 \le \ldots \le j_k$. However, it will be convenient to find another expression for the formal divergence with no restriction to the summation indices. According to Appendix 2,

$$\sum_i \sum_{j_1 \le j_2 \le \ldots \le j_k} \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i} = \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_k}} y^\sigma_{i_1 i_2 \ldots i_k i}, \tag{4}$$

where $y^\sigma_{i_1 i_2 \ldots i_k}$ on the right side stands for the canonical extension of the variables $y^\sigma_{i_1 i_2 \ldots i_k}$, $j_1 \le j_2 \le \ldots \le j_k$ to all values of the subscripts. With this convention, the formal derivative (3) can be expressed as

$$d_i g^i = \frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \ldots + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} y^\sigma_{j_1 j_2 \ldots j_s i}. \tag{5}$$

From expression (5) we immediately see that every solution $g^i$, defined on the set $W^r$ such that $r \ge s$, satisfies the system of partial differential equations

$$\frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_r}} + \frac{\partial g^{j_1}}{\partial y^\sigma_{i j_2 j_3 \ldots j_r}} + \frac{\partial g^{j_2}}{\partial y^\sigma_{j_1 i j_3 j_4 \ldots j_r}} + \ldots + \frac{\partial g^{j_{r-1}}}{\partial y^\sigma_{j_1 j_2 \ldots j_{r-2} i j_r}} + \frac{\partial g^{j_r}}{\partial y^\sigma_{j_1 j_2 \ldots j_{r-1} i}} = 0. \tag{6}$$

Our first aim will be to find solutions of this system.

The proof of the following lemma is based on the Young decomposition theory of tensor spaces.

**Lemma 1** (a) *Every solution* $g = g^i$ *of the system* (6) *is a polynomial function of the variables* $y^\sigma_{j_1 j_2 \ldots j_s}$.

(b) *If the system* (6) *has a solution* $g = g^i$ *of order* $r \ge s$, *then it also has a solution of order* $s$.

**Proof** (a) To prove Lemma 1 it is convenient to use multi-indices of the form $J = (j_1 j_2 \ldots j_r)$. First we show that condition (6) implies that the expression

$$\frac{\partial^n g^i}{\partial y^{\sigma_1}_{J_1} \partial y^{\sigma_2}_{J_2} \ldots \partial y^{\sigma_n}_{J_n}} \tag{7}$$

vanishes for all $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $J_1, J_2, \ldots, J_n$. This expression is indexed with $nr+1$ indices $q_l$, where $l = 1, 2, \ldots, n, n+1, n+2, \ldots, nr, nr+1$ and $1 \le q_l \le n$ (entries of the multi-indices and the index $i$). The (unique) cycle decomposition of the number $nr+1$ includes exactly one scheme, namely the scheme $(r+1, r, \dots, r)$ (one row with $r+1$ boxes, $n-1$ rows with $r$ boxes). The corresponding Young diagrams as well as (nontrivial) Young projectors are then necessarily of the form

(8)

| | |
|---|---|
| $J_1$ | $i$ |
| $J_2$ | |
| $J_3$ | |
| $\dots$ | |
| $J_n$ | |

The first row represents symmetrization in the entries of the multi-index $J_1$ and the index $i$. But according to (6), these Young symmetrizers annihilate (7), so the Young decomposition yields

$$\frac{\partial^n g^i}{\partial y^{\sigma_1}_{J_1} \partial y^{\sigma_2}_{J_2} \dots \partial y^{\sigma_n}_{J_n}} = 0. \tag{9}$$

Consequently, $g^i$ is polynomial in the variables $y^\sigma_J$.

(b) Consider the formal divergence equation (2) with the right-hand side $f = f(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_s})$, and its solution $g = g^i$ of order $r \geq s+1$. Then

$$\frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \dots + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \dots j_r}} y^\sigma_{j_1 j_2 \dots j_r i} = f, \tag{10}$$

and condition (6) is satisfied. Then by the first part of this proof

$$g^i = g^i_0 + g^i_1 + g^i_2 + \dots + g^i_{n-1}, \tag{11}$$

where $g^i_p$ is a homogeneous polynomial of degree $p$ in the variables $y^\sigma_{j_1 j_2 \dots j_r}$. Substituting from (11) into (10) we get, because $f$ does not depend on $y^\sigma_{j_1 j_2 \dots j_r}$,

$$\frac{\partial g^i_0}{\partial x^i} + \frac{\partial g^i_0}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i_0}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i_0}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \dots + \frac{\partial g^i_0}{\partial y^\sigma_{j_1 j_2 \dots j_{r-1}}} y^\sigma_{j_1 j_2 \dots j_{r-1} i} = f. \tag{12}$$

Repeating this procedure, we get some functions $h = h^i$, defined on $V^s$, satisfying

$$\frac{\partial h^i}{\partial x^i} + \frac{\partial h^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial h^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial h^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \dots + \frac{\partial h^i}{\partial y^\sigma_{j_1 j_2 \dots j_s}} y^\sigma_{j_1 j_2 \dots j_s i} = f. \tag{13}$$

$h^i$ is a solution of order $s$.

**Remark 1** If $g = g^i$ is a solution of order $r$ of the formal divergence equation (2), then equations (6) represent restrictions to the *coefficients* of the polynomials $g^i$.

**Remark 2** Every solution of the homogeneous formal divergence equation

(14) $$d_i g^i = 0$$

is defined on $U$. Indeed, according to Lemma 1, if (14) has a solution, then this solution is defined on $V$; thus

(15) $$\frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y_i^\sigma = 0,$$

hence $(\partial g^i / \partial y^\sigma) = 0$ and $g^i$ depends on $x^i$ only.

Let $s \geq 1$ and let $f : W^s \to \mathbf{R}$ be a differentiable function. Sometimes it is useful to divide the formal derivative $d_i f$ of the function *f* in two terms; by the *i-th cut formal derivative* of *f* we mean the function $d'_i f : W^s \to \mathbf{R}$ defined by

(1) $$d'_i f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma} y_i^\sigma + \frac{\partial f}{\partial y_{j_1}^\sigma} y_{j_1 i}^\sigma + \frac{\partial f}{\partial y_{j_1 j_2}^\sigma} y_{j_1 j_2 i}^\sigma + \ldots + \frac{\partial f}{\partial y_{j_1 j_2 \ldots j_{s-1}}^\sigma} y_{j_1 j_2 \ldots j_{s-1} i}^\sigma .$$

The *i*-th formal derivative, which is defined on $W^{s+1}$, is then expressed as

(2) $$d_i f = d'_i f + \frac{\partial f}{\partial y_{j_1 j_2 \ldots j_s}^\sigma} y_{j_1 j_2 \ldots j_s i}^\sigma .$$

The following assertion is a restatement of the definition of a solution of the formal divergence equation (2), Section 3.1.

**Lemma 2** *Let* $f : W^s \to \mathbf{R}$ *and* $g^i : W^s \to \mathbf{R}$ *be differentiable functions. The following conditions are equivalent:*

(a) *The functions* $g^i$ *satisfy the formal divergence equation.*

(b) *The functions* $g^i$ *satisfy the system*

(3) $$d'_i g^i = f$$

*and*

(4) $$\frac{\partial g^i}{\partial y_{j_1 j_2 \ldots j_s}^\sigma} + \frac{\partial g^{j_1}}{\partial y_{i j_2 j_3 \ldots j_s}^\sigma} + \frac{\partial g^{j_2}}{\partial y_{j_1 i j_3 j_4 \ldots j_s}^\sigma} + \ldots + \frac{\partial g^{j_s}}{\partial y_{j_1 j_2 \ldots j_{s-1} i}^\sigma} = 0.$$

**Proof** Immediate.

## 3.2 Integrability of formal divergence equations

We introduce the concepts, responsible for integrability of the formal divergence equation, and prove the integrability theorem.

To any function $f : W^s \to \mathbf{R}$ we assign an *n*-form $\lambda_f$ and an $(n+1)$-form $E_f$ on $W^s$, by

(1) $$\lambda_f = f\omega_0,$$

and

(2) $$E_f = E_\sigma(f)\omega^\sigma \wedge \omega_0,$$

where the components $E_\sigma(f)$ are defined by

(3) $$E_\sigma(f) = \frac{\partial f}{\partial y^\sigma} + \sum_{k=1}^{s} (-1)^k d_{p_1} d_{p_2} \ldots d_{p_k} \frac{\partial f}{\partial y^\sigma_{p_1 p_2 \cdots p_k}}.$$

We call $\lambda_f$ the *Lagrange form*, or the *Lagrangian*, and $E_f$ the *Euler-Lagrange form*, associated with *f*. The components $E_\sigma(f)$ are called the *Euler-Lagrange expressions*.

In the following lemma we use the *horizontalization homomorphism h* and the 1-contact homomorphism $p_1$, acting on modules of differential forms on the *r*-jet prolongation $W^r = J^r W$ of the fibred manifold *W* (see Chapter 2).

**Lemma 3** *For any function* $f : W^s \to \mathbf{R}$, *there exists an n-form* $\Theta_f$, *defined on* $W^{2s-1}$, *such that*

(a) $h\Theta_f = \lambda_f$.

(b) The form $p_1 d\Theta_f$ is $\omega^\sigma$-generated.

**Proof** We search for $\Theta_f$ of the form

(4) $$\begin{aligned}\Theta_f &= f\omega_0 \\ &\quad + (f^i{}_\sigma \omega^\sigma + f^{i j_1}_\sigma \omega^\sigma_{j_1} + f^{i j_1 j_2}_\sigma \omega^\sigma_{j_1 j_2} + \ldots + f^{i j_1 j_2 \cdots j_{s-1}}_\sigma \omega^\sigma_{j_1 j_2 \cdots j_{s-1}}) \wedge \omega_i,\end{aligned}$$

where the coefficients $f^{i\, j_1 j_2 \cdots j_k}_\sigma$ are supposed to be *symmetric* in the superscripts $i, j_1, j_2, \ldots, j_k$. Then condition (a) is obviously satisfied. Computing $p_1 d\Theta$ we have

(5) $$\begin{aligned}p_1 d\Theta_f &= df \wedge \omega_0 + (hdf^i{}_\sigma \wedge \omega^\sigma + f^i{}_\sigma d\omega^\sigma + hdf^{i\, j_1}_\sigma \wedge \omega^\sigma_{j_1} \\ &\quad + f^{i\, j_1}_\sigma d\omega^\sigma_{j_1} + hdf^{i\, j_1 j_2}_\sigma \wedge \omega^\sigma_{j_1 j_2} + f^{i\, j_1 j_2}_\sigma d\omega^\sigma_{j_1 j_2} \\ &\quad + \ldots + hdf^{i\, j_1 j_2 \cdots j_{s-1}}_\sigma \wedge \omega^\sigma_{j_1 j_2 \cdots j_{s-1}} + f^{i\, j_1 j_2 \cdots j_{s-1}}_\sigma d\omega^\sigma_{j_1 j_2 \cdots j_{s-1}}) \wedge \omega_i\end{aligned}$$

$$
\begin{aligned}
&= \left( \frac{\partial f}{\partial y^{\sigma}} \omega^{\sigma} + \frac{\partial f}{\partial y^{\sigma}_{j_1}} \omega^{\sigma}_{j_1} + \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2}} \omega^{\sigma}_{j_1 j_2} + \ldots + \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 \ldots j_s}} \omega^{\sigma}_{j_1 j_2 \ldots j_s} \right) \wedge \omega_0 \\
&+ (d_k f^{i}{}_{\sigma} dx^k \wedge \omega^{\sigma} + d_k f^{i\,j_1}_{\sigma} dx^k \wedge \omega^{\sigma}_{j_1} + d_k f^{i\,j_1 j_2}_{\sigma} dx^k \wedge \omega^{\sigma}_{j_1 j_2} \\
&+ \ldots + d_k f^{i\,j_1 j_2 \ldots j_{s-1}}_{\sigma} dx^k \wedge \omega^{\sigma}_{j_1 j_2 \ldots j_{s-1}}) \wedge \omega_i \\
&- (f^{i}{}_{\sigma} \omega^{\sigma}_{k} \wedge dx^k + f^{i\,j_1}_{\sigma} \omega^{\sigma}_{j_1 k} \wedge dx^k + f^{i\,j_1 j_2}_{\sigma} \omega^{\sigma}_{j_1 j_2 k} \wedge dx^k \\
&+ \ldots + f^{i\,j_1 j_2 \ldots j_{s-1}}_{\sigma} \omega^{\sigma}_{j_1 j_2 \ldots j_{s-1} k} \wedge dx^k) \wedge \omega_i .
\end{aligned}
$$

This expression can also be written as

(6)
$$
\begin{aligned}
p_1 d\Theta_f &= \left( \frac{\partial f}{\partial y^{\sigma}} - d_i f^{i}{}_{\sigma} \right) \omega^{\sigma} \wedge \omega_0 + \left( \frac{\partial f}{\partial y^{\sigma}_{j_1}} - d_i f^{i\,j_1}_{\sigma} - f^{j_1}{}_{\sigma} \right) \omega^{\sigma}_{j_1} \wedge \omega_0 \\
&+ \left( \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2}} - d_i f^{i\,j_1 j_2}_{\sigma} - f^{j_2\,j_1}_{\sigma} \right) \omega^{\sigma}_{j_1 j_2} \wedge \omega_0 \\
&+ \ldots + \left( \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 \ldots j_{s-1}}} - d_i f^{i\,j_1 j_2 \ldots j_{s-1}}_{\sigma} - f^{j_{r-1}\,j_1 j_2 \ldots j_{s-2}}_{\sigma} \right) \omega^{\sigma}_{j_1 j_2 \ldots j_{s-1}} \wedge \omega_0 \\
&+ \left( \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 \ldots j_s}} - f^{j_r\,j_1 j_2 \ldots j_{s-1}}_{\sigma} \right) \omega^{\sigma}_{j_1 j_2 \ldots j_{s-1} j_s} \wedge \omega_0 .
\end{aligned}
$$

But we can choose $f^{i}{}_{\sigma}$, $f^{i\,j_1}_{\sigma}$, $f^{i\,j_1 j_2}_{\sigma}$, …, $f^{i\,j_1 j_2 \ldots j_{s-1}}_{\sigma}$ from the conditions

(7)
$$
\begin{aligned}
&f^{j_s\,j_1 j_2 \ldots j_{s-1}}_{\sigma} = \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 \ldots j_s}}, \\
&f^{j_{r-1}\,j_1 j_2 \ldots j_{s-2}}_{\sigma} = \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 \ldots j_{s-1}}} - d_i f^{i\,j_1 j_2 \ldots j_{s-1}}_{\sigma} = \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 \ldots j_{s-1}}} - d_{i_1} \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 \ldots j_{s-1} i_1}}, \\
&\ldots \\
&f^{j_2\,j_1}_{\sigma} = \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2}} - d_i f^{i\,j_1 j_2}_{\sigma} = \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2}} - d_{i_1} \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 i_1}} + d_{i_1} d_{i_2} \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 i_1 i_2}} \\
&\qquad - \ldots + (-1)^{r-2} d_{i_1} d_{i_2} \ldots d_{i_{s-2}} \frac{\partial f}{\partial y^{\sigma}_{j_1 j_2 i_1 i_2 \ldots i_{s-2}}}, \\
&f^{j_1}{}_{\sigma} = \frac{\partial f}{\partial y^{\sigma}_{j_1}} - d_i f^{i\,j_1}_{\sigma} = \frac{\partial f}{\partial y^{\sigma}_{j_1}} - d_{i_1} \frac{\partial f}{\partial y^{\sigma}_{j_1 i_1}} + d_{i_1} d_{i_2} \frac{\partial f}{\partial y^{\sigma}_{j_1 i_1 i_2}} \\
&\qquad - \ldots + (-1)^{s-1} d_{i_1} d_{i_2} \ldots d_{i_{s-1}} \frac{\partial f}{\partial y^{\sigma}_{j_1 i_1 i_2 \ldots i_{s-1}}},
\end{aligned}
$$

and for this choice the form $p_1 d\Theta$ is $\omega^{\sigma}$-generated, proving (b).

Using formulas (4) and (7), we see that the form $\Theta = \Theta_f$, constructed in the proof, has the expression

$$\Theta_f = f\omega_0 + \sum_{k=0}^{s} \left( \sum_{l=0}^{s-k-1} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k p_1 p_2 \ldots p_l i}} \right) \omega^\sigma_{j_1 j_2 \ldots j_k} \wedge \omega_i . \tag{8}$$

This form obeys properties (a) and (b) of Lemma 3. We call $\Theta_f$ the *principal Lepage equivalent* of the function *f*, or of the Lagrange form $\lambda_f$. Computing $p_1 d\Theta_f$, we get the *Euler-Lagrange form*, associated with *f*,

$$p_1 d\Theta_f = E_f . \tag{9}$$

Now we are in a position to study integrability of the formal divergence equation; the proof includes the construction of the solutions.

**Theorem 1** *Let* $f : W^s \to \mathbf{R}$ *be a function. The following two conditions are equivalent*:

(a) *The formal divergence equation* $d_i g^i = f$ *has a solution defined on the set* $W^s$.

(b) *The Euler-Lagrange form, associated with f, vanishes,*

$$E_f = 0. \tag{10}$$

**Proof** 1. Suppose that condition (a) is satisfied and the formal divergence equation has a solution $g = g^i$, defined on $W^s$. Differentiating the function $d_i g^i$, we get the formulas

$$\frac{\partial d_i g^i}{\partial y^\sigma} = d_i \frac{\partial g^i}{\partial y^\sigma}, \tag{11}$$

and for every $k = 1, 2, \ldots, s$,

$$\begin{aligned} \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_k}} &= d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_k}} \\ &+ \frac{1}{k} \left( \frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_k}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 \ldots i_k}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 \ldots i_k}} + \ldots + \frac{\partial g^{i_k}}{\partial y^\sigma_{i_2 i_3 \ldots i_{k-1}}} \right). \end{aligned} \tag{12}$$

Using these formulas, we can compute the Euler-Lagrange expressions $E_\sigma(f) = E_\sigma(d_i g^i)$ in several steps. First, we have

$$\begin{aligned} &E_\sigma(d_i g^i) \\ &= d_{i_1} \left( \frac{\partial g^{i_1}}{\partial y^\sigma} - \frac{\partial d_i g^i}{\partial y^\sigma_{i_1}} + d_{i_2} \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2}} - \ldots + (-1)^s d_{i_2} d_{i_3} \ldots d_{i_s} \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}} \right) \end{aligned} \tag{13}$$

$$= d_{i_1} d_{i_2} \left( -\frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} + \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2}} - d_{i_3} \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 i_3}} + \ldots + (-1)^s d_{i_3} d_{i_4} \ldots d_{i_s} \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}} \right).$$

Second, using symmetrization,

$$\begin{aligned} E_\sigma(d_i g^i) &= d_{i_1} d_{i_2} \left( -\frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} + d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2}} + \frac{1}{2}\left( \frac{\partial g^{i_1}}{\partial y^\sigma_{i_2}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} \right) \right. \\ (14) \qquad & \left. - d_{i_3} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} + \ldots + (-1)^r d_{i_3} d_{i_4} \ldots d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 \ldots i_r}} \right) \\ &= d_{i_1} d_{i_2} d_{i_3} \left( \frac{\partial g^{i_3}}{\partial y^\sigma_{i_1 i_2}} - \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} + \ldots + (-1)^r d_{i_4} d_{i_5} \ldots d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 \ldots i_s}} \right). \end{aligned}$$

We continue this process and obtain after $s-1$ steps

$$(15) \qquad E_\sigma(d_i g^i) = (-1)^s d_{i_1} d_{i_2} \ldots d_{i_{s-1}} d_{i_s} d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}.$$

But since *f* is defined on $W^s$, the solution $g^i$ necessarily satisfies

$$(16) \qquad \frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s+1}}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 i_4 \ldots i_{s+1}}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 i_5 \ldots i_{s+1}}} + \ldots + \frac{\partial g^{i_{s+1}}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s-1} i_r i_1}} = 0,$$

proving that $E_\sigma(d_i g^i) = 0$.

2. Suppose that $E_\sigma(f) = 0$. We want to show that there exist functions $g^i : V^s \to \mathbf{R}$ such that $f = d_i g^i$. Let *I* be the fibred homotopy operator for differential forms on $V^{2s}$, associated with the projection $\pi^{2s} : V \to U$ (Chapter 2, Section 2.7). We have

$$(16) \qquad \Theta_f = Id\Theta_f + dI\Theta_f + \Theta_0 = Ip_1 d\Theta_f + Ip_2 d\Theta_f + dI\Theta_f + \Theta_0,$$

where $\Theta_0$ is an *n*-form, projectable on *U*. In this formula, $p_1 d\Theta_f = 0$ by hypothesis, $Ip_2 d\Theta_f$ is 1-contact, and since $d\Theta_0 = 0$ identically, we have $\Theta_0 = d\vartheta_0$ for some $\vartheta_0$ (on *U*). Moreover $h\Theta_f = hd(I\Theta_f + \vartheta_0) = f\omega_0$. Defining functions $g^i$ on $V^{2s}$ by the condition

$$(17) \qquad h(I\Theta_f + \vartheta_0) = g^i \omega_i,$$

we see we have constructed a solution of the formal divergence equation. Indeed, from (16), $hd(I\Theta_f + \vartheta_0) = hdh(I\Theta_f + \vartheta_0) = d_i g^i \cdot \omega_0 = f\omega_0$. Then, however, we may choose $g^i$ to be defined on $W^s$ as required (Section 3.1, Lemma 1).

If the formal divergence equation has a solution, then this solution is unique, up to a system of functions $g^i = g^i(x^j)$, such that $(\partial g^i / \partial x^i) = 0$.

**Remark 3** If a formal divergence equation $d_i g^i = f$ has a solution $g^i$, defined on the set $W^s$, then any other solution is given as $g^i + h^i$, where $h^i$ are functions on *U* such that $\partial h^i / \partial x^i = 0$ (see Section 3.1, Remark 2).

Condition $E_\lambda = 0$ (9) is called the *integrability condition* for the formal divergence equation. In terms of differential equations, this condition can equivalently be written as

(18) $$E_\sigma(f) = 0.$$

## 3.3 Projectable extensions of differential forms

Denote

(1) $$\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n,$$

and $\omega_i = i_{\partial/\partial x^i}\omega_0$, that is,

(2) $$\omega_i = \frac{1}{(n-1)!}\varepsilon_{ij_2j_3\ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \ldots \wedge dx^{j_n}.$$

Consider a $\pi^s$- horizontal $(n-1)$-form $\eta$ on $W^s$, expressed as

(3) $$\eta = g^i\omega_i = \frac{1}{(n-1)!} h_{j_2j_3\ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \ldots \wedge dx^{j_n}.$$

Note that from expression (2), the components of the form $\eta$ satisfy the transformation formulas

(4) $$h_{j_2j_3\ldots j_n} = \varepsilon_{ij_2j_3\ldots j_n} g^i, \quad g^k = \frac{1}{(n-1)!}\varepsilon^{kj_2j_3\ldots j_n} h_{j_2j_3\ldots j_n}.$$

In the following lemma we derive a formula for the derivatives of the functions $h_{j_2j_3\ldots j_n}$ and $g^k$; to this purpose a straightforward calculation is needed. Denote by Alt and Sym the *alternation* and *symmetrization* in the corresponding indices.

**Lemma 4** *The functions* $g^i$ *and* $h_{j_1j_2\ldots j_{n-1}}$ *satisfy*

(5) $$\begin{aligned} &\frac{1}{r+1}\varepsilon_{il_2l_3\ldots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\ldots k_s}} + \frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\ldots k_s}} + \frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\ldots k_s}} + \ldots + \frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\ldots k_{s-1}i}}\right) \\ &= \frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}} - \frac{s(n-1)}{s+1}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\delta^{k_1}_{l_2} \quad \mathrm{Sym}(k_1k_2\ldots k_s) \quad \mathrm{Alt}(l_2l_3\ldots l_n). \end{aligned}$$

**Proof** Formula (5) is an immediate consequence of equations (4). Differentiating we get

$$\frac{\partial g^i}{\partial y^\sigma_{k_1 k_2 \dots k_s}} = \frac{1}{(n-1)!} \varepsilon^{i j_2 j_3 \dots j_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{k_1 k_2 \dots k_s}}, \tag{6}$$

hence

$$\begin{aligned}
&\frac{1}{s+1} \varepsilon_{i l_2 l_3 \dots l_n} \left( \frac{\partial g^i}{\partial y^\sigma_{k_1 k_2 \dots k_s}} + \frac{\partial g^{k_1}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} + \frac{\partial g^{k_2}}{\partial y^\sigma_{k_1 i k_3 k_4 \dots k_s}} + \dots + \frac{\partial g^{k_s}}{\partial y^\sigma_{k_1 k_2 \dots k_{s-1} i}} \right) \\
&= \frac{1}{s+1} \frac{1}{(n-1)!} \varepsilon_{i l_2 l_3 \dots l_n} \varepsilon^{i j_2 j_3 \dots j_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{k_1 k_2 \dots k_s}} \\
&+ \frac{1}{s+1} \frac{1}{(n-1)!} \varepsilon_{i l_2 l_3 \dots l_n} \varepsilon^{k_1 j_2 j_3 \dots j_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} \\
&+ \frac{1}{s+1} \frac{1}{(n-1)!} \varepsilon_{i l_2 l_3 \dots l_n} \varepsilon^{k_2 j_2 j_3 \dots j_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{k_1 i k_3 k_4 \dots k_s}} \\
&+ \dots + \frac{1}{s+1} \frac{1}{(n-1)!} \varepsilon_{i l_2 l_3 \dots l_n} \varepsilon^{k_s j_2 j_3 \dots j_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{k_1 k_2 \dots k_{s-1} i}} \\
&= \frac{1}{s+1} \frac{\partial h_{l_2 l_3 \dots l_n}}{\partial y^\sigma_{k_1 k_2 \dots k_s}} + \frac{1}{s+1} \frac{n!}{(n-1)!} \Bigg( \delta^{k_1}_i \delta^{j_2}_{l_2} \delta^{j_3}_{l_3} \dots \delta^{j_n}_{l_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} \\
&+ \delta^{k_2}_i \delta^{j_2}_{l_2} \delta^{j_3}_{l_3} \dots \delta^{j_n}_{l_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{k_1 i k_3 k_4 \dots k_s}} + \dots + \delta^{k_s}_i \delta^{j_2}_{l_2} \delta^{j_3}_{l_3} \dots \delta^{j_n}_{l_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{k_1 k_2 \dots k_{s-1} i}} \Bigg) \\
&\operatorname{Alt}(i l_2 l_3 \dots l_n).
\end{aligned} \tag{7}$$

We calculate the alternations $\operatorname{Alt}(i l_2 l_3 \dots l_n)$ of the summands in the parentheses in two steps. Consider the first summand. Alternating in the indices $(l_2 l_3 \dots l_n)$ and then in $(i l_2 l_3 \dots l_n)$, we get

$$\begin{aligned}
&\delta^{k_1}_i \delta^{j_2}_{l_2} \delta^{j_3}_{l_3} \dots \delta^{j_n}_{l_n} \frac{\partial h_{j_2 j_3 \dots j_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} \quad \operatorname{Alt}(i l_2 l_3 \dots l_n) \\
&= \frac{1}{n} \left( \delta^{k_1}_i \frac{\partial h_{l_2 l_3 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} - \delta^{k_1}_{l_2} \frac{\partial h_{i l_3 l_4 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} - \delta^{k_1}_{l_3} \frac{\partial h_{l_2 i l_4 l_5 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} - \dots - \delta^{k_1}_{l_n} \frac{\partial h_{l_2 l_3 \dots l_{n-1} i}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} \right) \\
&= \frac{1}{n} \left( \frac{\partial h_{l_2 l_3 \dots l_n}}{\partial y^\sigma_{k_1 k_2 k_3 \dots k_s}} - \delta^{k_1}_{l_2} \frac{\partial h_{i l_3 l_4 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} - \delta^{k_1}_{l_3} \frac{\partial h_{l_2 i l_4 l_5 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} - \dots - \delta^{k_1}_{l_n} \frac{\partial h_{l_2 l_3 \dots l_{n-1} i}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} \right),
\end{aligned} \tag{8}$$

and similarly for the remaining terms. Altogether

$$
\begin{aligned}
&\frac{1}{s+1}\varepsilon_{il_2l_3\ldots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\ldots k_s}}+\frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\ldots k_s}}+\ldots+\frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\ldots k_{s-1}i}}\right)\\
&=\frac{1}{s+1}\left(\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}}+\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}-\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\right.\\
&-\ldots-\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}-\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\\
&-\ldots-\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}-\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}-\delta^{k_2}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}\\
(9)\qquad&-\ldots-\delta^{k_2}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}+\ldots+\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}\\
&\left.-\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}-\delta^{k_s}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}-\ldots-\delta^{k_s}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}\right)\\
&=\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}}-\frac{1}{s+1}\left(\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\ldots+\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\right.\\
&+\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}+\delta^{k_2}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}+\ldots+\delta^{k_2}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}\\
&\left.+\ldots+\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}+\delta^{k_s}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}+\ldots+\delta^{k_s}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}\right),
\end{aligned}
$$

and, with the help of alternations and symmetrizations,

$$
\begin{aligned}
&\frac{1}{s+1}\varepsilon_{il_2l_3\ldots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\ldots k_s}}+\frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\ldots k_s}}+\ldots+\frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\ldots k_{s-1}i}}\right)\\
&=\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}}\\
(10)\qquad&-\frac{n-1}{s+1}\frac{1}{n-1}\left(\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\ldots+\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\right.\\
&+\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}+\delta^{k_2}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}+\ldots+\delta^{k_2}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}\\
&\left.+\ldots+\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}+\delta^{k_s}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}+\ldots+\delta^{k_s}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}\right)
\end{aligned}
$$

$$= \frac{\partial h_{l_2 l_3 \dots l_n}}{\partial y^\sigma_{k_1 k_2 \dots k_s}}$$
$$- \frac{n-1}{s+1}\delta^{k_1}_{l_2} \frac{\partial h_{i l_3 l_4 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} - \frac{n-1}{s+1}\delta^{k_2}_{l_2} \frac{\partial h_{i l_3 l_4 \dots l_n}}{\partial y^\sigma_{i k_1 k_3 \dots k_s}} - \dots - \frac{n-1}{s+1}\delta^{k_s}_{l_2} \frac{\partial h_{i l_3 l_4 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_{s-1} k_1}}$$
$$\mathrm{Alt}(l_2 l_3 \dots l_n)$$
$$= \frac{\partial h_{l_2 l_3 \dots l_n}}{\partial y^\sigma_{k_1 k_2 \dots k_s}} - \frac{s(n-1)}{s+1}\delta^{k_1}_{l_2} \frac{\partial h_{i l_3 l_4 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} \quad \mathrm{Alt}(l_2 l_3 \dots l_n) \quad \mathrm{Sym}(k_1 k_2 \dots k_s).$$

Let $\eta$ be a $\pi^s$-horizontal form $\eta$, defined on $W^s$. A form $\mu$ on $W^{s-1}$ is said to be a $\pi^{s,s-1}$*-projectable extension* of $\eta$, if $\eta$ is equal to the horizontal components of $\mu$,

(11) $\eta = h\mu.$

Our objective now will be to find conditions for $\eta$ ensuring that $\mu$ does exist. Let $\eta$ be expressed in two bases of $(n-1)$-forms by formula (3).

**Theorem 2** *The following two conditions are equivalent*:
(a) $\eta$ *has a* $\pi^{s,s-1}$*-projectable extension*.
(b) *The components* $g^i$ *satisfy*

(12) $$\frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \dots j_s}} + \frac{\partial g^{j_1}}{\partial y^\sigma_{i j_2 j_3 \dots j_s}} + \frac{\partial g^{j_2}}{\partial y^\sigma_{j_1 i j_3 j_4 \dots j_s}} + \dots + \frac{\partial g^{j_s}}{\partial y^\sigma_{j_1 j_2 \dots j_{s-1} i}} = 0.$$

(c) *The components* $h_{i_1 i_2 \dots i_{n-1}}$ *satisfy*

(13) $$\frac{\partial h_{l_2 l_3 \dots l_n}}{\partial y^\sigma_{k_1 k_2 \dots k_s}} - \frac{r(n-1)}{r+1} \frac{\partial h_{i l_3 l_4 \dots l_n}}{\partial y^\sigma_{i k_2 k_3 \dots k_s}} \delta^{k_1}_{l_2} = 0 \quad \mathrm{Sym}(k_1 k_2 \dots k_s) \quad \mathrm{Alt}(l_2 l_3 \dots l_n).$$

**Proof** 1. To show that (a) implies (b), suppose that we have an $(n-1)$-form $\mu$, defined on $W^{s-1}$, such that $\eta = h\mu$. Then $hd\eta = d_i g^i \cdot \omega_0$, which is a form on $W^{s+1}$. But $(\pi^{s,s-1})^* d\mu = d(\pi^{s,s-1})^* \mu$ hence $hd\eta = hdh\mu = hd\mu$, so $hd\eta$ is $\pi^{s+1,s}$-projectable (with projection $hd\mu$). But

(14) $$\begin{aligned} hd\eta &= d_i g^i \cdot \omega_0 \\ &= \left( \frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \dots + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \dots j_s}} y^\sigma_{j_1 j_2 \dots j_s i} \right) \omega_0, \end{aligned}$$

so $\pi^{s+1,s}$-projectability implies (12).
2. (c) follows from (b) by Lemma 4.
3. Now we prove that condition (c) implies (a). Write $\eta$ as in (3),

$$\eta = \frac{1}{(n-1)!} h_{j_2 j_3 \dots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \dots \wedge dx^{j_n}. \tag{15}$$

By Lemma 1, Section 3.1, and formula (4), the functions $h_{j_2 j_3 \dots j_n}$ are polynomial in the variables $y^{\sigma}_{Jj}$, where $J$ is a multi-index of length $s-1$. Thus,

$$\begin{aligned} h_{i_1 i_2 \dots i_{n-1}} &= B_{i_1 i_2 \dots i_{n-1}} + B^{J_1 k_1}_{\sigma_1}{}_{i_1 i_2 \dots i_{n-1}} y^{\sigma_1}_{J_1 k_1} + B^{J_1 k_1}_{\sigma_1}{}^{J_2 k_2}_{\sigma_2}{}_{i_1 i_2 \dots i_{n-1}} y^{\sigma_1}_{J_1 k_1} y^{\sigma_2}_{J_2 k_2} \\ &+ \dots + B^{J_1 k_1}_{\sigma_1}{}^{J_2 k_2}_{\sigma_2} \dots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}_{i_1 i_2 \dots i_{n-1}} y^{\sigma_1}_{J_1 k_1} y^{\sigma_2}_{J_2 k_2} \dots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} \\ &+ B^{J_1 k_1}_{\sigma_1}{}^{J_2 k_2}_{\sigma_2} \dots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}^{J_{n-1} k_{n-1}}_{\sigma_{n-1}}{}_{i_1 i_2 \dots i_{n-1}} y^{\sigma_1}_{J_1 k_1} y^{\sigma_2}_{J_2 k_2} \dots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1} k_{n-1}}. \end{aligned} \tag{16}$$

The coefficients in this expression are supposed to be symmetric in the multi-indices ${}^{Jk}_{\sigma}$, ${}^{Lj}_{\nu}$. By hypothesis the polynomials (16) satisfy condition (13)

$$\frac{\partial h_{i_2 i_3 \dots i_n}}{\partial y^{\sigma}_{Jk}} - \frac{r(n-1)}{r+1} \frac{\partial h_{l i_3 i_4 \dots i_n}}{\partial y^{\sigma}_{Jl}} \delta^k_{i_2} = 0 \quad \mathrm{Sym}(Jk) \quad \mathrm{Alt}(i_2 i_3 \dots i_n), \tag{17}$$

which reduces to some conditions for the coefficients. To find these conditions, we compute

$$\begin{aligned} \frac{\partial h_{i_1 i_2 \dots i_{n-1}}}{\partial y^{\sigma}_{Jk}} &= B^{Jk}_{\sigma}{}_{i_1 i_2 \dots i_{n-1}} + 2 B^{Jk}_{\sigma}{}^{J_2 k_2}_{\sigma_2}{}_{i_1 i_2 \dots i_{n-1}} y^{\sigma_2}_{J_2 k_2} \\ &+ \dots + (n-2) B^{Jk}_{\sigma}{}^{J_2 k_2}_{\sigma_2} \dots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}_{i_1 i_2 \dots i_{n-1}} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \dots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} \\ &+ (n-1) B^{Jk}_{\sigma}{}^{J_2 k_2}_{\sigma_2} \dots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}^{J_{n-1} k_{n-1}}_{\sigma_{n-1}}{}_{i_1 i_2 \dots i_{n-1}} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \dots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1} k_{n-1}}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \frac{\partial h_{l i_2 i_3 \dots i_{n-1}}}{\partial y^{\sigma}_{Jl}} &= B^{Jl}_{\sigma}{}_{l i_2 i_3 \dots i_{n-1}} + 2 B^{Jl}_{\sigma}{}^{J_2 k_2}_{\sigma_2}{}_{l i_2 i_3 \dots i_{n-1}} y^{\sigma_2}_{J_2 k_2} \\ &+ \dots + (n-2) B^{Jl}_{\sigma}{}^{J_2 k_2}_{\sigma_2} \dots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}_{l i_2 i_3 \dots i_{n-1}} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \dots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} \\ &+ (n-1) B^{Jl}_{\sigma}{}^{J_2 k_2}_{\sigma_2} \dots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}^{J_{n-1} k_{n-1}}_{\sigma_{n-1}}{}_{l i_2 i_3 \dots i_{n-1}} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \dots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1} k_{n-1}}, \end{aligned} \tag{19}$$

from which we have, changing the index notation,

$$\begin{aligned} \frac{\partial h_{l i_3 i_4 \dots i_n}}{\partial y^{\sigma}_{Jl}} \delta^k_{i_2} &= B^{Jl}_{\sigma}{}_{l i_3 i_4 \dots i_n} \delta^k_{i_2} + 2 B^{Jl}_{\sigma}{}^{J_2 k_2}_{\sigma_2}{}_{l i_3 i_4 \dots i_n} \delta^k_{i_2} y^{\sigma_2}_{J_2 k_2} \\ &+ \dots + (n-2) B^{Jl}_{\sigma}{}^{J_2 k_2}_{\sigma_2} \dots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}_{l i_3 i_4 \dots i_n} \delta^k_{i_2} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \dots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} \\ &+ (n-1) B^{Jl}_{\sigma}{}^{J_2 k_2}_{\sigma_2} \dots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}^{J_{n-1} k_{n-1}}_{\sigma_{n-1}}{}_{l i_3 i_4 \dots i_n} \delta^k_{i_2} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \dots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1} k_{n-1}} \\ &\mathrm{Sym}(Jk) \quad \mathrm{Alt}(i_2 i_3 \dots i_n). \end{aligned} \tag{20}$$

Thus, comparing the coefficients in (20) and (18), condition (17) yields

$$
\begin{aligned}
&B_{\sigma}^{Jk}{}_{i_1 i_2 \dots i_{n-1}} = \frac{s(n-1)}{s+1} B_{\sigma}^{Jl}{}_{l i_2 i_3 \dots i_{n-1}} \delta_{i_1}^{k} \\
&B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2}{}_{i_1 i_2 \dots i_{n-1}} = \frac{s(n-1)}{s+1} B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2}{}_{l i_2 i_3 \dots i_{n-1}} \delta_{i_1}^{k} \\
&\dots \\
&B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2}{}_{\sigma_3}^{J_3 k_3} \dots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{i_1 i_2 \dots i_{n-1}} = \frac{s(n-1)}{s+1} B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2}{}_{\sigma_3}^{J_3 k_3} \dots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{l i_2 i_3 \dots i_{n-1}} \delta_{i_1}^{k}, \\
&B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2}{}_{\sigma_3}^{J_3 k_3} \dots {}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{i_1 i_2 \dots i_{n-1}} = \frac{s(n-1)}{s+1} B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2}{}_{\sigma_3}^{J_3 k_3} \dots {}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{l i_2 i_3 \dots i_{n-1}} \delta_{i_1}^{k}, \\
&\quad \mathrm{Sym}(Jk) \quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1}).
\end{aligned} \tag{21}
$$

On the other hand, any $(n-1)$-form $\mu$ on $W^{s-1}$ can be expressed as

$$\mu = \mu_0 + \omega_I^{\nu} \wedge \Phi_{\nu}^{I} + d\omega_I^{\nu} \wedge \Psi_{\nu}^{I}, \tag{22}$$

where

$$
\begin{aligned}
\mu_0 = {} & A_{i_1 i_2 \dots i_{n-1}} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}} \\
& + A_{\sigma_1}^{J_1}{}_{i_2 i_3 \dots i_{n-1}} dy_{J_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_{n-1}} \\
& + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \dots i_{n-1}} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_{n-1}} \\
& + \dots + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-2}}^{j_{n-2}}{}_{i_{n-1}} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge \dots \wedge dy_{J_{n-2}}^{\sigma_{n-2}} \wedge dx^{i_{n-1}} \\
& + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-1}}^{J_{n-1}} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge \dots \wedge dy_{J_{n-1}}^{\sigma_{n-1}},
\end{aligned} \tag{23}
$$

and the coefficients are *traceless* (Section 2.2, Theorem 3). Then $h\mu = h\mu_0$ because $h$ is an exterior algebra homomorphism, annihilating the contact forms $\omega^{\nu}$, and

$$
\begin{aligned}
h\mu = {} & (A_{i_1 i_2 \dots i_{n-1}} + A_{\sigma_1}^{J_1}{}_{i_2 i_3 \dots i_{n-1}} y_{J_1 i_1}^{\sigma_1} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \dots i_{n-1}} y_{J_1 i_1}^{\sigma_1} y_{J_2 i_2}^{\sigma_2} \\
& + \dots + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-2}}^{J_{n-2}}{}_{i_{n-1}} y_{J_1 i_1}^{\sigma_1} y_{J_2 i_2}^{\sigma_2} \dots y_{J_{n-2} i_{n-2}}^{\sigma_{n-2}} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-1}}^{j_{n-1}} y_{J_1 i_1}^{\sigma_1} y_{J_2 j_2}^{\sigma_2} \dots y_{J_{n-1} i_{n-1}}^{\sigma_{n-1}}) \\
& \cdot dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}}.
\end{aligned} \tag{24}
$$

Now comparing the coefficients in (24) and (16) we see that the equation $h\mu = \eta$ for $\pi^{s,s-1}$-projectable extensions of the form $\eta$ is equivalent with the system

$$
\begin{aligned}
&B_{i_1 i_2 \dots i_{n-1}} = A_{i_1 i_2 \dots i_{n-1}}, \\
&B_{\sigma_1}^{J_1 k_1}{}_{i_1 i_2 \dots i_{n-1}} = A_{\sigma_1}^{J_1}{}_{i_2 i_3 \dots i_{n-1}} \delta_{i_1}^{k_1} \quad \mathrm{Sym}(J_1 k_1) \quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1}), \\
&B_{\sigma_1}^{J_1 k_1}{}_{\sigma_2}^{j_2 k_2}{}_{i_1 i_2 \dots i_{n-1}} = A_{\sigma_1}^{J_1}{}_{\sigma_2}^{j_2}{}_{i_3 i_4 \dots i_{n-1}} \delta_{i_1}^{k_1} \delta_{i_2}^{k_2} \quad \mathrm{Sym}(J_1 k_1) \quad \mathrm{Sym}(J_2 k_2) \\
&\quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1}),
\end{aligned} \tag{25}
$$

$$\ldots$$

$$B^{J_1k_1\ J_2k_2\ \ldots\ J_{n-2}k_{n-2}}_{\sigma_1\ \ \sigma_2\ \ldots\ \sigma_{n-2}\ \ i_1i_2\ldots i_{n-1}} = A^{J_1\ J_2\ \ldots\ J_{n-2}}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_{n-2}\ i_{n-1}} \delta^{k_1}_{i_1}\delta^{k_2}_{i_2}\ldots\delta^{k_{n-2}}_{i_{n-2}} \quad \mathrm{Sym}(J_1k_1)$$
$$\mathrm{Sym}(J_2k_2) \quad \ldots \mathrm{Sym}(J_{n-2}k_{n-2}) \quad \mathrm{Alt}(i_1i_2\ldots i_{n-1}),$$
$$B^{J_1k_1\ J_2k_2\ \ldots\ J_{n-2}k_{n-2}\ J_{n-1}k_{n-1}}_{\sigma_1\ \ \sigma_2\ \ldots\ \sigma_{n-2}\ \ \sigma_{n-1}\ \ i_1i_2\ldots i_{n-1}} = A^{J_1\ J_2\ \ldots\ J_{n-2}\ j_{n-1}}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_{n-2}\ \sigma_{n-1}} \delta^{k_1}_{i_1}\delta^{k_2}_{i_2}\ldots\delta^{k_{n-2}}_{i_{n-2}}\delta^{k_{n-1}}_{i_{n-1}}$$
$$\mathrm{Sym}(j_1k_1)\mathrm{Sym}(j_2k_2) \quad \ldots \quad \mathrm{Sym}(j_{n-1}k_{n-1}) \quad \mathrm{Alt}(i_1i_2\ldots i_{n-1})$$

for unknown functions $A_{i_1i_2\ldots i_{n-1}}$, $A^{J_1}_{\sigma_1\ i_2i_3\ldots i_{n-1}}$, $A^{J_1\ J_2}_{\sigma_1\ \sigma_2\ i_3i_4\ldots i_{n-1}}$, $\ldots$, $A^{J_1\ J_2\ \ldots\ J_{n-2}}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_{n-2}\ i_{n-1}}$, and $A^{J_1\ J_2\ \ldots\ J_{n-2}\ J_{n-1}}_{\sigma_1\ \sigma_2\ \ldots\ \sigma_{n-2}\ \sigma_{n-1}}$.

We can now solve this system with the help of the trace decomposition theory, namely with the trace decomposition formula of the symmetric-alternating tensors; in what follows we use the notation of Appendix 8 and Appendix 9.

We consider each of equations (25) separately. The second equation is

(26) $$B^{Jk}_{\sigma\ i_1i_2\ldots i_{n-1}} = A^{J}_{\sigma\ i_2i_3\ldots i_{n-1}}\delta^k_{i_1} \quad \mathrm{Sym}(Jk) \quad \mathrm{Alt}(i_1i_2\ldots i_{n-1}).$$

Denoting $B = B^{J_1k_1}_{\sigma_1\ i_1i_2\ldots i_{n-1}}$ and $A = A^{J}_{\sigma\ i_2i_3\ldots i_{n-1}}\delta^k_{i_1}$, this equation can also be written as $B = \mathbf{q}\tilde{A}$ where $\tilde{A} = \tilde{A}^{J}_{\sigma\ i_2i_3\ldots i_{n-1}}$ is defined by

(27) $$A^{J}_{\sigma\ i_2i_3\ldots i_{n-1}} = \frac{s(n-1)}{s+1}\tilde{A}^{J}_{\sigma\ i_2i_3\ldots i_{n-1}}.$$

But $B$ satisfies the first condition (21), which can also be written as $B = \mathbf{q}\,\mathrm{tr}\,B$. Consequently, the trace decomposition formula yields $\tilde{A} = \mathrm{tr}\,\mathbf{q}\tilde{A} + \mathbf{q}\,\mathrm{tr}\,\tilde{A} = \mathrm{tr}\,B$ because $\tilde{A}$ is traceless; thus, we get a solution

(28) $$A = \frac{s(n-1)}{s+1}\tilde{A} = \frac{s(n-1)}{s+1}\mathrm{tr}\,B.$$

Next equation (25) is

(29) $$B^{J_1k_1\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ i_1i_2\ldots i_{n-1}} = A^{J_1\ J_2}_{\sigma_1\ \sigma_2\ i_3i_4\ldots i_{n-1}}\delta^{k_1}_{i_1}\delta^{k_2}_{i_2} \quad \mathrm{Sym}(J_1k_1) \quad \mathrm{Sym}(J_2k_2)$$
$$\mathrm{Alt}(i_1i_2\ldots i_{n-1}).$$

This equation can be understood as a condition for the trace decomposition of the tensor $B = B^{J_1k_1\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ i_1i_2\ldots i_{n-1}}$ (Appendix 9). According to conditions (21)

(30) $$B^{J_1k_1\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ i_1i_2\ldots i_{n-1}} = \frac{s(n-1)}{s+1}B^{J_1l\ J_2k_2}_{\sigma_1\ \ \sigma_2\ \ li_2i_3\ldots i_{n-1}}\delta^{k_1}_{i_1} \quad \mathrm{Sym}(J_1k_1)$$
$$\mathrm{Alt}(i_1i_2\ldots i_{n-1}).$$

Analogously

(31) $$B_{\sigma_1\ \ \sigma_2\ \ i_1 i_2 \dots i_{n-1}}^{J_1 k_1\ J_2 k_2} = \frac{s(n-1)}{s+1} B_{\sigma_1\ \ \sigma_2\ \ l i_2 i_3 \dots i_{n-1}}^{J_1 k_1\ J_2 l} \delta_{i_1}^{k_2} \quad \mathrm{Sym}(J_2 k_2) \quad \mathrm{Alt}(i_1 i_2 \dots i_{n-1}).$$

These conditions mean that *B* is a Kronecker tensor whose summands contain exactly one factor of the form $\delta_i^\alpha$, where $\alpha$ runs through $J_1 k_1$ and *i* through the set $\{i_1, i_2, \dots, i_{n-1}\}$, and exactly one factor $\delta_i^\beta$, where $\beta$ runs through $J_2 k_2$ and *i* through $\{i_1, i_2, \dots, i_{n-1}\}$; thus, *B* must be a linear combinations of the terms of the form $\delta_i^{j_1} \delta_l^{j_2}$, $\delta_i^{j_1} \delta_l^{k_2}$, $\delta_i^{k_1} \delta_l^{j_2}$, $\delta_i^{k_1} \delta_l^{k_2}$. From the complete trace decomposition theorem it now follows that the coefficients at these Kronecker tensors can be chosen traceless. This shows, however, that equation (29) has a solution $A_{\sigma_1\ \sigma_2\ i_3 i_4 \dots i_{n-1}}^{J_1\ J_2}$.

To complete the construction of the $\pi^{s,s-1}$-projectable extension $\mu$ of the form $\eta$, we proceed in the same way.

A remarkable property of solutions of the formal divergence equation is obtained when we combine Theorem 1 and Theorem 2: we show that the solutions can also be described as projectable extensions of forms on $W^s$.

**Theorem 3** *Let* $f : W^s \to \mathbf{R}$ *be a function, let* $g = g^i$ *be a system of functions, defined on* $W^s$*, and let* $\eta = g^i \omega_i$. *Then the following conditions are equivalent:*

(a) *The system* $g^i$ *is a solution of the formal divergence equation*

(32) $$d_i g^i = f$$

(b) *There exists a projectable extension* $\mu$ *of the form* $\eta$ *such that*

(33) $$hd\mu = f\omega_0.$$

**Proof** 1. If the functions $g^i$ solve the formal divergence equation $d_i g^i = f$, then condition (12) is satisfied and $\eta$ has a projectable extension $\mu$ (Theorem 2). Then $\eta = h\mu$, hence

(34) $$(\pi^{s+1,s})^* hd\mu = hdh\mu = hd\eta = d_i g^i \cdot \omega_0 = f\omega_0,$$

proving (33).

2. Conversely, suppose that $g^i \omega_i = h\mu$. Then a direct calculation yields $hd\mu = hdh\mu = d_1 g^i \cdot \omega_0$, hence (32) follows from (33).

## References

[K14] D. Krupka, The total divergence equation, Lobachevskii Journal of Mathematics 23 (2006) 71-93