6 Examples: Natural Lagrange structures

Examples presented in this chapter include typical variational functionals that appear as variational principles in the theory of geometric and physical fields. We begin by the discussion of the well-known *Hilbert variational functional* for the metric fields, first considered in Hilbert [H] in 1915, whose Euler-Lagrange equations are the *Einstein vacuum equations*. We give a manifold interpretation of this functional and show that its *second-order* Lagrangian, the *formal scalar curvature*, possesses a global *first order Lepage* equivalent. The Lagrangian used by Hilbert is an example of a *differential invariant* of a metric field (and its first and second derivatives). It should be pointed out, however, that the variational considerations as well as the resulting extremal equations do not depend on any assumptions with regard to the signature of underlying metric fields.

Our approach to the subject closely follow the preprint Krupka and Lenc [KL]. The theory of jets and differential invariants incl. applications is explained in Krupka and Janyska [KJ] (see also a general treatment of Kolar, Michor, Slovak [KMJ]). Variational principles with similar invariance properties were studied by Anderson [A1] in connection with the inverse variational problem. More general classes of *natural bundles* and *natural Lagrangians* that are differential invariants of *any* collection of tensor fields, or *any* geometric object fields, were introduced in Krupka and Trautman [KT] and Krupka [K3], [K10]. The claims in this chapter are *not* routine; the reader should provide a proof of them or consult the corresponding references.

For contemporary research in the theory of natural Lagrange structures we refer to Ferraris, Francaviglia, Palese and Winterroth [FFPW], Patak and Krupka [PK], Bloch, Krupka, Urban, Voicu, Volna, and Zenkov [Bl], Palese and Winterroth [PW] and the references therein. Extensive literature on the classical invariant theory, related with the subject, can be found in Kolar, Michor and Slovak [KMS] and Krupka and Janyska [KJ], however, this topic is outside the scope of this book. The variational functionals for submanifolds, whose underlying structures *differ* from fibred manifolds, are not considered in this book (cf. Urban and Krupka [UK3]).

6.1 The Hilbert variational functional

The modern geometric interpretation of variational principles in physics requires the knowledge of the structure of underlying fibred spaces as well as adequate (intrinsic and also coordinate) methods of the calculus of variations on these spaces. In this example we briefly consider the *Hilbert variational functional* for metric fields on a general n-dimensional manifold X, a well-known functional providing, for $n = 4$, the variational principle for the *Einstein vacuum equations* in the general relativity theory (Hilbert [H]). Note that the Hilbert variational principle does *not* restrict the topology of the underlying (*spacetime*) manifold *X*. If we require that the *topology* of spacetime should have its origin in *matter* and *physical fields*, then this principle should be completed with some other one.

In this example we follow the preprint Krupka and Lenc [KL]; the topic certainly needs further investigations. Our assertions are formulated without proof, which can however be easily reconstructed by means of the general theory. Basic knowledge of the concepts of Riemannian (and pseudo-Riemannian) geometry is supposed.

Let *X* be an *n*-dimensional smooth manifold, $T_2^0 X$ the vector bundle of tensors of type $(0,2)$ over *X*, and let $\tau : T_2^0 X \to X$ be the tensor bundle projection. $T_2^0 X$ contains the open set $\operatorname{Met} X$ of *symmetric, regular bilinear forms* on the tangent spaces at the points of *X*. Then the restriction of the tensor bundle projection τ defines a *fibred manifold structure* on the set $\operatorname{Met} X$ over the manifold *X*; we call this fibred manifold the *bundle of metrics* over *X*. Its sections are *metric fields* on the manifold *X*. Integral variational functionals for the metric fields are defined by *n*-forms on the *r*-jet prolongations $J^r \operatorname{Met} X$ of the fibred manifold $\operatorname{Met} X$.

Any chart (U,φ), $\varphi = (x^i)$, on *X* induces a chart (V,ψ), $\psi = (x^i, g_{ij})$, on $\operatorname{Met} X$, where $V = \tau^{-1}(U)$ and g_{ij} are functions on *V* defined by the decomposition $g = g_{ij} dx^i \otimes dx^j$ of the bilinear forms; the *coordinate functions* g_{ij} entering the chart (V,ψ) satisfy $1 \le i \le j \le n$. The associated fibred charts on the *r*-jet prolongations $J^r \operatorname{Met} X$ are then defined in a standard way. In particular, if $r = 2$, then the associated chart is denoted by (V^2, ψ^2), $\psi^2 = (x^i, g_{ij}, g_{ij,k}, g_{ij,kl})$, where $i \le j$, $k \le l$, and $g_{ij,k} = d_k g_{ij}$, $g_{ij,kl} = d_k d_l g_{ij}$; d_k is the *formal derivative operator*. We denote

$$
(1) \qquad
\begin{aligned}
&\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \\
&\omega_k = (-1)^{k-1} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \ldots \wedge dx^n, \\
&\omega_{ij} = dg_{ij} - g_{ij,p} dx^p, \\
&\omega_{ij,k} = dg_{ij,k} - g_{ij,kp} dx^p.
\end{aligned}
$$

Then the forms $dx^i, \omega_{ij}, \omega_{ij,k}, dg_{ij,kl}$ constitute the *contact basis* on the set V^2. We need some systems of functions on V^2. The functions

$$
(2) \qquad \Gamma^i_{jk} = \frac{1}{2} g^{im} (g_{mk,j} + g_{jm,k} - g_{jk,m}),
$$

where g^{im} are elements of the *inverse matrix* of the matrix g_{ij}, are called the *formal Christoffel symbols*; note that the derivative $g_{pj,k}$ can be reconstructed from Γ^i_{jk} by the formula $g_{pj,k} = g_{pi}\Gamma^i_{jk} + g_{ji}\Gamma^i_{pk}$. The expressions

$$
(3) \qquad R_{ik} = \Gamma^l_{ik,l} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^l_{il,k} - \Gamma^m_{il}\Gamma^l_{km}, \quad R = g^{ik} R_{ik},
$$

where $\Gamma^l_{ik,j}$ are the formal derivatives $d_j \Gamma^l_{ik}$, define the *formal Ricci tensor* with components R_{ik}, and a function $R : J^2 \operatorname{Met} X \to \mathbf{R}$, the *formal scalar* *curvature*. Every metric field $U \ni x \to g(x) \in \operatorname{Met} X$, defined on an open set in X, can be prolonged to the section $U \ni x \to J^2 g(x) \in J^2 \operatorname{Met} X$ of the second jet prolongation $J^2 \operatorname{Met} X$. Composing the second jet prolongation $J^2 g$ with the formal scalar curvature we get a real-valued function on U, $x \to (R \circ J^2 g)(x) = R(J^r_x g)$, the *scalar curvature of the metric* g, and a second-order Lagrangian

(4) $$\lambda = R\sqrt{|\det g_{ij}|} \cdot \omega_0 .$$

λ is called the *Hilbert Lagrangian*. The variational functional

(5) $$\Gamma_\Omega(\tau) \ni g \to \lambda_\Omega(\gamma) = \int_\Omega J^2 g^* \lambda \in \mathbf{R},$$

where Ω is any compact set in the domain of definition of the section γ, is the *Hilbert variational functional* for the metric fields on X.

We shall restate basic general theorems of the variational theory on fibred manifolds for this special case. It should be pointed out, however, that all these statements could also be proved *directly*, without reference to the general theory. Our first statement rephrases the existence theorem for Lepage equivalents of a given Lagrangian; we claim in addition, that the (*second-order*) Hilbert Lagrangian possesses a *first order* Lepage equivalent.

Recall that $\tau^{2,0}$ is the canonical jet projection of $J^2 \operatorname{Met} X$ onto $\operatorname{Met} X$, expressed as the mapping $(x^i, g_{ij}, g_{ij,k}, g_{ij,kl}) \to (x^i, g_{ij})$, and denote

(6) $$\mathcal{R} = R\sqrt{|\det g_{ij}|}.$$

$\mathcal{R}$ is the *component* of the Hilbert Lagrangian with respect to the chart on $J^2 \operatorname{Met} X$, associated with the chart (U, φ), $\varphi = (x^i)$.

Theorem 1 (Existence of Lepage equivalents) *There exists an n-form* Θ_H *on the first jet prolongation* $J^1 \operatorname{Met} X$ *with the following properties:*

(a) $h\Theta_H = \lambda$.

(b) $p_1 d\Theta_H$ *is* $\tau^{2,0}$*-horizontal*.

To prove Theorem 1 we can use the principal Lepage equivalent of a second-order Lagrangian (Section 4.5, Example 2), which is now given by

(7) $$\Theta_H = \mathcal{R}\omega_0 + \left(\left(\frac{\partial \mathcal{R}}{\partial g_{ij,k}} - d_l \frac{\partial \mathcal{R}}{\partial g_{ij,kl}} \right) \omega_{ij} + \frac{\partial \mathcal{R}}{\partial g_{ij,kl}} \omega_{ij,l} \right) \wedge \omega_k .$$

Substituting from (6), we get the *principal Lepage equivalent of the Hilbert Lagrangian*

(8) $$\begin{aligned} \Theta_H &= \sqrt{|\det g_{rs}|} g^{ip} (\Gamma^j_{ip} \Gamma^k_{jk} - \Gamma^j_{ik} \Gamma^k_{jp}) \omega_0 \\ &\quad + \sqrt{|\det g_{rs}|} (g^{jp} g^{iq} - g^{pq} g^{ij})(dg_{pq,j} + \Gamma^k_{pq} dg_{jk}) \wedge \omega_i . \end{aligned}$$

One can also prove Theorem 1 by searching for Θ_H in the form

(9) $$\Theta_H = \mathcal{R}\omega_0 + (f^{ijk}\omega_{ij} + f^{ijkl}\omega_{ij,l}) \wedge \omega_k,$$

with an invariant condition $f^{ijkl} = f^{ijlk}$. The following is another expression for Θ_H.

Theorem 2 *The form* Θ_H *satisfying conditions* (a) and (b) *has an expression*

(10) $$\Theta_H = -\mathcal{H}\omega_0 + \mathcal{P}^{ij,k} dg_{ij} \wedge \omega_k + d\eta,$$

where

(11) $$\begin{aligned} &\mathcal{H} = \sqrt{|\det g_{rs}|} \cdot g^{ij}(\Gamma^k_{ik}\Gamma^r_{jr} - \Gamma^k_{ij}\Gamma^r_{kr}), \\ &\mathcal{P}^{ij,k} = \frac{1}{2}\sqrt{|\det g_{rs}|}(-g^{ki}g^{sj}\Gamma^q_{qs} - g^{kj}g^{si}\Gamma^q_{qs} + g^{ks}g^{ij}\Gamma^q_{qs} \\ &\quad + g^{pi}g^{sj}\Gamma^k_{ps} + g^{pj}g^{si}\Gamma^k_{ps} - g^{ij}g^{ps}\Gamma^k_{ps}), \\ &\eta = \sqrt{|\det g_{rs}|}(g^{jl}\Gamma^k_{jl} - g^{kl}\Gamma^r_{rl})\omega_k. \end{aligned}$$

These explicit formulas show that the Lepage form Θ_H is of the first order. Since $h\Theta_H = \lambda$, the Hilbert variational functional (1) is a *first order* functional

(12) $$\Gamma_\Omega(\tau) \ni g \to \lambda_H(\gamma) = \int_\Omega J^1 g^* \Theta_H \in \mathbf{R}.$$

Existence of the Lepage equivalent Θ_H has a few immediate consequences. The most important one is the form of the first variation formula (Section 4.6). Recall this formula for any τ-projectable vector field Ξ on the fibred manifold $\operatorname{Met} X$, expressed by

(13) $$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi_{ij}\frac{\partial}{\partial g_{ij}}.$$

Then for every metric field g, defined on an open set in X, the Lie derivative $\partial_{J^1\Xi}\Theta_H$ is along $J^1 g$ expressed as

(14) $$J^1 g^* \partial_{J^1\Xi}\Theta_H = J^1 g^* i_{J^1\Xi} d\Theta_H + dJ^1 g^* i_{J^1\Xi}\Theta_H.$$

This is the basic (global) *infinitesimal first variation formula* for the Hilbert Lagrangian, allowing us to study its *extremals* and *conservation law equations*. The horizontal components $hi_{J^1\Xi}d\Theta_H$ and $hdJ^1 g^* i_{J^1\Xi}\Theta_H$, corresponding with formula (14) are

(15) $$hi_{J^1\Xi}d\Theta_H = \left(\frac{\partial \mathcal{R}}{\partial g_{ij}} - d_k \frac{\partial \mathcal{R}}{\partial g_{ij,k}} + d_k d_l \frac{\partial \mathcal{R}}{\partial g_{ij,kl}} \right)(\Xi_{ij} - g_{ij,p}\xi^p)\omega_0,$$

and

(16) $$hdi_{J^1\Xi}\Theta_H = d_i w^i \cdot \omega_0,$$

where

(17) $$w^i = \mathcal{R}\xi^i + \left(\frac{\partial \mathcal{R}}{\partial g_{kl,i}} + d_j \frac{\partial \mathcal{R}}{\partial g_{kl,ij}} \right)(\Xi_{kl} - g_{kl,p}\xi^p) + \frac{\partial \mathcal{R}}{\partial g_{kl,ij}}(\Xi_{klj} - g_{kl,jp}\xi^p).$$

Note that the horizontalization h in (15) and (16) characterizes the forms $i_{J^1\Xi}d\Theta_H$ and $di_{J^1\Xi}\Theta_H$ *along the* 1*-jet prolongations* $J^1 g$ of sections of the fibred manifold $\operatorname{Met} X$. Expression (15) represents the *Euler-Lagrange term*, and (16) is the *boundary term*.

Since from the definition of the r-jet prolongation of a vector field the expression $\Xi_{klj} - g_{kl,jp}\xi^p$ can be expresses as

(18) $$\begin{aligned} d_j(\Xi_{kl} - g_{kl,p}\xi^p) &= d_j\Xi_{kl} - g_{kl,pj}\xi^p - g_{kl,p}\frac{\partial \xi^p}{\partial x^j} \\ &= \Xi_{klj} - g_{kl,pj}\xi^p \end{aligned}$$

(see Section 1.7), we can also write formula (17) as

(19) $$\begin{aligned} w^i &= \mathcal{R}\xi^i + \left(\frac{\partial \mathcal{R}}{\partial g_{kl,i}} + d_j \frac{\partial \mathcal{R}}{\partial g_{kl,ij}} \right)(\Xi_{kl} - g_{kl,p}\xi^p) + \frac{\partial \mathcal{R}}{\partial g_{kl,ij}} d_j(\Xi_{kl} - g_{kl,p}\xi^p) \\ &= \mathcal{R}\xi^i + \frac{\partial \mathcal{R}}{\partial g_{kl,i}}(\Xi_{kl} - g_{kl,p}\xi^p) + d_j\left(\frac{\partial \mathcal{R}}{\partial g_{kl,ij}}(\Xi_{kl} - g_{kl,p}\xi^p) \right). \end{aligned}$$

The Lapage equivalent Θ_H determines the Euler-Lagrange equations:

Theorem 3 (Euler-Lagrange expressions, Noether currents) (a) *The Euler-Lagrange term in the first variation formula* (14) *has an expression*

(20) $$hi_{J^1\Xi}d\Theta_H = \left(\frac{1}{2} g_{ij}R - R_{ij} \right) g^{ir}g^{js}(\Xi_{rs} - g_{rs,p}\xi^p)\sqrt{|\det g_{rs}|}\omega_0.$$

(b) *The boundary term is given by the expression*

(21) $$\begin{aligned} w^i &= \mathcal{R}\xi^i + \sqrt{|\det g_{rs}|}(g^{jl}g^{pi} - g^{pj}g^{li})\Gamma^k_{pj}(\Xi_{kl} - g_{kl,m}\xi^m) \\ &\quad + \sqrt{|\det g_{rs}|}(g^{kj}g^{il} - g^{ij}g^{kl})(\Xi_{klj} - g_{kl,jm}\xi^m). \end{aligned}$$

The $(n+1)$-form defined by expression (20), characterizing *externals* of the Hilbert variational functionals, is the *Euler-Lagrange form*

(22) $$E(\lambda) = p_1 d\Theta_H = \sqrt{|\det g_{rs}|} E_{ij} g^{ir} g^{js} \omega_{rs} \wedge \omega_0,$$

where E_{ij} is the *formal Einstein tensor*,

(23) $$E_{ij} = \frac{1}{2} g_{ij} R - R_{ij},$$

The corresponding Euler-Lagrange equations are the *Einstein equations*

(24) $$E_{ij} \circ J^2 g = 0.$$

The $(n-1)$-form $i_{J^1\Xi}\Theta_H$ in (16) is the *Noether current* associated with the vector field Ξ.

A specific property of the Hilbert Lagrangian consists in its invariance under *all* diffeomorphisms of the fibred manifold $\operatorname{Met} X$, induced by diffeomorphisms of the underlying manifold *X*. Recall briefly the corresponding definitions (Krupka [K3]). Suppose we are given a diffeomorphism $\alpha : U \to \bar{U}$, where U and $\bar{U}$ are open subsets of *X*. First we wish to show that α lifts to a diffeomorphism α_{Met} of the set $\tau^{-1}(U)$ into $\tau^{-1}(\bar{U})$, and find equations of α_{Met}. If U and $\bar{U}$ are domains of definition of two charts, (U,φ), $\varphi = (x^i)$, and $(\bar{U},\bar{\varphi})$, $\bar{\varphi} = (\bar{x}^\sigma)$, then for any point $x \in U$, a metric $\bar{g}$ at the point $\alpha(x) \in \bar{U}$ is expressed as

(25) $$\bar{g} = \bar{g}_{\sigma\nu} \cdot dy^\sigma(\alpha(x)) \otimes dy^\nu(\alpha(x)),$$

where $\bar{g}_{\sigma\nu}$ are real numbers. Then setting

(26) $$\begin{aligned} T_2^0\alpha \cdot \bar{g} &= \bar{g}_{\sigma\nu} (\alpha * dy^\sigma)(x) \otimes (\alpha * dy^\nu)(x) \\ &= \bar{g}_{\sigma\nu} d(y^\sigma \circ \alpha)(x) \otimes (y^\nu \circ \alpha)(x) \\ &= \bar{g}_{\sigma\nu} \left(\frac{\partial (y^\sigma \alpha \varphi^{-1})}{\partial x^i} \right)_{\varphi(x)} \left(\frac{\partial (y^\nu \alpha \varphi^{-1})}{\partial x^j} \right)_{\varphi(x)} dx^i(x) \otimes dx^j(x), \end{aligned}$$

we get a metric $g = T_2^0\alpha \cdot \bar{g}$ at the point *x*. Thus, replacing α with α^{-1}, we get a diffeomorphism $\operatorname{Met}\alpha : \tau^{-1}(U) \to \tau^{-1}(\bar{U})$, defined in components as the correspondence

(27) $$\begin{aligned} &x^i \to x^i \alpha \varphi^{-1}(\varphi(x)), \\ &g_{ij} \to \bar{g}_{\sigma\nu} = g_{ij} \left(\frac{\partial (x^i \alpha^{-1} \bar{\varphi}^{-1})}{\partial \bar{x}^\sigma} \right)_{\varphi(\alpha(x))} \left(\frac{\partial (x^j \alpha^{-1} \bar{\varphi}^{-1})}{\partial \bar{x}^\nu} \right)_{\varphi(\alpha(x))}. \end{aligned}$$

This construction can be adapted to the local 1-parameter group α_t of a vector field ξ on *X*. To this purpose we may choose, for all sufficiently small t, $(\bar{U},\bar{\varphi}) = (U,\varphi)$. Express ξ as

(28) $$\xi = \xi^i \frac{\partial}{\partial x^i}.$$

Then the mapping $\mathrm{Met}\,\alpha$ (27) is replaced with the mapping expressed as

(29) $$\begin{aligned} &(t,x^i) \to x^i \alpha_t \varphi^{-1}(\varphi(x)) = x^i \alpha_t(x), \\ &(t,g_{ij}) \to \bar{g}_{rs} = g_{ij} \left(\frac{\partial (x^i \alpha_t^{-1} \varphi^{-1})}{\partial x^r} \right)_{\varphi(\alpha_t(x))} \left(\frac{\partial (x^j \alpha_t^{-1} \bar{\varphi}^{-1})}{\partial x^s} \right)_{\varphi(\alpha_t(x))}, \end{aligned}$$

representing the *canonical lift* $\mathrm{Met}\,\alpha_t$ of the flow α_t to the fibred manifold $\mathrm{Met}\,X$. The corresponding lift of the vector field ξ to the fibred manifold $\mathrm{Met}\,X$, denoted $\mathrm{Met}\,\xi$, is obtained by differentiating of the functions (29) at $t = 0$. Differentiating the mapping $(t,x^i) \to x^i \alpha_t(x)$ yields the component ξ^i of ξ. Since $\alpha_t^{-1} = \alpha_{-t}$ and $\alpha_0 = \mathrm{id}$, the second row in (29) yields the expression

(30) $$\begin{aligned} &g_{ij} \left(\frac{\partial}{\partial x^r} \left(\frac{d(x^i \alpha_{-t} \varphi^{-1})}{dt} \right)_0 \right)_{\varphi(x)} \delta_s^j + g_{ij} \delta_r^i \left(\frac{\partial}{\partial x^s} \left(\frac{d(x^j \alpha_{-t} \bar{\varphi}^{-1})}{dt} \right)_0 \right)_{\varphi(x)} \\ &= -g_{is} \left(\frac{\partial \xi^i}{\partial x^r}_0 \right)_{\varphi(x)} - g_{rj} \left(\frac{\partial \xi^j}{\partial x^s}_0 \right)_{\varphi(x)}. \end{aligned}$$

Thus, since the vector field $\mathrm{Met}\,\xi$ is determined by its flow, we have

(31) $$\mathrm{Met}\,\xi = \xi^i \frac{\partial}{\partial x^i} - \left(g_{is} \frac{\partial \xi^i}{\partial x^r} + g_{ri} \frac{\partial \xi^i}{\partial x^s} \right) \frac{\partial}{\partial g_{rs}}.$$

The Hilbert Lagrangian is easily seen to be diffeomorphism-invariant or, which is the same, a *differential invariant* (cf. Krupka and Janyska [KJ], Kolar, Michor and Slovak [KMS]). This property can also be expressed in terms of Lie derivatives.

Theorem 4 *For every vector field ξ, defined on an open set in X,*

(32) $$\partial_{J^2 \mathrm{Met}\,\xi} \lambda = 0.$$

Combining Theorem 4 and the first variation formula (14), where $\Xi = \mathrm{Met}\,\xi$ we obtain the identity

(33) $$J^1 g * i_{J^1 \mathrm{Met}\,\xi} d\Theta_H + d J^1 g * i_{J^1 \mathrm{Met}\,\xi} \Theta_H = 0$$

holding for all ξ and all γ. The meaning of this condition requires further analysis.

6.2 Natural Lagrange structures

The class of *natural Lagrange structures* represents a far-going generalisation of the Hilbert variational principle, discussed in the previous example. The *Lagrangians* for these Lagrange structures are defined on natural bundles by an invariance condition with respect to diffeomorphisms of the underlying manifold, analogous to property $\partial_{J^2 \mathrm{Met}\,\xi}\lambda = 0$, of the Hilbert Lagrangian λ (see Section 6.1, (32)). Conditions of this kind can be rephrased by saying that the Lagrangians should be *differential invariants* (Krupka and Janyska [KJ]); a specific feature of such a Lagrangian consists in its property to define a variational principle not only for one specific fibred manifold but rather for the *category* of locally isomorphic fibred manifolds. For the natural bundles and their generalisations – gauge natural bundles we refer to Kolar, Michor and Slovak [KMS].

Our brief exposition follows the general theory explained in Chapter 4 and two papers on natural Lagrange structures Krupka [K3] and [K10]. The relationship between natural Lagrangians and the inverse problem of the calculus of variations was studied by Anderson [A1].

By the *r-th differential group* of the Euclidean space $\mathbf{R}^n$ we mean the group L^r_n of invertible *r*-jets with source and target at the origin $0 \in \mathbf{R}^n$. An element of the group L^r_n is an *r*-jet $J^r_0\alpha$, whose representative is a diffeomorphisms $\alpha : U \to V$, where *U* and *V* are neighbourhoods of the origin and $\alpha(0) = 0$. The group operation $L^r_n \times L^r_n \ni (J^r_0\alpha, J^r_0\beta) \to J^r_0(\alpha \circ \beta) \in L^r_n$ is defined by the composition of jets. The *canonical* (global) *coordinates* $a^i_{j_1 j_2 \ldots j_k}$ on L^r_n are defined by the condition $a^i_{j_1 j_2 \ldots j_k}(J^r_0\alpha) = D_{j_1} D_{j_2} \ldots D_{j_k} \alpha^i(0)$, where $1 \le k \le r$, $1 \le j_1 \le j_2 \le \ldots \le j_k \le n$, and α^i are components of the diffeomorphism α. Since the group operation is polynomial, the differential group is a Lie group. Clearly, L^1_n can be canonically identified with the general linear group $GL_n(\mathbf{R})$.

Let *X* be a smooth manifold of dimension *n*. By an *r-frame* at a point $x \in X$ we mean an invertible *r*-jet $J^r_0\zeta$ with source $0 \in \mathbf{R}^n$ and target *x*. The set of *r*-frames, denoted $\mathcal{F}^r X$, has a natural smooth structure and is endowed with the canonical jet projection $\pi^r : \mathcal{F}^r X \to X$: Every chart (U, φ), $\varphi = (x^i)$, on *X* induces a chart $((\pi^r)^{-1}(U), \varphi^r)$, $\varphi^r = (x^i, \zeta^i_{j_1 j_2 \ldots j_k})$, on $\mathcal{F}^r X$ by $\zeta^i_{j_1 j_2 \ldots j_k}(J^r_0\zeta) = D_{j_1} D_{j_2} \ldots D_{j_k} \zeta^i(0)$, where $1 \le k \le r$, $1 \le j_1 \le j_2 \le \ldots \le j_k \le n$, and ζ^i are the components of ζ in the chart (U, φ). The mapping $\mathcal{F}^r X \times L^r_n \ni (J^r_0\zeta, J^r_0\alpha) \to J^r_0(\zeta \circ \alpha) \in \mathcal{F}^r X$ defines on $\mathcal{F}^r X$ the structure of a (right) *principal fibre bundle* with structure group L^r_n. $\mathcal{F}^r X$ is called the *bundle of r-frames* over *X*. If $r = 1$, then $\mathcal{F}^1 X$ can be canonically identified with the bundle of linear frames $\mathcal{F}X$.

As an example one can easily derive the equations, describing the structure of the principal L^2_n-bundle of 2-frames. The group multiplication in the differential group L^2_n is given by

(1)
$$a^i_j(A\circ B)=a^i_k(A)a^k_j(B),$$
$$a^i_{j_1j_2}(A\circ B)=a^i_{k_1k_2}(A)a^{k_1}_{j_1}(B)a^{k_2}_{j_2}(B)+a^i_k(A)a^k_{j_1j_2}(B),$$

where $A=J^2_0\alpha$, $B=J^2_0\beta$. The right action of L^2_n on $\mathcal{F}^2X$ is expressed by the formulas

(2)
$$\zeta^i_j(\zeta\circ A)=\zeta^i_k(\zeta)a^k_j(A),$$
$$\zeta^i_{j_1j_2}(\zeta\circ A)=\zeta^i_{k_1k_2}(\zeta)a^{k_1}_{j_1}(A)a^{k_2}_{j_2}(A)+\zeta^i_k(\zeta)a^k_{j_1j_2}(A).$$

We need some categories:

(a) $\mathcal{D}_n$ – the category of diffeomorphisms of smooth, *n*-dimensional manifolds,

(b) $\mathcal{PB}_n(G)$ – the category of homomorphisms of principal fibre bundles with structure group *G*, whose projections are morphisms of $\mathcal{D}_n$,

(c) $\mathcal{FB}_n(G)$ – the category of homomorphisms of fibre bundles, associated with principal fibre bundles from $\mathcal{PB}_n(G)$.

Let $\tau:\mathcal{D}_n\to\mathcal{PB}_n(G)$ be a *lifting*, that is, a covariant functor, assigning to an object *X* of the category $\mathcal{D}_n$ an object τX of $\mathcal{PB}_n(G)$ and to a morphism $f:U\to V$ of $\mathcal{D}_n$ a morphism $\tau f:\tau U\to\tau V$ of $\mathcal{PB}_n(G)$. Let *Q* be a manifold, endowed with a left action of the Lie group *G*. For any manifold *X* belonging to $\mathcal{D}_n$, *Q* defines a fibre bundle $\tau_Q X$ with type fibre *Q*, associated with τX. $f:U\to V$ also defines a morphism $\tau_Q f:\tau_Q U\to\tau_Q V$ of the category $\mathcal{FB}_n(G)$. The correspondence $X\to\tau_Q X$, $f\to\tau_Q f$ is a covariant functor from $\mathcal{D}_n$ to $\mathcal{FB}_n(G)$, called the *Q-lifting* associated with the lifting τ. This lifting is denoted by τ_Q.

In many applications *Q* is a space of tensors on the vector space $\mathbf{R}^n$. Then *Q* is endowed with the *tensor action* $GL_n(\mathbf{R})\times Q\ni(g,p)\to g\cdot p\in Q$. In this case the *Q*-lifting τ_Q assigns to a smooth *n*-dimensional manifold *X* the tensor bundle $\tau_Q X$ of *tensors of type Q* over *X* and to a morphism $f:U\to V$ of $\mathcal{D}_n$ the corresponding morphism $\tau_Q f:\tau_Q U\to\tau_Q V$ of the category $\mathcal{FB}_n(GL_n(\mathbf{R}))$.

In the calculus of variations we need the *jet prolongations* of these fibre bundles. Denote by T^r_nQ the set of *r*-jets with source $0\in\mathbf{R}^n$ and target in *Q*. T^r_nQ is endowed with the a action of the differential group L^{r+1}_n,

(3)
$$L^{r+1}_n\times T^r_nQ\ni(J^{r+1}_0\alpha,J^r_0\zeta)\to J^r_0((D\alpha\cdot\zeta)\circ\alpha^{-1})\in T^r_nQ$$

(Krupka [K3]). Calculating this mapping in a chart we easily find that formally, this jet formula represents *transformation properties* of the derivatives of a tensor field of type *Q*. The following interpretation is important for applications; namely, it possesses a tool how to construct *natural Lagrangians* for collections of tensor fields of a given type *Q*.

Lemma 1 *Let X be a smooth n-dimensional manifold.*

(a) *Formula* (3) *defines the structure of a fibre bundle with type fibre*

$T_n^r Q$, associated with the principal L_n^{r+1}-bundle $\mathcal{F}^{r+1}X$.

(b) *The correspondence $X \to J^r\tau_Q X$, $f \to J^r f_Q X$ is a covariant functor from the category $\mathcal{D}_n$ to the category $\mathcal{FB}_n(L_n^{r+1})$.*

The lifting $J^r\tau_Q$ is called the *r-jet prolongation* of the lifting τ_Q.

The notion of the r-jet prolongation can naturally be extended to *any* manifolds Q endowed with a left action of the general linear group $GL_n(\mathbf{R})$.

These notions represent the underlying general concepts of the theory of natural variational structures. Namely, let X be an n-dimensional manifold (an object of the category $\mathcal{D}_n$), Q a manifold endowed with a left action of the general linear group $L_n^1 = GL_n(\mathbf{R})$, $\tau_Q X$ the fibre bundle with base X and type fibre Q, associated with the bundle of frames $\mathcal{F}X$ (an object of the category $\mathcal{FB}_n(L_n^1)$), and let $J^r\tau_Q X$ be the r-jet prolongation of $\tau_Q X$ (an object of the category $\mathcal{FB}_n(L_n^{r+1})$). Let $J^r\tau_Q\xi$ be the *lift* of a vector field ξ, defined on X, to the bundle $J^r\tau_Q X$ (an object of $\mathcal{FB}_n(L_n^{r+1})$). We say that a Lagrangian λ, defined on $J^r\tau_Q X$ is *natural*, if for all vector fields ξ,

(4) $$\partial_{J^r\tau_Q\xi}\lambda = 0$$

Now let (Y,λ) be a variational structure of order r, let X be the base of the fibred manifold Y, and suppose without loss of generality that the form λ is a *Lagrangian*. We shall say that that the variational structure (Y,λ) is *natural*, if there exists a left L_n^1-manifold Q such that $Y = \tau_Q X$. Thus, roughly, a natural Lagrange structure consists of a *natural bundle* $Y = \tau_Q X$ and a *natural Lagrangian* on this natural bundle.

Examples 1. The variational structure $(\mathrm{Met}\,X,\lambda)$, where λ is the Hilbert Lagrangian (Section 6.1).

2. The Lagrangian for a covector field and a metric field in the general relativity theory, representing interaction of the electromagnetic and gravitational fields in the general relativity theory. The corresponding natural Lagrange structure is the pair (Y,λ), where the fibred manifold Y is the fibre product $\mathrm{Met}\,X \oplus T^*X$ over a manifold X; its sections are the pairs of tensor fields (g,A), locally expressible as

(5) $$g = g_{ij}dx^i \otimes dx^j, \quad A = A_i dx^i.$$

The Lagrangian is of the form $\lambda = \lambda_H + \lambda'$, where λ_H is the Hilbert Lagrangian and the term λ', describing the interaction of the *gravitational* and *electromagnetic* field is defined by the *interaction Lagrangian*

(6) $$\lambda' = g^{ij}g^{kl}(A_{i,k} - A_{k,i})(A_{j,l} - A_{l,j})\sqrt{|\det g_{rs}|}\,\omega_0.$$

In this formula $A_{i,k} = d_k A_i$ are *formal derivatives*. The Euler-Lagrange equations consists of two systems, the *Maxwell equations*, and the *Einstein equations* whose left-hand side is the Einstein tensor E_{ij} (23) and the right-hand side is the variational energy-momentum tensor of the electromagnetic field (cf. Bloch, Krupka, Urban, Voicu, Volna, Zenkov [Bl]).

3. An example of a *gauge-natural* variational structure is provided by the Hilbert-Young-Mills Lagrangian (see e.g. Patak and Krupka [PK]).

6.3 Connections

We give in this section an example of a first order natural Lagrange structure $(\mathscr{C}X,\lambda_{\mathscr{C}})$, whose underlying fibred manifold is *not* a tensor bundle.

Consider the vector space $Q=\mathbf{R}^n\otimes(\mathbf{R}^n)^*\otimes(\mathbf{R}^n)^*$ of tensors of type $(1,2)$ on the vector space $\mathbf{R}^n$, with the canonical coordinates Γ^i_{jk}. We shall refer to Γ^i_{jk} as the *formal Christoffel symbols*. Q is endowed with a *non-linear* left action of the differential group L^2_n, defined in charts by

$$\bar{\Gamma}^i_{jk}=a^i_p(b^q_j b^r_k\Gamma^p_{qr}+b^p_{jk}), \tag{1}$$

where a^i_j,a^i_{jk} are the canonical coordinates on L^2_n, and b^i_j, b^i_{jk} are functions on L^2_n defined by the formulas $a^i_p b^p_j=\delta^i_j$, $a^i_{pq}b^p_j+a^i_p a^s_q b^p_{js}=0$. Note that this action is defined by the *transformation equations* for the components of a connection. For *any* n-dimensional manifold X, the left action (1) defines in a standard way a fibre bundle over X with type fibre Q, associated with the principal L^2_n-bundle of 2-frames $\mathscr{F}^2X$, denoted $\mathscr{C}X=\mathscr{F}^2_QX$. We call this fibre bundle the *connection bundle*. Its sections are *connection fields*, or *connections* on the underlying manifold X. One can also assign to any diffeomorphism α of n-dimensional manifolds its lifting $\mathscr{F}^2\alpha$, an isomorphism of the corresponding bundles of 2-frames, and the associated lifting $\mathscr{F}^2_Q\alpha$, an isomorphism of the corresponding fibre bundles with type-fibre $\mathscr{C}\alpha=T^1_nQ$. Then the correspondence $X\to\mathscr{C}X$, $\alpha\to\mathscr{C}\alpha$ is a Q-lifting, associated with the 2-frame lifting $\mathscr{F}^2$ from the category $\mathscr{D}_n$ to $\mathscr{FB}_n(L^2_n)$.

The notion of the connection bundle was introduced in this way for the *symmetric* tensor product $Q=\mathbf{R}^n\otimes((\mathbf{R}^n)^*\odot(\mathbf{R}^n)^*)$ in the paper Krupka [K9], with the aim to study differential invariants of symmetric linear connections. The formal Christoffel symbols entering formula (1) are in general *not* symmetric.

Now the q-lifting $X\to\mathscr{C}X$, $\alpha\to\mathscr{C}\alpha$ induces in a standard way its r-jet prolongation liftings $X\to J^r\mathscr{C}X$, $\alpha\to J^r\mathscr{C}\alpha$ from $\mathscr{D}_n$ to $\mathscr{FB}_n(L^{r+2}_n)$. In this example we need the case $r=1$. If X is a fixed n-dimensional manifold with some local coordinates (x^i) are some local coordinates on X, then the associated fibred coordinates on $\mathscr{C}X$ are (x^i,Γ^i_{jk}), and the associated coordinates on $J^1\mathscr{C}X$ are $(x^i,\Gamma^i_{jk},\Gamma^i_{jk,l})$, where the coordinate functions $\Gamma^i_{jk,l}$ are defined by the formal derivative operator as $\Gamma^i_{jk,l}=d_l\Gamma^i_{jk}$.

Using these coordinates we set

$$R_{ik}=\Gamma^s_{ik,s}-\Gamma^s_{is,k}+\Gamma^s_{ik}\Gamma^m_{sm}-\Gamma^m_{is}\Gamma^s_{km} \tag{2}$$

and

(3) $$\lambda_{\mathscr{C}} = \sqrt{|\det R_{ij}|} \cdot \omega_0 .$$

The system of functions R_{ik} is called the *formal Ricci tensor*, and $\lambda_{\mathscr{C}}$ is a global horizontal n-form, defined on the fibred manifold $J^1\mathscr{C}X$. Formula (3) concludes the construction of a natural Lagrange structure $(\mathscr{C}X, \lambda_{\mathscr{C}})$.

We show that the principal Lepage equivalent of the of the Lagrangian $\lambda_{\mathscr{C}}$ is given by

(4) $$\Theta_{\mathscr{C}} = \sqrt{|\det R_{ij}|}\left(\omega_0 + \frac{1}{2}(R^{jk}\delta^l_i - R^{jl}\delta^k_i)\omega^i_{jk} \wedge \omega_l\right),$$

where

(5) $$\begin{aligned} &\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n , \\ &\omega_l = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{l-1} \wedge dx^{l+1} \wedge \ldots \wedge dx^n , \\ &\omega^i_{jk} = d\Gamma^i_{jk} - \Gamma^i_{jk,s} dx^s . \end{aligned}$$

Denote for further calculations $\upsilon = \det R_{rs}$ and $C = \sqrt{|\upsilon|}$. We shall consider the open set in the fibred manifold $J^1\mathscr{C}X$ defined by the condition $\upsilon \neq 0$. Differentiating we have

(6) $$\begin{aligned} \frac{\partial C}{\partial \Gamma^i_{jk}} &= \frac{1}{2\sqrt{|\upsilon|}} \operatorname{sgn}\upsilon \frac{\partial \upsilon}{\partial R_{pq}} \frac{\partial R_{pq}}{\partial \Gamma^i_{jk}} = \frac{1}{2\sqrt{|\upsilon|}} \operatorname{sgn}\upsilon \cdot \upsilon \cdot R^{pq} \frac{\partial R_{pq}}{\partial \Gamma^i_{jk}} \\ &= \frac{\sqrt{|\upsilon|}}{2} R^{pq} \cdot (\delta^s_i \delta^j_p \delta^k_q \Gamma^m_{sm} + \delta^m_i \delta^j_s \delta^k_m \Gamma^s_{pq} - \Gamma^s_{qm} \delta^m_i \delta^j_p \delta^k_s - \Gamma^m_{ps} \delta^s_i \delta^j_q \delta^k_m) \\ &= \frac{\sqrt{|\upsilon|}}{2} (R^{jk}\Gamma^m_{im} + \delta^k_i R^{pq}\Gamma^j_{pq} - R^{jq}\Gamma^k_{qi} - R^{pj}\Gamma^k_{pi}), \end{aligned}$$

and

(7) $$\begin{aligned} \frac{\partial C}{\partial \Gamma^i_{jk,l}} &= \frac{\sqrt{|\upsilon|}}{2} R^{pq} \frac{\partial R_{pq}}{\partial \Gamma^i_{jk,l}} = \frac{\sqrt{|\upsilon|}}{2} R^{pq} (\delta^s_i \delta^j_p \delta^k_q \delta^l_s - \delta^s_i \delta^j_p \delta^k_s \delta^l_q) \\ &= \frac{\sqrt{|\upsilon|}}{2} (R^{jk}\delta^l_i - R^{jl}\delta^k_i). \end{aligned}$$

Hence the principal Lepage equivalent is

(8) $$\Theta_{\mathscr{C}} = C\omega_0 + \frac{\partial C}{\partial \Gamma^i_{jk,l}} \omega^i_{jk} \wedge \omega_l = \sqrt{|\rho|}\left(\omega_0 + \frac{1}{2}(R^{jk}\delta^l_i - R^{jl}\delta^k_i)\omega^i_{jk} \wedge \omega_l\right)$$

as required.

Formula (4) can be used for explicit description of the properties of the variational functional

$$(9) \qquad \Gamma_\Omega(\tau_X) \ni \Gamma \to \int_\Omega J^1\Gamma^* \lambda_{\mathscr{C}} = \int_\Omega J^1\Gamma^* \Theta_{\mathscr{C}} \in \mathbf{R},$$

for connections Γ on an n-dimensional manifold X; in this formula τ_X is the projection of the fibred manifold $\mathscr{C}X$ onto X. In particular, we can determine the *Euler-Lagrange form* $p_1 d\Theta_{\mathscr{C}}$ for extremal connections and the corresponding Noether's currents. We do not analyse the resulting formulas here.

Remark A fundamental notion of the differential geometry of connections on a manifold X is the curvature tensor. From the point of view of the *variational geometry*, this notion can be represented by the *formal curvature tensor*

$$(10) \qquad R^l_{ikj} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{ik}\Gamma^m_{jm} - \Gamma^m_{ij}\Gamma^l_{km},$$

defined on the 1-jet prolongation $J^1\mathscr{C}X$ of the bundle of connections $\mathscr{C}X$. Note that the formal Ricci tensor (2) represents the trace of the formal curvature tensor (10) in the indices l and j; one can also consider a different variational functional for connection fields whose Lagrangian is based on the trace of R^l_{ikj} in the indices l and i, $\lambda = \sqrt{|\det R^s_{skj}|}\,\omega_0$.

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