5 Invariant variational structures

Let *X* be any manifold, *W* an open set in *X*, and let $\alpha : W \to X$ be a smooth mapping. A differential form η, defined on the set $\alpha(W)$ in *X*, is said to be *invariant* with respect to α, if the transformed form $\alpha^* \eta$ coincides with η, that is, if $\alpha^* \eta = \eta$ on the set $W \cap \alpha(W)$; in this case we also say that α is an *invariance transformation* of η. A vector field, whose local one-parameter group consists of invariance transformations of η, is called the *generator* of invariance transformations.

These definitions can naturally be extended to variational structures (Y,ρ) and to the integral variational functionals associated with them. Our objective in this section is to study invariance properties of the form ρ and other differential forms, associated with ρ, the Lagrangian λ, and the Euler-Lagrange form E_λ. The class of transformations we consider are automorphisms of fibred manifolds and their jet prolongations. This part of the variational theory represents a notable extension of the classical coordinate concepts and methods to topologically non-trivial fibred manifolds that cannot be covered by a single chart. The geometric coordinate-free structure of the infinitesimal first variation formula leads in several consequences, such as the geometric invariance criteria of the Lagrangians and the Euler-Lagrange forms, a global theorem on the conservation law equations, and the relationship between extremals and conservation laws. Resuming we can say that these results as a whole represent an extension of the classical Noether's theory to higher-order variational functionals on fibred manifolds (Noether [N]).

In this chapter we basically follow Trautman's formulation of the invariance theory based on the geometric understanding of the topic (Trautman [Tr1], [Tr2]). The concept of the *jet prolongation* of a vector field and its meaning for the geometric notion of a variation for invariance theory was discussed in Krupka [K6], [K1]. The fundamentals of the invariance theory for differential equations and the calculus of variations in Euclidean spaces developed along the classical lines can be found in Olver [O1]; however, in this work the Trautman's approach using geometric characteristics of the underlying transformations such as the the Lie derivatives, is not included. A complete treatment of the work of Noether on invariant variational principles is presented, also within the classical local framework, by Kosmann-Schwarzbach [K-S].

In this chapter we follow our previous notations. Throughout, *Y* is a fixed fibred manifold with base *X* and projection π. We set $\dim X = n$, $\dim Y = n + m$. $J^r Y$ denotes the *r-jet prolongation* of *Y*, and $\pi^{r,s}$ and π^r are the canonical jet projections. For any set $W \subset Y$ the set $(\pi^{r,0})^{-1}(W)$ is denoted by W^r. $\Omega_q^r W$ denotes the module of *q*-forms defined on W^r, $\Omega_{q,Y}^r W$ is the submodule of $\pi^{r,0}$-horizontal forms, and $\Omega^r W$ is the exterior algebra of differential forms on W^r. We use the *horizontalization morphism* of exterior algebras $h : \Omega^r W \to \Omega^{r+1} W$. The *r*-jet prolongation of a morphism α of the fibred manifold *Y* is denoted by $J^r \alpha$. Analogously, the *r*-jet prolongation of a π-projectable vector field is denoted by $J^r \Xi$.

5.1 Invariant differential forms

We present in this section some elementary remarks on the invariance of differential forms on smooth manifolds under diffeomorphisms. We prove two standard lemmas that are permanently used in the theory of invariant variational structures.

Let X be a smooth manifold, W an open set in X and $\alpha : W \to X$ a diffeomorphism. Let ρ be a p-form on X. We say that ρ is *invariant with respect to* α, if its pull-back $\alpha^* \rho$ coincides with α,

(1) $$\alpha^* \rho = \rho.$$

A diffeomorphism α, satisfying condition (1), is called the *invariance transformation* of ρ.

These definitions immediately transfer to vector fields. Let ξ be a vector field on X, α^ξ its flow, and α_t^ξ its local 1-parameter groups, defined by the condition $\alpha_t^\xi(x) = \alpha^\xi(t,x)$, where the points (t,x) belong to the domain of definition of α^ξ. We say that ξ is the *generator of invariance transformations* of ρ, if its local 1-parameter groups are invariance transformations of ρ, that is,

(2) $$(\alpha_t^\xi)^* \rho(x) = \rho(x)$$

for all points (t,x) from the domain of α^ξ.

Lemma 1 *For every point* (t,x) *from the domain of definition of the flow of the vector field* ξ,

(3) $$\frac{d}{dt}(\alpha_t^\xi)^* \rho(x) = ((\alpha_t^\xi)^* \partial_\xi \rho)(x).$$

Proof Let (t,x_0) be a point from the domain of α^ξ. Choose tangent vectors $\xi_1, \xi_2, \dots, \xi_p \in T_{x_0}X$ and consider the value of the form $(\alpha_t^\xi)^* \rho(x_0)$ on these tangent vectors. This gives rise to a real-valued function $t \to ((\alpha_t^\xi)^* \rho)(x_0)(\xi_1, \xi_2, \dots, \xi_p)$. Differentiating this function at a point t_0, we have

(4) $$\left(\frac{d}{dt}((\alpha_t^\xi)^* \rho)(x_0)(\xi_1, \xi_2, \dots, \xi_p) \right)_{t_0} = \left(\frac{d}{ds}((\alpha_{t_0+s}^\xi)^* \rho)(x_0)(\xi_1, \xi_2, \dots, \xi_p) \right)_0 .$$

But the flow satisfies the condition $\alpha_{t_0+s}^\xi = \alpha_s^\xi \circ \alpha_{t_0}^\xi$ so we have

$$\left(\frac{d}{dt}((\alpha_t^\xi)^*\rho)(x_0)(\xi_1,\xi_2,\ldots,\xi_p)\right)_{t_0}$$

$$=\left(\frac{d}{ds}((\alpha_{t_0}^\xi)^*(\alpha_s^\xi)^*\rho)(x_0)(\xi_1,\xi_2,\ldots,\xi_p)\right)_0$$

(5) $$=\left(\frac{d}{ds}((\alpha_s^\xi)^*\rho)(\alpha_{t_0}^\xi(x_0))(T\alpha_{t_0}^\xi\cdot\xi_1,T\alpha_{t_0}^\xi\cdot\xi_2,\ldots,T\alpha_{t_0}^\xi\cdot\xi_p)\right)_0$$

$$=\partial_\xi\rho(\alpha_{t_0}^\xi(x_0)(T\alpha_{t_0}^\xi\cdot\xi_1,T\alpha_{t_0}^\xi\cdot\xi_2,\ldots,T\alpha_{t_0}^\xi\cdot\xi_p)$$

$$=((\alpha_s^\xi)^*\partial_\xi\rho)(x_0)(\xi_1,\xi_2,\ldots,\xi_p).$$

This is formula (3).

Lemma 2 (Invariance lemma) *Let ξ be a vector field on X, and let ρ be a p-form on X. The following two conditions are equivalent:*

(a) *ξ generates invariance transformations of ρ.*

(b) *The Lie derivative of ρ by ξ vanishes,*

(6) $$\partial_\xi\rho=0.$$

Proof 1. If ξ generates invariance transformations of ρ, then we differentiate both sides of equation (2) with respect to *t* at $t=0$ and obtain formula (6).

2. If condition (6) is satisfied, then by Lemma 1,

(7) $$\frac{d}{dt}((\alpha_t^\xi)^*\rho)(x)=0$$

on the domain of the flow α^ξ. Thus, the curve $t\to((\alpha_t^\xi)^*\rho)(x)$ is independent of *t*, and since its domain is connected, its value is constant and must be equal to $((\alpha_0^\xi)^*\rho)(x)=\rho(x)$. This proves condition (2).

5.2 Invariant Lagrangians and conservation equations

Let *W* be an open set in *Y*, let λ be a Lagrangian of order *r* for *Y*, defined on $W^r\subset J^rY$. Consider an automorphism $\alpha:W\to Y$ of *Y*, and its *r*-jet prolongation $J^r\alpha:W^r\to J^rY$. We say that α is an *invariance transformation* of λ if $J^r\alpha^*\lambda=\lambda$. The *generator* of invariance transformations of λ is a π-projectable vector field on *Y* whose local one-parameter group consists of invariance transformations of λ.

In the following lemma we use fibred charts (V,ψ), $\psi=(x^i,y^\sigma)$, and our standard multi-index notation. Recall that the *contact* 1-forms ω_J^σ, locally generating the *contact ideal*, are the 1-forms, defined by the formula $\omega_J^\sigma=dy_J^\sigma-y_{Jj}^\sigma dx^j$ (Section 2.1, Theorem 1).

Lemma 3 *Suppose we have a vector field* Z *on* J^rY *. The following two conditions are equivalent:*

(a) *For every fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *on Y, every* σ *, and every multi-index J such that* $0 \le |J| \le r-1$, *the form* $\partial_{\mathrm{Z}}\omega_J^\sigma$ *is a contact form*.

(b) *There exists a* π *-projectable vector field* Ξ *such that* $\mathrm{Z} = J^r\Xi$ *.*

Proof Write $\omega_J^\tau = dy_J^\tau - y_{Jj}^\tau dx^j$ and

(1) $$\mathrm{Z} = \zeta^i \frac{\partial}{\partial x^i} + \mathrm{Z}_I^\sigma \frac{\partial}{\partial y_I^\sigma}.$$

Then

(2) $$\begin{aligned}\partial_{\mathrm{Z}}\omega_J^\tau &= i_{\mathrm{Z}} d\omega_J^\tau + d i_{\mathrm{Z}}\omega_J^\tau = -i_{\mathrm{Z}}(dy_{Jj}^\tau \wedge dx^j) + d i_{\mathrm{Z}}(dy_J^\tau - y_{Jl}^\tau dx^l)\\ &= -\mathrm{Z}_{Jj}^\tau dx^j + \zeta^j dy_{Jj}^\tau + d(\mathrm{Z}_J^\tau - y_{Jl}^\tau \zeta^l)\\ &= -\mathrm{Z}_{Jj}^\tau dx^j + \zeta^j dy_{Jj}^\tau + d\mathrm{Z}_J^\tau - \zeta^l dy_{Jl}^\tau - y_{Jl}^\tau d\zeta^l\\ &= -\mathrm{Z}_{Jj}^\tau dx^j + d\mathrm{Z}_J^\tau - y_{Jl}^\tau d\zeta^l\\ &= (-\mathrm{Z}_{Jj}^\tau + d_j \mathrm{Z}_J^\tau - y_{Jl}^\tau d_j\zeta^l)dx^j + \frac{\partial \mathrm{Z}_J^\tau}{\partial y_K^\lambda}\omega_K^\lambda,\end{aligned}$$

and our assertion follows from Section 1.7, Lemma 8.

Lemma 4 *Let* λ *be a Lagrangian of order r for Y.*

(a) *A* π *-projectable vector field* Ξ *on Y generates invariance transformations of* λ *if and only if*

(3) $$\partial_{J^r\Xi}\lambda = 0.$$

(b) *Generators of invariance transformations of* λ *constitute a subalgebra of the algebra of vector fields on* J^rY *.*

Proof (a) This is a trivial consequence of definitions.

(b) Any two generators satisfy $[J^r\Xi_1, J^r\Xi_2] = J^r[\Xi_1,\Xi_2]$ (Section 1.7, Lemma 11). Then, however,

(4) $$\partial_{J^r[\Xi_1,\Xi_2]}\lambda = \partial_{[J^r\Xi_1, J^r\Xi_2]}\lambda = \partial_{J^r\Xi_1}\partial_{J^r\Xi_2}\lambda - \partial_{J^r\Xi_2}\partial_{J^r\Xi_1}\lambda = 0.$$

We keep terminology used by Trautman [T1], [T2] and call equation (3), the *Noether equation*. This equation represents a relation between the Lagrangian λ and the generator Ξ of invariance transformation. Given λ, we can use the Noether equation to determine the generators Ξ. On the other hand, given a Lie algebra of π -projectable vector fields Ξ, one can use the corresponding Noether equations *to determine* invariant Lagrangians λ .

Theorem 1 *Suppose that a Lagrangian* λ *is invariant with respect to a* π *-projectable vector field* Ξ *. Then for any Lepage equivalent* ρ *of* λ

(5) $$hi_{J^r\Xi}d\rho + hdi_{J^r\Xi}\rho = 0,$$

or, which is the same,

(6) $$J^r\gamma * i_{J^r\Xi}d\rho + dJ^r\gamma * i_{J^r\Xi}\rho = 0$$

for every section γ *of* Y.

Proof From Section 4.6, Theorem 7,

(7) $$h\partial_{J^r\Xi}\rho = \partial_{J^{r+1}\Xi}h\rho = \partial_{J^{r+1}\Xi}\lambda = hi_{J^r\Xi}d\rho + hdi_{J^r\Xi}\rho$$

which implies (5).

Remark 1 According to Section 4.3, Theorem 3, condition (5) reduces locally to

(8) $$hi_{J^r\Xi}d\Theta_\lambda + hdi_{J^r\Xi}\Theta_\lambda = 0,$$

where Θ_λ is the principal Lepage equivalent of the Lagrangian form λ .

By a *conserved current* for a section $\gamma \in \Gamma_\Omega(\pi)$ we mean any $(n-1)$-form $\eta \in \Omega_n^r W$ such that

(9) $$dJ^s\gamma * \eta = 0.$$

We call formula (9) the *conservation law equation*; it is also called a *conservation law* for the section γ .

The following assertion says that *extremals* of invariant Lagrangians satisfy, in addition to the Euler-Lagrange equations, also some other conditions, expressed by the *conservation law equations*.

Theorem 2 (First theorem of Emmy Noether) *Let* $\lambda \in \Omega_{n,X}^r W$ *be a Lagrangian,* ρ *a Lepage equivalent of* λ *defined on* J^sY *, and let* γ *be an extremal. Then for every generator* Ξ *of invariance transformations of* λ

(10) $$dJ^s\gamma * i_{J^s\Xi}\rho = 0.$$

Proof The proof is based on the first variation formula (Section 4.6, Theorem 7, (c)), and is trivial. Indeed, we have

(11) $$J^r\gamma * \partial_{J^r\Xi}\lambda = J^s\gamma * i_{J^s\Xi}d\rho + dJ^s\gamma * i_{J^s\Xi}\rho,$$

and since the left-hand side vanishes, by invariance, and the first summand on the right-hand side also vanishes, because γ is an extremal, we get formula (10) as required.

Note that (global) condition (10) can also be written in a different way, by means of locally defined principal Lepage equivalents Θ_λ of the Lagrangian λ. From the structure theorem on Lepage forms we know that, locally, $\rho = \Theta_\lambda + d\nu + \mu$, where ν is a contact form, and μ is a contact form of order of contactness ≥ 2. Then $dJ^s\gamma * i_{J^s\Xi}\rho = dJ^s\gamma * (i_{J^s\Xi}\Theta_\lambda + i_{J^s\Xi}d\nu + i_{J^s\Xi}\mu)$. But the form $i_{J^s\Xi}\mu$ is contact; moreover, $i_{J^s\Xi}d\nu = \partial_{J^s\Xi}\nu - di_{J^s\Xi}\nu$, from which we deduce that

(12) $$J^s\gamma * i_{J^s\Xi}\mu = 0, \quad dJ^s\gamma * i_{J^s\Xi}d\nu = dJ^s\gamma * \partial_{J^s\Xi}\nu - dJ^s\gamma * di_{J^s\Xi}\nu = 0.$$

Consequently, under the hypothesis of Theorem 1, condition

(13) $$dJ^s\gamma * i_{J^s\Xi}\Theta_\lambda = 0$$

holds over coordinate neighbourhoods of fibred charts on *Y*.

One can also use invariance of variational functionals in a different way. Namely, the infinitesimal first variation formula shows that the property of a Lagrangian to be invariant reduces the number of the Euler-Lagrange equations.

Theorem 3 *If λ is invariant, ρ is a Lepage equivalent of λ, and γ a section satisfying the conservation law equation*

(14) $$dJ^r\gamma * i_{J^r\Xi}\rho = 0,$$

then for any fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, the associated Euler-Lagrange expressions are linearly dependent along γ.

Proof The infinitesimal first variation formula gives

(15) $$J^r\gamma * i_{J^r\Xi}d\rho = J^r\gamma * i_{J^r\Xi}p_1 d\rho = J^r\gamma * i_{J^r\Xi}E_{h\rho} = 0.$$

Consequently, in the chart (V,ψ), $\psi = (x^i, y^\sigma)$, for the given vector field Ξ, the Euler-Lagrange expressions of the Lagrangian $\lambda = h\rho$ satisfy (15) and are linearly dependent along γ.

Example (Conservation law equations) In the following example we consider the product fibred manifold $Y = X \times \mathbf{R}^m$. Denote by y^σ the canonical coordinates on $\mathbf{R}^m$, and by x^i, y^σ some coordinates on *Y*. Consider the translation vector fields

(16) $$\Xi_\tau = \frac{\partial}{\partial y^\tau}.$$

One can easily determine the *r*-jet prolongations of these vertical vector fields. We get

(17) $$J^r\Xi_\tau = \frac{\partial}{\partial y^\tau}.$$

Invariance conditions for a Lagrangian $\lambda = \mathscr{L}\omega_0$ are $\partial_{J^r\Xi_\tau}\lambda = i_{J^r\Xi_\tau}d\lambda = 0$, that is,

(18) $$\frac{\partial \mathscr{L}}{\partial y^\tau} = 0.$$

In classical variational calculus condition (18) is sometimes called the *Routh condition*. The principal Lepage equivalent is

(19) $$\Theta_{\mathscr{L}} = \mathscr{L}\omega_0 + \sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1j_2\dots j_kp_1p_2\dots p_li}}\right)\omega^\sigma_{j_1j_2\dots j_k}\wedge\omega_i,$$

and its contraction by $J^r\Xi_\tau$ is

(20) $$i_{J^{r+1}\Xi}\Theta_{\lambda_\rho} = \sum_{l=0}^{r-1}(-1)^l d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial \mathscr{L}}{\partial y^\tau_{p_1p_2\dots p_li}}\omega_i.$$

Therefore, the invariance condition $J^{r+1}\gamma * E_\tau(\lambda)\omega_0 + dJ^{r+1}\gamma * i_{J^{r+1}\Xi}\Theta_{\lambda_\rho} = 0$ reduces to

(21) $$E_\tau(\lambda) - \sum_{l=0}^{r-1}(-1)^l d_i d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial \mathscr{L}}{\partial y^\tau_{p_1p_2\dots p_li}} = 0.$$

In particular, if γ satisfies the conservation law equation

(22) $$\sum_{l=0}^{r-1}(-1)^l d_i d_{p_1}d_{p_2}\dots d_{p_l}\frac{\partial \mathscr{L}}{\partial y^\tau_{p_1p_2\dots p_li}} = 0,$$

it also solves the Euler-Lagrange equation

(23) $$E_\tau(\lambda)\circ J^{r+1}\gamma = 0.$$

In particular, if λ is invariant with respect to all translation vector fields Ξ_τ, then the system of the Euler-Lagrange equations is equivalent with the system of the conservation law equations.

Remark 2 It should be pointed out that in general, the principal Lepage equivalent Θ_λ, considered as a form depending on the Lagrangian λ, does not satisfy the invariance condition $\partial_{J^r\Xi}\Theta_\lambda = \Theta_{\partial_{J^r\Xi}\lambda}$.

Remark 3 The geometric structure of the first Noether's theorem may be explained as follows. Let *Y* be any manifold, ρ a differential form on *Y*. If ξ is a vector field on *Y*. such that the Lie derivative $\partial_\xi\rho$ vanishes,

$\partial_\xi \rho = 0$, then by the *Cartan's formula*, ρ and ξ satisfy $i_\xi d\rho + di_\xi \rho = 0$. Then for any mapping $f: X \to Y$ satisfying the "Euler-Lagrange equation" $f * i_\xi d\rho = 0$, the identity $f * i_\xi d\rho + df * i_\xi \rho = 0$ yields the "conservation law equation" $df * i_\xi \rho = 0$.

Remark 4 (Invariance with respect to a Lie group action) The first theorem of Emmy Noether as explained in Theorem 2 is concerned with variational integrals, invariant with respect to 1-parameter groups of automorphisms of underlying manifolds Y. Clearly, the same theorem applies to invariance with respect to group actions of (finite-dimensional) Lie groups G on Y. The corresponding conservation law equations $dJ^s\gamma * i_{J^s\Xi}\rho = 0$ (14) represent a system, in which the vector fields Ξ are fundamental vector fields, defined by the Lie algebra of G. Thus, we get the system of k equations on J^sY, where k is the dimension of G.

Remark 5 (Second theorem of Emmy Noether) Some variational functionals admit broad classes of invariance transformations that cannot be characterized as Lie group actions. These transformations depend rather on arbitrary functions than on finite number of real parameters. Consequences of invariance of this kind are known as the second Noether's theorem (cf. Olver [O1], where the systems possessing the second Noether's theorem are characterized as abnormal). However, also this type of invariance can sometimes be understood as invariance with respect to a (finite-dimensional) Lie group; namely, this situation arises when the given Lagrangian is a differential invariant (Krupka and Trautman [KT] and Krupka [K10]; see also Chapter 6 of this book).

5.3 Invariant Euler-Lagrange forms

Let $\alpha: W \to Y$ be an automorphism of *Y*, and let ε be a source form on J^sY. We say that α is an *invariance transformation* of ε, if $J^s\alpha * \varepsilon = \varepsilon$. The *generator* of invariance transformations of ε is a π-projectable vector field on *Y* whose local one-parameter group consists of invariance transformations of ε.

Lemma 5 (Noether-Bessel-Hagen equation) *Let ε be a source form of order s for Y.*

(a) *A π-projectable vector field Ξ on Y is the generator of invariance transformations of ε if and only if*

(1) $$\partial_{J^r\Xi}\varepsilon = 0.$$

(b) *Generators of invariance transformations of ε constitute a subalgebra of the algebra of vector fields on J^rY.*

Proof The same as the proof of Lemma 4, Section 5.2.

Equation (1) is a geometric version of what is known in the classical calculus of variations as the *Noether-Bessel-Hagen equation*.

Let λ be a Lagrangian of order *r* for *Y*, let α be any automorphism of *Y*, and let E_λ be the Euler-Lagrange form of λ. Using the identity

(2) $$J^{2r}\alpha^* E_\lambda = E_{J^r\alpha^*\lambda}$$

(Section 4.5, Theorem 6), we easily obtain the following statement.

Lemma 6 (a) *Every invariance transformation of λ is an invariance transformation of the Euler-Lagrange form E_λ.*

(b) *For every invariance transformation α of E_λ, the Lagrangian $\lambda - J^r\alpha^*\lambda$ is variationally trivial.*

Proof (a) This follows from (2): if $J^r\alpha^*\lambda = 0$, then $J^{2r}\alpha^* E_\lambda = 0$.

(b) This is again an immediate consequence of (2): if $J^{2r}\alpha^* E_\lambda = 0$ then $E_{J^r\alpha^*\lambda} = 0$.

We can generalize the Noether's theorem to invariance transformations of the Euler-Lagrange form. However, since the proof is based on the theorem on the kernel of the Euler-Lagrange mapping, the assertion we obtain is of local character. We denote by Θ_λ the principal Lepage equivalent of λ.

Theorem 4 *Let λ be a Lagrangian of order r, let γ be an extremal, and let Ξ be a generator of invariance transformations of the Euler-Lagrange form E_λ. Then for every point $y_0 \in Y$ there exists a fibred chart (V,ψ) at y_0 and an $(n-1)$-form η, defined on V^{r-1}, such that on $\pi(V)$*

(3) $$dJ^{2r}\gamma^*(i_{J^s\Xi}\Theta_\lambda + \eta) = 0.$$

Proof Under the hypothesis of Theorem 4, from Section 4.10, Theorem 1, from formula $\partial_{J^{2r}\Xi} E_\lambda = E_{\partial_{J^r\Xi}\lambda}$ we obtain $E_{\partial_{J^r\Xi}\lambda} = 0$. Thus, the Lagrangian $\partial_{J^r\Xi}\lambda$ belongs to the kernel of the Euler-Lagrange mapping, so it must be of the form $\partial_{J^r\Xi}\lambda = hd\eta$ over sufficiently small open sets *V* in *Y* such that (V,ψ) is a fibred chart (Section 4.8, Theorem 9). Then, however, from the infinitesimal first variation formula over *V*, expression

(4) $$J^r\gamma^*\partial_{J^r\Xi}\lambda = J^{2r-1}\gamma^* i_{J^s\Xi} d\Theta_\lambda + dJ^{2r-1}\gamma^* i_{J^s\Xi}\Theta_\lambda,$$

reduces to

(5) $$J^r\gamma^* hd\eta = dJ^s\gamma^* i_{J^s\Xi}\Theta_\lambda.$$

Since $J^r\gamma^* hd\eta = J^r\gamma^* d\eta = dJ^r\gamma^*\eta$, this proves formula (3).

Remark 6 If $r = 1$, then the principal Lepage equivalent Θ_λ is globally well defined. Moreover, it follows from the properties of the Euler-Lagrange mapping that the form η may be taken as a globally defined form on *Y*.

5.4 Symmetries of extremals and Jacobi vector fields

Let λ be a Lagrangian of order r for a fibred manifold Y, and let γ be an extremal of λ; thus, we suppose that γ satisfies the Euler-Lagrange equation

(1) $$E_\lambda \circ J^{2r}\gamma = 0.$$

Consider an automorphism $\alpha : W \to Y$ of Y with projection α_0, and its r-jet prolongation $J^r\alpha : W^r \to J^rY$. We say that α is a *symmetry* of γ, if the section $\alpha\gamma\alpha_0^{-1}$ is also a solution of the Euler-Lagrange equations, that is,

(2) $$E_\lambda \circ J^{2r}(\alpha\gamma\alpha_0^{-1}) = 0.$$

We say that a π-projectable vector field Ξ is the *generator of symmetries* of γ, or *generates symmetries* of γ, if its local one-parameter group consists of symmetries of γ.

We need a lemma on pushforward vector fields. Consider a vector field ξ and a diffeomorphism $\alpha : W \to X$, defined on an open set $W \subset X$. By the *pushforward vector field* of ξ with respect to α we mean the vector field $\xi^{(\alpha)}$ defined on W by

(3) $$\xi^{(\alpha)}(x) = T_{\alpha^{-1}(x)}\alpha \cdot \xi(\alpha^{-1}(x)).$$

Lemma 7 *Let X be a manifold, W an open set in X, Z a vector field on X, $\alpha : W \to X$ a diffeomorphism, and ρ a p-form. Then*

(4) $$i_\xi \alpha^* \rho = \alpha^* i_{\xi^{(\alpha)}}\rho.$$

Proof We have, with standard notation,

(5) $$\begin{aligned}(i_\xi\alpha^*\rho)(x)(\xi_1,\xi_2,\dots,\xi_p) &= \alpha^*\rho(x)(\xi(x),\xi_1,\xi_2,\dots,\xi_p)\\ &= \rho(\alpha(x))(T_x\alpha\cdot\xi(x),T_x\alpha\cdot\xi_1,T_x\alpha\cdot\xi_2,\dots,T_x\alpha\cdot\xi_p)\\ &= \rho(\alpha(x))(\xi^{(\alpha)}(\alpha(x)),T_x\alpha\cdot\xi_1,T_x\alpha\cdot\xi_2,\dots,T_x\alpha\cdot\xi_p)\\ &= i_{\xi^{(\alpha)}(\alpha(x))}\rho(\alpha(x))(T_x\alpha\cdot\xi_1,T_x\alpha\cdot\xi_2,\dots,T_x\alpha\cdot\xi_p)\\ &= \alpha^*(i_{\xi^{(\alpha)}}\rho)(x)(\xi_1,\xi_2,\dots,\xi_p).\end{aligned}$$

This is exactly formula (4).

The following theorem says that invariance transformations of the Euler-Lagrange form E_λ permute extremals of the variational structure (λ, Y) and give us examples of symmetries.

Theorem 5 *An invariance transformation of the Euler-Lagrange form* E_λ *is a symmetry of every extremal* γ.

Proof 1. Let $\alpha : W \to Y$ be any automorphism of Y with projection $\alpha_0 : \pi(W) \to X$. Let Z be any π-projectable vector field with projection Z_0. We show that the pushforward vector field

$$\mathrm{Z}^{(\alpha)} = T\alpha \cdot \mathrm{Z} \circ \alpha^{-1} \tag{5}$$

is π-projectable, with projection $\mathrm{Z}_0^{(\alpha_0)} = T\alpha_0 \cdot \mathrm{Z}_0 \circ \alpha_0^{-1}$. Indeed, for every $y \in \alpha(W)$

$$\begin{aligned} T_y\pi \cdot \mathrm{Z}^{(\alpha)}(y) &= T_y\pi \cdot T_{\alpha^{-1}(y)}\alpha \cdot \mathrm{Z}(\alpha^{-1}(y)) = T_{\alpha^{-1}(y)}(\pi\alpha) \cdot \mathrm{Z}(\alpha^{-1}(y)) \\ &= T_{\pi(\alpha^{-1}(y))}\alpha_0 \cdot T_{\alpha^{-1}(y)}\pi \cdot \mathrm{Z}(\alpha^{-1}(y)) = T_{\alpha_0^{-1}\pi(y)}\alpha_0 \cdot \mathrm{Z}_0(\pi\alpha^{-1}(y)) \\ &= T_{\alpha_0^{-1}\pi(y)}\alpha_0 \cdot \mathrm{Z}_0(\alpha_0^{-1}\pi(y)) = \mathrm{Z}^{(\alpha_0)}(\pi(y)), \end{aligned} \tag{6}$$

proving that $\mathrm{Z}^{(\alpha)}$ is projectable and its projection is $\mathrm{Z}_0^{(\alpha_0)}$.

Let β_t denote the local 1-parameter group of Z, and let $\beta_{0,t}$ be its projection. Then since

$$\begin{aligned} \left(\frac{d}{dt}\alpha\beta_t\alpha^{-1}(y)\right)_0 &= T_{\alpha^{-1}(y)}\alpha \cdot \left(\frac{d}{dt}\beta_t\alpha^{-1}(y)\right)_0 \\ &= T_{\alpha^{-1}(y)}\alpha \cdot \mathrm{Z}(\alpha^{-1}(y)) = \mathrm{Z}^{(\alpha)}(y), \end{aligned} \tag{7}$$

$\alpha\beta_t\alpha^{-1}$ is the 1-parameter group of $\mathrm{Z}^{(\alpha)}$. The 1-parameter group of the projection $\mathrm{Z}_0^{(\alpha_0)}$ is defined by $\pi\alpha\beta_t\alpha^{-1} = \alpha\pi\beta_t\alpha^{-1} = \alpha\beta_{0,t}\pi\alpha^{-1} = \alpha\beta_{0,t}\alpha_0^{-1}\pi$ and is equal to $\alpha\beta_{0,t}\alpha_0^{-1}$.

Since $\mathrm{Z}^{(\alpha)}$ is projectable, its s-jet prolongation $J^s\mathrm{Z}^{(\alpha)}$ is defined. Since we know the 1-parameter groups of $\mathrm{Z}^{(\alpha)}$, then $J^s\mathrm{Z}^{(\alpha)}$ at a point $J^s_x\gamma$ is given by differentiation of the curve $t \to J^s_{\alpha_0\beta_{0,t}\alpha_0^{-1}(x)}(\alpha\beta_t\alpha^{-1})\gamma(\alpha_0\beta_{0,t}^{-1}\alpha_0^{-1})$ at $t = 0$,

$$J^s\mathrm{Z}^{(\alpha)}(J^s_x\gamma) = \left(\frac{d}{dt}J^s_{\alpha_0\beta_{0,t}\alpha_0^{-1}(x)}(\alpha\beta_t\alpha^{-1})\gamma(\alpha_0\beta_{0,t}^{-1}\alpha_0^{-1})\right)_0. \tag{8}$$

It can be easily seen that the vector field $J^s\mathrm{Z}^{(\alpha)}$ can be determined by

$$J^s\mathrm{Z}^{(\alpha)} = TJ^s\alpha \cdot J^s\mathrm{Z} \circ J^s\alpha^{-1}. \tag{9}$$

We determine the right-hand side at a point $J^s_x\gamma \in J^s\alpha(W^s)$. Using standard differentiations we have

$$T_{J^s\alpha^{-1}(J^r_x\gamma)}J^s\alpha \cdot J^s\mathrm{Z}(J^s\alpha^{-1}(J^r_x\gamma)) = \left(\frac{d}{dt}J^s\alpha(J^s\beta_t(J^s\alpha^{-1}(J^r_x\gamma)))\right)_0. \tag{10}$$

The curve $t \to J^s\alpha(J^s\beta_t(J^s\alpha^{-1}(J^r_x\gamma)))$ can be expressed from the definition of s-jet prolongation of a fibred automorphism (see Section 1.4). We have

$$\begin{aligned}(11)\qquad & J^s\alpha(J^s\beta_t(J^s\alpha^{-1}(J^r_x\gamma))) = J^s\alpha(J^s\beta_t(J^s_{\alpha_0^{-1}(x)}\alpha^{-1}\gamma\alpha_0)))\\ &= J^s\alpha(J^s_{\beta_{0,t}\alpha_0^{-1}(x)}\beta_t\alpha^{-1}\gamma\alpha_0\beta_{0,t}^{-1}))\\ &= J^s_{\alpha_0\beta_{0,t}\alpha_0^{-1}(x)}(\alpha\beta_t\alpha^{-1})\gamma(\alpha_0\beta_{0,t}^{-1}\alpha_0^{-1}).\end{aligned}$$

Differentiating this curve we get the vector field $J^s\Xi^{(\alpha)}$ (9).

2. Let W be the domain of α. We have by definition for every point $J^s_x\gamma \in W^r$, $J^s\alpha(J^s_x\gamma) = J^s_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}$. Then $(J^s\alpha\circ J^s\gamma)(x) = (J^s\alpha\gamma\alpha_0^{-1}\circ\alpha_0)(x)$, and we can write on the domain $\alpha_0(\pi(W))$ of the section $\alpha\gamma\alpha_0^{-1}$

$$(12)\qquad J^s\alpha\circ J^s\gamma\circ\alpha_0^{-1} = J^s\alpha\gamma\alpha_0^{-1}.$$

Consider the Euler-Lagrange form E_λ, the n-form $i_{J^s\Xi}E_\lambda$ that appears in the first variation formula and its values along the section $J^s\alpha\gamma\alpha_0^{-1}$. We have

$$(13)\qquad (J^s\alpha\gamma\alpha_0^{-1})^*i_{J^s\Xi}E_\lambda = (\alpha_0^{-1})^*(J^s\gamma)^*(J^s\alpha)^*i_{J^s\Xi}E_\lambda$$

on the domain $\alpha_0(\pi(W))$ of the section $\alpha\gamma\alpha_0^{-1}$. We can easily find an expression for the form $(J^s\alpha)^*i_{J^s\Xi}E_\lambda$ on W^r. Choose any tangent vectors $\Xi_1,\Xi_2,\ldots,\Xi_n$ at the point $J^s_x\gamma\in W^r$. Then

$$\begin{aligned}(14)\qquad & ((J^s\alpha)^*i_{J^s\Xi}E_\lambda)(J^s_x\gamma)(\Xi_1,\Xi_2,\ldots\Xi_n)\\ &= E_\lambda(J^s\alpha(J^s_x\gamma))(J^s\Xi(J^s\alpha(J^s_x\gamma)),TJ^s\alpha\cdot\Xi_1,TJ^s\alpha\cdot\Xi_2,\ldots,TJ^s\alpha\cdot\Xi_n).\end{aligned}$$

Writing $J^s\Xi(J^s\alpha(J^s_x\gamma)) = TJ^s\alpha\cdot TJ^s\alpha^{-1}\cdot J^s\Xi(J^s\alpha(J^s_x\gamma))$, we get from (9)

$$(15)\qquad T_{J^s\alpha(J^s_x\gamma)}J^s\alpha^{-1}\cdot J^s\Xi(J^s\alpha(J^s_x\gamma)) = J^s\Xi^{(\alpha^{-1})}(J^s_x\gamma)$$

and

$$\begin{aligned}(16)\qquad & ((J^s\alpha)^*i_{J^s\Xi}E_\lambda)(J^s_x\gamma)(\Xi_1,\Xi_2,\ldots\Xi_n)\\ &= E_\lambda(J^s\alpha(J^s_x\gamma))(TJ^s\alpha\cdot J^s\Xi^{(\alpha^{-1})}(J^s_x\gamma),TJ^s\alpha\cdot\Xi_1,\ldots,TJ^s\alpha\cdot\Xi_n)\\ &= (J^s\alpha)^*E_\lambda(J^s_x\gamma)(J^s\Xi^{(\alpha^{-1})}(J^s_x\gamma),\Xi_1,\Xi_2,\ldots,\Xi_n)\\ &= i_{J^s\Xi^{(\alpha^{-1})}(J^s_x\gamma)}(J^s\alpha)^*E_\lambda(J^s_x\gamma)(\Xi_1,\Xi_2,\ldots,\Xi_n)\\ &= (i_{J^s\Xi^{(\alpha^{-1})}}(J^s\alpha)^*E_\lambda(J^s_x\gamma))(\Xi_1,\Xi_2,\ldots,\Xi_n),\end{aligned}$$

or, which is the same,

$$(17)\qquad J^s\alpha^*i_{J^s\Xi}E_\lambda = i_{J^s\Xi^{(\alpha^{-1})}}J^s\alpha^*E_\lambda.$$

3. Now we can show that if γ is an extremal and α is an invariance transformation of E_λ, then for any Z

(18) $(J^s\alpha\gamma\alpha_0^{-1})^* i_{J^s\mathrm{Z}} E_\lambda = 0.$

Since by hypothesis, $(J^s\alpha)^* E_\lambda = E_\lambda$, (17) implies $J^s\alpha^* i_{J^s\mathrm{Z}} E_\lambda = i_{J^s\mathrm{Z}(\alpha^{-1})} E_\lambda$, thus, along $J^s\gamma$,

(19) $J^s\gamma^* J^s\alpha^* i_{J^s\mathrm{Z}} E_\lambda = 0.$

But the left-hand side is, from (12)

$$\begin{aligned} J^s\gamma^* J^s\alpha^* i_{J^s\mathrm{Z}} E_\lambda &= (J^s\alpha \circ J^s\gamma)^* i_{J^s\mathrm{Z}} E_\lambda \\ &= (J^s\alpha\gamma\alpha_0^{-1} \circ \alpha_0)^* i_{J^s\mathrm{Z}} E_\lambda \\ &= \alpha_0^* J^s\alpha\gamma\alpha_0^{-1*} i_{J^s\mathrm{Z}} E_\lambda, \end{aligned} \tag{20}$$

proving (18) as well as Theorem 5.

The following theorem describes properties of individual extremals.

Theorem 6 *Let λ be a Lagrangian of order r, let s be the order of the Euler-Lagrange form E_λ, and let γ be an extremal. Then a π-projectable vector field Ξ generates symmetries of γ if and only if*

(20) $E_{\partial_{J^r\Xi}\lambda} \circ J^s\gamma = 0.$

Proof 1. Suppose we have an extremal γ and a vector field Ξ generating symmetries of γ; we prove that condition (20) is satisfied. We proceed in several steps.

Denote by α_t and $\alpha_{0,t}$ the 1-parameter group of Ξ and its projection, respectively. Using formulas (13) and (17) and invariance of the Euler-Lagrange mapping (Section 4.5, Theorem 6) we get

$$\begin{aligned} (J^s\alpha_t\gamma\alpha_{0,t}^{-1})^* i_{J^s\mathrm{Z}} E_\lambda &= (\alpha_{0,t}^{-1})^* (J^s\gamma)^* i_{J^s\mathrm{Z}(\alpha_{-t})} (J^s\alpha_t^* E_\lambda) \\ &= (\alpha_{0,t}^{-1})^* J^s\gamma^* i_{J^s\mathrm{Z}(\alpha_{-t})} E_{J^s\alpha_t^*\lambda}. \end{aligned} \tag{21}$$

Since the left-hand side vanishes by hypothesis, the right-hand side yields

(22) $J^s\gamma^* i_{J^s\mathrm{Z}(\alpha_t)} E_{J^r\alpha_t^*\lambda} = 0.$

We want to differentiate the form $i_{J^s\mathrm{Z}(\alpha_{-t})} E_{J^s\alpha_t^*\lambda}$ with respect to *t* at $t = 0$ and then consider the resulting form along the prolongation $J^s\gamma$ of the extremal γ. To perform differentiation, note that the derivative of $i_{J^s\mathrm{Z}(\alpha_{-t})} E_{J^s\alpha_t^*\lambda}$

at $t=0$ is the Lie derivative of the form $i_{J^s Z} E_\lambda$ by the vector field $J^s Z$. Indeed, for every point $J^r_x \delta$ belonging to the domain of $J^s \alpha_t$ for sufficiently small t, and any tangent vectors $\Xi_1, \Xi_2, \dots, \Xi_n$ at $J^r_x \delta$,

(23)
$$\begin{aligned}&(J^s \alpha_t^* i_{J^s Z} E_\lambda)(J^r_x \delta)(\Xi_1, \Xi_2, \dots, \Xi_n)\\ &\quad = E_\lambda(J^s \alpha_t(J^r_x \delta))(J^s Z(J^s \alpha_t(J^r_x \delta)), TJ^s \alpha_t \cdot \Xi_1, \dots, TJ^s \alpha_t \cdot \Xi_n).\end{aligned}$$

Substituting

(24)
$$\begin{aligned}&J^s Z(J^s \alpha_t(J^r_x \delta))\\ &\quad = TJ^s \alpha_t \cdot TJ^s \alpha_t^{-1} \cdot J^s Z(J^s \alpha_t(J^r_x \delta))\\ &\quad = TJ^s \alpha_t \cdot J^s Z^{(\alpha_{-t})}(J^r_x \delta)\end{aligned}$$

from (15), we have

(25)
$$\begin{aligned}&(J^s \alpha_t^* i_{J^s Z} E_\lambda)(J^r_x \delta)(\Xi_1, \Xi_2, \dots, \Xi_n)\\ &\quad = J^s \alpha_t^* E_\lambda(J^r_x \delta)(J^s Z^{(\alpha_{-t})}(J^r_x \delta)), \Xi_1, \Xi_2, \dots, \Xi_n).\\ &\quad = i_{J^s Z^{(\alpha_{-t})}(J^r_x \delta)} E_{J^r \alpha_t^* \lambda}(J^r_x \delta)(\Xi_1, \Xi_2, \dots, \Xi_n)\end{aligned}$$

hence

(26)
$$J^s \alpha_t^* i_{J^s Z} E_\lambda = i_{J^s Z^{(\alpha_{-t})}} E_{J^r \alpha_t^* \lambda}.$$

This formula proves that the derivative with respect to t at $t=0$ of the right-hand side is exactly the Lie derivative of the form $i_{J^s Z} E_\lambda$ with respect to the vector field $J^r \Xi$.

Then, however, since

(27)
$$\frac{d}{dt} J^s \alpha_t^* i_{J^s Z} E_\lambda = J^s \alpha_t^* \partial_\Xi i_{J^s Z} E_\lambda = \frac{d}{dt} i_{J^s Z^{(\alpha_{-t})}} E_{J^r \alpha_t^* \lambda}$$

(Lemma 1), so we have along the extremal γ, from (22),

(28)
$$\begin{aligned}&J^r \gamma * J^s \alpha_t^* \partial_\Xi i_{J^s Z} E_\lambda = J^r \gamma * \frac{d}{dt} J^s \alpha_t^* i_{J^s Z} E_\lambda\\ &\quad = J^r \gamma * \frac{d}{dt} i_{J^s Z^{(\alpha_{-t})}} E_{J^r \alpha_t^* \lambda}\\ &\quad = 0.\end{aligned}$$

On the other hand, using the Cartan's formula for the Lie derivative of a differential form (see Appendix 5, (9)), we have

$$
\begin{aligned}
\partial_{J^s\Xi} i_{J^s Z} E_\lambda &= i_{J^s\Xi} d i_{J^s Z} E_\lambda + d i_{J^s\Xi} i_{J^s Z} E_\lambda \\
&= i_{J^s\Xi} (\partial_{J^s Z} E_\lambda - i_{J^s Z} dE_\lambda) - d i_{J^s Z} i_{J^s\Xi} E_\lambda \\
&= i_{J^s\Xi} \partial_{J^s Z} E_\lambda - i_{J^s\Xi} i_{J^s Z} dE_\lambda - \partial_{J^s Z} i_{J^s\Xi} E_\lambda + i_{J^s Z} d i_{J^s\Xi} E_\lambda \\
&= i_{J^s\Xi} \partial_{J^s Z} E_\lambda - i_{J^s\Xi} i_{J^s Z} dE_\lambda - \partial_{J^s Z} i_{J^s\Xi} E_\lambda + i_{J^s Z} (E_{\partial_{J^r\Xi}\lambda} - i_{J^s\Xi} dE_\lambda) \\
&= i_{J^s\Xi} \partial_{J^s Z} E_\lambda - \partial_{J^s Z} i_{J^s\Xi} E_\lambda + i_{J^s Z} E_{\partial_{J^r\Xi}\lambda},
\end{aligned} \tag{29}
$$

and from the Lie bracket formula

$$
i_{[J^s Z, J^s\Xi]} E_\lambda = \partial_{J^s Z} i_{J^s\Xi} E_\lambda - i_{J^s\Xi} \partial_{J^s Z} E_\lambda \tag{30}
$$

we get

$$
\partial_{J^s\Xi} i_{J^s Z} E_\lambda = -i_{[J^s Z, J^s\Xi]} E_\lambda + i_{J^s Z} E_{\partial_{J^r\Xi}\lambda}. \tag{31}
$$

Now since γ is an extremal and Ξ generates symmetries of γ, we have $J^s\gamma * i_{[J^s Z, J^s\Xi]} E_\lambda = 0$ and from equation (28), $J^s\gamma * \partial_{J^s\Xi} i_{J^s Z} E_\lambda = 0$, thus, $J^s\gamma * i_{J^s Z} E_{\partial_{J^r\Xi}\lambda} = 0$ as required.

2. Conversely, suppose that we have an extremal γ and a vector field Ξ such that condition $E_{\partial_{J^r\Xi}\lambda} \circ J^s\gamma = 0$ (20) holds. We want to show that Ξ generates symmetries of γ, that is,

$$
\alpha_{0,t}^* J^s(\alpha_t \gamma \alpha_{0,t}^{-1}) * i_{J^s Z} E_\lambda = 0, \tag{32}
$$

where α_t is the local 1-parameter group of Ξ and Z is any π-projectable vector field.

According to Section 4.10, Theorem 11, condition (20) implies

$$
J^s\gamma * i_{J^s Z} E_{\partial_{J^r\Xi}\lambda} = J^s\gamma * i_{J^s Z} \partial_{J^s\Xi} E_\lambda = 0 \tag{33}
$$

for all π-projectable vector fields Z. Thus, at any point $J^r_x\gamma$

$$
i_{J^s Z(J^r_x\gamma)} \partial_{J^s\Xi} E_\lambda (J^r_x\gamma) = 0 \tag{34}
$$

therefore, $\partial_{J^s\Xi} E_\lambda (J^r_x\gamma) = 0$ because the Euler-Lagrange form is 1-contact. Thus by Section 5.1, Lemma 2,

$$
(J^s\alpha_t) * E_\lambda (J^s_x\gamma) = E_\lambda (J^s_x\gamma). \tag{35}
$$

Contracting the left-hand side by $J^s Z(J^r_x\gamma)$ and using Lemma 7,

$$
J^r\gamma * i_{J^s Z} (J^s\alpha_t) * E_\lambda = J^r\gamma * (J^s\alpha_t) * i_{J^s Z(\alpha_{-t})} E_\lambda \tag{36}
$$

$$= (J^s\alpha_t \circ J^r\gamma)^* i_{J^s Z^{(\alpha_{-t})}} E_\lambda = (J^s \alpha_t \gamma \alpha_{0,t}^{-1} \circ \alpha_{0,t})^* i_{J^s Z^{(\alpha_{-t})}} E_\lambda$$

$$= (\alpha_{0,t})^* (J^s \alpha_t \gamma \alpha_{0,t}^{-1})^* i_{J^s Z^{(\alpha_{-t})}} E_\lambda = \alpha_{0,t}^* J^s (\alpha_t \gamma \alpha_{0,t}^{-1})^* i_{J^s Z} E_\lambda .$$

Since the contraction of the right hand side vanishes, because γ is an extremal, we have $\alpha_{0,t}^* J^s (\alpha_t \gamma \alpha_{0,t}^{-1})^* i_{J^s Z} E_\lambda = 0$, proving (32).

Remark 7 Properties of the systems of partial differential equations, described in this section, namely their invariance properties, strongly rely on the variational origin of these systems. The structure of these equations, esp. their invariance properties, indicates possibilities of applying specific methods of solving these equations. Clearly, these specific topics need further research.

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