4 Variational structures

In this chapter a complete treatment of the foundations of the calculus of variations on fibred manifolds is presented. Using the calculus of differential forms as the main tool, the aim is to study higher-order integral variational functionals of the form $\gamma \to \int J^r\gamma * \rho$, depending on sections γ of a fibred manifold Y, where ρ is a general differential form on the jet manifold J^rY and $J^r\gamma$ is the r-jet prolongation γ. The *horizontal* forms ρ are the *Lagrangians*.

In Sections 4.1 – 4.7 we consider variations (deformations) of sections of Y as *vector fields*, permuting the set of sections, and the *prolongations* of these vector fields to the jet manifolds J^rY. The variations are applied to the functionals in a geometric way by means of the *Lepage forms*, (Krupka [K13], [K1]). The main idea can be introduced by means of the *Cartan's formula* for the Lie derivative of a differential form η on a manifold Z, $\partial_\xi \eta = i_\xi d\eta + di_\xi \eta$, where i_ξ is the contraction by a vector field ξ and d is the exterior derivative operator. For any manifold X and any mapping $f : X \to Z$, the Lie derivative satisfies $f * \partial_\xi \eta = f * i_\xi d\eta + df * i_\xi \eta$. Replacing Z with the r-jet prolongation J^rY and η with ρ, we prove that the form ρ in the variational functional $\gamma \to \int J^r\gamma * \rho$ may be chosen in such a way that the Cartan's formula for ρ becomes a geometric version of the classical *first variation formula*. These forms are the *Lepage forms*; a structure theorem we prove implies that for different underlying manifold structures and order of their jet prolongations, this concept generalizes the well-known *Cartan form* in classical mechanics (Cartan [C]), the *Poincaré-Cartan forms* in the first order field theory (Garcia [G]), the so-called *fundamental forms* (Krupka [K2], [K13], Betounes [B]), the 2*nd-order generalisation of the Poincaré-Cartan form* (Krupka [K13]), the *Carathéodory form* (Crampin and Saunders [CS]), and the *Hilbert form* in Finsler geometry (Crampin and Saunders [CS], Krupka [K7]). For survey research we refer to Krupka, Krupková and Saunders [KKS1], [KKS2].

The first variation formula, expressed by means of a Lepage form ρ, leads to the concept of the *Euler-Lagrange form*, a global differential form, defined by means of the exterior derivative $d\rho$ (cf. Krupka [K13] and also Goldschmidt and Sternberg [GS], where the Euler-Lagrange form is interpreted as a vector-valued form). The coordinate components of the Euler-Lagrange form coincide with the *Euler-Lagrange expressions* of the classical variational calculus, and its classical analogue is simply the collection of the Euler-Lagrange expression. The corresponding *Euler-Lagrange equations* for *extremals* of a variational functional are then related to each fibered chart, and should be analysed in any concrete case from local and global viewpoints.

The first variation formula also gives rise to the *Euler-Lagrange mapping*, assigning to a Lagrangian its Euler-Lagrange form. The domain and image of this mapping are some Abelian groups of differential forms. A complete treatment of the local theory is presented in Sections 4.9 – 4.11, using the fibred homotopy operator as the basic tool. First the *Vainberg-Tonti formula*, allowing us to assign a Lagrangian to *any* source form, is considered (Vainberg [V], Tonti [To]), and is extended to the higher-order variational theory (Krupka [K8], [K16]). The theorem on the Euler-Lagrange equations of the *Vainberg-Tonti Lagrangian*, proved in Section 4.9, determining the *image* of the Euler-Lagrange mapping in terms of the (local variationality) *Helmholtz* conditions, is a basic instrument for the *local inverse variational problem*, treated in Sections 4.10 and 4.11 (Anderson and Duchamp [AD], Krupka [K11]).

Specific research directions in the variational geometry have been developed for several decades. Different aspects of the local inverse problem are given extensive investigation in Anderson and Thompson [AT], Bloch, Krupka, Urban, Voicu, Volna and Zenkov [Bl], Bucataru [Bu], Crampin [Cr], Krupka and Saunders [KS], Krupková and Prince [KrP], Olver [O2], Sarlet, Crampin and Martinez [SCM], Urban and Krupka [UK2] and many others. Remarks on the history of the inverse problem can be found in Havas [H]; original sources are Helmholtz [He] (the inverse problem for systems of second-order ordinary differential equations) and Sonin [So] and Douglas [Do] (for variational integrating factors).

The theorem on the kernel of the Euler-Lagrange mapping is proved in Section 4.10 on the basis of the formal divergence equations (Chapter 3) and the approach initiated in Krupka [K12], Krupka and Musilová [KM].

Our basic notation in this chapter follows Chapter 2 and Chapter 3: Y is a fixed fibred manifold with orientable base manifold X and projection π, and $\dim X = n$, $\dim Y = n + m$. $J^r Y$ is the r-jet prolongation of Y, $\pi^{r,s}$ and π^r are the canonical jet projections. For any set $W \subset Y$ we denote $W^r = (\pi^{r,0})^{-1}(W)$. $\Omega_q^r W$ is the module of q-forms defined on W^r. Sometimes, when no misunderstanding may possibly arise, to simplify formulas we do not distinguish between the differential forms ρ, defined on the base manifold X of a fibred manifold $\pi^s : J^s Y \to X$ and its canonical lifting $(\pi^s)^* \rho$ to the jet manifold $J^s Y$. Similarly, the Lie derivative $\partial_{J^r\Xi} \rho$ and contraction $i_{J^r\Xi} \rho$ are denoted simply by $\partial_\Xi \rho$ and $i_\Xi \rho$.

Since the subject of this chapter is the *higher order* calculus of variations, some proofs of our statements include extensive coordinate calculations; in order not to make difficult the understanding we prefer to present them as complete as possible.

4.1 Variational structures on fibred manifolds

By a *variational structure* we shall mean a pair (Y, ρ), where Y is a fibred manifold over an n-dimensional manifold X with projection π and ρ is an n-form on the r-jet prolongation $J^r Y$.

Suppose that we have a variational structure (Y, ρ). Let Ω be a compact, n-dimensional submanifold of X with boundary (a *piece* of X). Denote by $\Gamma_\Omega(\pi)$ the set of differentiable sections of π over Ω (of a fixed order of differentiability). Then for any section $\gamma \in \Gamma_\Omega(\pi)$ of Y, the pull-back $J^r\gamma^* \rho$ by the r-jet prolongation $J^r\gamma$ is an n-form on a neighbourhood of the piece Ω. Integrating the n-form $J^r\gamma^* \rho$ on Ω, we get a function $\Gamma_\Omega(\pi) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$, defined by

$$(1) \qquad \rho_\Omega(\gamma) = \int_\Omega J^r\gamma^* \rho.$$

ρ_Ω is called the *variational functional*, associated with (Y,ρ) (over Ω).

The variational functional of the form (1) is referred to as the *integral variational functional, associated* with ρ.

If *W* is an open set in *Y*, considered as a fibred manifold with projection $\pi|_W$, then restricting the *n*-form ρ to $W^r \subset J^rY$ we get a variational structure (W,ρ). The corresponding variational functional is the restriction of the variational functional (1) to the set $\Gamma_\Omega(\pi|_W) \subset \Gamma_\Omega(\pi)$. Elements of this set are sections whose values lie in *W*.

On the other hand, *any n*-form ρ on the set W^r defines a variational structure (W,ρ). The corresponding variational functional is given by

(2) $$\Gamma_\Omega(\pi|_W) \ni \gamma \to \rho_\Omega(\gamma) = \int_\Omega J^r\gamma^*\rho \in \mathbf{R}.$$

If $W = Y$, then $\Gamma_\Omega(\pi|_W) = \Gamma_\Omega(\pi)$ and formula (2) reduces to (1).

Let *W* be an open set in *Y*. For every *r* we denote by $\Omega^r_{n,X}W$ the submodule of the module of *q*-forms Ω^r_nW, consisting of π^r*-horizontal* forms. Elements of the set $\Omega^r_{n,X}W$ are called *Lagrangians* (of order *r*) for the fibred manifold *Y*.

Let $\rho \in \Omega^r_nW$. There exists a unique Lagrangian $\lambda_\rho \in \Omega^{r+1}_{n,X}W$ such that

(3) $$J^{r+1}\gamma^*\lambda_\rho = J^r\gamma^*\rho$$

for all sections γ of *Y*. The *n*-form λ_ρ can alternatively be defined by the first canonical decomposition to the form ρ (Chapter 2, Section 2.4)

(4) $$(\pi^{r+1,r})^*\rho = h\rho + p_1\rho + p_2\rho + \ldots + p_n\rho$$

as the *horizontal component* of ρ,

(5) $$\lambda_\rho = h\rho.$$

λ_ρ is a Lagrangian, said to be *associated with* ρ. Property (3) says that the variational functional ρ_Ω can also be expressed as

(6) $$\rho_\Omega(\gamma) = \int_\Omega J^{r+1}\gamma^*\lambda_\rho.$$

We give the chart expressions of ρ and $h\rho$ in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, on *Y* (or, more exactly, in the associated charts on J^rY and $J^{r+1}Y$). Recall that in multi-index notation the *contact basis* of 1-forms on V^r (and analogously on V^{r+1}) is defined to be the basis $(dx^i, \omega^\sigma_J, dy^\sigma_I)$, where the multi-indices satisfy $0 \le |J| \le r-1$, $|I| = r$, and

(7) $$\omega^\sigma_J = dy^\sigma_J - y^\sigma_{Jj}dx^j.$$

We also associate with the given chart the *n*-form

(8) $$\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n,$$

(considered on $U = \pi(V) \subset X$, and also on V^r), sometimes called the *local volume form*, associated with (V,ψ).

According to the trace decomposition theorem (Section 2.2, Theorem 3), ρ has an expression

(9) $$\rho = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} d\omega_J^\sigma \wedge \Psi_\sigma^J + \rho_0,$$

where

(10) $$\begin{aligned} \rho_0 &= A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n} \\ &+ A_{\sigma_1}^{J_1}{}_{i_2 i_3 \ldots i_n} dy_{J_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_n} \\ &+ A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \ldots i_n} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \ldots \wedge dx^{i_n} \\ &+ \ldots + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \ldots {}_{\sigma_{n-1}}^{J_{n-1}}{}_{i_n} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge \ldots \wedge dy_{J_{n-1}}^{\sigma_{n-1}} \wedge dx^{i_n} \\ &+ A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \ldots {}_{\sigma_n}^{J_n} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge \ldots \wedge dy_{J_n}^{\sigma_n}, \end{aligned}$$

and the coefficients $A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \ldots {}_{\sigma_s}^{J_s}{}_{i_{s+1} i_{s+2} \ldots i_n}$ are *traceless*. Then $h\rho = h\rho_0$ because h is an exterior algebra homomorphism, annihilating the contact forms ω_J^σ and $d\omega_J^\sigma$. Thus,

(11) $$\begin{aligned} \lambda_\rho &= (A_{i_1 i_2 \ldots i_n} + A_{\sigma_1}^{J_1}{}_{i_2 i_3 \ldots i_n} y_{J_1 i_1}^{\sigma_1} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \ldots i_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 i_2}^{\sigma_2} \\ &+ \ldots + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \ldots {}_{\sigma_{n-1}}^{J_{n-1}}{}_{i_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 i_2}^{\sigma_2} \ldots y_{J_{n-1} i_{n-1}}^{\sigma_{n-1}} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \ldots {}_{\sigma_n}^{J_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 j_2}^{\sigma_2} \ldots y_{J_n i_n}^{\sigma_n}) \\ &\cdot dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n}. \end{aligned}$$

Using the local volume form (8) we also write

(12) $$\lambda_\rho = \mathscr{L}\omega_0,$$

where

(13) $$\begin{aligned} \mathscr{L} &= \varepsilon^{i_1 i_2 \ldots i_n} (A_{i_1 i_2 \ldots i_n} + A_{\sigma_1}^{J_1}{}_{i_2 i_3 \ldots i_n} y_{J_1 i_1}^{\sigma_1} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \ldots i_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 i_2}^{\sigma_2} \\ &+ \ldots + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \ldots {}_{\sigma_{n-1}}^{J_{n-1}}{}_{i_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 i_2}^{\sigma_2} \ldots y_{J_{n-1} i_{n-1}}^{\sigma_{n-1}} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \ldots {}_{\sigma_n}^{J_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 j_2}^{\sigma_2} \ldots y_{J_n i_n}^{\sigma_n}). \end{aligned}$$

$\mathscr{L}$ is a function on V^{r+1} called the *Lagrange function*, associated with ρ (or with the Lagrangian λ_ρ).

Remark 1 Sometimes the integration domain Ω in the variational functional ρ_Ω is not fixed, but is arbitrary. Then formula (2) defines a *family* of variational functionals labelled by Ω. This situation usually appears in variational principles in physics.

Remark 2 Orientability of the base X of the fibred manifold Y is not an essential assumption; replacing differential forms by *twisted base differential forms*, one can also develop the variational theory for *non-orientable* bases X (Krupka [K10]). Variational functionals, defined on fibred manifolds over non-orientable bases, may appear in the general relativity theory and field theory, and in the variational theory for submanifolds.

Remark 3 (The structure of Lagrange functions) Formulas (12) and (13) describe *general structure* of the Lagrangians, associated with the class of variational functionals (2). The Lagrange functions $\mathscr{L}$ that appear in chart descriptions of the Lagrangians are multi-linear, symmetric functions of the variables y_I^σ, where $|I| = r+1$.

Remark 4 (Lagrangians) Let ρ be an n-form belonging to the submodule $\Omega_{n,X}^r W \subset \Omega_n^r W$ of π^r*-horizontal forms*, expressed as

(14) $$\rho = \frac{1}{n!} A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n}.$$

Then since $dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n} = \varepsilon^{i_1 i_2 \ldots i_n} \omega_0$, one can equivalently write

(15) $$\rho = \mathscr{L}\omega_0,$$

where the Lagrange function $\mathscr{L}$ is given by

(16) $$\mathscr{L} = \frac{1}{n!} A_{i_1 i_2 \ldots i_n} \varepsilon^{i_1 i_2 \ldots i_n}.$$

The following lemma describes all n-forms $\rho \in \Omega_n^r W$, whose associated Lagrangians belong to the same module $\Omega_n^r W$, that is, are of order r.

Lemma 1 *For a form $\rho \in \Omega_n^r W$ the following two conditions are equivalent:*

(1) *The Lagrangian λ_ρ is defined on W^r.*

(2) *In any fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, ρ has an expression*

(17) $$\rho = \mathscr{L}\omega_0 + \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J| = r-1} d\omega_J^\sigma \wedge \Psi_\sigma^J$$

for some function $\mathscr{L}$ and some forms Φ_σ^J and Ψ_σ^J.

Proof This follows from (5) and (13).

4.2 Variational derivatives

Let U be an open subset of X, $\gamma : U \to Y$ be a section, and let Ξ be a π-projectable vector field on an open set $W \subset Y$ such that $\gamma(U) \subset W$. If

α_t is the local 1-parameter group of Ξ, and $\alpha_{(0)t}$ its π-projection, then

(1) $$\gamma_t = \alpha_t \gamma \alpha_{(0)t}^{-1}$$

is a 1-parameter family of *sections* of *Y*, depending differentiably on the parameter *t*: Indeed, since $\pi\alpha_t = \alpha_{(0)t}\pi$, we have

(2) $$\pi\gamma_t(x) = \pi\alpha_t\gamma\alpha_{(0)t}^{-1}(x) = \alpha_{(0)t}\pi\gamma\alpha_{(0)t}^{-1}(x) = \alpha_{(0)t}\alpha_{(0)t}^{-1}(x) = x$$

on the domain of γ_t, so γ_t is a section for each *t*. The family γ_t is called the *variation*, or *deformation*, of the section γ, *induced* by the vector field Ξ.

Recall that a *vector field along* γ is a mapping $\Xi: U \to TY$ such that $\Xi(x) \in T_{\gamma(x)}Y$ for every point $x \in U$. Given Ξ, formula

(3) $$\xi = T\pi \cdot \Xi$$

then defines a vector field ξ on *U*, called the π*-projection* of Ξ.

The following theorem says that every vector field along a section γ can be extended to a π-projectable vector field, defined on a neighbourhood of the image of γ in *Y*. Moreover, the *r*-jet prolongation of the extended vector field, considered along $J^r\gamma$, is independent of the extension.

Theorem 1 *Let* γ *be a section of Y defined on an open set* $U \subset X$, *let* Ξ *be a vector field along* γ.

(a) *There exists a* π*-projectable vector field* $\tilde{\Xi}$, *defined on a neighbourhood of the set* $\gamma(U)$, *such that for each* $x \in U$

(4) $$\tilde{\Xi}(\gamma(x)) = \Xi(x).$$

(b) *Any two* π*-projectable vector fields* Ξ_1, Ξ_2, *defined on a neighbourhood of* $\gamma(U)$, *such that* $\Xi_1(\gamma(x)) = \Xi_2(\gamma(x))$ *for all* $x \in U$, *satisfy*

(5) $$J^r\Xi_1(J^r_x\gamma) = J^r\Xi_2(J^r_x\gamma).$$

Proof (a) Choose $x_0 \in U$ and a fibred chart (V_0, ψ_0), $\psi_0 = (x_0^i, y_0^\sigma)$, at the point $\gamma(x_0) \in Y$, such that $\pi(V_0) \subset U$ and $\gamma(\pi(V_0)) \subset V_0$. Ξ has in this chart an expression

(6) $$\Xi(\gamma(x)) = \xi^i(x)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(x)} + \Xi^\sigma(x)\left(\frac{\partial}{\partial y^\sigma}\right)_{\gamma(x)}.$$

on $\pi(V_0)$. Set for any $y \in V_0$, $\tilde{\xi}^i(y) = \xi^i(\pi(y))$, $\tilde{\Xi}^\sigma(y) = \Xi^\sigma(\pi(y))$, and define a vector field $\tilde{\Xi}$ on V_0 by

(7) $$\tilde{\Xi} = \tilde{\xi}^i \frac{\partial}{\partial x^i} + \tilde{\Xi}^\sigma \frac{\partial}{\partial y^\sigma}.$$

The vector field $\tilde{\tilde{\Xi}}$ satisfies $\tilde{\tilde{\Xi}}(\gamma(x)) = \Xi(\gamma(x))$ on $\pi(V_0)$.

Applying this construction to every point of the domain of definition U of Ξ we may suppose that we have families of fibred charts (V_ι, ψ_ι), $\psi_\iota = (x_\iota^i, y_\iota^\sigma)$, and vector fields $\tilde{\tilde{\Xi}}_\iota$, where ι runs through an index set I, such that $\pi(V_\iota) \subset U$, $\gamma(\pi(V_\iota)) \subset V_\iota$ for every $\iota \in I$, $\tilde{\tilde{\Xi}}_\iota$ is defined on V_ι, and $\tilde{\tilde{\Xi}}_\iota(\gamma(x))) = \tilde{\tilde{\Xi}}(\gamma(x)))$ for all $\pi(V_\iota)$.

Let $\{\chi_\iota\}_{\iota \in I}$ be a partition of unity, subordinate to the covering $\{V_\iota\}_{\iota \in I}$ of the set $\gamma(U) \subset Y$. Setting

$$\tilde{\tilde{\Xi}} = \sum_{\iota \in I} \chi_\iota \tilde{\tilde{\Xi}}_\iota, \tag{8}$$

we get a vector field on the open set $V = \bigcup V_\iota$. For any $x \in U$ the point $\gamma(x)$ belongs to some of the sets V_ι, thus, $\gamma(U) \subset V$. The value of $\tilde{\tilde{\Xi}}(\gamma(x))$ is

$$\begin{aligned} \tilde{\tilde{\Xi}}(\gamma(x)) &= \sum_{\iota \in I} \chi_\iota(\gamma(x)) \tilde{\tilde{\Xi}}_\iota(\gamma(x)) = \left(\sum_{\iota \in I} \chi_\iota(\gamma(x)) \right) \Xi(\gamma(x)) \\ &= \Xi(\gamma(x)) \end{aligned} \tag{9}$$

because $\{\chi_\iota\}_{\iota \in I}$ is a partition of unity.

(b) It is sufficient to verify equality (5) in a chart. Suppose that

$$\Xi_1 = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}, \quad \Xi_2 = \zeta^i \frac{\partial}{\partial x^i} + \mathrm{Z}^\sigma \frac{\partial}{\partial y^\sigma} \tag{10}$$

and

$$\xi^i = \zeta^i, \quad \Xi^\sigma \circ \gamma = \mathrm{Z}^\sigma \circ \gamma . \tag{11}$$

Then from the formulas

$$\begin{aligned} \Xi^\sigma_{j_1 j_2 \dots j_k} &= d_{j_k} \Xi^\sigma_{j_1 j_2 \dots j_{k-1}} - y^\sigma_{j_1 j_2 \dots j_{k-1} i} \frac{\partial \xi^i}{\partial x^{j_k}}, \\ \mathrm{Z}^\sigma_{j_1 j_2 \dots j_k} &= d_{j_k} \mathrm{Z}^\sigma_{j_1 j_2 \dots j_{k-1}} - y^\sigma_{j_1 j_2 \dots j_{k-1} i} \frac{\partial \zeta^i}{\partial x^{j_k}} \end{aligned} \tag{12}$$

for the components of $J^r \Xi_1$ and $J^r \Xi_2$ (Section 1.7, Lemma 10), and from the formal derivative formula (11), Section 2.1 we observe that the left-hand sides in (12) are polynomials in the variables $y^\sigma_{j_1 j_2 \dots j_s}$, $1 \le s \le r$. Therefore, condition (11) applies to the coefficients of these polynomials and we get $\Xi^\sigma_{j_1 j_2 \dots j_k} \circ J^r \gamma = \mathrm{Z}^\sigma_{j_1 j_2 \dots j_k} \circ J^r \gamma$.

A π-projectable vector field $\tilde{\tilde{\Xi}}$, satisfying condition (a) of Theorem 1, is called a *π–projectable extension* of Ξ. Using (b) and *any* π–projectable extension $\tilde{\tilde{\Xi}}$, we may define, for the given section γ,

$$(13) \qquad J^r\Xi(J^r_x\gamma) = J^r\tilde{\Xi}(J^r_x\gamma).$$

Then $J^r\Xi$ is a vector field along the r-jet prolongation $J^r\gamma$ of γ; we call this vector field the *r-jet prolongation* of the vector field (along γ) Ξ.

Variations ("deformations") of sections induce the corresponding variations ("deformations") of the variational functionals. Let $\rho \in \Omega^r_n W$ be a form, $\Omega \subset \pi(W)$ a piece of *X*. Choose a section $\gamma \in \Gamma_\Omega(\pi|_W)$ and a π-projectable vector field Ξ on *W*, and consider the variation (1) of γ, induced by Ξ. Since the domain of γ_t contains Ω for all sufficiently small *t*, the value of the variational functional $\Gamma_\Omega(\pi|_W) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$ at γ_t is defined, and we get a real-valued function, defined on a neighbourhood $(-\varepsilon,\varepsilon)$ of the point $0 \in \mathbf{R}$,

$$(14) \qquad (-\varepsilon,\varepsilon) \ni t \to \rho_{\alpha_{(0)t}(\Omega)}(\alpha_t\gamma\alpha^{-1}_{(0)t}) = \int_{\alpha_{(0)t}(\Omega)} J^r(\alpha_t\gamma\alpha^{-1}_{(0)t})^*\rho \in \mathbf{R}.$$

It is easily seen that this function is differentiable. Since

$$(15) \qquad J^r(\alpha_t\gamma\alpha^{-1}_{(0)t}))^*\rho = (\alpha^{-1}_{(0)t})^*(J^r\gamma)^*(J^r\alpha_t)^*\rho,$$

where $J^r\alpha_t$ is the local 1-paremeter group of the r-jet prolongation $J^r\Xi$ of the vector field Ξ, we have, using properties of the pull-back operation and the theorem on transformation of the integration domain,

$$(16) \qquad \int_{\alpha_{(0)t}(\Omega)} (J^r(\alpha_t\gamma\alpha^{-1}_{(0)t}))^*\rho = \int_\Omega J^r\gamma^*(J^r\alpha_t)^*\rho.$$

Thus, since the piece Ω is compact, differentiability of the function (14) follows from the theorem on differentiation of an integral, depending upon a parameter.

Differentiating (14) at $t = 0$ one obtains, using (16) and the definition of the Lie derivative,

$$(17) \qquad \left(\frac{d}{dt}\rho_\Omega(\alpha_t\gamma\alpha^{-1}_{(0)t})\right)_0 = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\rho.$$

Note that this expression can be written, in the notation introduced by formula (2), Section 4.1, as

$$(18) \qquad (\partial_{J^r\Xi}\rho)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\rho.$$

The number (18) is called the *variation* of the integral variational functional ρ_Ω *at the point* γ*, induced by the vector field* Ξ.

This formula shows that the function $\Gamma_\Omega(\pi|_W) \ni \gamma \to (\partial_{J^r\Xi}\lambda)_\Omega(\gamma) \in \mathbf{R}$ is the variational functional (over Ω), defined by the form $\partial_{J^r\Xi}\rho$. We call this functional the *variational derivative*, or the *first variation* of the variational functional ρ_Ω by the vector field Ξ.

Formula (18) admits a direct generalization. If Z is another π-projectable vector field on *W*, then the *second variational derivative*, or the *second variation*, of the variational functional ρ_Ω by the vector fields Ξ and Z, is the mapping $\Gamma_\Omega(\pi|_W) \ni \gamma \to (\partial_{J^r Z}\partial_{J^r \Xi}\rho)_\Omega(\gamma) \in \mathbf{R}$, defined by

$$(19) \qquad (\partial_{J^r Z}\partial_{J^r \Xi}\rho)_\Omega(\gamma) = \int_\Omega J^r\gamma^* \partial_{J^r Z}\partial_{J^r \Xi}\rho.$$

It is now obvious how *higher-order variational derivatives* are defined: one should simply apply the Lie derivative (with respect to different vector fields) several times.

A section $\gamma \in \Gamma_\Omega(\pi|_W)$ is called a *stable point* of the variational functional λ_Ω with respect to its variation Ξ, if

$$(20) \qquad (\partial_{J^r \Xi}\rho)_\Omega(\gamma) = 0.$$

In practice, one usually requires that a section be a stable point with respect to a *family* of its variations, defined by the problem considered.

Formula (18) can also be expressed in terms of the Lagrangian $\lambda_\rho = h\rho$, the horizontal component of ρ. Since for any π-projectable vector field Ξ the Lie derivative by its *r*-jet prolongation $J^r\Xi$ commutes with the horizontalisation,

$$(21) \qquad h\partial_{J^r \Xi}\rho = \partial_{J^r \Xi}h\rho$$

(see Section 2.5, Theorem 9, (d)), the first variation of the integral variational functional ρ_Ω at a point $\gamma \in \Gamma_\Omega(\pi|_W)$, induced by the vector field Ξ, can be written as

$$(22) \qquad (\partial_{J^r \Xi}\rho)_\Omega(\gamma) = \int_\Omega J^{r+1}\gamma^* \partial_{J^{r+1}\Xi}\lambda_\rho.$$

4.3 Lepage forms

In this section we introduce a class of *n*-forms ρ on the *r*-jet prolongation J^rY of the fibred manifold *Y*, defining variational structures (W,ρ) by imposing certain conditions on the exterior derivative $d\rho$. Properties of these forms determine the structure of the Lie derivatives $\partial_{J^r \Xi}\rho$, where Ξ is a π-projectable vector field on *Y*, and of the integrands of the variational functionals $\gamma \to (\partial_{J^r \Xi}\rho)_\Omega(\gamma)$ (Section 4.2, (18)). Roughly speaking, we study those forms ρ for which the well-known Cartan's formula $\partial_{J^r \Xi}\rho = i_{J^r \Xi}d\rho + di_{J^r \Xi}\rho$ of the calculus of forms becomes an *infinitesimal analogue* of the integral first variation formula, known from the classical calculus of variations on Euclidean spaces.

First we summarize some useful notation related with a chart (U,φ), $\varphi=(x^i)$, on an n-dimensional manifold X. Denote

$$\begin{aligned}
&\omega_0=\frac{1}{n!}\varepsilon_{i_1 i_2\ldots i_n}dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_n},\\
(1)\qquad &\omega_{k_1}=\frac{1}{1!(n-1)!}\varepsilon_{k_1 i_2 i_3\ldots i_n}dx^{i_2}\wedge dx^{i_3}\wedge\ldots\wedge dx^{i_n},\\
&\omega_{k_1k_2}=\frac{1}{2!(n-2)!}\varepsilon_{k_1 k_2 i_3 i_4\ldots i_n}dx^{i_3}\wedge dx^{i_4}\wedge\ldots\wedge dx^{i_n}.
\end{aligned}$$

The inverse transformation formulas are

$$\begin{aligned}
&dx^{l_1}\wedge dx^{l_2}\wedge\ldots\wedge dx^{l_n}=\varepsilon^{l_1 l_2\ldots l_n}\omega_0,\\
(2)\qquad &dx^{l_2}\wedge dx^{l_3}\wedge\ldots\wedge dx^{l_n}=\varepsilon^{k_1 l_2 l_3\ldots l_n}\omega_{k_1},\\
&dx^{l_3}\wedge dx^{l_4}\wedge\ldots\wedge dx^{i_n}=\varepsilon^{k_1 k_2 l_3 l_4\ldots l_n}\omega_{k_1k_2}
\end{aligned}$$

(cf. Appendix 10). Also note that ω_{jk} can be written as

$$\begin{aligned}
(3)\qquad \omega_{jk}&=i_{\partial/\partial x^j}i_{\partial/\partial x^k}\omega_0\\
&=(-1)^{j+k}dx^1\wedge dx^2\wedge\ldots\wedge dx^{j-1}\wedge dx^{j+1}\wedge\ldots\wedge dx^{k-1}\wedge\ldots\wedge dx^n,
\end{aligned}$$

whenever $j<k$. Then

$$(4)\qquad dx^l\wedge\omega_{jk}=\delta^l_j\omega_k-\delta^l_k\omega_j,$$

which is an immediate consequence of definitions: since se have the identity $\omega_k=(-1)^{k-1}dx^l\wedge dx^1\wedge dx^2\wedge\ldots\wedge dx^{k-1}\wedge dx^{k+1}\wedge\ldots\wedge dx$, then

$$(5)\qquad i_{\partial/\partial x^j}(dx^l\wedge\omega_k)=\begin{cases}\delta^l_k i_{\partial/\partial x^j}\omega_0=\delta^l_k\omega_j,\\ \delta^l_j\omega_k-dx^l\wedge i_{\partial/\partial x^j}\omega_k=\delta^l_j\omega_k-dx^l\wedge\omega_{jk}.\end{cases}$$

We prove three lemmas characterizing the structure of n-forms on the r-jet prolongation J^rY.

Lemma 2 *An n-form* ρ *on* $W^r\subset J^rY$ *has in a fibred chart* (V,ψ), $\psi=(x^i,y^\sigma)$, *an expression*

$$(6)\qquad \rho=\rho_0+\tilde{\rho}+d\eta$$

with the following properties:

(a) *The n-form* ρ_0 *is generated by the contact forms* ω^σ_J, *where* $0\le|J|\le r-1$, *that is,*

(7) $$\rho_0 = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J,$$

where

(8) $$\Phi_\sigma^J = \Phi_{\sigma\,(1)}^J + \Phi_{\sigma\,(2)}^J + \tilde{\Phi}_\sigma^J,$$

the forms $\Phi_{\sigma\,(1)}^J$ *are generated by the contact forms* ω_J^σ, $0 \le |J| \le r-1$, $\Phi_{\sigma\,(2)}^J$ *are generated by* $d\omega_I^\sigma$ *with* $|I| = r-1$, *and*

(9) $$\begin{aligned}\tilde{\Phi}_\sigma^J &= \tilde{\Phi}_{\sigma\; i_1 i_2 \dots i_{n-1}}^J dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}_{\sigma\;\sigma_1\; i_2 i_3 \dots i_{n-1}}^{J\;I_1} dy_{I_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}_{\sigma\;\sigma_1\;\sigma_2\; i_3 i_4 \dots i_{n-1}}^{J\;I_1\;I_2} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_{n-1}} \\ &+ \dots + \tilde{\Phi}_{\sigma\;\sigma_1\;\sigma_2 \dots \sigma_{n-2}\; i_{n-1}}^{J\;I_1\;I_2 \dots I_{n-2}} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge \dots \wedge dy_{I_{n-2}}^{\sigma_{n-2}} \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}_{\sigma\;\sigma_1\;\sigma_2 \dots \sigma_{n-1}}^{J\;I_1\;I_2 \dots I_{n-1}} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge \dots \wedge dy_{I_{n-1}}^{\sigma_{n-1}},\end{aligned}$$

where the multi-indices are of length $|I_1|, |I_2|, \dots, |I_{n-1}| = r$ *and all the coefficients* $\tilde{\Phi}_{\sigma\;\sigma_1\; i_2 i_3 \dots i_{n-1}}^{J\;I_1}$, $\tilde{\Phi}_{\sigma\;\sigma_1\;\sigma_2\; i_3 i_4 \dots i_{n-1}}^{J\;I_1\;I_2}$, $\dots$, $\tilde{\Phi}_{\sigma\;\sigma_1\;\sigma_2 \dots \sigma_{n-2}\; i_{n-1}}^{J\;I_1\;I_2 \dots I_{n-2}}$ *are traceless.*

(b) η *is a contact* $(n-1)$*-form such that*

(10) $$\eta = \sum_{|I| = r-1} \omega_I^\sigma \wedge \Psi_\sigma^I,$$

where the forms Ψ_σ^I *do not contain any exterior factor* ω_J^σ *such that* $0 \le |J| \le r-1$.

(c) $\tilde{\rho}$ *has an expression*

(11) $$\begin{aligned}\tilde{\rho} &= A_{i_1 i_2 \dots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n} \\ &+ A_{\sigma_1\; i_2 i_3 \dots i_n}^{I_1} dy_{I_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n} \\ &+ A_{\sigma_1\;\sigma_2\; i_3 i_4 \dots i_n}^{I_1\;I_2} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_n} \\ &+ \dots + A_{\sigma_1\;\sigma_2 \dots \sigma_{n-1}\; i_n}^{I_1\;I_2 \dots I_{n-1}} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge \dots \wedge dy_{I_{n-1}}^{\sigma_{n-1}} \wedge dx^{i_n} \\ &+ A_{\sigma_1\;\sigma_2 \dots \sigma_n}^{I_1\;I_2 \dots I_n} dy_{I_1}^{\sigma_1} \wedge dy_{I_2}^{\sigma_2} \wedge \dots \wedge dy_{I_n}^{\sigma_n},\end{aligned}$$

where $|I_1|, |I_2|, \dots, |I_n| = r$ *and all the coefficients* $A_{\sigma_1\; i_2 i_3 \dots i_n}^{I_1}$, $A_{\sigma_1\;\sigma_2\; i_3 i_4 \dots i_n}^{I_1\;I_2}$, $\dots$, $A_{\sigma_1\;\sigma_2 \dots \sigma_{n-1}\; i_n}^{I_1\;I_2 \dots I_{n-1}}$ *are traceless.*

Proof From the trace decomposition theorem (Section 2.2, Theorem 3), ρ can be written as

(12) $$\rho = \rho_{(1)} + \rho_{(2)} + \tilde{\rho},$$

where $\rho_{(1)}$ includes all ω_J^σ-generated terms, where $0 \le |J| \le r-1$, $\rho_{(2)}$ includes all $d\omega_I^\sigma$-generated terms with $|J| = r-1$ (and does not contain any exterior factor ω_J^σ), and $\tilde{\rho}$ is expressed by (11). Then

$$\rho_{(2)} = \sum_{|I|=r-1} d\omega_I^\sigma \wedge \Psi_\sigma^I = d\left(\sum_{|I|=r-1} \omega_I^\sigma \wedge \Psi_\sigma^I\right) - \sum_{|I|=r-1} \omega_I^\sigma \wedge d\Psi_\sigma^I, \tag{13}$$

so we get

$$\begin{aligned} \rho &= \rho_{(1)} - \sum_{|I|=r-1} \omega_I^\sigma \wedge d\Psi_\sigma^I + d\left(\sum_{|I|=r-1} \omega_I^\sigma \wedge \Psi_\sigma^I\right) + \tilde{\rho} \\ &= \rho_0 + d\left(\sum_{|I|=r-1} \omega_I^\sigma \wedge \Psi_\sigma^I\right) + \tilde{\rho}. \end{aligned} \tag{14}$$

proving Lemma 2.

Our next aim will be to find the chart expression for the horizontal and 1-contact components of the n-form

$$\tau = \rho_0 + \tilde{\rho} \tag{15}$$

from Lemma 2.

Lemma 3 *Suppose that* τ *has an expression* (7) *and* (11).
(a) *The horizontal component* $h\tau$ *is given by*

$$\begin{aligned} h\tau = (&A_{i_1 i_2 \dots i_n} + A_{\sigma_1}^{I_1}{}_{i_2 i_3 \dots i_n} y_{I_1 i_1}^{\sigma_1} + A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \\ &+ \dots + A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &+ A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2} \dots {}_{\sigma_n}^{I_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_n i_n}^{\sigma_n}) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}. \end{aligned} \tag{16}$$

(b) *The* 1*-contact component* $p_1\tau$ *is given by*

$$\begin{aligned} p_1\tau = &\sum_{0 \le |J| \le r-1} (\tilde{\Phi}_\sigma^J{}_{i_2 i_3 \dots i_n} + \tilde{\Phi}_\sigma^J{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_2 i_2}^{\sigma_2} + \tilde{\Phi}_\sigma^J{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \dots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\ &+ \dots + \tilde{\Phi}_\sigma^J{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3} \dots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &+ \tilde{\Phi}_\sigma^J{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3} \dots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) \omega_J^\sigma \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n} \\ &+ \sum_{|I|=r} (A_\sigma^I{}_{i_2 i_3 \dots i_n} + 2A_{\sigma_1}^I{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \dots i_n} y_{I_2 i_2}^{\sigma_2} + 3A_\sigma^I{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \dots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\ &+ \dots + (n-1) A_\sigma^I{}_{\sigma_2}^{I_2} \dots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &+ n A_\sigma^I{}_{\sigma_2}^{I_2} \dots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) \omega_I^\sigma \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}. \end{aligned} \tag{17}$$

Proof (a) Clearly, $h\tau = h\tilde{\rho}$ and (16) follows.
(b) The form $p_1\tau$ is given by

$$p_1\tau = \sum_{0\le|J|\le r-1} \omega_J^\sigma \wedge h\Phi_\sigma^J + p_1\tilde{\rho}. \tag{18}$$

Then

$$\begin{aligned} h\tilde{\Phi}_\sigma^J &= (\tilde{\Phi}_{\sigma\ i_1 i_2 \dots i_{n-1}}^{J} + \tilde{\Phi}_{\sigma\ \sigma_1\ i_2 i_3 \dots i_{n-1}}^{J\ I_1} y_{I_1 i_1}^{\sigma_1} + \tilde{\Phi}_{\sigma\ \sigma_1\ \sigma_2\ i_3 i_4 \dots i_{n-1}}^{J\ I_1\ I_2} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \\ &\quad + \dots + \tilde{\Phi}_{\sigma\ \sigma_1\ \sigma_2 \dots \sigma_{n-2}\ i_{n-1}}^{J\ I_1\ I_2 \dots I_{n-2}} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_{n-2} i_{n-2}}^{\sigma_{n-2}} \\ &\quad + \tilde{\Phi}_{\sigma\ \sigma_1\ \sigma_2 \dots \sigma_{n-1}}^{J\ I_1\ I_2 \dots I_{n-1}} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}}) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}} \\ &= (\tilde{\Phi}_{\sigma\ i_2 i_3 \dots i_n}^{J} + \tilde{\Phi}_{\sigma\ \sigma_2\ i_3 i_4 \dots i_n}^{J\ I_2} y_{I_2 i_2}^{\sigma_2} + \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3\ i_4 i_5 \dots i_n}^{J\ I_2\ I_3} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\ &\quad + \dots + \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3 \dots \sigma_{n-1}\ i_n}^{J\ I_2\ I_3 \dots I_{n-1}} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &\quad + \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3 \dots \sigma_n}^{J\ I_2\ I_3 \dots I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} p_1\tilde{\rho} &= (A_{\sigma_1\ i_2 i_3 \dots i_n}^{I_1} + 2A_{\sigma_1\ \sigma_2\ i_3 i_4 \dots i_n}^{I_1\ I_2} y_{I_2 i_2}^{\sigma_2} + 3A_{\sigma_1\ \sigma_2\ \sigma_3\ i_4 i_5 \dots i_n}^{I_1\ I_2\ I_3} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\ &\quad + \dots + (n-1) A_{\sigma_1\ \sigma_2 \dots \sigma_{n-1}\ i_n}^{I_1\ I_2 \dots I_{n-1}} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &\quad + nA_{\sigma_1\ \sigma_2 \dots \sigma_n}^{I_1\ I_2 \dots I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) \omega_{I_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n} \\ &= \sum_{|I|=r} (A_{\sigma\ i_2 i_3 \dots i_n}^{I} + 2A_{\sigma_1\ \sigma_2\ i_3 i_4 \dots i_n}^{I\ I_2} y_{I_2 i_2}^{\sigma_2} + 3A_{\sigma\ \sigma_2\ \sigma_3\ i_4 i_5 \dots i_n}^{I\ I_2\ I_3} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\ &\quad + \dots + (n-1) A_{\sigma\ \sigma_2 \dots \sigma_{n-1}\ i_n}^{I\ I_2 \dots I_{n-1}} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &\quad + nA_{\sigma\ \sigma_2 \dots \sigma_n}^{I\ I_2 \dots I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_n i_n}^{\sigma_n}) \omega_I^\sigma \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}. \end{aligned} \tag{20}$$

(17) now follows from (19) and (20).

Now we find the chart expression for the pull-back $(\pi^{r+1,r})^*\rho$. According to Lemma 2

$$(\pi^{r+1,r})^*\rho = h\tilde{\rho} + p_1(\rho_0 + \tilde{\rho}) + d\eta + \mu, \tag{21}$$

where $h\tilde{\rho} = h\tau$ and $p_1\rho_0 + p_1\tilde{\rho}$ are given by Lemma 3, and the order of contactness of μ is ≥ 2. We define f_0 and $f_\sigma^{J\ i}$ by the formulas

$$h\tilde{\rho} = f_0\omega_0, \quad p_1(\rho_0 + \tilde{\rho}) = \sum_{0\le|J|\le r} f_\sigma^{J\ i} \omega_J^\sigma \wedge \omega_i. \tag{22}$$

Explicitly,

$$\begin{aligned} f_0 &= \varepsilon^{i_1 i_2 \dots i_n} (A_{i_1 i_2 \dots i_n} + A_{\sigma_1\ i_2 i_3 \dots i_n}^{I_1} y_{I_1 i_1}^{\sigma_1} + A_{\sigma_1\ \sigma_2\ i_3 i_4 \dots i_n}^{I_1\ I_2} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \\ &\quad + \dots + A_{\sigma_1\ \sigma_2 \dots \sigma_{n-1}\ i_n}^{I_1\ I_2 \dots I_{n-1}} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} + A_{\sigma_1\ \sigma_2 \dots \sigma_n}^{I_1\ I_2 \dots I_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \dots y_{I_n i_n}^{\sigma_n}), \end{aligned} \tag{23}$$

and, since $\varepsilon^{i i_2 i_3 \dots i_n} \omega_i = dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}$,

$$f_\sigma^{J\ i} = \varepsilon^{ii_2i_3\dots i_n}(\tilde{\Phi}_{\sigma\ \ i_2i_3\dots i_n}^{J} + \tilde{\Phi}_{\sigma\ \sigma_2\ \ i_3i_4\dots i_n}^{J\ I_2} y_{I_2i_2}^{\sigma_2} + \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3\ \ i_4i_5\dots i_n}^{J\ I_2\ I_3} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}$$
$$+\dots + \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3\dots\sigma_{n-1}\ i_n}^{J\ I_2\ I_3\ \ \ I_{n-1}} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}\dots y_{I_{n-1}i_{n-1}}^{\sigma_{n-1}} + \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3\dots\sigma_n}^{J\ I_2\ I_3\ \ \ I_n} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}\dots y_{I_ni_n}^{\sigma_n}), \tag{24}$$

and

$$f_\sigma^{I\ i} = \varepsilon^{ii_2i_3\dots i_n}(A_{\sigma\ \ i_2i_3\dots i_n}^{I} + 2A_{\sigma\ \sigma_2\ \ i_3i_4\dots i_n}^{I\ I_2} y_{I_2i_2}^{\sigma_2} + 3A_{\sigma\ \sigma_2\ \sigma_3\ \ i_4i_5\dots i_n}^{I\ I_2\ I_3} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}$$
$$+\dots + (n-1)A_{\sigma\ \sigma_2\dots\sigma_{n-1}\ i_n}^{I\ I_2\ \ \ I_{n-1}} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}\dots y_{I_{n-1}i_{n-1}}^{\sigma_{n-1}}$$
$$+ nA_{\sigma\ \sigma_2\dots\sigma_n}^{I\ I_2\ \ \ I_n} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}\dots y_{I_ni_n}^{\sigma_n}), \tag{25}$$

where $0 \le |J| \le r-1$ and $|I| = r$.

We further decompose the forms $f_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i$.

Lemma 4 *For $k \ge 1$ the forms $\omega_{j_1j_2\dots j_k}^\sigma \wedge \omega_i$ can be decomposed as*

$$\omega_{l_1l_2\dots l_k}^\sigma \wedge \omega_i = \frac{1}{k+1}(\omega_{l_1l_2\dots l_k}^\sigma \wedge \omega_i$$
$$+ \omega_{il_2l_3\dots l_k}^\sigma \wedge \omega_{l_1} + \omega_{l_1il_3l_4\dots l_k}^\sigma \wedge \omega_{l_2} + \dots + \omega_{l_1l_2\dots l_{k-1}i}^\sigma \wedge \omega_{l_k})$$
$$+ \frac{1}{k+1}((\omega_{l_1l_2\dots l_k}^\sigma \wedge \omega_i - \omega_{il_2l_3\dots l_k}^\sigma \wedge \omega_{l_1}) + (\omega_{l_1l_2\dots l_k}^\sigma \wedge \omega_i - \omega_{l_1il_3l_4\dots l_k}^\sigma \wedge \omega_{l_2})$$
$$+ \dots + (\omega_{l_1l_2\dots l_k}^\sigma \wedge \omega_i - \omega_{l_1l_2\dots l_{k-1}i}^\sigma \wedge \omega_{l_k})). \tag{26}$$

The forms $\omega_{l_1l_2\dots l_k}^\sigma \wedge \omega_i - \omega_{l_1l_2\dots l_{p-1}il_{p+1}\dots l_{k-1}l_k}^\sigma \wedge \omega_{l_p}$ are closed and can be expressed as

$$\omega_{l_1l_2\dots l_k}^\sigma \wedge \omega_i - \omega_{l_1l_2\dots l_{p-1}il_{p+1}\dots l_{k-1}l_k}^\sigma \wedge \omega_{l_p} = d(\omega_{l_1l_2\dots l_{p-1}l_{p+1}\dots l_{k-1}l_k}^\sigma \wedge \omega_{il_p}). \tag{27}$$

Proof Indeed, from (4)

$$d\omega_{l_1l_2\dots l_{p-1}l_{p+1}\dots l_{k-1}l_k}^\sigma \wedge \omega_{l_pi} = -\omega_{l_1l_2\dots l_{p-1}l_{p+1}\dots l_{k-1}l_kj}^\sigma \wedge dx^j \wedge \omega_{l_pi}$$
$$= -\omega_{l_1l_2\dots l_{p-1}l_{p+1}\dots l_{k-1}l_kj}^\sigma \wedge dx^j \wedge \omega_{l_pi} = \omega_{l_1l_2\dots l_{p-1}l_{p+1}\dots l_{k-1}l_kj}^\sigma \wedge (\delta_i^j\omega_{l_p} - \delta_{l_p}^j\omega_i)$$
$$= -\omega_{l_1l_2\dots l_{p-1}l_{p+1}\dots l_{k-1}l_kl_p}^\sigma \wedge \omega_i + \omega_{l_1l_2\dots l_{p-1}l_{p+1}\dots l_{k-1}l_ki}^\sigma \wedge \omega_{l_p}. \tag{28}$$

Now we are in a position to prove the following theorem on the structure of *n*-forms on W^r.

Theorem 2 *Let $\rho \in \Omega_n^r W$. For every fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, the pull-back $(\pi^{r+1,r})^*\rho$ has an expression*

$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + \sum_{0\le|J|\le r} P_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i + d\eta + \mu, \tag{29}$$

where the components $P_\sigma^{J\,i}$ *are symmetric in the superscripts,* η *is a* 1-*contact form, and* μ *is a contact form whose order of contactness is* ≥ 2. *The functions* $P_\sigma^{I\,i}$ *such that* $|I| = r$ *satisfy*

$$(30) \qquad P_\sigma^{I\,i} = \frac{\partial f_0}{\partial y_{Ii}^\sigma}.$$

The forms $f_0\omega_0$, $\sum P_\sigma^{J\,i}\omega_J^\sigma \wedge \omega_i$, *and* μ *in this decomposition are unique.*

Proof We use formulas (21) and (22) and apply Lemma 4 to the forms $f_\sigma^{J\,i}\omega_J^\sigma \wedge \omega_i$. Write with explicit index notation $f_\sigma^{J\,i} = P_\sigma^{j_1 j_2 \dots j_k\,i}$. We have the decomposition

$$(31) \qquad f_\sigma^{j_1 j_2 \dots j_k\,i} = P_\sigma^{j_1 j_2 \dots j_k\,i} + Q_\sigma^{j_1 j_2 \dots j_k\,i},$$

where $P_\sigma^{j_1 j_2 \dots j_k\,i} = f_\sigma^{j_1 j_2 \dots j_k\,i}$ $\mathrm{Sym}(j_1 j_2 \dots j_k i)$ is the symmetric component, and $Q_\sigma^{j_1 j_2 \dots j_k\,i}$ is the complementary component of the system $f_\sigma^{j_1 j_2 \dots j_k\,i}$. We have, for each k, $1 \leq k \leq r$,

$$(32) \qquad \begin{aligned} & f_\sigma^{j_1 j_2 \dots j_k\,i}\omega_{j_1 j_2 \dots j_k}^\sigma \wedge \omega_i \\ & = P_\sigma^{j_1 j_2 \dots j_k\,i}\omega_{j_1 j_2 \dots j_k}^\sigma \wedge \omega_i - \frac{1}{k+1}Q_\sigma^{j_1 j_2 \dots j_k\,i} d(\omega_{j_2 j_3 \dots j_k}^\sigma \wedge \omega_{j_1 i} \\ & + \omega_{j_1 j_3 j_4 \dots j_k}^\sigma \wedge \omega_{j_2 i} + \dots + \omega_{j_1 j_2 \dots j_{k-1}}^\sigma \wedge \omega_{j_k i}) \\ & = P_\sigma^{j_1 j_2 \dots j_k\,i}\omega_{j_1 j_2 \dots j_k}^\sigma \wedge \omega_i - \frac{1}{k+1} d(Q_\sigma^{j_1 j_2 \dots j_k\,i}(\omega_{j_2 j_3 \dots j_k}^\sigma \wedge \omega_{j_1 i} \\ & + \omega_{j_1 j_3 j_4 \dots j_k}^\sigma \wedge \omega_{j_2 i} + \dots + \omega_{j_1 j_2 \dots j_{k-1}}^\sigma \wedge \omega_{j_k i})) \\ & + \frac{1}{k+1} dQ_\sigma^{j_1 j_2 \dots j_k\,i} \wedge (\omega_{j_2 j_3 \dots j_k}^\sigma \wedge \omega_{j_1 i} + \omega_{j_1 j_3 j_4 \dots j_k}^\sigma \wedge \omega_{j_2 i} \\ & + \dots + \omega_{j_1 j_2 \dots j_{k-1}}^\sigma \wedge \omega_{j_k i}). \end{aligned}$$

The exterior derivative $dQ_\sigma^{j_1 j_2 \dots j_k\,i}$, when lifted to the set V^{r+2}, can be decomposed as

$$(33) \qquad \begin{aligned} (\pi^{r+2,r+1})^* dQ_\sigma^{j_1 j_2 \dots j_k\,i} &= hdQ_\sigma^{j_1 j_2 \dots j_k\,i} + pdQ_\sigma^{j_1 j_2 \dots j_k\,i} \\ &= d_p Q_\sigma^{j_1 j_2 \dots j_k\,i} dx^p + pdQ_\sigma^{j_1 j_2 \dots j_k\,i}. \end{aligned}$$

Substituting from (33) back to (32) we get 1-contact and a 2-contact summands. The 1-contact are equal to

$$(34) \qquad \begin{aligned} & hdQ_\sigma^{j_1 j_2 \dots j_k\,i} \wedge (\omega_{j_2 j_3 \dots j_k}^\sigma \wedge \omega_{j_1 i} + \omega_{j_1 j_3 j_4 \dots j_k}^\sigma \wedge \omega_{j_2 i} + \dots + \omega_{j_1 j_2 \dots j_{k-1}}^\sigma \wedge \omega_{j_k i}) \\ & = -d_p Q_\sigma^{j_1 j_2 \dots j_k\,i}(\omega_{j_2 j_3 \dots j_k}^\sigma \wedge dx^p \wedge \omega_{j_1 i} + \omega_{j_1 j_3 j_4 \dots j_k}^\sigma \wedge dx^p \wedge \omega_{j_2 i} \\ & + \dots + \omega_{j_1 j_2 \dots j_{k-1}}^\sigma \wedge dx^p \wedge \omega_{j_k i}) \end{aligned}$$

$$= -d_p Q_\sigma^{j_1 j_2 \dots j_k\ i}(\omega^\sigma_{j_2 j_3 \dots j_k} \wedge (\delta^p_{j_1}\omega_i - \delta^p_i \omega_{j_1})$$
$$+ \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge (\delta^p_{j_2}\omega_i - \delta^p_i \omega_{j_2}) + \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge (\delta^p_{j_k}\omega_i - \delta^p_i \omega_{j_k}))$$
$$= -(d_p Q_\sigma^{p j_2 j_3 \dots j_k\ i}\omega^\sigma_{j_2 j_3 \dots j_k} + d_p Q_\sigma^{j_1 p j_3 j_4 \dots j_k\ i}\omega^\sigma_{j_1 j_3 j_4 \dots j_k}$$
$$+ \dots + d_p Q_\sigma^{j_1 j_2 \dots j_{k-1} p\ i}\omega^\sigma_{j_1 j_2 \dots j_{k-1}})\omega_i + d_p Q_\sigma^{j_1 j_2 \dots j_k\ p}(\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1}$$
$$+ \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2} + \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k})$$
$$= -k d_p (Q_\sigma^{p j_2 j_3 \dots j_k\ i} - Q_\sigma^{i j_2 j_3 \dots j_k\ p})\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_i.$$

Note that from the definition of the functions $Q_\sigma^{p j_2 j_3 \dots j_k\ i}$ and from formula (24) we easily see that this form is $\pi^{r+2,r+1}$-projectable. Thus, returning to (32), we have on V^{r+1}

(35)
$$f_\sigma^{j_1 j_2 \dots j_k\ i}\omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i = P_\sigma^{j_1 j_2 \dots j_k\ i}\omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i$$
$$- \frac{k}{k+1} d_p (Q_\sigma^{p j_2 j_3 \dots j_k\ i} - Q_\sigma^{i j_2 j_3 \dots j_k\ p})\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_i$$
$$- \frac{1}{k+1} d(Q_\sigma^{j_1 j_2 \dots j_k\ i}(\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1 i} + \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2 i}$$
$$+ \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k i}))$$
$$+ \frac{1}{k+1} p d Q_\sigma^{j_1 j_2 \dots j_k\ i} \wedge (\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_{j_1 i} + \omega^\sigma_{j_1 j_3 j_4 \dots j_k} \wedge \omega_{j_2 i}$$
$$+ \dots + \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_{j_k i}).$$

This sum replaces $f_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i$, where $|J| = k$, with the symmetrized term $P_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i$, a term $d_p(Q_\sigma^{p j_2 j_3 \dots j_k\ i} - Q_\sigma^{i j_2 j_3 \dots j_k\ p})\omega^\sigma_{j_2 j_3 \dots j_k} \wedge \omega_i$ containing $\omega^\sigma_J \wedge \omega_i$ with $|J| = k-1$, a closed form, and a 2-contact term.

Using these expressions in (21), written as

$$(\pi^{r+1,r})^* \rho = f_0 \omega_0 + \sum_{0 \le |J| \le r} f_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i + d\eta + \mu, \tag{36}$$

we can redefine the coefficients and get

$$(\pi^{r+1,r})^* \rho = f_0 \omega_0 + \sum_{0 \le |J| \le r-1} f_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i + \sum_{|J| \le r} P_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i + d\eta + \mu. \tag{37}$$

After r steps we get (29).

To prove (30), we differentiate (23) and compare the result with (25).

It remains to prove uniqueness of the decomposition (29). Supposing that $(\pi^{r+1,r})^* \rho = 0$ we immediately obtain $f_0 \omega_0 = 0$ and $\mu = 0$ hence

$$\sum_{0 \le |J| \le r} P_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i + d\eta = 0. \tag{38}$$

Differentiating (38) and taking into account the 1-contact component of the resulting $(n+1)$-form,

(39)
$$\sum_{0\le|J|\le r} p_1(dP_\sigma^{J\,i}\wedge\omega_J^\sigma\wedge\omega_i - P_\sigma^{J\,i}\omega_{Ji}^\sigma\wedge\omega_0)$$
$$= -\sum_{0\le|J|\le r}(d_iP_\sigma^{J\,i}\wedge\omega_J^\sigma - P_\sigma^{J\,i}\omega_{Ji}^\sigma)\wedge\omega_0 = 0,$$

which is only possible when $P_\sigma^{J\,i}=0$ because $P_\sigma^{J\,i}$ are symmetric in the superscripts.

In the following lemma we consider vector fields on any fibred manifold Y with base X and projection π.

Lemma 5 *Let ξ be a vector field on X. There exists a π-projectable vector field $\tilde{\xi}$ on Y whose π-projection is ξ.*

Proof We can construct $\tilde{\xi}$ by means of an atlas on Y, consisting of fibred charts, and a subordinate partition of unity (cf. Theorem 1, Section 4.2).

Now we study properties of differential n-forms ρ, defined on $W^r \subset J^rY$, which play a key role in global variational geometry. To this purpose we write the decomposition formula (29) as

(40)
$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + P_\sigma{}^i\omega^\sigma\wedge\omega_i + \sum_{k=1}^{r} P_\sigma^{j_1j_2\dots j_k\,i}\omega_{j_1j_2\dots j_k}^\sigma\wedge\omega_i + d\eta + \mu,$$

where

(41)
$$P_\sigma^{j_1j_2\dots j_r\,i} = \frac{\partial f_0}{\partial y_{j_1j_2\dots j_ri}^\sigma}.$$

Lemma 6 *Let $\rho\in\Omega_n^rW$. The following three conditions are equivalent:*

(a) *$p_1d\rho$ is a $\pi^{r+1,0}$-horizontal $(n+1)$-form.*

(b) *For each $\pi^{r,0}$-vertical vector field ξ on W^r,*

(42)
$$hi_\xi d\rho = 0.$$

(c) *The pull-back $(\pi^{r+1,r})^*\rho$ has the chart expression* (40)*, such that the coefficients satisfy*

(43)
$$\frac{\partial f_0}{\partial y_{j_1j_2\dots j_k}^\sigma} - d_iP_\sigma^{j_1j_2\dots j_k\,i} - P_\sigma^{j_1j_2\dots j_{k-1}\,j_k} = 0, \quad k = 1,2,\dots,r.$$

(d) *$p_1d\rho$ belongs to the ideal on the exterior algebra on W^{r+1}, locally generated by the forms ω^σ.*

Proof 1. Let Ξ be a vector field on W^r, $\tilde{\Xi}$ a vector field on W^{r+1} such that $T\pi^{r+1,r}\cdot\tilde{\Xi}=\Xi\circ\pi^{r+1,r}$ (Lemma 5). Then $i_{\tilde{\Xi}}(\pi^{s+1,s})^*d\rho=(\pi^{s+1,s})^*i_{\Xi}d\rho$, and the forms on both sides can canonically be decomposed into their contact components. We have

$$(44)\qquad i_{\tilde{\Xi}}p_1d\rho+i_{\tilde{\Xi}}p_2d\rho+\ldots+i_{\tilde{\Xi}}p_{n+1}d\rho=hi_{\Xi}d\rho+p_1i_{\Xi}d\rho+\ldots+p_ni_{\Xi}d\rho.$$

Comparing the horizontal components on both sides we get

$$(45)\qquad hi_{\tilde{\Xi}}p_1d\rho=(\pi^{r+2,r+1})^*hi_{\Xi}d\rho.$$

Let $p_1d\rho$ be $\pi^{r+1,0}$-horizontal. Then if Ξ is $\pi^{r,0}$-vertical, $\tilde{\Xi}$ is $\pi^{r+1,0}$-vertical, and we get $hi_{\tilde{\Xi}}p_1d\rho=(\pi^{r+2,r+1})^*hi_{\Xi}d\rho=0$, which implies, by injectivity of the mapping $(\pi^{r+2,r+1})^*$, that $hi_{\Xi}d\rho=0$.

Conversely, let $hi_{\Xi}d\rho=0$ for each $\pi^{r,0}$-vertical vector field ξ. Then by (45), $hi_{\tilde{\Xi}}p_1d\rho=i_{\tilde{\Xi}}p_1d\rho=0$ for all $\pi^{r+1,r}$-projectable, $\pi^{r+1,0}$-vertical vector fields $\tilde{\Xi}$. If in a fibred chart,

$$(46)\qquad \tilde{\Xi}=\sum_{k=1}^{r}\Xi^{\sigma}_{j_1j_2\ldots j_k}\frac{\partial}{\partial y^{\sigma}_{j_1j_2\ldots j_k}}$$

and

$$(47)\qquad p_1d\rho=\sum_{k=0}^{r}A_{\sigma}^{j_1j_2\ldots j_k}\omega^{\sigma}_{j_1j_2\ldots j_k}\wedge\omega_0,$$

then we get

$$(48)\qquad A_{\sigma}^{j_1j_2\ldots j_k}=0,\quad 1\le k\le r,$$

proving $\pi^{r+1,0}$-horizontality of $p_1d\rho$. This proves that conditions (a) and (b) are equivalent.

2. Express $(\pi^{r+1,r})^*\rho$ in a fibred chart by (40). Then

$$(49)\qquad \begin{aligned} p_1d\rho&=\left(\frac{\partial f_0}{\partial y^{\sigma}}-d_iP_{\sigma}^{\ i}\right)\omega^{\sigma}\wedge\omega_0+\\ &+\sum_{k=1}^{r}\left(\frac{\partial f_0}{\partial y^{\sigma}_{j_1j_2\ldots j_k}}-d_iP_{\sigma}^{j_1j_2\ldots j_k\ i}-P_{\sigma}^{j_1j_2\ldots j_{k-1}\ j_k}\right)\omega^{\sigma}_{j_1j_2\ldots j_k}\wedge\omega_0\\ &+\left(\frac{\partial f_0}{\partial y^{\sigma}_{j_1j_2\ldots j_{r+1}}}-P_{\sigma}^{j_1j_2\ldots j_r\ j_{r+1}}\right)\omega^{\sigma}_{j_1j_2\ldots j_rj_{r+1}}\wedge\omega_0\end{aligned}$$

Formula (49) proves equivalence of conditions (a) and (c).

3. Conditions (a) and (d) are obviously equivalent.

Any form $\rho \in \Omega_n^r W$ such that the 1-contact form $p_1 d\rho$ is $\pi^{r+1,0}$-horizontal, is called a *Lepage form*. Lepage forms may equivalently be defined by any of the equivalent conditions of Lemma 6.

Remark 5 (Existence of Lepage forms) It is easily seen that the system (43) has always a solution, and the solution is unique. Indeed,

$$\begin{aligned} P_\sigma^{j_1 j_2 \dots j_{k-1}\ j_k} &= \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_{i_1} P_\sigma^{j_1 j_2 \dots j_k\ i_1} \\ &= \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_{i_1}\left(\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k i_1}} - d_{i_2} P_\sigma^{j_1 j_2 \dots j_k i_1\ i_2}\right) \\ &= \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_{i_1} \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k i_1}} + d_{i_1} d_{i_2} P_\sigma^{j_1 j_2 \dots j_{k-1} i_1\ i_2} \\ &= \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k}} - d_{i_1} \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k i_1}} + d_{i_1} d_{i_2}\left(\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_{k-1} i_1 i_2}} - d_{i_3} P_\sigma^{j_1 j_2 \dots j_{k-1} i_1 i_2\ i_3}\right) \\ &= \dots = \sum_{l=0}^{r+1-k} (-1)^l d_{i_1} d_{i_2} \dots d_{i_l} \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k i_1 i_2 \dots i_l}}, \end{aligned} \tag{50}$$

so the coefficients $P_\sigma^{\ j_1}$, $P_\sigma^{j_1 j_2 \dots j_{k-1}\ j_k}$ are completely determined by the function f_0. In particular, Lepage forms always exist over fibred coordinate neighbourhoods. One can also interpret this result in such a way that to any form $\rho \in \Omega_n^r W$ and any fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, on W, one can always assign a Lepage form, belonging to the module $\Omega_n^{r+1} V$. Note that we have already considered conditions (43) in connection with the integrability condition for formal differential equations (cf. Section 3.2, Lemma 3).

Theorem 3 *A form $\rho \in \Omega_n^r W$ is a Lepage form if and only if for every fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y such that $V \subset W$, $(\pi^{r+1,r})^* \rho$ has an expression*

$$(\pi^{r+1,r})^* \rho = \Theta + d\eta + \mu, \tag{51}$$

where

$$\Theta = f_0 \omega_0 + \sum_{k=0}^{r} \left(\sum_{l=0}^{r-k} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \dots j_k p_1 p_2 \dots p_l i}} \right) \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i, \tag{52}$$

f_0 is a function, defined by the chart expression $h\rho = f_0 \omega_0$, η is a 1-contact form, and μ is a contact form whose order of contactness is ≥ 2.

Proof Suppose we have a Lepage form ρ expressed by (40) where conditions (43) are satisfied, and consider conditions (20). Then repeating (50) we get formula (52). The converse follows from (49) and (40).

The n-form Θ defined by (52), is sometimes called the *principal component* of the Lepage form ρ with respect to the fibred chart (V,ψ). Note that Θ depends only on the Lagrangian $h\rho = \lambda_\rho$ associated with ρ; the forms Θ constructed this way are defined only locally, but their horizontal components define a global form.

Remark 6 Equations (43) include conditions ensuring that the order of the functions $P_\sigma^{j_1 j_2 \ldots j_k\ i}$ does not exceed the order of f_0. We obtained these conditions using polynomiality of the expression on the left-hand side in the jet variables $y^\sigma_{j_1 j_2 \ldots j_k}$, $k > r+1$. Similarly, when Θ is expressed by (52), the order restrictions apply to f_0 since the coefficients at $\omega^\sigma_{j_1 j_2 \ldots j_k} \wedge \omega_i$ should be of order $\leq r+1$.

4.4 Euler-Lagrange forms

We defined in Section 4.3 a Lepage form $\rho \in \Omega^r_n W$ by a condition on the exterior derivative $\rho \in \Omega^r_n W$, derived from the fibred manifold structure on Y. Namely, we required that the 1-contact component $p_1 d\rho$ should belong to the ideal of forms, defined on W^{r+1}, generated in any fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, by the contact 1-forms ω^σ. Now we study properties of the exterior derivative $d\rho$. We express a Lepage form ρ as in formula (50), Section 4.3.

Theorem 4 *If $\rho \in \Omega^r_n W$ is a Lepage form, then the form $(\pi^{r+1,r})^* d\rho$ has an expression*

(1) $$(\pi^{r+1,r})^* d\rho = E + F,$$

where E is a 1-contact, $(\pi^{r+1,0})$-horizontal $(n+1)$-form, and F is a form whose order of contactness is ≥ 2. E is unique and has the chart expression

(2) $$E = \left(\frac{\partial f_0}{\partial y^\sigma} - \sum_{l=1}^{r+1} (-1)^{l-1} d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \right) \omega^\sigma \wedge \omega_0.$$

Proof For any ρ, $E = p_1 d\rho$, and $F = p_2 d\rho + p_3 d\rho + \ldots + p_{n+1} d\rho$. But for a Lepage form ρ,

(3) $$E = p_1 d\Theta = \left(\frac{\partial f_0}{\partial y^\sigma} - d_i P_\sigma^{\ i} \right) \omega^\sigma \wedge \omega_0,$$

where by Section 4.3, (50),

(4) $$P_\sigma^{\ i} = \sum_{l=0}^{s} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \ldots p_l i}}.$$

This proves formula (2).

Note that similarly as the form Θ, E depends only on the Lagrangian $\lambda_\rho = f_0\omega_0$, associated with Θ. The $(n+1)$-form E is called the *Euler-Lagrange form*, associated with the Lepage form ρ, or with the Lagrangian $\lambda_\rho = f_0\omega_0$. The components of E

(5) $$E_\sigma(f_0) = \frac{\partial f_0}{\partial y^\sigma} - \sum_{l=1}^{r+1}(-1)^{l-1} d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \dots p_l}}$$

are called the *Euler-Lagrange expressions* of the function f_0, or of the Lagrangian λ_ρ (in the given fibred chart).

4.5 Lepage equivalents and the Euler-Lagrange mapping

Our aim now will be to study Lepage forms with fixed horizontal components – the Lagrangians. As before, denote by $\Omega^r_{n,X}W$ the submodule of the module $\Omega^r_n W$, formed by π^r *-horizontal n-forms* (*Lagrangians of order r* for *Y*). Clearly, the set $\Omega^r_{n,X}W$ contains the Lagrangians λ_η, associated with the *n*-forms $\eta \in \Omega^{r-1}_n W$, defined on W^{r-1}.

The following is an existence theorem of Lepage forms whose horizontal component is given.

Theorem 5 *To any Lagrangian* $\lambda \in \Omega^r_{n,X}W$ *there exists an integer* $s \le 2r-1$ *and a Lepage form* $\rho \in \Omega^s_n W$ *of order or contactness* ≤ 1 *such that*

(1) $$h\rho = \lambda.$$

Proof We show that the theorem is true for $s = 2r-1$. Choose an atlas $\{(V_\iota, \psi_\iota)\}$ on *Y*, consisting of fibred charts (V_ι, ψ_ι), $\psi_\iota = (x^i_\iota, y^\sigma_\iota)$, and a partition of unity $\{\chi_\iota\}$, subordinate to the covering $\{V_\iota\}$ of the fibred manifold *Y*. The functions χ_ι define (global) Lagrangians $\chi_\iota\lambda \in \Omega^r_{n,X}W$. We have in the chart (V_ι, ψ_ι)

(2) $$\lambda = \mathscr{L}_\iota \omega_{0,\iota},$$

where $\omega_{0,\iota} = dx^1_\iota \wedge dx^2_\iota \wedge \dots \wedge dx^n_\iota$. Then we set for each ι

(3) $$\begin{aligned}\Theta_\iota &= \chi_\iota \mathscr{L}_\iota \omega_{0,\iota} \\ &+ \sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial(\chi_\iota \mathscr{L}_\iota)}{\partial y^{\sigma}_{(\iota)\, j_1 j_2 \dots j_k p_1 p_2 \dots p_l i}}\right)\omega^\sigma_{j_1 j_2 \dots j_k,\iota} \wedge \omega_{0,\iota},\end{aligned}$$

where $\omega^{\sigma}_{j_1 j_2 \dots j_k, \iota} = dy^{\sigma}_{j_1 j_2 \dots j_k, \iota} - y^{\sigma}_{j_1 j_2 \dots j_k l, \iota} dx^{l}_{\iota}$. Thus, Θ_{ι} is the principal Lepage equivalent of the Lagrangian $\lambda = \mathscr{L}_{\iota}\omega_{0,\iota}$. Since the family $\{\chi_{\iota}\}$ is locally finite, the family $\{\Theta_{\iota}\}$ is also locally finite, thus the sum $\rho = \sum \Theta_{\iota}$ is defined. Then we have $p_1 d\rho = \sum p_1 d\Theta_{\iota}$, thus, ρ is a Lepage form, because each of the forms Θ_{ι} is Lepage. It remains to show that $h\rho = \lambda$. We have $h\rho = \sum h\Theta_{\iota} = \sum \chi_{\iota}\mathscr{L}_{\iota}\omega_{0,\iota}$. To compute this expression choose a fibred chart (V,ψ), $\psi = (x^i, y^{\sigma})$, such that the intersection $V \cap V_{\iota}$ is non-void for only finitely many indices ι. Using this chart, we have $\lambda = \mathscr{L}_{\iota}\omega_{0,\iota} = \mathscr{L}_{\iota}\omega_0$ on $V \cap V_{\iota}$ and, since

(4) $$\omega_{0,\iota} = \det\left(\frac{\partial x^i_{\iota}}{\partial x^j}\right) \cdot \omega_0,$$

then

(5) $$\mathscr{L}_{\iota} \det\left(\frac{\partial x^i_{\iota}}{\partial x^j}\right) = \mathscr{L}.$$

Consequently,

(6) $$h\rho = \sum \chi_{\iota}\mathscr{L}_{\iota}\omega_{0,\iota} = \sum \chi_{\iota}\mathscr{L}_{\iota} \det\left(\frac{\partial x^i_{\iota}}{\partial x^j}\right) \cdot \omega_0 = (\sum \chi_{\iota})\mathscr{L}\omega_0 = \mathscr{L}\omega_0$$

because $\sum \chi_{\iota} = 1$.

Let $\lambda \in \Omega^r_{n,X}W$ be a Lagrangian. A Lepage form $\rho \in \Omega^s_n W$ such that $h\rho = \lambda$ (possibly up to a canonical jet projection) is called a *Lepage equivalent* of λ.

If λ is expressed in a fibred chart (V,ψ), $\psi = (x^i, y^{\sigma})$, as

(7) $$\lambda = \mathscr{L}\omega_0,$$

then the form

(8) $$\Theta_{\mathscr{L}} = \mathscr{L}\omega_0 + \sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{j_1 j_2 \dots j_k p_1 p_2 \dots p_l i}}\right)\omega^{\sigma}_{j_1 j_2 \dots j_k} \wedge \omega_i$$

is called the *principal Lepage equivalent* of λ for the fibred chart (V,ψ). This form is in general defined on the set $V^{2r-1} \subset W^{2r-1}$.

Remark 7 The Lepage equivalent constructed in the proof of Theorem 5 is $\pi^{2r-1,r-1}$-horizontal, and its order of contactness is ≤ 1.

Remark 8 Theorem 5 says that the class of variational functionals, associated with the variational structures (W,ρ), introduced in Section 4.1, remains the same when we restrict ourselves to *Lepage* forms ρ. Thus, from now on, we may suppose without loss of generality that the variational functionals

(9) $$\Gamma_\Omega(\pi|_W) \ni \gamma \to \rho_\Omega(\gamma) = \int_\Omega J^r\gamma^* \rho \in \mathbf{R}$$

are defined by Lepage forms.

We give two basic examples of Lepage equivalents of Lagrangians.

Example 1 (Lepage forms of order 1) If $\lambda = \mathscr{L}\omega_0$ is a Lagrangian of order 1, then its principal Lepage equivalent is given by

(10) $$\Theta_\lambda = \mathscr{L}\omega_0 + \frac{\partial \mathscr{L}}{\partial y_i^\sigma}\omega^\sigma \wedge \omega_i.$$

The form (10) is called, due to Garcia [G], the *Poincare-Cartan form*. Its invariance with respect to transformations of fibred charts can be proved by a direct calculation (see Example 2).

Example 2 (Lepage forms of order 2) The principal Lepage equivalent of a second-order Lagrangian $\lambda = \mathscr{L}\omega_0$ is given by

(11) $$\Theta_\mathscr{L} = \mathscr{L}\omega_0 + \left(\frac{\partial \mathscr{L}}{\partial y_i^\sigma} - d_j\frac{\partial \mathscr{L}}{\partial y_{ij}^\sigma}\right)\omega^\sigma \wedge \omega_i + \frac{\partial \mathscr{L}}{\partial y_{ij}^\sigma}\omega_j^\sigma \wedge \omega_i$$

(Krupka [K13]). We show that in this case $\Theta_\mathscr{L}$ is invariant with respect to all transformations of fibred coordinates. It is sufficient to show that $\Theta_\mathscr{L}$ can be introduced in a unique way by invariant conditions. We define a form Θ on W^3 by the following three conditions:

(a) Θ is a Lepage form, that is $p_1 d\Theta$ is $\pi^{3,0}$-horizontal.

(b) The horizontal component of Θ coincides with the given Lagrangian λ; this condition reads $h\Theta = \lambda$.

To state the third condition, we assign to any fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, the contact forms $\omega_j^\sigma \wedge \omega_i$. One can easily derive the transformation properties of these forms. For any other fibred chart $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$, the local volume elements satisfy on the intersection $V \cap \bar{V}$

(12) $$\omega_0 = \det\left(\frac{\partial x^p}{\partial \bar{x}^p}\right)\bar{\omega}_0.$$

Using this formula, we get

(13) $$\omega_i = i_{\partial/\partial x^i}\omega_0 = \frac{\partial \bar{x}^l}{\partial x^i}\det\left(\frac{\partial x^p}{\partial \bar{x}^q}\right)\cdot i_{\partial/\partial \bar{x}^l}\bar{\omega}_0 = \frac{\partial \bar{x}^l}{\partial x^i}\det\left(\frac{\partial x^p}{\partial \bar{x}^q}\right)\cdot \bar{\omega}_l.$$

On the other hand we know that

$$\omega_j^\sigma = \frac{\partial y_j^\sigma}{\partial \bar{y}^\tau}\bar{\omega}^\tau + \frac{\partial y_j^\sigma}{\partial \bar{y}_j^\tau}\bar{\omega}_j^\tau = \frac{\partial y_j^\sigma}{\partial \bar{y}^\tau}\bar{\omega}^\tau + \frac{\partial y^\sigma}{\partial \bar{y}^\tau}\frac{\partial \bar{x}^l}{\partial x^j}\bar{\omega}_l^\tau \tag{14}$$

(Section 2.1, Theorem 1, Section 1.4, Example 5). These formulas imply

$$\begin{aligned}\omega_j^\sigma \wedge \omega_i &= \det\left(\frac{\partial x^p}{\partial \bar{x}^q}\right)\frac{\partial y_j^\sigma}{\partial \bar{y}^\tau}\frac{\partial \bar{x}^l}{\partial x^i}\bar{\omega}^\tau \wedge \bar{\omega}_l \\ &+ \det\left(\frac{\partial x^p}{\partial \bar{x}^q}\right)\frac{\partial y^\sigma}{\partial \bar{y}^\tau}\frac{\partial \bar{x}^l}{\partial x^i}\frac{\partial \bar{x}^k}{\partial x^j}\bar{\omega}_k^\tau \wedge \bar{\omega}_l .\end{aligned} \tag{15}$$

In particular, the forms $\omega_i^\sigma \wedge \omega_j + \omega_j^\sigma \wedge \omega_i$ locally generate a submodule of the module $\Omega_n^3(W^3)$. For the purpose of this example we denote this submodule by $\Theta_{n,1}^3(W^3)$. Now we require

(c) $\Theta \in \Theta_{n,1}^3(W^3)$.

Conditions (a), (b) and (c) uniquely define an n-form on W^3, and this n-form is obviously the form $\Theta_{\mathscr{L}}$ (11). Consequently, the principal Lepage equivalent $\Theta_{\mathscr{L}}$ of a 2nd order Lagrangian λ is globally well-defined. We usually write Θ_λ instead of $\Theta_{\mathscr{L}}$.

Choosing for any Lagrangian $\lambda \in \Omega_{n,X}^r W$ a Lepage equivalent ρ of λ, we can construct the Euler-Lagrange form E associated to ρ (Section 4.4, (2)); this $(n+1)$-form depends on λ only. We denote this form by E_λ and call it the *Euler-Lagrange form*, *associated with* λ. Clearly, E_λ may be defined by (local) principal Lepage equivalents $\Theta_{\mathscr{L}}$. Denoting by $\Omega_{n+1,Y}^{2r-1}W$ the module of $\pi^{2r-1,0}$*-horizontal* $(n+1)$*-forms* on W^{2r-1}, we get the mapping

$$\Omega_{n,X}^r W \ni \lambda \to E_\lambda \in \Omega_{n+1,Y}^r W \tag{16}$$

called the *Euler-Lagrange mapping*.

Remark 9 We can summarize basic motivations and properties of the Lepage forms by means of their relationship to the Euler-Lagrange forms. Denote by $\mathrm{Lep}_n^r W$ the vector subspace of the real vector space $\Omega_n^r W$, whose elements are Lepage forms. Taking into account properties of the exterior derivative of a Lepage form we see that the Euler-Lagrange mapping makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Lep}_n^r W & \xrightarrow{h} & \Omega_{n,X}^{r+1} W \\ \downarrow d & & \downarrow E \\ \Omega_{n+1}^{r+1} W & \xrightarrow{p_1} & \Omega_{n,Y}^{2(r+1)} W \end{array} \tag{17}$$

Basic motivation for the notion of a Lepage form is the construction of this diagram. Its commutativity demonstrates the relationship of the Euler-

Lagrange mapping and the exterior derivative of differential forms, just in the spirit of the work of Lepage [Le]. (17) shows that the Euler-Lagrange form has its origin in the exterior derivative operator.

The following theorem describes the behaviour of the Euler-Lagrange mapping under automorphisms of the underlying fibred manifold; it says that transformed Lagrangians have transformed Euler-Lagrange forms.

Theorem 6 *For each Lagrangian* $\lambda \in \Omega^r_{n,X}W$ *and each automorphism* α *of* Y

(18) $$J^{2r}\alpha^* E_\lambda = E_{J^{2r}\alpha^*\lambda}.$$

Proof To prove (18), we apply Theorem 4 of Section 4.4 to Lepage equivalents. Let $\rho_\lambda \in \Omega^s_n W$ be any Lepage equivalent of λ. Then

(19) $$(\pi^{s+1,s})^* d\rho = E_\lambda + F_\lambda.$$

It is easily seen that the pull-back $J^s\alpha^*\rho$ is a Lepage form whose Lagrangian is $hJ^s\alpha^*\rho = J^{s+1}\alpha^* h\rho = J^{s+1}\alpha^*\lambda$. Then from standard commutativity of the pull-back and the exterior derivative we have

(20) $$(\pi^{s+1,s})^* dJ^s\alpha^*\rho = (\pi^{s+1,s})^* J^s\alpha^* d\rho = J^{s+1}\alpha^*(\pi^{s+1,s})^* d\rho,$$

from which we conclude that $J^{s+1}\alpha^* E_\lambda + J^{s+1}\alpha^* F_\lambda = E_{J^s\alpha^*\lambda} + F_{J^s\alpha^*\lambda}$. Theorem 6 now follows from the uniqueness of the 1-contact component of these forms.

4.6 The first variation formula

Suppose that we have a variational structure (W,ρ), where W is an open set in a fibred manifold Y with n-dimensional base X, and ρ is a *Lepage form* on the set $W^r \subset J^rY$. Recall that for any piece Ω of X, and any open set $W \subset Y$, the Lepage form ρ defines the variational functional $\Gamma_W(\pi|_U) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$ by

(1) $$\rho_\Omega(\gamma) = \int_\Omega J^r\gamma^*\rho$$

(Section 4.1, (2)). The *first variation* of ρ_Ω by a π-projectable vector field Ξ is the variational functional $\Gamma_\Omega(\pi|_U) \ni \gamma \to (\partial_{J^r\Xi}\rho)_\Omega(\gamma) \in \mathbf{R}$, where

(2) $$(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = \int_\Omega J^r\gamma^* \partial_{J^r\Xi}\rho$$

(Section 4.2, (14)). As before, denote by λ_ρ the *horizontal component* of an

n-form ρ, that is the *Lagrangian*, associated with ρ. For Lepage forms, the following theorem on the structure of the integrand in the first variation (2) is just a restatement of definitions.

Theorem 7 *Let $\rho \in \Omega_n^r W$ be a Lepage form, Ξ a π-projectable vector field on W.*

(a) *The Lie derivative $\partial_{J^r\Xi}\rho$ can be expressed as*

$$\partial_{J^r\Xi}\rho = i_{J^r\Xi}d\rho + di_{J^r\Xi}\rho. \tag{3}$$

(b) *If Ξ is π-vertical, then*

$$\partial_{J^{r+1}\Xi}\lambda_\rho = i_{J^{r+1}\Xi}E_{\lambda_\rho} + hdi_{J^r\Xi}\rho. \tag{4}$$

(c) *For any section γ of Y with values in W,*

$$J^r\gamma^*\partial_{J^r\Xi}\rho = J^{r+1}\gamma^* i_{J^{r+1}\Xi}E_{\lambda_\rho} + dJ^r\gamma^* i_{J^r\Xi}\rho \tag{5}$$

(d) *For every piece Ω of X and every section γ of Y defined on Ω,*

$$\int_\Omega J^r\gamma^*\partial_{J^r\Xi}\rho = \int_\Omega J^{r+1}\gamma^* i_{J^{r+1}\Xi}E_\lambda + \int_{\partial\Omega} J^{r+1}\gamma^* i_{J^{r+1}\Xi}\rho. \tag{6}$$

Proof (a) This is a standard Lie derivative formula.

(b) If Ξ π-vertical, then since $h\partial_{J^r\Xi}\rho = \partial_{J^r\Xi}h\rho$, we have from (3) $h\partial_{J^r\Xi}\rho = i_{J^r\Xi}p_1 d\rho + hdi_{J^r\Xi}\rho$, but $p_1 d\rho = E_{\lambda_\rho}$ because ρ is a Lepage form.

(c) Formula (4) can be proved by a straightforward calculation:

$$\begin{aligned} J^r\gamma^*\partial_{J^r\Xi}\rho &= J^r\gamma^* i_{J^r\Xi}d\rho + J^r\gamma^* di_{J^r\Xi}\rho \\ &= J^{r+1}\gamma^* hi_{J^r\Xi}d\rho + J^r\gamma^* di_{J^r\Xi}\rho \\ &= J^{r+1}\gamma^* i_{J^{r+1}\Xi}p_1 d\rho + J^{r+1}\gamma^* i_{J^r\Xi}p_2 d\rho + J^r\gamma^* di_{J^r\Xi}\rho \\ &= J^{r+1}\gamma^* i_{J^{r+1}\Xi}E_{\lambda_\rho} + J^r\gamma^* di_{J^r\Xi}\rho. \end{aligned} \tag{7}$$

(d) Integrating (5) and using the Stokes' theorem on integration of closed $(n-1)$-forms on pieces of n-dimensional manifolds we get (6).

Any of the formulas (3), (4) and (5) is called, in the context of the variational theory on fibred manifolds, the *infinitesimal first variation formula*; (6) is the *integral first variation formula*.

Remark 10 Note that the infinitesimal first variation formulas in Theorem 7 have no analogue in the classical formulation of the calculus of variations. These formulas are based on the concept of a (global) Lepage form as well as on the use of (invariant) geometric operations such as the Lie derivative, exterior derivative and contraction of a form by a vector field, describing the variation procedure.

Remark 11 Theorem 7 can be used to obtain the corresponding formulas for higher variational derivatives (see Section 4.2).

4.7 Extremals

Let $U \subset X$ be an open set, $\gamma : U \to W$ a section, and let $\Xi : U \to TY$ be a vector field along the section γ; in our standard notation, γ is an element of the set $\Gamma_\Omega(\pi|_U)$. The *support* of the vector field Ξ is defined to be the set $\operatorname{supp}\Xi = \operatorname{cl}\{x \in U \mid \Xi(x) \neq 0\}$ (cl means *closure*). We know that each differentiable vector field Ξ along γ can be differentiably prolonged to a π-projectable vector field $\tilde{\Xi}$ defined on a neighbourhood of the set $\gamma(U)$ in W (Section 4.2, Theorem 1). $\tilde{\Xi}$ satisfies

(1) $$\tilde{\Xi} \circ \gamma = \Xi.$$

This property of vector fields along sections will be used in the definition of extremal sections, which can be introduced as follows.

Consider a Lepage form $\rho \in \Omega_n^r W$, and fix a piece Ω of X. We shall say that a section $\gamma \in \Gamma_\Omega(\pi|_U)$ is an *extremal* of the variational functional $\Gamma_\Omega(\pi|_U) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$ on Ω, if for all π-projectable vector fields Ξ, such that $\operatorname{supp}(\Xi \circ \gamma) \subset \Omega$,

(2) $$\int_\Omega J^r\gamma^* \partial_{J^r\Xi}\rho = 0.$$

Condition (2) can also be expressed as $(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = 0$. γ is called an *extremal of the Lagrange structure* (W, ρ), or simply an *extremal*, if it is an extremal of the variational functional ρ_Ω for *every* Ω in the domain of definition of γ.

In this sense the extremals can also be defined as those sections γ for which the values $\rho_\Omega(\gamma)$ of the variational functional ρ_Ω are not sensitive to small compact deformations of γ.

In the following necessary and sufficient conditions for a section to be an extremal, we use the *Euler-Lagrange form* E_{λ_ρ}, associated with the Lagrangian $\lambda_\rho = h\rho$, written in a fibred chart as

(3) $$E_{\lambda_\rho} = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0,$$

where the components $E_\sigma(\mathscr{L})$ are the *Euler-Lagrange expressions* (Section 4.4). Explicitly, if $h\rho = \mathscr{L}\omega_0$, then

(4) $$E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - \sum_{l=1}^{r+1} (-1)^{l-1} d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}}.$$