Theorem 8 *Let* $\rho \in \Omega_n^r W$ *be a Lepage form. Let* $\gamma : U \to W$ *a section, and* $\Omega \subset U$ *be a piece of* X. *The following conditions are equivalent:*

(a) γ *is an extremal on* Ω.

(b) *For every* π *-vertical vector field* Ξ *defined on a neighbourhood of* $\gamma(U)$, *such that* $\operatorname{supp}(\Xi \circ \gamma) \subset \Omega$,

(5) $$J^r\gamma^* i_{J^r\Xi} d\rho = 0.$$

(c) *The Euler-Lagrange form associated with the Lagrangian* $\lambda_\rho = h\rho$ *vanishes along* $J^{r+1}\gamma$, *i.e.,*

(6) $$E_{\lambda_\rho} \circ J^{r+1}\gamma = 0.$$

(d) *For every fibred chart* (V, ψ), $\psi = (x^i, y^\sigma)$, *such that* $\pi(V) \subset U$ *and* $\gamma(\pi(V)) \subset V$, γ *satisfies the system of partial differential equations*

(7) $$E_\sigma(\mathscr{L}_\rho) \circ J^{r+1}\gamma = 0, \quad 1 \le \sigma \le m.$$

Proof 1. We show that (a) implies (b). By Theorem 7, (d), for any piece Ω of X and any π-vertical vector field Ξ such that $\operatorname{supp}(\Xi \circ \gamma) \subset \Omega$,

(8) $$\int_\Omega J^r\gamma^* \partial_{J^r\Xi}\rho = \int_\Omega J^r\gamma^* i_{J^r\Xi} d\rho,$$

because the vector field $J^r\Xi$ vanishes along the boundary $\partial\Omega$. Then

(9) $$\int_\Omega J^r\gamma^* i_{J^r\Xi} d\rho = \int_\Omega J^{r+1}\gamma^* (\pi^{r+1,r})^* i_{J^r\Xi} d\rho = \int_\Omega J^{r+1}\gamma^* i_{J^{r+1}\Xi} p_1 d\rho,$$

where $p_1 d\rho = E_{h\rho}$ is the Euler-Lagrange form.

If Ω is contained in a coordinate neighbourhood, the support $\operatorname{supp}(\Xi \circ \gamma) \subset \Omega$ lies in the same coordinate neighbourhood. Writing $\Xi = \Xi^\sigma \cdot \partial / \partial y^\sigma$ and $p_1 d\rho = E_\sigma(\mathscr{L}_\rho)\omega^\sigma \wedge \omega_0$ then $i_{J^{r+1}\Xi} p_1 d\rho = E_\sigma(\mathscr{L}_\rho)\Xi^\sigma \omega_0$ and

(10) $$J^r\gamma^* i_{J^r\Xi} d\rho = (E_\sigma(\mathscr{L}_\rho) \circ J^{r+1}\gamma) \cdot (\Xi^\sigma \circ \gamma) \cdot \omega_0.$$

Now supposing that $J^r\gamma^* i_{J^r\Xi} d\rho \ne 0$ for some π-vertical vector field Ξ, the first variation formula

(11) $$\int_\Omega J^r\gamma^* i_{J^r\Xi} d\rho = \int_\Omega (E_\sigma(\mathscr{L}_\rho) \circ J^{r+1}\gamma) \cdot (\Xi^\sigma \circ \gamma) \cdot \omega_0$$

would give us a contradiction

(12) $$\int_\Omega J^3\gamma^* \partial_{J^r\Xi}\rho \ne 0.$$

Thus, (a) implies (b).

2. (c) is an immediate consequence of condition (b). Indeed, we can write for Ξ π-vertical

$$J^r\gamma^*i_{J^r\Xi}d\rho = (\pi^{r+1,r}\circ J^{r+1}\gamma)^*i_{J^r\Xi}d\rho = J^{r+1}\gamma^*(\pi^{r+1,r})^*i_{J^r\Xi}d\rho$$
$$= J^{r+1}\gamma^*i_{J^{r+1}\Xi}(\pi^{r+1,r})^*d\rho = J^{r+1}\gamma^*i_{J^{r+1}\Xi}p_1d\rho = J^{r+1}\gamma^*i_{J^{r+1}\Xi}E_{\lambda_\rho}. \tag{13}$$

3. (d) is just a restatement of (b) for the components of the form E_{λ_ρ}.

4. We apply Theorem 7, (d).

Equations (7) are called the *Euler-Lagrange equations*; these equations are indeed related to the chosen fibred chart (V,ψ), $\psi=(x^i,y^\sigma)$. However, since the Euler-Lagrange expressions are components of a (global) differential form, the Euler-Lagrange form, the solutions are independent of fibred charts.

If a Lagrangian $\lambda\in\Omega^r_{n,X}W$ is given and ρ is a Lepage equivalent of λ of order $s=2r-1$ (4.5, Theorem 5), then the Euler-Lagrange equations are of order $\leq 2r$.

Remark 12 For a fixed fibred chart (V,ψ), $\psi=(x^i,y^\sigma)$, the Euler-Lagrange equations represent a system of partial differential equations of order $r+1$ for unknown functions $(x^i)\to\gamma^\sigma(x^i)$, where $1\leq i\leq n$ and $1\leq\sigma\leq m$. This fact is due to the origin of the Lagrange function $\mathscr{L}$ that comes from a Lepage form, which is of order *r*. If we start with a given Lagrangian of order *r*, then the Euler-Lagrange equations are of order $2r$. To get an extremal γ on a piece $\Omega\subset X$ we have to solve this system for every fibred chart (V_ι,ψ_ι), $\psi=(x^i_\iota,y^\sigma_\iota)$, from a collection of fibred charts, such that the sets $\pi(V_\iota)$ cover Ω; then the solutions $(x^i_\iota)\to\gamma^\sigma_\iota(x^i_\iota)$ should be used to find a section γ such that $\gamma^\sigma_\iota=y^\sigma_\iota\gamma\varphi^{-1}_\iota$ for all indices ι.

Remark 13 Properties of nonlinear equations (7) depend on the form ρ; their *global* structure is can also be understood by means of condition (5). This condition says that a section γ is an extremal if and only if its *r*-jet prolongation is an *integral mapping* of an ideal of forms generated by the family of *n*-forms $i_{J^s\Xi}d\rho$. Using fibre chart formulas one can find explicit expressions for local generators of the ideal.

4.8 Trivial Lagrangians

Consider the Euler-Lagrange mapping, assigning to a Lagrangian its Euler-Lagrange form

$$\Omega^r_{n,X}W\ni\lambda\to E_\lambda\in\Omega^{2r}_{n+1,Y}W \tag{1}$$

(Section 4.5, (16)). The domain and the range of this mapping have the structure of Abelian groups (and real vector spaces), and the Euler-Lagrange mapping is a homomorphism of these Abelian groups. The purpose of this section is to describe the *kernel* of the Euler-Lagrange mapping. Elements of the kernel are the Lagrangians $\lambda \in \Omega^r_{n,X}W$ such that

(2) $$E_\lambda = 0,$$

are called (*variationally*) *trivial*, or *null*.

Trivial Lagrangians can locally be characterised as formal divergences or some closed forms.

Theorem 9 *Let* $\lambda \in \Omega^r_{n,X}W$ *be a Lagrangian. The following conditions are equivalent:*

(a) λ *is variationally trivial.*

(b) *For any fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *there exist functions* $g^i : V^r \to \mathbf{R}$, *such that on* V^r, $\lambda = \mathscr{L}\omega_0$, *where*

(3) $$\mathscr{L} = d_i g^i.$$

(c) *For every fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *such that* $V \subset W$, *there exists an* $(n-1)$*-form* $\mu \in \Omega^{r-1}_{n-1}V$ *such that on* V^r

(4) $$\lambda = hd\mu.$$

Proof 1. We show that (a) is equivalent with (b). Suppose that we have a variationally trivial Lagrangian $\lambda \in \Omega^r_n W$. Write for any fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, $\lambda = \mathscr{L}\omega_0$. Since by hypothesis the Euler-Lagrange expressions $E_\sigma(\mathscr{L})$ vanish, consequently, by Section 3.2, Theorem 1, $\mathscr{L} = d_i g^i$ for some functions g^i on V^r. The converse follows from the same Theorem.

2. Equivalence of (a) and (c) follows from Section 3.3, Theorem 3.

In general, Theorem 9 does not ensure existence of a *globally defined* form μ or $d\mu$. However, for first order Lagrangians local triviality already induces global variationality.

Corollary 1 *A first order Lagrange form* $\lambda \in \Omega^1_{n,X}W$ *is variationally trivial if and only if there exists an n-form* $\eta \in \Omega^0_n W$ *such that*

(5) $$\lambda = h\eta$$

and

(6) $$d\eta = 0.$$

Proof By Theorem 9, for any two points $y_1, y_2 \in W$ there exist two $(n-1)$-forms $\mu_1, \mu_2 \in Y$, defined on a neighbourhood of y_1 and y_2, such that $hd\mu_1 = \lambda$ and $hd\mu_2 = \lambda$, respectively. Then $hd(\mu_1 - \mu_2) = 0$ on the intersection of the corresponding neighbourhoods in W^1. But the horizontalization h, considered on forms on $J^0Y = Y$, is injective. Consequently, condition $hd(\mu_1 - \mu_2) = 0$ implies $d(\mu_1 - \mu_2) = 0$, so there exists an n-form $\eta \in \Omega_n^0 W$ whose restriction agrees with $d\mu_1$ and $d\mu_2$. Clearly, $d\eta = 0$.

4.9 Source forms and the Vainberg-Tonti Lagrangians

A 1-contact $(n+1)$-form $\varepsilon \in \Omega_{n+1,Y}^s W$, where s is a non-negative integer. is called a *source form* (Takens [T]). From this definition it follows that ε has in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, an expression

$$(1) \qquad \varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0,$$

where the components ε_σ depend on the jet coordinates x^i, y^σ, $y_{j_1}^\sigma$, $y_{j_1 j_2}^\sigma$, $\dots$, $y_{j_1 j_2 \dots j_s}^\sigma$. Clearly, every Euler-Lagrange form E_λ is a source form, thus, the set of source forms contains the Euler-Lagrange forms as a subset.

We assign to any source form ε a family of Lagrangians as follows. Let ε be defined on W^s, and let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fibred chart on Y, such that $V \subset W$, and the set $\psi(V)$ is star-shaped. Denote by I the fibred homotopy operator on V^s (Section 2.7). Then $I\varepsilon$ is a π^s-horizontal form, that is, a *Lagrangian* for Y, defined on V^s. This Lagrangian, denoted

$$(2) \qquad \lambda_\varepsilon = I\varepsilon,$$

is called the *Vainberg-Tonti Lagrangian*, associated with the source form ε (and the fibred chart (V,ψ)) (cf. Vainberg [V], Tonti [To]).

Recall that $I\varepsilon$ is defined by the fibred homotopy $\chi_s : [0,1] \times V^s \to V^s$, where $\chi_s(t, (x^i, y^\sigma, y_{j_1}^\sigma, y_{j_1 j_2}^\sigma, \dots, y_{j_1 j_2 \dots j_s}^\sigma)) = (x^i, ty^\sigma, ty_{j_1}^\sigma, ty_{j_1 j_2}^\sigma, \dots, ty_{j_1 j_2 \dots j_s}^\sigma)$. Since χ_s satisfies $\chi_s^* \varepsilon = (\varepsilon_\sigma \circ \chi_s)(t\omega^\sigma + y^\sigma dt) \wedge \omega_0$, we have, integrating the coefficient in this expression at dt,

$$(3) \qquad \lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0,$$

where

$$(4) \qquad \mathscr{L}_\varepsilon = y^\sigma \int_0^1 \varepsilon_\sigma \circ \chi_s \cdot dt,$$

or, which is the same,

$$(5) \quad \mathscr{L}_\varepsilon(x^i, y^\sigma, y_{j_1}^\sigma, \dots, y_{j_1 j_2 \dots j_s}^\sigma) = y^\sigma \int_0^1 \varepsilon_\sigma(x^i, ty^\nu, ty_{j_1}^\nu, \dots, ty_{j_1 j_2 \dots j_s}^\nu) dt.$$

We can find the chart expression for the Euler-Lagrange form E_{λ_ε} of the Vainberg-Tonti Lagrangian λ_ε; recall that

$$(6) \qquad E_{\lambda_\varepsilon} = E_\sigma(\mathscr{L}_\varepsilon) \omega^\sigma \wedge \omega_0,$$

where

$$E_\sigma(\mathscr{L}_\varepsilon) = \sum_{l=0}^{s} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \mathscr{L}_\varepsilon}{\partial y^\sigma_{p_1 p_2 \dots p_l}}. \tag{7}$$

To this purpose we derive two formulas for the formal derivative operator d_i. The formulas are completely parallel with the well-known classical Leibniz rules for partial derivatives of the product of functions.

Lemma 7 (a) *For every function f on* V^p

$$d_i(f \circ \chi_p) = d_i f \circ \chi_{p+1}. \tag{8}$$

(b) *For every function f on* V^s *and a collection of functions* $g^{p_1 p_2 \dots p_k}$ *on* V^s*, symmetric in the superscripts,*

$$\begin{aligned} & d_{p_1} d_{p_2} \dots d_{p_k}(f \cdot g^{p_1 p_2 \dots p_k}) \\ & = \sum_{i=0}^{k} \binom{k}{i} d_{p_1} d_{p_2} \dots d_{p_i} f \cdot d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_k} g^{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_k}. \end{aligned} \tag{9}$$

Proof (a) Formula (8) is an easy consequence of definitions.

(b) The proof is standard. We have

$$\begin{aligned} & d_{p_1}(f \cdot g^{p_1}) = d_{p_1} f \cdot g^{p_1} + f \cdot d_{p_1} g^{p_1} \\ & = \binom{1}{0} d_{p_1} f \cdot g^{p_1} + \binom{1}{1} f \cdot d_{p_1} g^{p_1}, \\ & d_{p_1} d_{p_2}(f \cdot g^{p_1 p_2}) = d_{p_2}(d_{p_1} f \cdot g^{p_1 p_2} + f \cdot d_{p_1} g^{p_1 p_2}) \\ & = d_{p_2} d_{p_1} f \cdot g^{p_1 p_2} + d_{p_1} f \cdot d_{p_2} g^{p_1 p_2} + d_{p_2} f \cdot d_{p_1} g^{p_1 p_2} + f \cdot d_{p_1} d_{p_1} g^{p_1 p_2} \\ & = \binom{2}{0} d_{p_2} d_{p_1} f \cdot g^{p_1 p_2} + \binom{2}{1} d_{p_1} f \cdot d_{p_2} g^{p_1 p_2} + \binom{2}{2} f \cdot d_{p_1} d_{p_1} g^{p_1 p_2}. \end{aligned} \tag{10}$$

Then, supposing that

$$\begin{aligned} & d_{p_1} d_{p_2} \dots d_{p_{k-1}}(f \cdot g^{p_1 p_2 \dots p_{k-1}}) \\ & = \sum_{i=0}^{k-1} \binom{k-1}{i} d_{p_1} d_{p_2} \dots d_{p_i} f \cdot d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_{k-1}} g^{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_{k-1}}, \end{aligned} \tag{11}$$

we have (8)

$$\begin{aligned} & d_{p_1} d_{p_2} \dots d_{p_{k-1}} d_{p_k}(f \cdot g^{p_1 p_2 \dots p_{k-1} p_k}) \\ & = f \cdot d_{p_1} d_{p_2} \dots d_{p_{k-1}} d_{p_k} g^{p_1 p_2 \dots p_{k-1} p_k} \\ & + \left(\binom{k-1}{0} + \binom{k-1}{1} \right) d_{p_1} f \cdot d_{p_2} d_{p_3} \dots d_{p_k} g^{p_1 p_2 \dots p_{k-1} p_k} \end{aligned} \tag{12}$$

$$+\left(\binom{k-1}{1}+\binom{k-1}{2}\right)d_{p_1}d_{p_2}f\cdot d_{p_3}d_{p_4}\ldots d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k}$$
$$+\ldots+\left(\binom{k-1}{k-2}+\binom{k-1}{k-1}\right)d_{p_1}d_{p_2}\ldots d_{p_{k-1}}\cdot d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k}$$
$$+\binom{k-1}{k-1}d_{p_k}d_{p_1}d_{p_2}\ldots d_{p_{k-1}}\cdot g^{p_1p_2\ldots p_{k-1}p_k}$$

and

$$\binom{k-1}{p}+\binom{k-1}{p+1}=\binom{k}{p+1}, \tag{13}$$

thus,

$$\begin{aligned} &d_{p_1}d_{p_2}\ldots d_{p_{k-1}}d_{p_k}(f\cdot g^{p_1p_2\ldots p_{k-1}p_k})\\ &\quad=\binom{k}{0}f\cdot d_{p_1}d_{p_2}\ldots d_{p_{k-1}}d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k}\\ &\quad+\binom{k}{1}d_{p_1}f\cdot d_{p_2}d_{p_3}\ldots d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k}\\ &\quad+\binom{k}{2}d_{p_1}d_{p_2}f\cdot d_{p_3}d_{p_4}\ldots d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k}\\ &\quad+\ldots+\binom{k}{k-1}d_{p_1}d_{p_2}\ldots d_{p_{k-1}}\cdot d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k}\\ &\quad+\binom{k}{k}d_{p_k}d_{p_1}d_{p_2}\ldots d_{p_{k-1}}\cdot g^{p_1p_2\ldots p_{k-1}p_k}. \end{aligned} \tag{14}$$

which is formula (8).

The Vainberg-Tonti Lagrangian $\lambda_\varepsilon=\mathscr{L}_\varepsilon\omega_0$ allows us to assign to *any* source form $\varepsilon=\varepsilon_\sigma\omega^\sigma\wedge\omega_0$ a variational functional and the corresponding Euler-Lagrange form of this functional, with the Euler-Lagrange expressions $E_\sigma(\mathscr{L}_\varepsilon)$. We shall determine the functions $E_\sigma(\mathscr{L}_\varepsilon)$ and compare them with the components ε_σ of the source form.

Theorem 10 *The Euler-Lagrange expressions of the Vainberg-Tonti Lagrangian λ_ε of a source form $\varepsilon=\varepsilon_\sigma\omega^\sigma\wedge\omega_0$ are*

$$E_\sigma(\mathscr{L}_\varepsilon)=\varepsilon_\sigma-\sum_{k=0}^{s}y^\nu_{q_1q_2\ldots q_k}\int_0^1 H_{\sigma\ \nu}^{\ q_1q_2\ldots q_k}(\varepsilon)\circ\chi_{2s}\cdot t\,dt, \tag{15}$$

where for every $k=0,1,2,\ldots,s$

$$\begin{aligned} H_{\sigma\nu}^{q_1q_2\ldots q_k}(\varepsilon)&=\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\ldots q_k}}-(-1)^k\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\ldots q_k}}\\ &-\sum_{l=k+1}^{s}(-1)^l\binom{l}{k}d_{p_{k+1}}d_{p_{k+2}}\ldots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\ldots q_kp_{k+1}p_{k+2}\ldots p_l}}. \end{aligned} \tag{16}$$

Proof We find a formula for the difference $\varepsilon_\sigma - E_\sigma(\mathscr{L}_\varepsilon)$. To simplify the formulas, we denote the homotopy χ_{s+l-i} simply by χ. Calculating the derivatives we have

$$\frac{\partial \mathscr{L}_\varepsilon}{\partial y^\sigma} = \int_0^1 \varepsilon_\sigma \circ \chi \cdot dt + y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot t\, dt, \tag{17}$$

and, by Lemma 7, (8) and (9), for every l, $1 \le l \le s$,

$$\begin{aligned} & d_{p_l} \dots d_{p_2} d_{p_1} \frac{\partial \mathscr{L}_\varepsilon}{\partial y^\sigma_{p_1 p_2 \dots p_l}} \\ & = d_{p_l} \dots d_{p_2} d_{p_1} \left(y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_l}} \circ \chi \cdot t\, dt \right) \\ & = \sum_{i=0}^{l} \binom{l}{i} d_{p_1} d_{p_2} \dots d_{p_i} y^\nu \cdot d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\, dt \\ & = \sum_{i=0}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \dots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\, dt. \end{aligned} \tag{18}$$

Then by (17) and (18),

$$\begin{aligned} E_\sigma(\mathscr{L}_\varepsilon) = & \int_0^1 \varepsilon_\sigma \circ \chi \cdot dt + y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot t\, dt \\ & + \sum_{l=1}^{s} (-1)^l \sum_{i=0}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \dots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\, dt. \end{aligned} \tag{19}$$

On the other hand,

$$\begin{aligned} \varepsilon_\sigma & = \int_0^1 \frac{d}{dt} (\varepsilon_\sigma \circ \chi \cdot t) dt \\ & = \int_0^1 \frac{d(\varepsilon_\sigma \circ \chi)}{dt} \cdot t\, dt + \int_0^1 \varepsilon_\sigma \circ \chi \cdot dt \\ & = \sum_{i=0}^{s} \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1 p_2 \dots p_i}} \circ \chi \cdot y^\nu_{p_1 p_2 \dots p_i} \cdot t\, dt + \int_0^1 \varepsilon_\sigma \circ \chi \cdot dt, \end{aligned} \tag{20}$$

hence

$$\begin{aligned} \varepsilon_\sigma - E_\sigma(\mathscr{L}_\varepsilon) = & \sum_{i=0}^{s} \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1 p_2 \dots p_i}} \circ \chi \cdot y^\nu_{p_1 p_2 \dots p_i} \cdot t\, dt - y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot t\, dt \\ & - \sum_{l=1}^{s} (-1)^l \sum_{i=0}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \dots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\, dt \end{aligned} \tag{21}$$

$$= \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu} \circ \chi \cdot y^\nu \cdot t\,dt - y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot t\,dt$$
$$- \sum_{l=1}^{s} (-1)^l \binom{l}{0} y^\nu \cdot \int_0^1 d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_l}} \circ \chi \cdot t\,dt$$
$$+ \sum_{i=1}^{s} \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1 p_2 \dots p_i}} \circ \chi \cdot y^\nu_{p_1 p_2 \dots p_i} \cdot t\,dt$$
$$- \sum_{l=1}^{s} (-1)^l \sum_{i=1}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \dots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\,dt.$$

We change summation in the double sum, replacing the summation through the pairs (l,i) with the summation through (i,l). Summation through (l,i) can be expressed by the scheme

$$(22) \qquad \begin{array}{l} (1,1) \\ (2,1),(2,2) \\ (3,1),(3,2),(3,3) \\ \dots \\ (s,1),(s,2),(s,3),\dots,(s-1,s),(s,s) \end{array}$$

Then it is easily seen that the same summation, but represented by the pairs, (i,l), is expressed by the scheme

$$(23) \qquad \begin{array}{l} (1,1),(1,2),(1,3),\dots,(1,s-1),(1,s) \\ (2,2),(2,3),\dots,(2,s-1),(2,s) \\ \dots \\ (s-1,s-1),(s-1,s) \\ (s,s) \end{array}$$

Consider the double sum in (21) (the summation through now becomes

$$\sum_{l=1}^{s} (-1)^l \sum_{i=1}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \dots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\,dt$$
$$(24) \qquad = \sum_{i=1}^{s} (-1)^l \sum_{l=i}^{s} \binom{l}{i} y^\nu_{p_1 p_2 \dots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\,dt$$
$$= \sum_{i=1}^{s} (-1)^i y^\nu_{p_1 p_2 \dots p_i} \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_l}} \circ \chi \cdot t\,dt$$
$$+ \sum_{i=1}^{s} (-1)^l \sum_{l=i+1}^{s} \binom{l}{i} y^\nu_{p_1 p_2 \dots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\,dt.$$

Returning to (21) we get

$$
\begin{aligned}
\varepsilon_\sigma - E_\sigma(\mathscr{L}_\varepsilon) &= \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu} \circ \chi \cdot y^\nu \cdot t\,dt - y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot t\,dt \\
&- \sum_{l=1}^{s} (-1)^l y^\nu \cdot \int_0^1 d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_l}} \circ \chi \cdot t\,dt \\
&+ \sum_{i=1}^{s} \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1 p_2 \dots p_i}} \circ \chi \cdot y^\nu_{p_1 p_2 \dots p_i} \cdot t\,dt \\
&- \sum_{i=1}^{s} (-1)^i y^\nu_{p_1 p_2 \dots p_i} \cdot \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_l}} \circ \chi \cdot t\,dt \\
&- \sum_{i=1}^{s} \sum_{l=i+1}^{s} (-1)^l \binom{l}{i} y^\nu_{p_1 p_2 \dots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \circ \chi \cdot t\,dt \\
&= y^\nu \int_0^1 \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} - \sum_{l=1}^{s} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_l}} \right) \circ \chi \cdot tdt \\
&+ y^\nu_{p_1 p_2 \dots p_i} \sum_{i=1}^{s} \int_0^1 \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1 p_2 \dots p_i}} - (-1)^i \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_l}} \right. \\
&\left. - \sum_{l=i+1}^{s} (-1)^l \binom{l}{i} \cdot d_{p_{i+1}} d_{p_{i+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \dots p_i p_{i+1} p_{i+2} \dots p_l}} \right) \circ \chi \cdot tdt.
\end{aligned} \tag{25}
$$

This formula proves Theorem 10.

The functions $H_{\sigma\nu}^{q_1 q_2 \dots q_k}(\varepsilon)$ (16) are called the *Helmholtz expressions*, associated with the source form ε.

It will be instructive to write up the Helmholtz expressions for lower-order source forms.

Remark 14 The Helmholtz expressions for the source forms of order $s = 3$ with components ε_σ are

$$
\begin{aligned}
H_{\sigma\nu}^{ijk}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ijk}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ijk}}, \\
H_{\sigma\nu}^{ij}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} + 3 d_k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ijk}}, \\
H_{\sigma\nu}^{i}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{i}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{i}} - 2 d_j \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} + 3 d_j d_k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ijk}}, \\
H_{\sigma\nu}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + d_i \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{i}} - d_i d_j \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} + d_i d_j d_k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ijk}}.
\end{aligned} \tag{26}
$$

Remark 15 Theorem 10 describes the difference between the given source form and the Euler-Lagrange form of the Vainberg-Tonti Lagrangian; we see, in particular, that responsibility for the difference lies on the properties of the source form, and is characterized by the Helmholtz expressions.

Lemma 8 *Let* $\lambda = \mathscr{L}\omega_0$ *be a Lagrangian, and let* Θ_λ *be its principal Lepage equivalent. Then the Vainberg-Tonti Lagrangian of the Euler-Lagrange form* $E_\lambda = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$,

(27) $$\lambda_{E_\lambda} = IE_\lambda,$$

satisfies

(28) $$\lambda_{E_\lambda} = \lambda - hd(I\Theta_\lambda + \mu_0).$$

Proof Using the fibred homotopy operator I, we can express the principal Lepage equivalent Θ_λ of λ as $\Theta_\lambda = Id\Theta_\lambda + dI\Theta_\lambda + \Theta_0$. Then the horizontal component is

(29) $$\begin{aligned} h\Theta_\lambda &= hId\Theta_\lambda + hdI\Theta_\lambda + h\Theta_0 = hIp_1 d\Theta_\lambda + hd(I\Theta_\lambda + \mu_0) \\ &= IE_\lambda + hd(I\Theta_\lambda + \mu_0) \end{aligned}$$

for some $(n-1)$-form μ_0 on X such that $\Theta = d\mu_0$, where $\lambda = h\Theta_\lambda$ and IE_λ is the Vainberg-Tonti Lagrangian.

Note that, in particular, formula (28) shows that the Vainberg-Tonti Lagrangian differs from the given Lagrangian λ by the term $hd(I\Theta_\lambda + \mu_0)$ that belongs to the kernel of the Euler-Lagrange mapping. This demonstrates that the Euler-Lagrange forms of λ and the Vainberg-Tonti Lagrangian λ_{E_λ} coincide.

Remark 16 (Euler-Lagrange source forms) Using homotopies and properties of formal divergence expressions (Chapter 3), we can give an elementary proof of Lemma 8, based on direct calculations. Namely, we prove that the Vainberg-Tonti Lagrangian of a source form $\varepsilon = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$ which *is* the Euler-Lagrange form of a Lagrangian $\lambda = \mathscr{L}\omega_0$, is given by

(30) $$y^\sigma \int_0^1 E_\sigma(\mathscr{L}) \circ \chi \cdot dt = \mathscr{L} + d_i \Psi^i.$$

First note that for any family of functions g^i on V^s, the formal divergence $d_i g^i$ satisfies the integral homotopy formula

(31) $$\int_0^1 d_i g^i \circ \chi \cdot dt = d_i \int_0^1 g^i \circ \chi \cdot dt.$$

Indeed, we have

$$
(32)\qquad \begin{aligned} d_i(g^i\circ\chi) &= \frac{\partial(g^i\circ\chi)}{\partial x^i} + \sum_{l=0}^{s}\frac{\partial(g^i\circ\chi)}{\partial y^\sigma_{p_1p_2\dots p_l}} y^\sigma_{p_1p_2\dots p_l k} \\ &= \left(\frac{\partial g^i}{\partial x^i} + \sum_{l=0}^{s}\frac{\partial g^i}{\partial y^\sigma_{p_1p_2\dots p_l}} y^\sigma_{p_1p_2\dots p_l k}\right)\circ\chi, \end{aligned}
$$

and formula (31) arises by integration.

Consider the Euler-Lagrange expressions $E_\sigma(\mathscr{L})$ of a Lagrangian of order r expressed as $\lambda = \mathscr{L}\omega_0$,

$$
(33)\qquad \begin{aligned} E_\sigma(\mathscr{L}) &= \frac{\partial\mathscr{L}}{\partial y^\sigma} - \sum_{l=1}^{r}(-1)^{l-1} d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial\mathscr{L}}{\partial y^\sigma_{p_1p_2\dots p_l}} \\ &= \frac{\partial\mathscr{L}}{\partial y^\sigma} - d_{p_1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{p_1}} + d_{p_1}d_{p_2}\frac{\partial\mathscr{L}}{\partial y^\sigma_{p_1p_2}} - \dots + (-1)^r d_{p_1}d_{p_2}\dots d_{p_r}\frac{\partial\mathscr{L}}{\partial y^\sigma_{p_1p_2\dots p_r}}, \end{aligned}
$$

and set

$$
(34)\qquad \begin{aligned} \Phi^{i_1}_\sigma &= \frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1}} - d_{p_2}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1p_2}} + d_{p_2}d_{p_3}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1p_2p_3}} \\ &\quad - \dots + (-1)^{r-1} d_{p_2}d_{p_3}\dots d_{p_r}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1p_2p_3\dots p_r}}, \\ \Phi^{i_1i_2}_\sigma &= \frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2}} - d_{p_3}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2p_3}} - d_{p_3}d_{p_4}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2p_3p_4}} \\ &\quad - \dots + (-1)^{r-1} d_{p_3}d_{p_4}\dots d_{p_r}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2p_3p_4\dots p_r}}, \\ &\dots \\ \Phi^{i_1i_2\dots i_k}_\sigma &= \frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2\dots i_k}} - d_{p_{k+1}}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2\dots i_k p_{k+1}}} - d_{p_{k+1}}d_{p_{k+2}}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2\dots i_k p_{k+1}p_{k+2}}} \\ &\quad - \dots + (-1)^{r-1} d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_r}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2\dots i_k p_{k+1}p_{k+2}\dots p_r}}, \\ &\dots \\ \Phi^{i_1i_2\dots i_{r-1}}_\sigma &= \frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2\dots i_{r-1}}} - d_{p_r}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2\dots i_{r-1}p_r}}, \\ \Phi^{i_1i_2\dots i_r}_\sigma &= \frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1i_2\dots i_r}}. \end{aligned}
$$

It is immediately seen that these functions, entering the Euler-Lagrange expression $E_\sigma(\mathscr{L})$ (33), satisfy the recurrence formula

(35) $$\Phi_\sigma^{i_1 i_2 \dots i_k} = \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \dots i_k}} - d_{p_{k+1}} \Phi_\sigma^{i_1 i_2 \dots i_k p_{k+1}}.$$

Using properties of the homotopy χ,

(36) $$\frac{d\mathscr{L}\circ\chi}{dt} = \frac{\partial \mathscr{L}}{\partial y^\sigma}\circ\chi\cdot y^\sigma + \sum_{l=1}^{r}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \dots p_l}}\circ\chi\cdot y^\sigma_{p_1 p_2 \dots p_l k}.$$

Hence, denoting $\mathscr{L}_0(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r}) = \mathscr{L}(x^i, 0,0,0,\dots,0)$, we get for the Vainberg-Tonti Lagrangian

(37) $$\begin{aligned}
&y^\sigma \int_0^1 E_\sigma(\mathscr{L})\circ\chi\cdot dt \\
&= y^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma}\circ\chi\cdot dt - y^\sigma \int_0^1 d_i \Phi_\sigma^i \circ\chi\cdot dt \\
&= \int_0^1 \left(\frac{d\mathscr{L}\circ\chi}{dt} - \sum_{l=1}^{r}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \dots p_l}}\circ\chi\cdot y^\sigma_{p_1 p_2 \dots p_l k}\right) dt - y^\sigma \int_0^1 d_i \Phi_\sigma^i \circ\chi\cdot dt \\
&= \mathscr{L} - \mathscr{L}_0 - \sum_{l=1}^{r} y^\sigma_{p_1 p_2 \dots p_l}\int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \dots p_l}}\circ\chi\cdot dt - y^\sigma d_i \int_0^1 \Phi_\sigma^i \circ\chi\cdot dt \\
&= \mathscr{L} - \mathscr{L}_0 - \sum_{l=1}^{r} y^\sigma_{p_1 p_2 \dots p_l}\int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \dots p_l}}\circ\chi\cdot dt \\
&+ y^\sigma_i \int_0^1 \Phi_\sigma^i \circ\chi\cdot dt - d_i\left(y^\sigma \int_0^1 \Phi_\sigma^i \circ\chi\cdot dt\right) \\
&\approx \mathscr{L} - \mathscr{L}_0 + y^\sigma_i \int_0^1 \left(\Phi_\sigma^i - \frac{\partial \mathscr{L}}{\partial y^\sigma_i}\right)\circ\chi\cdot dt \\
&- \sum_{l=2}^{r} y^\sigma_{p_1 p_2 \dots p_l}\int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \dots p_l}}\circ\chi\cdot dt.
\end{aligned}$$

The symbol $\approx$, replacing the equality sign $=$, means that we have omitted a formal divergence expression, annihilating the Euler-Lagrange expressions of the Vainberg-Tonti Lagrangian.

In formula (37)

(38) $$\begin{aligned}
&y^\sigma_i \int_0^1 \left(\Phi_\sigma^i - \frac{\partial \mathscr{L}}{\partial y^\sigma_i}\right)\circ\chi\cdot dt - y^\sigma_{p_1 p_2}\int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2}}\circ\chi\cdot dt \\
&= -y^\sigma_i \int_0^1 d_p \Phi_\sigma^{ip}\circ\chi\cdot dt - y^\sigma_{p_1 p_2}\int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2}}\circ\chi\cdot dt
\end{aligned}$$

$$= -y_i^\sigma d_p \int_0^1 \Phi_\sigma^{ip} \circ \chi \cdot dt - y_{p_1 p_2}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \circ \chi \cdot dt$$

$$= -d_p \left(y_i^\sigma \int_0^1 \Phi_\sigma^{ip} \circ \chi \cdot dt \right) + y_{ip}^\sigma \int_0^1 \Phi_\sigma^{ip} \circ \chi \cdot dt - y_{p_1 p_2}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \circ \chi \cdot dt$$

$$\approx y_{p_1 p_2}^\sigma \int_0^1 \left(\Phi_\sigma^{p_1 p_2} - \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \right) \circ \chi \cdot dt$$

thus,

$$\begin{aligned} y^\sigma \int_0^1 E_\sigma(\mathscr{L}) \circ \chi \cdot dt &\approx \mathscr{L} - \mathscr{L}_0 + y_i^\sigma \int_0^1 \left(\Phi_\sigma^i - \frac{\partial \mathscr{L}}{\partial y_i^\sigma} \right) \circ \chi \cdot dt \\ &- y_{p_1 p_2}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \circ \chi \cdot dt - \sum_{l=3}^{r} y_{p_1 p_2 \cdots p_l}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2 \cdots p_l}^\sigma} \circ \chi \cdot dt \\ &\approx \mathscr{L} - \mathscr{L}_0 + y_{p_1 p_2}^\sigma \int_0^1 \left(\Phi_\sigma^{p_1 p_2} - \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \right) \circ \chi \cdot dt \\ &- \sum_{l=3}^{r} y_{p_1 p_2 \cdots p_l}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2 \cdots p_l}^\sigma} \circ \chi \cdot dt. \end{aligned} \tag{39}$$

Repeating these decompositions we finally get the terms

$$\begin{aligned} & y_{p_1 p_2 \cdots p_{r-1}}^\sigma \int_0^1 \left(\Phi_\sigma^{p_1 p_2 \cdots p_{r-1}} - \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2 \cdots p_{r-1}}^\sigma} \right) \circ \chi \cdot dt - y_{p_1 p_2 \cdots p_r}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2 \cdots p_r}^\sigma} \circ \chi \cdot dt \\ &= -y_{p_1 p_2 \cdots p_{r-1}}^\sigma d_{p_r} \int_0^1 \Phi_\sigma^{p_1 p_2 \cdots p_r} \circ \chi \cdot dt - y_{p_1 p_2 \cdots p_r}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2 \cdots p_r}^\sigma} \circ \chi \cdot dt \\ &= -d_{p_r} \left(y_{p_1 p_2 \cdots p_{r-1}}^\sigma \int_0^1 \Phi_\sigma^{p_1 p_2 \cdots p_r} \circ \chi \cdot dt \right) + y_{p_1 p_2 \cdots p_r}^\sigma \int_0^1 \Phi_\sigma^{p_1 p_2 \cdots p_r} \circ \chi \cdot dt \\ &- y_{p_1 p_2 \cdots p_r}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2 \cdots p_r}^\sigma} \circ \chi \cdot dt \\ &= -d_{p_r} \left(y_{p_1 p_2 \cdots p_{r-1}}^\sigma \int_0^1 \Phi_\sigma^{p_1 p_2 \cdots p_r} \circ \chi \cdot dt \right). \end{aligned} \tag{40}$$

Since $\mathscr{L}_0$ is always, as a function of x^i only, of the formal divergence type, this proves that

$$y^\sigma \int_0^1 E_\sigma(\mathscr{L}) \circ \chi \cdot dt \approx \mathscr{L}, \tag{41}$$

proving formula (30).

4.10 The inverse problem of the calculus of variations

Our objective in this section is to study the *image* of the Euler-Lagrange mapping $\Omega^r_{n,X}W \ni \lambda \to E_\lambda \in \Omega^r_{n+1,Y}W$, considered as a subset of the set of source forms $\varepsilon \in \Omega^s_{n+1,Y}W$ (Section 4.9). The problem is to find a criterion for a source form to belong to the subset of the Euler-Lagrange forms.

First we show that the image of the Euler-Lagrange mapping is closed under the Lie derivative with respect to projectable vector fields.

Theorem 11 (Invariance of the image) *Let* $\lambda \in \Omega^r_{n,X}W$. *Then for any* π *-projectable vector field* Ξ *on W the Lie derivative* $\partial_{J^r\Xi}\lambda$ *belongs to the module* $\Omega^r_{n,X}W$ *and*

(1) $$\partial_{J^{2r}\Xi}E_\lambda = E_{\partial_{J^r\Xi}\lambda}.$$

Proof Since $\lambda \in \Omega^r_{n,X}W$, then $\partial_{J^r\Xi}\lambda \in \Omega^r_{n,X}W$. If ρ_λ is a Lepage equivalent of λ, and $\rho_{\partial_{J^r\Xi}\lambda}$ is a Lepage equivalent of the Lagrangian $\partial_{J^r\Xi}\lambda$, both defined on the set W^s, then, with the notation of Section 4.3, Theorem 3, $\rho_\lambda = \Theta_\lambda + d\eta + \mu$, $\rho_{\partial_{J^r\Xi}\lambda} = \Theta_{\partial_{J^r\Xi}\lambda} + d\eta' + \mu'$, and

(2) $$\partial_{J^s\Xi}\rho_\lambda = \partial_{J^s\Xi}\Theta_\lambda + d\partial_{J^s\Xi}\eta + \partial_{J^s\Xi}\mu.$$

The form $\partial_{J^s\Xi}\rho_\lambda$ has the horizontal component $h\partial_{J^s\Xi}\rho_\lambda = \partial_{J^{s+1}\Xi}h\rho_\lambda = \partial_{J^r\Xi}\lambda$, and is a Lepage form, because $p_1 d\partial_{J^s\Xi}\rho_\lambda = p_1 d\partial_{J^s\Xi}\Theta_\lambda = p_1\partial_{J^s\Xi}d\Theta_\lambda$ and the Lie derivative $\partial_{J^s\Xi}$ preserves contact forms (Section 2.5, Theorem 9). Thus, the forms $\rho_{\partial_{J^r\Xi}\lambda}$ and $\partial_{J^s\Xi}\rho_\lambda$ are both Lepage forms, and have the same Lagrangians. Consequently, their Euler-Lagrange forms agree, $\partial_{J^{2r}\Xi}E_\lambda = E_{\partial_{J^r\Xi}\lambda}$.

Rephrasing formula (1) we see that the Lie derivative of an Euler-Lagrange form by a vector field $J^{2r}\Xi$, where Ξ is a π-projectable vector field, permutes the set of Euler-Lagrange forms; the corresponding Lagrangians are also related by the Lie derivative operation.

Consider a source form $\varepsilon \in \Omega^s_{n+1,Y}W$. We say that ε is *variational*, if

(3) $$\varepsilon = E_\lambda$$

for some Lagrangian $\lambda \in \Omega^r_{n,X}W$. ε is said to be *locally variational*, if there exists an atlas on *Y*, consisting of fibred charts, such that for each chart (V,ψ), $\psi = (x^i, y^\sigma)$, from this atlas, the restriction of ε to V^s is variational.

The *inverse problem* of the calculus of variations, or the *variationality problem* for source forms, consists in finding conditions under which there exists a Lagrangian λ, satisfying equation (3); if these conditions are satisfied, then the problem is to find *all* Lagrangians for the source form ε. The *local inverse problem*, or *local variationality problem*, for a source form ε

consists in finding existence (integrability) conditions and solutions $\mathscr{L}$ of the system of partial differential equations

$$(4) \qquad \varepsilon_\sigma = \frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{l=1}^{r} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \dots p_l}}$$

with given functions $\varepsilon_\sigma = \varepsilon_\sigma(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_s})$ on the left-hand side (cf. Section 4.4, Theorem 4).

Let *r* be a fixed positive integer. We shall characterize the subspace of the vector space of source forms, which is in general larger than the image of the Euler-Lagrange mapping, namely, the subspace of *locally variational forms* (Krupka [K11]). Our next theorem states the relationship between the exterior derivative operator and the concept of variationality. It also indicates the meaning of *Lepage forms* for the inverse problem.

Theorem 12 (Local variationality od source forms) *Let* $\varepsilon \in \Omega^s_{n+1,Y} W$ *be a source form. The following two conditions are equivalent:*

(a) ε *is locally variational.*

(b) *For every point* $y \in W$ *there exist an integer r, a fibre chart* (V, ψ), $\psi = (x^i, y^\sigma)$, *at y and a form* $F \in \Omega^r_{n+1} V$ *of order of contactness* 2 *such that on* V^r

$$(5) \qquad d(\varepsilon + F) = 0.$$

Proof 1. Suppose that ε is locally variational, and choose a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, such that ε is variational on *V*; then $\varepsilon = E_\lambda$ for some Lagrangian $\lambda \in \Omega^r_{n,X} V$. Let Θ_λ denote the principal Lepage equivalent of λ, and set $F = p_2 d\Theta_\lambda$. Then $d(\varepsilon + F) = dd\Theta_\lambda = 0$.

2. Conversely, if for some fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, condition $d(\varepsilon + F) = 0$ holds on V^s, then $\varepsilon + F = d\rho$ for some ρ. ρ is obviously a Lepage form, hence $\varepsilon = p_1 d\rho$, so ε is a locally variational form whose Lagrangian is $h\rho$.

Remark 17 Theorem 12 indicates possible *geometric interpretation* of the exterior derivative $d\varepsilon$. Namely, formula (5) says that the variationality condition means that the *class* of $d\varepsilon$ modulo $(n+2)$-forms whose order of contactness is greater than 1, vanishes if and only if ε is locally variational. Developing this point of view to *q*-forms of *any* degree *q* leads to an idea to characterize the Euler-Lagrange mapping as a morphism in a sutable sheaf sequence of classes of forms (a “variational sequence”).

Properties of the form *F* in Theorem 1 can be further specified. Namely, for a given Lagrangian λ of order *r*, *F* can be determined from the exterior derivative of the principal Lepage equivalent Θ_λ (Section 4.5, (8)), and is $\pi^{2r-1,s-1}$-horizontal.

The following lemma is needed in the proof of another theorem on the local inverse of the calculus of variations.

Lemma 9 *Let U be an open set in $\mathbf{R}^n$ such that for each point $x_0 = (x_0^1, x_0^2, \dots, x_0^n)$ the segment $\{(tx_0^1, tx_0^2, \dots, tx_0^n) \mid t \in [0,1]\}$ belongs to U. Let $f: U \to \mathbf{R}$ be a function such that*

(6) $$\int_0^1 F(tx_0^1, tx_0^2, \dots, tx_0^n)\, dt = 0$$

for all points $(x_0^1, x_0^2, \dots, x_0^n) \in U$. Then $F = 0$.

Proof If (6) is true, then for any $s \in [0,1]$, $(sx_0^1, sx_0^2, \dots, sx_0^n) \in U$, thus,

(7) $$\int_0^1 F(tsx_0^1, tsx_0^2, \dots, tsx_0^n)\, dt = 0.$$

Differentiating with respect to s

(8) $$\int_0^1 \left(\frac{\partial F}{\partial x^k}\right)_{tsx_0} tsx_0^k\, dt = 0,$$

so at $s = 1$

(9) $$\int_0^1 \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t\, dt = 0.$$

On the other hand,

(10) $$\begin{aligned} &\frac{d}{dt}(tF(tx_0^1, tx_0^2, \dots, tx_0^n)) \\ &= F(tx_0^1, tx_0^2, \dots, tx_0^n) + t\frac{d}{dt}F(tx_0^1, tx_0^2, \dots, tx_0^n) \\ &= F(tx_0^1, tx_0^2, \dots, tx_0^n) + \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t. \end{aligned}$$

Integrating we have

(11) $$\begin{aligned} &F(x_0^1, x_0^2, \dots, x_0^n) \\ &= \int_0^1 F(tx_0^1, tx_0^2, \dots, tx_0^n)\, dt + \int_0^1 \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t\, dt \\ &= 0. \end{aligned}$$

Consider now the local inverse problem of the calculus of variations. We wish to find integrability conditions for the system of partial differential equations (4) and describe all solutions $\mathscr{L}$ of this system in an explicit form. To characterize locally variational forms, we need the *Helmholtz expressions* $H_{\sigma}{}^{q_1 q_2 \dots q_k}_{\nu}(\varepsilon)$ (Section 4.9, (16) and Remark 14)). Recall that

$$H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)=\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\dots q_k}}-(-1)^k\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_k}}$$
(12)
$$-\sum_{l=k+1}^{s}(-1)^l\binom{l}{k}d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}},$$

where $k=0,1,2,\dots,s$, and s is the order or ε .

Theorem 13 *Let V be an open star-shaped set in the Euclidean space* $\mathbf{R}^m$ *, and let* $\varepsilon_\sigma:V^s\to\mathbf{R}$ *be differentiable functions. The following two conditions are equivalent:*

(a) *Equation*

(13) $$\varepsilon_\sigma=\frac{\partial\mathscr{L}}{\partial y^\sigma}+\sum_{l=1}^{s}(-1)^l d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial\mathscr{L}}{\partial y^\sigma_{p_1p_2\dots p_l}}$$

has a solution $\mathscr{L}:V^s\to\mathbf{R}$.

(b) *For all* $k=0,1,2,\dots,s$ *, the function* ε_σ *satisfy*

(14) $$H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)=0$$

Proof 1. Suppose that the system (13) has a solution $\mathscr{L}$, defined on the set V^r. Then ε is the Euler-Lagrange form $E_\sigma(\mathscr{L})\omega^\sigma\wedge\omega_0$ of the Lagrangian $\lambda=\mathscr{L}\omega_0$; we may suppose without loss of generality that the Helmholtz expressions (12) are of order $s=2r$. Since the Lagrangian λ and the Vainberg-Tonti Lagrangian have the same Euler-Lagrange form (Section 4.9, Lemma 8), the Helmholtz expressions satisfy

(15) $$\int_0^1\sum_{k=0}^{2r}(y^\nu_{q_1q_2\dots q_k}H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon))\circ\chi\cdot dt=0$$

(Section 4.9, Theorem 10) hence, from Lemma 9,

(16) $$\sum_{k=0}^{2r}y^\nu_{q_1q_2\dots q_k}H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)=0.$$

Since by hypothesis ε is variational, that is, $\varepsilon=E_\lambda$ for some Lagrangian λ , then for any π -projectable vector field Ξ , $\partial_{J^{2r}\Xi}\varepsilon=\partial_{J^{2r}\Xi}E_\lambda=E_{\partial_{J^{2r}\Xi}\lambda}$ (Theorem 11) hence the form $\partial_{J^{2r}\Xi}\varepsilon$ is also variational. Thus, the Helmholtz expressions satisfy for all projectable vector fields Ξ ,

(17) $$\sum_{k=0}^{2r}y^\nu_{q_1q_2\dots q_k}H_{\sigma\nu}^{q_1q_2\dots q_k}(\partial_{J^{2r}\Xi}\varepsilon)=0$$

We shall show that this condition implies $H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)=0$.

Consider condition (17) for different choices of the vector field Ξ. It is sufficient to consider π-*vertical* vector fields, whose components do not depend on y^τ, that is,

(18) $$\Xi = \Xi^\sigma \frac{\partial}{\partial y^\sigma},$$

where $\Xi^\sigma = \Xi^\sigma(x^k)$. Then the components of the *r*-jet prolongation $J^r\Xi$ are

(19) $$\Xi^\sigma_{j_1 j_2 \dots j_k} = \frac{\partial^k \Xi^\sigma}{\partial x^{j_1} \partial x^{j_2} \dots \partial x^{j_k}}.$$

Writing $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ and using properties of the vector field Ξ, the Lie derivative $\partial_{J^{2r}\Xi}\varepsilon$, standing in (17), is given by

(20) $$\partial_{J^{2r}\Xi}\varepsilon = \partial_{J^{2s}\Xi}\varepsilon_\sigma \cdot \omega^\sigma \wedge \omega_0 = \sum_{k=0}^{2r} \frac{\partial \varepsilon_\sigma}{\partial y^\kappa_{j_1 j_2 \dots j_k}} \Xi^\kappa_{j_1 j_2 \dots j_k} \cdot \omega^\sigma \wedge \omega_0.$$

We denote

(21) $$\varepsilon' = \partial_{J^{2r}\Xi}\varepsilon, \quad \varepsilon'_\sigma = \sum_{k=0}^{2r} \frac{\partial \varepsilon_\sigma}{\partial y^\kappa_{j_1 j_2 \dots j_k}} \Xi^\kappa_{j_1 j_2 \dots j_k}.$$

Choose the vector field Ξ in the form

(22) $$\Xi = \frac{\partial}{\partial y^\tau},$$

where τ is any fixed integer. In components,

(23) $$\Xi^\sigma = \begin{cases} 1, & \sigma = \tau, \\ 0, & \sigma \neq \tau. \end{cases}$$

Then the *r*-jet prolongation $J^r\Xi$ has the components $\Xi^\sigma_{j_1 j_2 \dots j_r} = 0$, and the expression

(24) $$J^r\Xi = \frac{\partial}{\partial y^\tau}.$$

The Lie derivative (20) yields

(25) $$\varepsilon' = \frac{\partial \varepsilon_\sigma}{\partial y^\kappa} \Xi^\kappa \omega^\sigma \wedge \omega_0 = \frac{\partial \varepsilon_\sigma}{\partial y^\tau} \omega^\sigma \wedge \omega_0.$$

Thus, for the vector field (22),

(26) $$\varepsilon'_\sigma = \frac{\partial \varepsilon_\sigma}{\partial y^\tau}.$$

The Helmholtz expression $H_{\sigma}{}^{q_1q_2\ldots q_k}_{\nu}(\varepsilon')$ for the source form ε' can be written as

(27) $$\begin{aligned} H_{\sigma\nu}^{q_1q_2\ldots q_k}(\varepsilon') &= \frac{\partial}{\partial y^\nu_{q_1q_2\ldots q_k}}\frac{\partial \varepsilon_\sigma}{\partial y^\tau} - (-1)^k \frac{\partial}{\partial y^\sigma_{q_1q_2\ldots q_k}}\frac{\partial \varepsilon_\nu}{\partial y^\tau} \\ &\quad - \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial}{\partial y^\sigma_{q_1q_2\ldots q_k p_{k+1} p_{k+2}\ldots p_l}}\frac{\partial \varepsilon_\nu}{\partial y^\tau} \\ &= \frac{\partial H_{\sigma\nu}^{q_1q_2\ldots q_k}(\varepsilon)}{\partial y^\tau}, \end{aligned}$$

because the differential operators $\partial/\partial y^\tau$ and d_k commute. Condition (17) now implies

(28) $$\begin{aligned} \sum_{k=0}^{2r} y^\nu_{q_1q_2\ldots q_k} H_{\sigma\nu}^{q_1q_2\ldots q_k}(\partial_{J^{2r}\Xi}\varepsilon) &= \sum_{k=0}^{2r} y^\nu_{q_1q_2\ldots q_k} \frac{\partial H_{\sigma\nu}^{q_1q_2\ldots q_k}(\varepsilon)}{\partial y^\tau} \\ &= \sum_{k=0}^{2r} \frac{\partial (y^\nu_{q_1q_2\ldots q_k} H_{\sigma\nu}^{q_1q_2\ldots q_k}(\varepsilon))}{\partial y^\tau} - H_{\sigma\tau}(\varepsilon) \\ &= -H_{\sigma\tau}(\varepsilon) = 0. \end{aligned}$$

Consequently, (17) reduces to

(29) $$\sum_{k=1}^{2r} y^\nu_{q_1q_2\ldots q_k} H_{\sigma\nu}^{q_1q_2\ldots q_k}(\varepsilon) = 0.$$

Then by Theorem 11

(30) $$\sum_{k=1}^{2r} y^\nu_{q_1q_2\ldots q_k} H_{\sigma\nu}^{q_1q_2\ldots q_k}(\partial_{J^{2r}\Xi}\varepsilon) = 0.$$

Now consider equation (17) for the vector field

(31) $$\Xi = x^i \frac{\partial}{\partial y^\tau},$$

where i and τ are fixed integers. In components,

(32) $$\Xi^\sigma = \begin{cases} x^i, & \sigma = \tau, \\ 0, & \sigma \neq \tau. \end{cases}$$

Then the r-jet prolongation $J^r\Xi$ has the components

$$\Xi^\sigma_j = d_j\Xi^\sigma = \begin{cases} 1, & \sigma=\tau, j=i, \\ 0, & \sigma=\tau, j\neq i, \\ 0, & \sigma\neq\tau, \end{cases} \quad \Xi^\sigma_{j_1j_2\dots j_k}=0, \quad k\geq 2, \tag{33}$$

hence

$$J^r\Xi = x^i\frac{\partial}{\partial y^\tau}+\frac{\partial}{\partial y^\tau_i}. \tag{34}$$

The Lie derivative (20) yields

$$\varepsilon' = \left(\frac{\partial\varepsilon_\sigma}{\partial y^\kappa}\Xi^\kappa+\frac{\partial\varepsilon_\sigma}{\partial y^\kappa_j}\Xi^\kappa_{j_k}\right)\omega^\sigma\wedge\omega_0 = \left(\frac{\partial\varepsilon_\sigma}{\partial y^\tau}x^i+\frac{\partial\varepsilon_\sigma}{\partial y^\tau_i}\right)\omega^\sigma\wedge\omega_0. \tag{35}$$

Consequently, using the vector field (31),

$$\varepsilon'_\sigma = \frac{\partial\varepsilon_\sigma}{\partial y^\tau}x^i+\frac{\partial\varepsilon_\sigma}{\partial y^\tau_i}. \tag{36}$$

The Helmholtz expressions for ε'_σ become

$$\begin{aligned} &H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon') \\ &= \frac{\partial}{\partial y^\nu_{q_1q_2\dots q_k}}\left(\frac{\partial\varepsilon_\sigma}{\partial y^\tau}x^i+\frac{\partial\varepsilon_\sigma}{\partial y^\tau_i}\right)-(-1)^k\frac{\partial}{\partial y^\sigma_{q_1q_2\dots q_k}}\left(\frac{\partial\varepsilon_\nu}{\partial y^\tau}x^i+\frac{\partial\varepsilon_\nu}{\partial y^\tau_i}\right) \\ &-\sum_{l=k+1}^{2r}(-1)^l\binom{l}{k}d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_l}\frac{\partial}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}\left(\frac{\partial\varepsilon_\nu}{\partial y^\tau}x^i+\frac{\partial\varepsilon_\nu}{\partial y^\tau_i}\right) \\ &= \frac{\partial}{\partial y^\tau}\left(x^i\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\dots q_k}}-(-1)^k\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_k}}\right)\right. \\ &\left.-\sum_{l=k+1}^{2r}(-1)^l\binom{l}{k}d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_l}\left(x^i\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}\right)\right) \\ &+\frac{\partial}{\partial y^\nu_{q_1q_2\dots q_k}}\frac{\partial\varepsilon_\sigma}{\partial y^\tau_i}-(-1)^k\frac{\partial}{\partial y^\sigma_{q_1q_2\dots q_k}}\frac{\partial\varepsilon_\nu}{\partial y^\tau_i} \\ &-\sum_{l=k+1}^{2r}(-1)^l\binom{l}{k}d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_l}\frac{\partial}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}\frac{\partial\varepsilon_\nu}{\partial y^\tau_i}. \end{aligned} \tag{37}$$

In this expression

(38)
$$
\begin{aligned}
&d_{p_{k+1}} d_{p_{k+2}} \dots d_{p_l} \left(x^i \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \right) \\
&= d_{p_{k+2}} d_{p_{k+3}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k i p_{k+2} p_{k+3} \dots p_l}} \\
&+ d_{p_{k+2}} d_{p_{k+3}} \dots d_{p_l} \left(x^i d_{p_{k+1}} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \right).
\end{aligned}
$$

Note that for any function f, the formal derivative satisfies

(39)
$$
d_p \frac{\partial f}{\partial y^\tau_i} = \frac{\partial d_p f}{\partial y^\tau_i} - \frac{\partial f}{\partial y^\tau} \delta^i_p .
$$

Applying this rule we find

(40)
$$
\begin{aligned}
&d_{p_{k+1}} d_{p_{k+2}} \dots d_{p_l} \frac{\partial}{\partial y^\tau_i} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \\
&= \dots = \frac{\partial}{\partial y^\tau_i} d_{p_{k+3}} d_{p_{k+4}} \dots d_{p_l} d_{p_{k+2}} d_{p_{k+1}} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \\
&- (l-k) \frac{\partial}{\partial y^\tau} d_{p_{k+2}} d_{p_{k+3}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k i p_{k+2} p_{k+3} \dots p_l}} .
\end{aligned}
$$

Returning to (37)

(41)
$$
\begin{aligned}
&H^{q_1 q_2 \dots q_k}_{\sigma\nu}(\varepsilon') = \frac{\partial}{\partial y^\tau} \Bigg(x^i \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{q_1 q_2 \dots q_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k}} \right) \\
&- \Bigg(\sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} (l-k) d_{p_{k+2}} d_{p_{k+3}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k i p_{k+2} p_{k+3} \dots p_l}} \\
&+ x^i d_{p_{k+1}} d_{p_{k+2}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \Bigg) \Bigg) \\
&+ \frac{\partial}{\partial y^\tau_i} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{q_1 q_2 \dots q_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k}} \right) \\
&- \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} \Bigg(\frac{\partial}{\partial y^\tau_i} d_{p_{k+1}} d_{p_{k+2}} d_{p_{k+3}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k p_{k+1} p_{k+2} \dots p_l}} \\
&- (l-k) \frac{\partial}{\partial y^\tau} d_{p_{k+2}} d_{p_{k+3}} \dots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \dots q_k i p_{k+2} p_{k+3} \dots p_l}} \Bigg).
\end{aligned}
$$

Therefore

$$(42)\qquad H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon') = x^i \frac{\partial H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)}{\partial y^\tau} + \frac{\partial H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)}{\partial y_i^\tau}.$$

Now (17) is expressed as

$$(43)\qquad \begin{aligned} &\sum_{k=1}^{2r} y^\nu_{q_1q_2\dots q_k}\left(x^i \frac{\partial H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)}{\partial y^\tau} + \frac{\partial H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)}{\partial y_i^\tau} \right) \\ &\quad = \sum_{k=1}^{2r} y^\nu_{q_1q_2\dots q_k} x^i \frac{\partial H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)}{\partial y^\tau} + \sum_{k=1}^{2r} y^\nu_{q_1q_2\dots q_k} \frac{\partial H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)}{\partial y_i^\tau} \\ &\quad = x^i \frac{\partial}{\partial y^\tau} \sum_{k=1}^{2r} y^\nu_{q_1q_2\dots q_k} H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon) \\ &\quad + \sum_{k=1}^{2r} \frac{\partial}{\partial y_i^\tau}(y^\nu_{q_1q_2\dots q_k} H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)) - H_{\sigma\tau}^{\ i}(\varepsilon) = -H_{\sigma\tau}^{\ i}(\varepsilon) = 0. \end{aligned}$$

The proof can be completed by induction. To this purpose one should assume that $H_{\sigma\nu} = 0$, $H_{\sigma\nu}^{q_1} = 0$, $H_{\sigma\nu}^{q_1q_2} = 0$, $\dots$, $H_{\sigma\nu}^{q_1q_2\dots q_p} = 0$ for some p (induction hypothesis). Then conditions (29) and (30) are replaced with

$$(44)\qquad \sum_{k=p+1}^{2r} y^\nu_{q_1q_2\dots q_k} H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon) = 0$$

and

$$(45)\qquad \sum_{k=p+1}^{2r} y^\nu_{q_1q_2\dots q_k} H_{\sigma\nu}^{q_1q_2\dots q_k}(\partial_{J^{2r}\Xi}\varepsilon) = 0,$$

where the vector fields Ξ are of the form

$$(46)\qquad \Xi^\sigma = \begin{cases} x^{k_1}x^{k_2}{}_{\dots}x^{k_p}, & \sigma = \tau, \\ 0, & \sigma \neq \tau. \end{cases}$$

2. We prove that (b) implies (a). Suppose that a system of functions ε_σ satisfies conditions (14) and denote by $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ the corresponding source form. Then the Euler-Lagrange expressions of the Vainberg-Tonti Lagrangian $\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0$,

$$(47)\qquad E_\sigma(\mathscr{L}_\varepsilon) = \varepsilon_\sigma - \sum_{k=0}^{s} \int (y^\nu_{q_1q_2\dots q_k} H_{\sigma\nu}^{q_1q_2\dots q_k}(\varepsilon)) \circ \chi \cdot dt,$$

reduce to ε_σ (Section 4.9, Theorem 10). Thus $\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0$.

Remark 18 One can easily prove condition $H_{\sigma\nu}(\varepsilon)=0$ (48) in Theorem 13 by means of the *integrability criterion* for formal divergence equations (Section 3.2, Theorem 1). Consider the inverse problem equation

(49)
$$\begin{aligned}\varepsilon_\sigma &= \frac{\partial \mathscr{L}}{\partial y^\sigma} - d_{p_1}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1}} + d_{p_1}d_{p_2}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2}}\\ &\quad -\ldots+(-1)^{r-1}d_{p_1}d_{p_2}\ldots d_{p_{r-1}}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2\cdots p_{r-1}}} + (-1)^r d_{p_1}d_{p_2}\ldots d_{p_r}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2\cdots p_r}}\end{aligned}$$

and suppose it has a solution $\mathscr{L}$. Denoting

(50)
$$\begin{aligned}\Phi^{p_1}_\sigma &= \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1}} + d_{p_2}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2}} - \ldots + (-1)^{r-1}d_{p_2}d_{p_3}\ldots d_{p_{r-1}}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2\cdots p_{r-1}}}\\ &\quad + (-1)^r d_{p_2}d_{p_3}\ldots d_{p_r}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2\cdots p_r}},\end{aligned}$$

we get the formal divergence equation

(51)
$$\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma} = -d_{p_1}\Phi^{p_1}_\sigma .$$

Since by hypothesis there exists a solution, the integrability condition for this equation is satisfied, that is,

(52)
$$E_\tau\left(\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma}\right) = 0.$$

Explicitly, since the derivative d_i and the partial derivative $\partial/\partial y^\tau$ commute,

(53)
$$\begin{aligned}E_\tau\left(\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma}\right) &= \frac{\partial \varepsilon_\sigma}{\partial y^\tau} - d_{p_1}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1}} + d_{p_1}d_{p_2}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2}}\\ &\quad -\ldots+(-1)^{r-1}d_{p_1}d_{p_2}\ldots d_{p_{r-1}}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2\cdots p_{r-1}}}\\ &\quad +(-1)^r d_{p_1}d_{p_2}\ldots d_{p_r}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2\cdots p_r}} - \frac{\partial \varepsilon_\tau}{\partial y^\sigma} = 0.\end{aligned}$$

Comparing this formula with (12) we see we get exactly $H_{\sigma\,\tau}(\varepsilon)=0$.

We end this section with two remarks on the inverse problem for systems of differential equations.

Remark 19 (Variationality of differential equations) The concept of local variationality can be applied to the systems of partial differential equations. Fixing the functions ε_σ, we sometimes say, without aspiration to rigour, that the *system of partial differential equations*

(54) $$\varepsilon_\sigma(x^i, y^\tau, y^\tau_{j_1}, y^\tau_{j_1 j_2}, \dots, y^\tau_{j_1 j_2 \dots j_s}) = 0$$

is *variational*, it its left-hand sides coincide with the Euler-Lagrange equations of some Lagrangian. It is clear, however, that this concept of is not well defined; indeed, setting $\varepsilon'_\nu = \Phi^\nu_\sigma \varepsilon_\nu$ with any functions Φ^ν_σ such that $\det \Phi^\nu_\sigma \neq 0$, we get two equivalent systems $\varepsilon_\sigma = 0$ and $\varepsilon'_\nu = 0$, but it may happen that the first one is variational and the second one is not. If (5) is *not* variational and there exists Φ^ν_σ such that the equivalent system $\Phi^\nu_\sigma \varepsilon_\nu = 0$ *is* variational, we say that Φ^ν_σ are *variational integrators* for the system (5). It should be noted, however, that this terminology is also used in a different context of differential equations, expressed in a *contravariant* form.

Remark 20 (Sonin, Helmholtz, Douglas) The inverse problem of the calculus of variations was first considered in 1886 for *one* second-order ordinary differential equation by Sonin (see Sonin [So]; for this reference the author is indebted to V.D. Skarzhinski). He proved that *every* second order equation has a Lagrangian. It should be pointed out that in this paper the *variational multiplier*, in contemporary terminology, was used as a natural factor ensuring *covariance* of the considered equation. The variationality of *systems* of second-order ordinary differential equations, expressed in the covariant form, was studied by Helmholtz in 1887 and subsequently by many followers (Helmholtz [He]; see also Havas [H], where further references can be found). The systems of second-order ordinary differential equations, solved with respect to the second derivatives, were considered by Douglas in 1940 with the techniques of variational multipliers (see Douglas [Do], and e.g. Anderson and Thompson [AT], Bucataru [Bu], Crampin [Cr], Sarlet, Crampin and Martinez [SCM]).

4.11 Local variationality of second-order source forms

In this section we shall primarily be concerned with the *second-order* source forms and second-order systems of partial differential equations. The aim is to present a solution of the inverse problem of the calculus of variations for this class of source forms entirely by means of the theory of Lepage forms (Section 4.10, Theorem 12), and elementary integration theory of exterior differential systems.

Suppose we are given a second-order source form ε on $W^2 \subset J^2Y$, expressed in a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, as

(1) $$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0 .$$

Consider the system of partial differential equations

(2) $$\frac{\partial \mathscr{L}}{\partial y^\sigma} - d_p \frac{\partial \mathscr{L}}{\partial y_p^\sigma} + d_p d_q \frac{\partial \mathscr{L}}{\partial y_{pq}^\sigma} = \varepsilon_\sigma$$

for an unknown Lagrangian $\mathscr{L}$ of order 2. Clearly the left-hand sides of these equations are exactly the Euler-Lagrange expressions $E_\sigma(\mathscr{L})$ of the Lagrangian $\mathscr{L}$. The problem we consider is twofold: (a) to find the *variationality* (*integrability) conditions* for ε, ensuring existence of a solution $\mathscr{L}$, and (b) to find all solutions provided the integrability conditions are satisfied.

The following theorem, following from the theory of the Vainberg-Tonti Lagrangians, states that a second order variational source form always admits a *first order* Lagrangian; it seems that this extension of the well-known statement of the calculus of variations of simple integrals to the general multiple-integral problems in new. Note that the result restricts the class of locally variational forms to the source forms, depending on the second derivative variables *linearly*.

Theorem 14 *If a second-order source form* ε *, defined on* $W^2 \subset J^2Y$ *, is locally variational, then for every point* $y \in W$ *there exists a fibred chart* (V,ψ) *,* $\psi = (x^i, y^\sigma)$ *, at y and a first order Lagrangian* $\lambda_0 = \mathscr{L}_0 \omega_0$ *, defined on* V^1 *, such that*

(3) $$E_\sigma(\mathscr{L}_0) = \varepsilon_\sigma .$$

Proof If ε is variational, then by hypothesis the form $\varepsilon_\sigma \omega^\sigma \wedge \omega_i$ has a *second-order* Lagrangian $\lambda = \mathscr{L}\omega_0$ (the Vainberg-Tonti Lagrangian). The Euler-Lagrange form associated with λ is then given by

(4) $$E_\lambda = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0 ,$$

where

(5) $$\varepsilon_\sigma = E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - d_i \frac{\partial \mathscr{L}}{\partial y_i^\sigma} + d_i d_j \frac{\partial \mathscr{L}}{\partial y_{ij}^\sigma} .$$

One can find an explicit formula for the Euler-Lagrange expression (5); this expression does not depend on y_{ijk}^σ and y_{ijkl}^σ. Introducing (for this proof) the *cut formal derivative* of a function $f = f(x^i, y^\sigma, y_j^\sigma, y_{jk}^\sigma)$ as the function

(6) $$d'_j f = \frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial y^\sigma} y_j^\sigma + \frac{\partial f}{\partial y_i^\sigma} y_{ij}^\sigma$$

(see Section 3.1), we easily find

$$E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - d'_i \frac{\partial \mathscr{L}}{\partial y_i^\sigma} + d'_i d'_j \frac{\partial \mathscr{L}}{\partial y_{ij}^\sigma} + 2d'_i \frac{\partial^2 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{kl}^\nu} y_{klj}^\nu$$

(7)
$$+ \left(\frac{\partial^2 \mathscr{L}}{\partial y_j^\nu \partial y_{kl}^\sigma} - \frac{\partial^2 \mathscr{L}}{\partial y_j^\sigma \partial y_{kl}^\nu} \right) y_{klj}^\nu + \frac{\partial^3 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{kl}^\nu \partial y_{pq}^\tau} y_{pqi}^\tau y_{klj}^\nu$$

$$+ \frac{\partial^2 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{kl}^\nu} y_{klij}^\nu .$$

However, this function does not depend on y_{klj}^ν and y_{klij}^ν. Hence $\mathscr{L}$ must satisfy, among others,

(8)
$$\frac{\partial^2 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{kl}^\nu} = 0 \quad \mathrm{Sym}(klij).$$

But this condition implies

(9)
$$\frac{\partial^2 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{kl}^\sigma} + \frac{\partial^2 \mathscr{L}}{\partial y_{il}^\sigma \partial y_{jk}^\sigma} + \frac{\partial^2 \mathscr{L}}{\partial y_{ik}^\sigma \partial y_{jl}^\sigma} = 0,$$

then, for any two fixed indices i, j,

(10)
$$\frac{\partial^2 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{ii}^\sigma} = 0, \quad \frac{\partial^2 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{ii}^\sigma} = 0, \quad \frac{\partial^2 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{ij}^\sigma} + 2 \frac{\partial^2 \mathscr{L}}{\partial y_{ii}^\sigma \partial y_{jj}^\sigma} = 0,$$

hence, differentiating,

(11)
$$\frac{\partial^3 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{ij}^\sigma \partial y_{ij}^\sigma} = 0, \quad \frac{\partial^3 \mathscr{L}}{\partial y_{ij}^\sigma \partial y_{ii}^\sigma \partial y_{jj}^\sigma} = 0.$$

In particular, $\mathscr{L}$ must be a polynomial function of y_{jj}^σ, quadratic in each of the variables y_{jj}^σ. We can write

(12)
$$\mathscr{L} = \mathscr{L}_0 + \sum_{p \geq 1} \mathscr{L}_p ,$$

where $\mathscr{L}_0 = \mathscr{L}_0(x^k, y^\sigma, y_j^\sigma)$ is a function independent of y_{ij}^ν and $\mathscr{L}_p$ is a homogeneous polynomial of degree p,

(13)
$$\mathscr{L}_p = P_{\sigma_1 \ \sigma_2 \ \ldots \ \sigma_p}^{i_1 j_1 \ i_2 j_2 \ \ldots \ i_p j_p} y_{i_1 j_1}^{\sigma_1} y_{i_2 j_2}^{\sigma_2} \ldots y_{i_p j_p}^{\sigma_p} .$$

Substituting from this formula into (7),

$$E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}_0}{\partial y^\sigma} - d'_i \frac{\partial \mathscr{L}_0}{\partial y^\sigma_i}$$

(14)
$$+ \sum_{p \geq 1} \left(\frac{\partial \mathscr{L}_p}{\partial y^\sigma} - d'_i \frac{\partial \mathscr{L}_p}{\partial y^\sigma_i} + d'_i d'_j \frac{\partial \mathscr{L}_p}{\partial y^\sigma_{ij}} + 2d'_i \frac{\partial^2 \mathscr{L}_p}{\partial y^\sigma_{ij} \partial y^\nu_{kl}} y^\nu_{klj} \right.$$
$$\left. + \left(\frac{\partial^2 \mathscr{L}_p}{\partial y^\nu_j \partial y^\sigma_{kl}} - \frac{\partial^2 \mathscr{L}_p}{\partial y^\sigma_j \partial y^\nu_{kl}} \right) y^\nu_{klj} + \frac{\partial^3 \mathscr{L}_p}{\partial y^\sigma_{ij} \partial y^\nu_{kl} \partial y^\tau_{pq}} y^\tau_{pqi} y^\nu_{klj} + \frac{\partial^2 \mathscr{L}_p}{\partial y^\sigma_{ij} \partial y^\nu_{kl}} y^\nu_{klij} \right).$$

But the left-hand side does not depend on y^ν_{ijk} and y^ν_{ijkl}, so setting $y^\nu_{ijk} = 0$ and $y^\nu_{ijkl} = 0$ we get

(15)
$$E_\sigma(\mathscr{L}) = E_\sigma(\mathscr{L}_0) = \frac{\partial \mathscr{L}_0}{\partial y^\sigma} - d'_i \frac{\partial \mathscr{L}_0}{\partial y^\sigma_i}.$$

Replacing the cut formal derivative d'_i with d_i, this formula shows that the Euler-Lagrange expressions $E_\sigma(\mathscr{L}_0)$ of the first order Lagrangian $\lambda_0 = \mathscr{L}_0 \omega_0$ coincide with the components ε_σ of the source form ε. This proves Theorem 14.

Corollary 1 *Suppose that a second-order source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_i$ is variational. Then the components ε_σ depend linearly on the second derivative variables y^ν_{ij}, that is*

(16)
$$\varepsilon_\sigma = A_\sigma + B_{\sigma\nu}^{\ ij} y^\nu_{ij},$$

where the functions A_σ and $B_{\sigma\nu}^{\ ij}$ do not depend on the variables y^ν_{ij}.

Now we wish to find a criterion for a second-order source form ε (1) to be locally variational. As a main tool in the proof we use the concept of a Lepage form and the basic theorem on locally variational source forms (Section 4.10, Theorem 12).

Theorem 15 (Local variationality of source forms) *Let $\varepsilon \in \Omega^2_{n+1,Y} W$ be a source form. The following two conditions are equivalent:*

(a) *ε is locally variational.*

(b) *For every point $y \in W$ there exist an integer r and a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, at y, such that $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$, and the components ε_σ satisfy*

(17)
$$\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} = 0, \quad \frac{\partial \varepsilon_\sigma}{\partial y^\nu_j} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_j} - d_i \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} \right) = 0,$$
$$\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} - \frac{1}{2} d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_j} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_j} \right) = 0.$$

(c) *For every point* $y \in W$ *there exist an integer r, a fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *at y and a form* $F \in \Omega^r_{n+1}V$ *of order of contactness* ≤ 2 *such that on* V^r

(18) $$d(\varepsilon + F) = 0.$$

Proof 1. If (a) holds, then (b) is obtained by a direct calculation. Indeed, suppose that $\varepsilon_\sigma = E_\sigma(\mathscr{L})$ are the Euler-Lagrange expressions of a first order Lagrangian $\lambda = \mathscr{L}\omega_0$; then

(19) $$E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - \frac{\partial^2 \mathscr{L}}{\partial x^i \partial y^\sigma_i} - \frac{\partial^2 \mathscr{L}}{\partial y^\tau \partial y^\sigma_i} y^\tau_i - \frac{\partial^2 \mathscr{L}}{\partial y^\tau_i \partial y^\sigma_j} y^\tau_{ij}.$$

Differentiating we have

(20) $$\begin{aligned} \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} &= -\frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial y^\tau_p \partial y^\sigma_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\tau_q \partial y^\sigma_p} \right), \\ \frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} &= \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y^\nu_q} - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\nu_q \partial y^\sigma_s} - \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_q}, \\ \frac{\partial \varepsilon_\sigma}{\partial y^\nu} &= \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma} - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_s}, \end{aligned}$$

from which we get

(21) $$\begin{aligned} \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} &= -\frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial y^\nu_p \partial y^\sigma_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\nu_q \partial y^\sigma_p} \right) \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial y^\sigma_p \partial y^\nu_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\sigma_q \partial y^\nu_p} \right) = 0, \end{aligned}$$

and

(22) $$\begin{aligned} &\frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_q} - 2 d_p \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{qp}} \\ &= \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y^\nu_q} - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\nu_q \partial y^\sigma_s} - \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_q} \\ &\quad - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\sigma_q \partial y^\nu_s} - \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y^\nu_q} + d_p \left(\frac{\partial^2 \mathscr{L}}{\partial y^\sigma_p \partial y^\nu_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\sigma_q \partial y^\nu_p} \right) = 0, \end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + d_p \frac{\partial \varepsilon_\nu}{\partial y_p^\sigma} - d_p d_q \frac{\partial \varepsilon_\nu}{\partial y_{pq}^\sigma} \\
&= \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma} - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y_s^\sigma} - \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma} + d_s \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y_s^\nu} \\
&+ d_q \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y_q^\sigma} - d_q d_s \frac{\partial^2 \mathscr{L}}{\partial y_q^\sigma \partial y_s^\nu} - d_q \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y_q^\nu} \\
&+ \frac{1}{2} d_p d_q \left(\frac{\partial^2 \mathscr{L}}{\partial y_p^\sigma \partial y_q^\nu} + \frac{\partial^2 \mathscr{L}}{\partial y_q^\sigma \partial y_p^\nu} \right) = 0.
\end{aligned} \tag{23}$$

2. Suppose that the components $\varepsilon_\sigma = E_\sigma(\mathscr{L})$ of the Euler-Lagrange expressions of $\lambda = \mathscr{L}\omega_0$ satisfy condition (b). Setting

$$F = -\left(\frac{1}{4} \left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \omega^\nu + \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \omega_j^\nu \right) \wedge \omega^\sigma \wedge \omega_i, \tag{24}$$

we get by a straightforward calculation, using the canonical decomposition of forms into their horizontal and contact components and the identities $d\omega^\nu = -\omega_l^\nu \wedge dx^l$, $d\omega_j^\nu = -\omega_{jl}^\nu \wedge dx^l$, and $dx^l \wedge \omega_i = \delta_i^l \omega_0$,

$$\begin{aligned}
dF &= -\frac{1}{4} d\left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \wedge \omega^\nu \wedge \omega^\sigma \wedge \omega_i \\
&- \frac{1}{4} \left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) d\omega^\nu \wedge \omega^\sigma \wedge \omega_i \\
&- d \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \wedge \omega_j^\nu \wedge \omega^\sigma \wedge \omega_i - \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} d\omega_j^\nu \wedge \omega^\sigma \wedge \omega_i \\
&+ \left(\frac{1}{4} \left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \omega^\nu + \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \omega_j^\nu \right) \wedge d\omega^\sigma \wedge \omega_i \\
&= -\frac{1}{4} d_i \left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \omega^\nu \wedge \omega^\sigma \wedge \omega_0 \\
&+ \left(\frac{1}{2} \left(\frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} - \frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} \right) - d_j \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \right) \omega_i^\nu \wedge \omega^\sigma \wedge \omega_0 \\
&- \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \omega_{ij}^\nu \wedge \omega^\sigma \wedge \omega_0 - \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \omega_j^\nu \wedge \omega_i^\sigma \wedge \omega_0 \\
&- \frac{1}{4} pd \left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \wedge \omega^\nu \wedge \omega^\sigma \wedge \omega_i - pd \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \wedge \omega_j^\nu \wedge \omega^\sigma \wedge \omega_i.
\end{aligned} \tag{25}$$

Consequently, since

$$d\varepsilon = \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu} \omega^\nu + \frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} \omega_i^\nu + \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \omega_{ij}^\nu \right) \wedge \omega^\sigma \wedge \omega_0, \tag{26}$$

the exterior derivative $d(\varepsilon + F)$ is expressed as

$$\begin{aligned} d(\varepsilon + F) &= \frac{1}{4}\left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} - \frac{1}{2} d_i \left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \right) \omega^\nu \wedge \omega^\sigma \wedge \omega_0 \\ &+ \left(\frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) - d_j \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \right) \omega_i^\nu \wedge \omega^\sigma \wedge \omega_0 \\ &- \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_{ij}^\sigma} \right) \omega_j^\nu \wedge \omega_i^\sigma \wedge \omega_0 \\ &- \frac{1}{4} pd \left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \wedge \omega^\nu \wedge \omega^\sigma \wedge \omega_i - pd \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \wedge \omega_j^\nu \wedge \omega^\sigma \wedge \omega_i . \end{aligned} \tag{27}$$

Thus, by hypothesis (b),

$$\begin{aligned} d(\varepsilon + F) &= -\frac{1}{4} pd \left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \wedge \omega^\nu \wedge \omega^\sigma \wedge \omega_i \\ &- pd \frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} \wedge \omega_j^\nu \wedge \omega^\sigma \wedge \omega_i . \end{aligned} \tag{28}$$

2. Suppose that the functions ε_σ satisfy condition (b). Substituting from (17) to $d\varepsilon$ we have

$$\begin{aligned} d\varepsilon &= \left(\frac{1}{4} d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y_j^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_j^\sigma} \right) \omega^\nu + \frac{1}{2} d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_{ij}^\sigma} \right) \omega_i^\nu \right. \\ &\left. + \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \omega_i^\nu + \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_{ij}^\sigma} \right) \omega_{ij}^\nu \right) \wedge \omega^\sigma \wedge \omega_0 \\ &= \left(\frac{1}{4} d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y_j^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_j^\sigma} \right) \omega^\nu + \frac{1}{2} \left(d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_{ij}^\sigma} \right) + \frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma} \right) \omega_i^\nu \right. \\ &\left. + \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_{ij}^\sigma} \right) \omega_{ij}^\nu \right) \wedge \omega^\sigma \wedge \omega_0 . \end{aligned} \tag{29}$$

On the other hand, we can recognize in formula (29) some terms in the form of an exterior derivative. Observe that

(30)
$$\begin{aligned} p_2 d\left(\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge\omega^\sigma\wedge\omega_j\right) &= d_j\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge\omega^\sigma\wedge\omega_0\\ &+\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)d\omega^\nu_i\wedge\omega^\sigma\wedge\omega_j-\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge d\omega^\sigma\wedge\omega_j\\ &= d_j\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge\omega^\sigma\wedge\omega_0\\ &=\left(d_j\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i+\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_{ij}\right)\wedge\omega^\sigma\wedge\omega_0, \end{aligned}$$

and

(31)
$$\begin{aligned} &p_2 d\left(\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu\wedge\omega^\sigma\wedge\omega_i\right)\\ &=\left(d_i\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu+2\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu_i\right)\wedge\omega^\sigma\wedge\omega_0. \end{aligned}$$

Thus, $d\varepsilon$ is expressible as

(32)
$$\begin{aligned} d\varepsilon &= \frac{1}{4}p_2 d\left(\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu\wedge\omega^\sigma\wedge\omega_i\right)\\ &+\frac{1}{2}p_2 d\left(\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge\omega^\sigma\wedge\omega_j\right). \end{aligned}$$

Setting

(33)
$$F=-\frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu-\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\right)\wedge\omega^\sigma\wedge\omega_j$$

and $\rho=\varepsilon+F$ we get assertion (c).

3. To show that condition (c) implies (a) we can repeat the proof of Theorem 12 for source forms of order 2. Suppose that for some fibred chart (V,ψ), $\psi=(x^i,y^\sigma)$, on the fibred manifold *Y* condition $d(\varepsilon+F)=0$ holds on V^2. Integrating we get $\varepsilon+F=d\eta$ for some *n*-form η. But since $\varepsilon=p_1 d\eta$, the form η is a Lepage form, therefore, so ε must be a locally variational form whose Lagrangian is $h\eta$.

Remark 21 In the proof of Theorem 15 we have assigned to a second-order source form $\varepsilon=\varepsilon_\sigma\omega^\sigma\wedge\omega_0$ the form $\rho=\varepsilon+F$, defined by the requirement $d\rho=0$. The solution

$$\rho = \varepsilon_\sigma \omega^\sigma \wedge \omega_0 \tag{34}$$
$$-\frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma}\right)\omega^\nu - \left(\frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_{ij}^\sigma}\right)\omega_i^\nu\right) \wedge \omega^\sigma \wedge \omega_j$$

extends the source form by a form of order of contactness ≥ 2. This construction, involving the exterior derivative operator, is closely related to the variationality of the form ε, and can be considered as a motivation for possible generalizations of the geometric theory of Lepage differential *n*-forms to $(n+1)$-forms and differential forms of higher degree (cf. Krupka, Krupková and Saunders [KKS2]). This notable construction also indicates the possibility to interpret a source forms an interpretation as a *class* of forms modulo contact forms; this idea has been developed by the theory of variational sequences (Krupka [K19]).

Theorem 16 (First order Lepage forms) *Let $\rho \in \Omega_n^1 V$ be an n-form. The following two conditions are equivalent:*

(a) *$\rho \in \Omega_n^1 V$ is a Lepage form.*

(b) *There exists a first order Lagrangian $\lambda \in \Omega_{n,X}^1 V$, an n-form κ of order of contactness ≥ 2 and a contact $(n-1)$-form τ, such that*

$$\rho = \Theta_\lambda + \kappa + d\tau. \tag{35}$$

Proof 1. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibred chart on *Y*, and let ρ be a first order Lepage form, defined on the set V^1. Then the form $\varepsilon = p_1 d\rho$ is a *second-order* Euler-Lagrange form, defined on V^2, associated to the *second-order* Lagrangian $h\rho$ – the *horizontal component* of ρ. On the other hand, it follows from Theorem 14 that ε has a *first order* Lagrangian λ; denoting by Θ_λ the principal Lepage equivalent of λ, we have $\varepsilon = p_1 d\Theta_\lambda$ hence

$$p_1 d\rho = p_1 d\Theta_\lambda. \tag{36}$$

Consequently, $p_1 d(\rho - \Theta_\lambda) = 0$ and by the theorem on the kernel of the Euler-Lagrange mapping (Section 4.8, Theorem 9, (c)), there exists an $(n-1)$-form μ, defined on V^1, such that $h(\rho - \Theta_\lambda) = hd\mu$ hence

$$\rho - \Theta_\lambda = \eta + d\mu \tag{37}$$

for some contact form η such that $p_1 d\eta = 0$. Therefore, η satisfies two conditions

$$h\eta = 0, \quad p_1 d\eta = 0. \tag{38}$$

The first one implies that $\eta = \omega^\sigma \wedge \Phi_\sigma + d\omega^\sigma \wedge \Psi_\sigma$ for some forms Φ_σ and Ψ_σ (Section 2.3, Theorem 7, (b). We can also write

$$\eta = \omega^\sigma \wedge (\Phi_\sigma + d\Psi_\sigma) + d(\omega^\sigma \wedge \Psi_\sigma) \tag{39}$$

for some forms Φ_σ and Ψ_σ. Setting $\tau_\sigma = \Phi_\sigma + d\Psi_\sigma$, the second condition (38) implies

(40) $$p_1 d\eta = -\omega_l^\sigma \wedge dx^l \wedge h\tau_\sigma - \omega^\sigma \wedge hd\tau_\sigma = 0.$$

We want to show that this condition implies $h\tau_\sigma = 0$. Indeed, for any $\pi^{2,0}$-vertical vector field

(41) $$\Xi = \Xi_i^\sigma \frac{\partial}{\partial y_i^\sigma} + \Xi_{ij}^\sigma \frac{\partial}{\partial y_{ij}^\sigma}$$

condition (41) yields $\Xi_l^\sigma dx^l \wedge h\tau_\sigma = 0$. Writing $h\tau_\sigma = A_\sigma^i \omega_i$, this condition implies $\Xi_l^\sigma A_\sigma^i dx^l \wedge \omega_i = \Xi_l^\sigma A_\sigma^l \omega_0 = 0$ hence $A_\sigma^l = 0$. Thus $h\tau_\sigma = 0$. Substituting from this result to (40) and (38) we see that assertion (a) implies (b).

2. The converse is obvious.

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