7 Elementary sheaf theory

The purpose of this chapter is to explain selected topics of the sheaf theory over paracompact, Hausdorff topological spaces. The choice of questions we consider are predetermined by the global variational theory over (topologically nontrivial) fibred manifolds, namely by the problem how to characterize differences between the local and global properties of the Euler-Lagrange mapping, between *locally* and *globally trivial Lagrangians*, and *locally and globally variational source forms*. To this purpose the central topic we follow is the *abstract De Rham theorem* and its consequences. In the context of this book, the cohomology of abstract sheaves should be compared with the cohomology of the associated complexes of global sections, and the cohomology of underlying smooth manifolds.

This chapter requires basic knowledge of the point-set topology; to help the reader some parts of the topology of local homeomorphisms have been included. Our treatment, intended for larger audience of readers who are not specialists in algebraic topology and sheaf theory, includes all proofs and their technical details, and from this point of view is wider than similar advanced texts in specialized monograph literature.

The main reference covering the choice of material needed in this book is Wells [We]; for different aspects of the sheaf theory, esp. the cohomology, we also refer to Bott and Tu [BT], Bredon [Br], Godement [Go], Lee [L] and Warner [W].

7.1 Sheaf spaces

Recall that a continuous mapping $\sigma : S \to X$ of a topological space S into a topological space X is called a *local homeomorphism*, if every point $s \in S$ has a neighbourhood V such that the set $\sigma(V)$ is open set in X and the restricted mapping $\sigma|_V$ is a homeomorphism of V onto $\sigma(V)$.

By a *sheaf space structure* on a topological space S we mean a topological space X together with a *surjective* local homeomorphism $\sigma : S \to X$. The topological space S endowed with a sheaf space structure is called a *sheaf space* or an *étalé space*. X is the *base space*, and σ is the *projection* of the sheaf space S. For every point $x \in X$, the set $S_x = \sigma^{-1}(x)$ is called the *fibre* over x. We denote a sheaf space by $\sigma : S \to X$ or just by S when no misunderstanding may possibly arise.

A mapping $\gamma : Y \to S$, where Y is a subset of X, is called a *section* of the topological space$S$ over Y (or more precisely, a section of the projection σ), if $\gamma(x) \in S_x$ for all points $x \in Y$. Obviously, γ is a section if and only if

(1) $$\sigma \circ \gamma = \mathrm{id}_Y .$$

If $Y = X$, γ is a *global section*. The set of sections (resp. *continuous* sections), defined on a set *U* is denoted by $(\operatorname{Sec} S)U$ (resp. $(\operatorname{Sec}^{(c)} S)U$, and also $\Gamma(U,S)$). The union of the sets $(\operatorname{Sec} S)U$ (resp. $(\operatorname{Sec}^{(c)} S)U$) through $U \subset X$ is denoted by $\operatorname{Sec} S$ (resp. $\operatorname{Sec}^{(c)} S$).

Lemma 1 (a) *A local homeomorphism is an open mapping.*

(b) *The restriction of a local homeomorphism to a topological subspace is a local homeomorphism.*

(c) *The composition of two local homeomorphisms is a local homeomorphism.*

Proof (a) Let $\sigma : S \to X$ be a local homeomorphism. Any open subset *V* of *S* is expressible as tue union $\bigcup V_\iota$, where V_ι is an open set such that $\sigma|_{V_\iota}$ is a homeomorphism. Then the set $\sigma(V) = \bigcup \sigma(V_\iota)$ must be open as the union of open sets.

(b) Let $T \subset S$ be a subspace and $V \subset S$ an open set such that $\sigma|_V$ is a homeomorphism. Then $V \cap T = V \cap (\sigma|_V)^{-1}(\sigma(T)) = (\sigma|_V)^{-1}(\sigma(V) \cap \sigma(T))$, and $\sigma(V \cap T) = \sigma(V) \cap \sigma(T)$. Thus the image of the open set $\sigma(V \cap T) \subset T$ by $\sigma|_T$ is open in $\sigma(T)$. Since $\sigma|_T|_{V\cap T} = \sigma|_{V \cap T}$ is a continuous bijection and is an open mapping hence a homeomorphism, $\sigma|_T|_{V\cap T}$ is a homeomorphism.

(c) The proof is immediate.

Lemma 2 *Let S be a sheaf space with base X and projection* σ.

(a) *To every point* $s \in S$ *there exists a neighbourhood U of the point* $x = \sigma(s)$ *in X and a continuous section* $\gamma : U \to S$ *such that* $\gamma(x) = s$.

(b) *Let* γ *be a continuous section of S, defined on an open subset of X. Then to every point x from the domain of* γ *and every neighbourhood V of* $\gamma(x)$ *such that* $\sigma|_V$ *is a homeomorphism, there exists a neighbourhood U of X such that* $\gamma(U) \subset V$ *and* $\gamma|_U = (\sigma|_V)^{-1}|_U$.

(c) *If U and V are open sets in X and* $\gamma : U \to S$ *and* $\delta : V \to S$ *are continuous sections, then the set* $\{x \in U \cap V \mid \gamma(x) = \delta(x)\}$ *is open.*

(d) *Every continuous section of S, defined on an open set in X, is an open mapping.*

Proof (a) We choose a neighbourhood *V* of *s* such that $\sigma|_V$ is a homeomorphism and set $U = \sigma(V)$, $\gamma = (\sigma|_V)^{-1}$.

(b) By continuity of γ, we choose a neighbourhood *U* of *x* such that $\gamma(U) \subset V$, and apply the mapping $\gamma = (\sigma|_V)^{-1}$ to both sides of the identity $\sigma|_V \circ \gamma|_U = \mathrm{id}_U$. We get $\gamma|_U = (\sigma|_V)^{-1}$.

(c) We may suppose that $\{x \in U \cap V \mid \gamma(x) = \delta(x)\} \neq \emptyset$. Choose a point $x_0 \in U \cap V$, and a neighbourhood *W* of the point $\gamma(x) = \delta(x)$ such that $\sigma(W)$ is open and $\sigma|_W$ is a homeomorphism. By condition (b), x_0 has a neighbourhood U_0 such that $\gamma(U_0) \subset V$ and $\gamma|_{U_0} \subset (\sigma|_V)^{-1}|_{U_0}$. Analogously x_0 has a neighbourhood of V_0 such that $\delta(V_0) \subset W$ and $\delta|_{V_0} \subset (\sigma|_W)^{-1}|_{V_0}$. Thus $\gamma|_{U_0 \cap V_0} \subset (\sigma|_W)^{-1}|_{U_0 \cap V_0} = \delta|_{U_0 \cap V_0}$ proving (c).

(d) Let *U* be an open set in *X*, $\gamma : U \to S$ a continuous section. It is sufficient to show that the set $\gamma(U) \subset S$ is open. To every point $x \in U$ we assign a neighbourhood $V_{\gamma(x)}$ of the point $\gamma(x)$ such that $\sigma(V_{\gamma(x)})$ is open and the mapping $\sigma|_{V_{\gamma(x)}}$ is a homeomorphism, and a neighbourhood U_x of the point x such that $U_x \subset U$, $\gamma(U_x) \subset V_{\gamma(x)}$, and $\gamma|_{U_x} = (\sigma_{\gamma(x)})^{-1}|_{U_x}$ (see Part (b) of this lemma). Then since $(\sigma_{\gamma(x)})^{-1} : \sigma(V_{\gamma(x)}) \to V_{\gamma(x)} \subset S$ is a homeomorphism, $\gamma(U_x)$ is open in S, and we have $\gamma(U) = \gamma(\bigcup U_x) = \bigcup \gamma(U_x)$, which is an open set.

Remark 1 Suppose that S is a *Hausdorff* space. Let $\gamma : U \to S$ and $\delta : V \to S$ be two continuous sections, defined on open sets U and V in X, such that $U \cap V \neq \emptyset$ and $\gamma(x_0) = \delta(x_0)$ at a point $x_0 \in U \cap V$. Then $\gamma = \delta$ on the connected component of $U \cap V$ containing x_0. Indeed, since S is Hausdorff, the set $U_0 = \{x \in U \cap V \mid \gamma(x) = \delta(x)\}$ is closed. Since by Lemma 2, (c) the set U_0 is open, it must be equal to the connected component of the point x_0. This remark shows that if a sheaf space S is Hausdorff, it satisfies the *principle of analytic continuation*. On the other hand if the principle of analytic continuation is not valid, S cannot be Hausdorff.

Suppose that we have a *set* S, a topological space X and a mapping $\sigma : S \to X$. Then there exists at most one topology on S for which σ is a local homeomorphism. Indeed, if τ_1 and τ_2 are two such topologies, $s \in S$ a point, $V \in \tau_1$ and $W \in \tau_2$ open sets such that $\sigma|_V$ and $\sigma|_W$ are homeomorphisms, then $U = \sigma(V) \cap \sigma(W)$ is a neighbourhood of the point $x = \sigma(s)$, and $\sigma^{-1}(U)$ is a neighbourhood of the point s both in τ_1 and τ_2. This implies, in particular, that the identity mapping id_S is a homeomorphism.

Let S be a sheaf space with base X and projection σ. Beside its own topology, the set S may be endowed with the *final topology*, associated with the family of continuous sections, defined on open subsets of X.

Lemma 3 *Let S be a sheaf space with base X and projection σ.*

(a) *The open sets $V \subset S$ such that $\sigma|_V$ is a homeomorphism form a basis of the topology of S.*

(b) *The topology of S coincides with the final topology, associated with the set $\mathrm{Sec}^{(c)} S$ of continuous sections of S.*

(c) *The topology induced on fibres of S is the discrete topology.*

Proof (a) This is an immediate consequence of the definition of a local homeomorphism.

(b) If a subset W of S is an open set in the topology of S, then for every continuous section γ of S, $\gamma^{-1}(W)$ is an open subset of X hence by definition, W is open in the final topology. Conversely, let W be open in the final topology. For any section $\gamma : U \to S$, $\gamma(\gamma^{-1}(W)) \subset W \cap \gamma(U) \subset W$. If the section γ is continuous, then by the definition of the final topology, $\gamma^{-1}(W)$ is an open set; moreover, since γ is open in the topology of S (Lemma 2, (d)), the set $\gamma(\gamma^{-1}(W))$ is open in the topology of S. But by Lemma 2, (a), the sets $\gamma(\gamma^{-1}(W))$ cover W which implies that W is open in the topological space S.

(c) This assertion is evident.

Let $\sigma : S \to X$ and $\tau : T \to Y$ be two sheaf spaces. Recall that a mapping $f : S \to T$ is said to be *projectable*, if

(2) $$\tau \circ f = f_0 \circ \sigma$$

for some mapping $f_0 : X \to Y$. Obviously, the same can be expressed by saying that there exists f_0 such that the diagram

(3) $$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow \sigma & & \downarrow \tau \\ X & \xrightarrow{f_0} & Y \end{array}$$

commutes. If f_0 exists, it follows from condition (2) that it is unique. If *f* is continuous, then the mapping f_0 is also continuous since it is always expressible on open sets as $f_0 = \tau \circ f \circ \gamma$ for some continuous sections γ of the topological space *S*.

A continuous projectable mapping $f : S \to T$ is called a *morphism* of the sheaf space *S* into the sheaf space *T*, or just a *sheaf space morphism*.

Lemma 4 *Let $\sigma : S \to X$ and $\tau : T \to Y$ be sheaf spaces, $f : S \to T$ a surjective mapping and $f_0 : X \to Y$ its projection. Then f is a local homeomorphism if and only if f_0 is a local homeomorphism.*

Proof Let $x \in X$ be a point, γ a continuous section of *S* defined on a neighbourhood of *x*. Choose a neighbourhood *W* of the point $f(\gamma(x))$ such that $\tau|_W$ is a homeomorphism, a neighbourhood *V* of $\gamma(x)$ such that $f(V) \subset W$, and a neighbourhood *U* of *x* such that $U \subset \sigma(V)$ and $\gamma|_U$ is a homeomorphism. Then $\tau|_W \circ f|_V \circ \gamma|_U = (\tau \circ f \circ \gamma)|_U$, and from condition (2), $(\tau \circ f \circ \gamma)|_U = (f_0 \circ \sigma \circ \gamma)|_U = f_0|_U$ proving Lemma 4.

Denote by f_x the *restriction* of a mapping $f : S \to T$ to the fibre S_x over a point $x \in X$. If $X = Y$, we have the following assertion.

Corollary 1 *Let $\sigma : S \to X$ and $\tau : T \to X$ be two sheaf spaces, and let $f : S \to T$ be a projectable mapping whose projection is the identity mapping* id_X.

(a) *f is a local homeomorphism.*

(b) *f is injective (resp. surjective) if and only if f_x is injective (resp. surjective) for each $x \in X$.*

Proof (a) This follows from Lemma 4.

(b) These assertions follow immediately from the definitions.

Let $\sigma : S \to X$ and $\tau : T \to Y$ be two sheaf spaces. The Cartesian product $S \times T$ together with the mapping $\sigma \times \tau : S \times T \to X \times Y$ defined by the formula $(\sigma \times \tau)(s,t) = (\sigma(s), \tau(t))$ is a sheaf space, called the *product* of *S* and *T*. If $X = Y$, we define a subset of the Cartesian product $S \times T$ by $S \times_X T = \{(s,t) \in S \times T \mid \sigma(t) = \tau(s)\}$, and a mapping $\sigma \times_X \tau : S \times_X T \to X$ by

$(\sigma \times_X \tau)(s,t) = \sigma(s) = \tau(t)$. If we consider the set $S \times_X T$ with the induced topology, the mapping $\sigma \times_X \tau$ defines on $S \times_X T$ the structure of a sheaf space, called the *fibre product* of the sheaf spaces S and T.

Let $\sigma : S \to X$, $\sigma' : S' \to X$ and $\tau : T \to Y$, $\tau' : T' \to Y$ be sheaf spaces. Let $f : S \to T$ and $f' : S' \to T'$ be two projectable mappings over the same projection $f_0 : X \to Y$. For every point (s,s') we define a mapping $f \times_X f' : S \times S' \to T \times T'$ by $(f \times_X f')(s,s') = (f(s), f'(s'))$. This gives rise to the following commutative diagram

$$\begin{array}{ccc} S \times_X S' & \xrightarrow{\iota} & S \times S' \\ \downarrow f \times_X f' & & \downarrow f \times f' \\ T \times_Y T' & \xrightarrow{\kappa} & T \times T' \end{array} \tag{4}$$

where the horizontal arrows denote the canonical inclusions. The mapping $f \times_X f'$ is called the *fibre product* of f and f'. It is easily seen that if f and f' are *continuous*, then the fibre product $f \times_X f'$ is also continuous: indeed, for any open set U in $T \times_Y T'$ there exists an open set V in $T \times T'$ such that $U = \kappa^{-1}(V)$; since

$$\begin{aligned} (f \times_X f')^{-1}(U) &= (f \times_X f')^{-1}(\kappa^{-1}(V)) \\ &= (\kappa \circ (f \times_X f'))^{-1}(V) = ((f \times f') \circ \iota)^{-1}(V) \end{aligned} \tag{5}$$

is an open set in $S \times_X S'$, the mapping $f \times_X f'$ must be continuous.

We give some examples of sheaf spaces; using these examples we also discuss properties of the topology of sheaf spaces.

Examples 1. Continuous global sections of a sheaf space need not necessarily exist. Consider for example the *real line* $\mathbf{R} = \mathbf{R}^1$ and the *unit circle* $S^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$. The mapping $\sigma : \mathbf{R} \to S^1$, defined by the formula $\sigma(s) = (\cos 2\pi s, \sin 2\pi s)$ is a surjective local homeomorphism. It is easily seen that σ has *no* continuous global section. Suppose the opposite. Then if γ is a continuous global section, $\gamma(S^1) \subset \mathbf{R}$ is a non-void compact and open set in $\mathbf{R}$ hence coincides with $\mathbf{R}$. However, this is a contradiction since $\mathbf{R}$ is non-compact.

2. Let $S^2 = \{(x,y) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbf{R}^3$, and consider an equivalence relation $\sim$ on S^2 "$(x,y,z) \sim (x',y',z')$ *if either* $(x,y,z) = (x',y',z')$ *or* $(x,y,z) \sim -(x',y',z')$". The quotient space $S^2/\sim$ is called the *real projective plane* and is denoted by RP^2. The quotient projection $\sigma : S^2 \to RP^2$ is a sheaf space. The set RP^2 can be identified with the set of straight lines in $\mathbf{R}^3$ passing through the origin.

3. A local homeomorphism admitting a *global* continuous section is *not* necessarily a homeomorphism: Define a subspace $S = \{(x,r) \in \mathbf{R}^2 \mid r = 0,1\}$ of $\mathbf{R}^2$ and a mapping $\sigma : S \to \mathbf{R}$ by the condition $\sigma(x,r) = x$. Then the

mapping $\gamma:\mathbf{R}\to S$ defined by $\gamma(x,0)=x$ is a global continuous section of S but σ is not a homeomorphism.

4. Consider the subspace $S=\{(x,r)\in\mathbf{R}^2 \mid r=-1,1\}$ of $\mathbf{R}^2$, two points $a,b\in\mathbf{R}$ such that $a<b$, and a partitions of S defined by the subsets $\{(x,-1)\}$, $\{(x,1)\}$ if $x\le a$, $x\ge b$, and $\{(x,-1),(x,1)\}$ if $a<x<b$ (one- and two-element subsets). Let $\sim$ be an equivalence relation on S defined by this partition, and denote $X=X/\sim$. The quotient mapping of S onto X is a surjective local homeomorphism; the quotient space X is *not* Hausdorff. Further, assigning to each of the sets $\{(x,-1)\}$, $\{(x,1)\}$, $\{\{x,-1\},\{x,-1\}\}$ the point $x\in\mathbf{R}$, we obtain a local homeomorphim of X onto the real line $\mathbf{R}$.

5. The topological subspace S of $\mathbf{R}^3$, defined in a parametric form as $S=\{(x,y,z)\in\mathbf{R}^3 \mid x=\cos t, y=\sin t, z=t, t\in\mathbf{R}\}$ (the *helix*), together with the restriction of the Cartesian projection $\pi:\mathbf{R}^3\to\mathbf{R}^2$ to S is a local homeomorphism of S onto the circle S^1 (Example 1). This example shows that for a general local homeomorphism $\sigma:S\to X$ the topology of S does *not* necessarily coincide with the *initial* topology of the topology of X by the mapping σ.

6. If $\sigma:S\to X$ is a sheaf space and Y is an open subset of X, then the restriction $\sigma|_{\sigma^{-1}(Y)}:\sigma^{-1}(Y)\to Y$ is a sheaf space.

7. The Cartesian projection $\pi:X\times Q\to X$, where X is a topological space and Q is a non-void set endowed with the *discrete* topology, is a sheaf space.

8. Using the notation of Example 1 we obtain a surjective local homeomorphism $\sigma\times\sigma$ of the real plane $\mathbf{R}^2$ onto the *torus* $S^1\times S^1$.

7.2 Abelian sheaf spaces

An *Abelian sheaf space structure* on a topological space S consists of a sheaf space structure with base X and projection σ such that for every point $x\in X$ the fibre S_x over x is an Abelian group and the subtraction mapping $S\times_X S\ni(s,t)\to s-t\in S$ is continuous. A topological space S, endowed with an Abelian sheaf space structure is called an *Abelian sheaf space*. We usually denote an Abelian sheaf space $\sigma:S\to X$, or simply by S. Sometimes, when no misunderstanding may arise, we call an Abelian sheaf space just a *sheaf space*.

A *sheaf subspace* of the Abelian sheaf space S is an open set $T\subset S$ such that for every point $x\in X$, the intersection $T\cap S_x$ is a subgroup of the Abelian group S_x.

The Abelian sheaf space structure on a topological space S induces the Abelian group structure on sections of S. The *zero section* is the mapping $\theta:X\to S$, assigning to a point $x\in X$ the neutral element of the Abelian group S_x. Clearly, θ is a *global continuous section* of S: if $x_0\in X$ is a point and γ is any continuous section over a neighbourhood U of x_0, then $\theta(x)=\gamma(x)-\gamma(x)$ on U, which implies that θ is expressible as the composition of two continuous mappings $U\ni x\to(\gamma(x),\gamma(x))\in S\times_X S$ and

$S \times_X S \ni (s,t) \to s - t \in S$. The open set $\theta(X)$ is called the *zero sheaf subspace* of *S*. For any two sections γ and δ, defined on the same set in *X*, one can naturally define the *sum* $\gamma + \delta$ and the *opposite* $-\gamma$ of the section γ. Thus, the set of sections over an open subset of *X* has an Abelian group structure. If the sections γ and δ are continuous, then $\gamma + \delta$ and $-\gamma$ are also continuous.

For any subspace *Y* of the base space *X*, the restriction of the projection σ to the set $\sigma^{-1}(Y)$ is a sheaf subspace of the Abelian sheaf space *S* with base *Y*, called the *restriction* of *S* to *Y*.

Remark 2 If a local homeomorphism admits an Abelian sheaf space structure, then it necessarily admits a continuous global section (the zero section). Conversely, local homeomorphisms, which do not admit a global continuous section, do not admit an Abelian sheaf space structure.

Examples 9. In this example we construct a sheaf space of Abelian groups, the *skyscraper sheaf space*, whose topology is *not* Hausdorff. Denote by $\mathbf{Z}$ the set of integers in the set of real numbers $\mathbf{R}$. Let *X* be a Hausdorff space, x_0 a point of *X*, and let *S* be a subset of the Cartesian product $X \times \mathbf{Z}$, defined as $S = (X \setminus \{x_0\}) \times \{0\}) \cup (\{x_0\} \times \mathbf{Z})$. The subsets of *S* of the form $U \times \{x_0\}$, where *U* is an open set in *X* and $\{x_0\} \notin U$, and $((V \setminus \{x_0\} \times \{0\}) \bigcup \{(x_0, z)\}$, where *V* is open in *X*, $x_0 \in V$ and $z \in \mathbf{Z}$, is a basis for a topology on *S*. In this topology the restriction of the first Cartesian projection is a local homeomorphism of *S* onto *X*. For any two different points $z_1, z_2 \in \mathbf{Z}$, every neighbourhood of the point $(x_0, z_1) \in S$ (resp. $(x_0, z_2) \in S$) contains a neighbourhood $((V_1 \setminus \{x_0\} \times \{0\}) \bigcup \{(x_0, z_1)\}$ of the point $(x_0, z_1) \in S$ (resp. $((V_2 \setminus \{x_0\} \times \{0\}) \bigcup \{(x_0, z_2)\}$ of $(x_0, z_2) \in S$), whose intersection is $((V_1 \cap V_2) \setminus \{x_0\}) \times \{0\}$. Assuming $(V_1 \cap V_2) \setminus \{x_0\} = \emptyset$, we get a neighbourhood $V_1 \cap V_2$ of $\{x_0\}$ equal to $\{x_0\}$. Thus, if $\{x_0\}$ is *not* an isolated point, *S* is *not* Hausdorff.

10. The restriction of the Cartesian projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ to the helix (Section 7.1, Example 5) is a surjective local homeomorphism of *S* onto the unit circle S^1. This local homeomorphism cannot be endowed with a sheaf structure because it does not admit a continuous global section.

11. Consider a topological space *X* and an Abelian group *G* with *discrete topology*. The Cartesian product $X \times G$, endowed with the product topology, and the first Cartesian projection is a sheaf space, called the *constant sheaf space* over *X with fibre G*. We usually denote this sheaf by G_X. If *U* is an open set in *X* and $\gamma : U \to G_X$ a continuous section, then the restriction of γ to any connected open subset *V* of *U* is constant, that is, of the form $V \ni x \to \gamma(x) = (x, g) \in G_X$ for some $g \in G$. Since the continuous image of a connected subspace is connected, the second Cartesian projection $\mathrm{pr}_2 \circ \gamma(V) \in G$ consists of a single point. In particular, every continuous section of a constant sheaf space is constant on connected components of the base, that is, *locally constant*.

12. The *trivial sheaf space of Abelian groups* over a topological space

X is defined as X together with the identity homeomorphism $\mathrm{id}_X : X \to X$, and trivial Abelian group structure on every fibre $\{x\} = \mathrm{id}_X^{-1}(x)$. Thus, the trivial sheaf space is the sheaf space 0_X.

13. Let T be a sheaf space of Abelian groups with base X and projection τ, and let R and S be two sheaf subspaces of T. For every point $x \in X$, $R_x + S_x$ is a subgroup of the Abelian group T_x. We set

(6) $$R + S = \bigcup_{x \in X} (R_x + S_x).$$

$R + S$ is an open subset of T: if $t \in R + S$, then $t = r + s$, where $r \in R$ and $s \in S$, and because R (resp. S) is a sheaf subspace of T, r (resp. s) has a neighbourhood U (resp. V) in R (resp. S) such that τ restricted to U (resp. V) is a homeomorphism. But both R and S are open in T. Thus $U + V$ is open in T, proving that $R + S$ is open in T. Therefore, $R + S$ is a sheaf subspace of T. We call this subspace the *sum* of R and S.

Let S and T be two Abelian sheaf spaces over a topological space X, σ and τ the corresponding projections. A projectable continuous mapping $f : S \to T$ over the identity mapping id_X is called a *morphism of Abelian sheaf spaces*, if for every point $x \in X$ the restriction $f_x = f|_{\sigma^{-1}(x)}$ to the fibre over x is a morphism of Abelian groups. A morphism $f : S \to T$ of Abelian sheaf spaces such that both f and f^{-1} are bijections, is called an *isomorphism* of Abelian sheaf spaces. The mapping id_S is the *identity morphism* of S. To simplify terminology, we sometimes call morphisms of Abelian sheaf spaces just *morphisms of sheaf spaces*, of *sheaf space morphisms*.

The composite $f \circ g$ of two morphisms of Abelian sheaf spaces is again a morphism of Abelian sheaf spaces.

Consider a sheaf space morphism $f : S \to T$ and set

(7) $$\mathrm{Ker} f = \{ s \in S \mid f(s) = 0 \}, \quad \mathrm{Im} f = f(S).$$

Obviously, these sets can be expressed as

(8) $$\mathrm{Ker} f = \bigcup_{x \in X} \mathrm{Ker} f_x, \quad \mathrm{Im} f = \bigcup_{x \in X} \mathrm{Im} f_x.$$

Lemma 5 *Let S and T be two Abelian sheaf spaces over a topological space X with projections σ and τ, $f : S \to T$ a sheaf space morphism.*

(a) $\mathrm{Ker} f$ *is a sheaf subspace of S.*

(b) $\mathrm{Im} f = f(S)$ *is a sheaf subspace of T.*

Proof (a) Since $\mathrm{Ker} f = f^{-1}(0(X))$, where $0(X)$ is the zero sheaf subspace of T, which is an open set in T, the set $\mathrm{Ker} f$ is open in S. Since $\sigma(\mathrm{Ker} f) = X$ and for each $x \in X$, $\mathrm{Ker} f \cap S_x$ is a subgroup of S_x, $\mathrm{Ker} f$ is a sheaf subspace of S.

(b) By Lemma 1, (b), the restriction of the projection τ to $f(S)$ is a local homeomorphism. The image of $\tau|_{f(S)}$ is given by $\tau(f(S)) = \sigma)S) = X$. For each point $x \in X$ the set $f(S) \cap T_x$ is a subgroup of T_x. The commutative diagram

$$\begin{array}{ccc} f(S) \times_X f(S) & \longrightarrow & T \times_X T \\ \downarrow & & \downarrow \\ f(S) & \longrightarrow & T \end{array} \tag{9}$$

in which the horizontal arrows are inclusions and the vertical arrows are subtractions (in fibres), shows that the subtractions $f(S) \times_X f(S) \to f(S)$ are continuous.

The sheaf subspace $\operatorname{Ker} f$ (resp. $\operatorname{Im} f$) is called the *kernel* (resp. *image*) of the morphism of Abelian sheaf spaces $f : S \to T$.

Let $\sigma : S \to X$ be a sheaf space, T a sheaf subspace of S. Consider an equivalence relation on S "$s_1 \sim s_2$ *if* $\sigma(s_1) = \sigma(s_2)$ *and* $s_1 - s_2 \in T$ ". Let S/T be the quotient space (endowed with the quotient topology), and let ρ denote the quotient projection; if $[s]$ is the class of an element $s \in S$, then $\rho(s) = [s]$. Define a mapping $\tau : S/T \to X$ by $\tau([s]) = \sigma(s)$. Since ρ is surjective, τ is a unique mapping such that

$$\tau \circ \rho = \sigma. \tag{10}$$

Since the composite $\tau \circ \rho = \sigma$ is continuous, τ is also continuous.

Note that for every point $x \in X$ the fibre $\tau^{-1}(x) = (S/T)_x = S_x / T_x$ has the structure of an Abelian group. We wish to show that the quotient S/T has the structure of a sheaf space over X with projection τ, and ρ is a morphism of Abelian sheaf spaces.

It is easily seen that the quotient mapping is open. Let $V \subset S$ be an open set. To show that $\rho(V)$ is open in the quotient topology means to show that $V' = \rho^{-1}(\rho(V))$ is open in the topology of S. But $V' = V + (\sigma|_T)^{-1}(\sigma(V))$. Since through every point of T passes a continuous section, defined on an open subset of $\sigma(V)$, the set V' is expressible as a union of open sets arising as images of continuous sections (Lemma 2, (d)). Thus ρ is open.

We show that ρ is a local homeomorphism. Clearly, if $s \in S$ is a point and V is its neighbourhood such that $\sigma|_V$ is a bijection, then $\sigma|_V = \tau|_W \circ \rho|_V$, where $W = \rho(V)$; since $\rho|_V : V \to W$ is surjective, both $\tau|_W$ and $\rho|_V$ must be bijective. Hence $(\sigma|_V)^{-1} \circ \tau|_W \circ \rho|_V = \mathrm{id}_V$. Thus $(\sigma|_V)^{-1} = (\rho|_V)^{-1} \circ (\tau|_W)^{-1}$ and $\rho|_V \circ (\sigma|_V)^{-1} \circ \tau|_W = \mathrm{id}_W$. But W is open since the quotient mapping ρ is open and $(\rho|_V)^{-1} = (\sigma|_V)^{-1} \circ \tau|_W$, which is a continuous mapping. This proves that $\rho|_V$ is a homeomorphism. Now it is easy to conclude that the mapping τ is a local homeomorphism: we take the sets W and V as above and write $\tau|_W = \sigma|_V \circ (\rho|_V)^{-1}$.

It remains to check that the subtraction in S/T is continuous. We have a commutative diagram

$$
(11) \qquad \begin{array}{ccc} S \times_X S & \xrightarrow{\varphi} & S \\ \downarrow \rho \times_X \rho & & \downarrow \rho \\ (S/T) \times_X (S/T) & \xrightarrow{\psi} & S/T \end{array}
$$

in which φ denotes the mapping $(s_1, s_2) \to s_1 - s_2$ and ψ is the mapping $([s_1],[s_2]) \to [s_1 - s_2])$, and $\rho \times_X \rho$ is the fibre product. But ρ, φ and $\rho \times_X \rho$ are local homeomorphisms, so from Lemma 4 we conclude that ψ is also a local homeomorphism.

The Abelian sheaf space S/T is called the *quotient sheaf space* of the sheaf space *S* by *T*. The morphism of Abelian sheaf spaces $\rho : S \to S/T$ is the *quotient projection*.

7.3 Sections of Abelian sheaf spaces

Suppose that we have an Abelian sheaf space *S* with base *X* and projection σ. Consider the correspondence $U \to \mathrm{Sec}^{(c)} U$, denoted by $\mathrm{Sec}^{(c)}$, assigning to every non-empty open set *U* in *X* the Abelian group $\mathrm{Sec}^{(c)} U$ of continuous sections over *U*. We extend this correspondence to the whole topology of *X* by assigning to the empty set $\emptyset$ the trivial one-point Abelian group 0. To any open sets *U*, *V* in *X* such that $U \subset V$ we assign a group morphism $s_{VU} : (\mathrm{Sec}^{(c)} S)V \to (\mathrm{Sec}^{(c)} S)U$ defined by

$$
(1) \qquad s_{VU} \circ \gamma = \gamma|_U
$$

(the *restriction* of the continuous section γ to the set *U*). We get a family $\{(\mathrm{Sec}^{(c)} S)U\}$, labelled by the set *U*, and a family $\{s_{VU}\}$, labelled by the sets *U* and *V*. s_{VU} are called *restriction mappings*, or *restrictions* of the Abelian sheaf space *S*.

We say that two continuous sections $\gamma, \delta \in (\mathrm{Sec}^{(c)} S)U$ *coincide locally*, if there exists an open covering $\{U_\iota\}_{\iota \in I}$ of *U* such that $s_{UU_\iota}(\gamma) = s_{UU_\iota}(\delta)$ for each ι from the indexing set *I*. A family $\{\gamma_\iota\}_{\iota \in I}$ of continuous sections $\gamma_\iota \in (\mathrm{Sec}^{(c)} S)U_\iota$ is said to be *compatible*, if $s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) = s_{U_\kappa, U_\iota \cap U_\kappa}(\gamma_\kappa)$ for all indices $\iota, \kappa \in I$. We say that the family of sections $\{\gamma_\iota\}_{\iota \in I}$ *locally generates* a section $\gamma \in (\mathrm{Sec}^{(c)} S)U$, where $U = \bigcup U_\iota$, if $s_{UU_\iota}(\gamma) = \gamma_\iota$ for all $\iota \in I$; we also say that γ is *locally generated* by the family $\{\gamma_\iota\}_{\iota \in I}$. A family of continuous sections, locally generating a continuous section, is compatible.

The following are basic properties of the restriction mappings s_{VU} and the Abelian groups $(\mathrm{Sec}^{(c)} S)U$.

Lemma 6 *The correspondence* $\mathrm{Sec}^{(c)} S$ *has the following properties:*

(1) $(\mathrm{Sec}^{(c)} S)\emptyset = 0$.

(2) $s_{UU} = \mathrm{id}_U$ *for every open set U in X.*

(3) $s_{WU} = s_{VU} \circ s_{WV}$ *for all open sets U, V, W such that* $U \subset V \subset W$.

(4) *If two continuous sections* γ *and* δ *coincide locally, then* $\gamma=\delta$.

(5) *Every compatible family of continuous sections of S locally generates a continuous section of S.*

Proof (1) holds by definition, and assertions (2) and (3) are immediate. We prove condition (4). Let $\{U_\iota\}_{\iota\in I}$ be a family of open sets in *X*, $U=\bigcup U_\iota$, $\gamma_1,\gamma_2\in(\mathrm{Sec}^{(c)}S)U$ two sections such that the restrictions satisfy $\gamma_1|_{U_\iota}=\gamma_2|_{U_\iota}$ for all ι . Let $x\in U$. Then by hypothesis there exists an index ι such that $x\in U_\iota$; consequently, $\gamma_1(x)=\gamma_1|_{U_\iota}(x)=\gamma_2|_{U_\iota}(x)=\gamma_2(x)$, and since the point *x* is arbitrary, we have $\gamma_1=\gamma_2$ proving (4). Now we prove condition (5). Let $\{\gamma_\iota\}_{\iota\in I}$ be a family such that $\gamma_\iota\in(\mathrm{Sec}^{(c)}S)U_\iota$ and $\gamma_\iota|_{U_\iota\cap U_\kappa}=\gamma_\kappa|_{U_\iota\cap U_\kappa}$ for all indices $\iota,\kappa\in I$. Let $x\in U$ be a point. Then there exists an index ι such that $x\in U_\iota$; we choose ι and set $\gamma(x)=\gamma_\iota(x)$. If also $x\in U_\kappa$, then $\gamma_\iota|_{U_\iota\cap U_\kappa}(x)=\gamma_\kappa|_{U_\iota\cap U_\kappa}(x)$ hence $\gamma(x)=\gamma_\kappa(x)$, so the value $\gamma(x)$ is defined independently of the choice of the index ι . It follows from the definition that γ , defined in this way, is continuous on U_ι for every ι hence on *U*, thus, $\gamma\in(\mathrm{Sec}^{(c)}S)U$ proving (5).

The correspondence $\mathrm{Sec}^{(c)}S$, assigning to an open set $U\subset X$ the Abelian group $(\mathrm{Sec}^{(c)}S)U$, is called the *sheaf of continuous sections* of the Abelian sheaf space *S*, or just the *Abelian sheaf*, *associated* with *S*.

Let $\sigma:S\to X$ and $\tau:T\to X$ be two Abelian sheaf spaces over the same base space *X*, $f:S\to T$ a sheaf space morphism. Consider the associated Abelian sheaves $\mathrm{Sec}^{(c)}S$ and $\mathrm{Sec}^{(c)}T$, and denote by $\{s_{VU}\}$ and $\{t_{VU}\}$ the corresponding families of restrictions in these sheaves. If γ is a continuous section of *S*, $\gamma\in(\mathrm{Sec}^{(c)}S)U$, then $f\circ\gamma\in(\mathrm{Sec}^{(c)}T)U$. Setting

(2) $$f_U(\gamma)=f\circ\gamma,$$

we obtain an Abelian group morphism $f_U:(\mathrm{Sec}^{(c)}S)U\to(\mathrm{Sec}^{(c)}T)U$. Obviously, for every pair of open sets $U,V\subset X$ such that $U\subset V$, the diagram

(3) $$\begin{array}{ccc} (\mathrm{Sec}^{(c)}S)V & \xrightarrow{f_V} & S \\ \downarrow s_{VU} & & \downarrow t_{VU} \\ (\mathrm{Sec}^{(c)}S)U & \xrightarrow{f_U} & S/T \end{array}$$

commutes. The family $f=\{f_U\}$, labelled by *U*, is called the *Abelian sheaf morphism* of the sheaf $\mathrm{Sec}^{(c)}S$ into the sheaf $\mathrm{Sec}^{(c)}T$, *associated* with the Abelian sheaf space morphism $f:S\to T$. We usually denote the associated Abelian sheaf morphism by $f:\mathrm{Sec}^{(c)}S\to\mathrm{Sec}^{(c)}T$.

Now we study the sheaves associated with a sheaf subspace of an Abelian sheaf space, and the sheaves associated with the kernel and the image of an Abelian sheaf space morphism. Recall that the kernel $\mathrm{Ker}\,f$ and the image $\mathrm{Im}\,f$ of a sheaf space morphism $f:S\to T$ is a sheaf subspace of *S* and *T*, respectively.

Lemma 7 (a) *S is a sheaf subspace of an Abelian sheaf space T if and only if the Abelian group* $(\mathrm{Sec}^{(c)} S)U$ *is a subgroup of* $(\mathrm{Sec}^{(c)} T)U$ *for every open set U in X.*

(b) *Let* $\sigma : S \to X$ *and* $\tau : T \to X$ *be two Abelian sheaf spaces,* $f : S \to T$ *an Abelian sheaf space morphism, and let* $\gamma \in (\mathrm{Sec}^{(c)} S)U$ *. Then* $\gamma \in (\mathrm{Sec}^{(c)} \mathrm{Ker}\, f)U$ *if and only if* $\mathsf{f}_U(\gamma) = 0$.

(c) *Let* $\sigma : S \to X$ *and* $\tau : T \to X$ *be two Abelian sheaf spaces, let* $f : S \to T$ *be a sheaf space morphism, and let* $\delta \in (\mathrm{Sec}^{(c)} T)U$ *be a continuous section. Then* $\delta \in (\mathrm{Sec}^{(c)} \mathrm{Im}\, f)U$ *if and only if it is locally generated by a family of continuous sections* $\{\mathsf{f}_{U_\iota}(\gamma_\iota)\}_{\iota\in I}$ *, where* $\gamma_\iota \in (\mathrm{Sec}^{(c)} S)U_\iota$ *, and the family* $\{U_\iota\}_{\iota\in I}$ *is an open covering of U.*

Proof (a) If S is a sheaf subspace of T, then S is open in the sheaf space T, and $S_x = S \cap T_x \subset T_x$ is a subgroup for every $x \in X$. If $\gamma \in (\mathrm{Sec}^{(c)} S)U$, then γ is continuous in T because S is open. Thus, $\gamma \in (\mathrm{Sec}^{(c)} T)U$, and $(\mathrm{Sec}^{(c)} S)U$ must be a subgroup of $(\mathrm{Sec}^{(c)} T)U$. Conversely, let $x \in X$, $s_1, s_2 \in S_x$, and let $\gamma_1, \gamma_2 \in (\mathrm{Sec}^{(c)} S)U_x$ be continuous sections defined on a neighbourhood U_x of x such that $\gamma_1(x) = s_1$, $\gamma_2(x) = s_2$ (Lemma 2, (a)). The union of the sets U_x coincides with U which implies that U is open. Moreover since $\gamma_1 + \gamma_2 \in (\mathrm{Sec}^{(c)} S)U$ then $s_1 + s_2 = \gamma_1(x) + \gamma_2(x) = (\gamma_1 + \gamma_2)(x) \in S_x$.

(b) This is a trivial consequence of (2).

(c) Let $\delta \in (\mathrm{Sec}^{(c)} \mathrm{Im}\, f)U$, and let $x \in X$. Then $\delta(x) = f(\gamma_x(x))$ for some continuous section γ_x, defined on a neighbourhood U_x of x such that $U_x \subset U$ (Lemma 2, (b)). We may assume, shrinking U_x if necessary, that both δ and γ_x are homeomorphisms on U_x. Then $s_{UU_x}(\delta) = f \circ \gamma_x = \mathsf{f}_{U_x}(\gamma_x)$, so the family $\{\mathsf{f}_{U_x}(\gamma_x)\}_{x\in U}$ locally generates δ. The converse is obvious.

Remark 3 Lemma 7, (c) does not assure that for a continuous section $\delta \in (\mathrm{Sec}^{(c)} \mathrm{Im}\, f)U$, there always exists a continuous section $\gamma \in (\mathrm{Sec}^{(c)} S)U$ such that $\delta = \mathsf{f}_U(\gamma)$.

In accordance with Lemma 7, (a) given a sheaf subspace S of an Abelian sheaf T, we define a *subsheaf* of the sheaf $\mathrm{Sec}^{(c)} T$ as the correspondence $U \to (\mathrm{Sec}^{(c)} S)U$, and write $\mathrm{Sec}^{(c)} S \subset \mathrm{Sec}^{(c)} T$. If $f : S \to T$ is a sheaf space morphism, then the *kernel* (resp. the *image*) of the sheaf morphism $\mathsf{f} : \mathrm{Sec}^{(c)} S \to \mathrm{Sec}^{(c)} T$ is defined to be the Abelian sheaf, associated with the sheaf space $\mathrm{Ker}\, f$ (resp. $\mathrm{Im}\, f$); that is, we set

(4) $$\mathrm{Ker}\,\mathsf{f} = \mathrm{Sec}^{(c)} \mathrm{Ker}\, f, \quad \mathrm{Im}\,\mathsf{f} = \mathrm{Sec}^{(c)} \mathrm{Im}\, f.$$

7.4 Abelian presheaves

We can use properties (1), (2) and (3) of the sets of sections of an Abelian sheaf space (Section 7.3, Lemma 6) to introduce the concept of an *Abelian presheaf*. Diagram (3) will then be used to define Abelian presheaf morphisms. Properties (4) and (5) will be required to define *complete presheaves*, that is, (abstract) *sheaves*.

Let *X* be a topological space, $\mathcal{S}$ a correspondence assigning to an open set $U \subset X$ an Abelian group $\mathcal{S}U$ and to every pair of open sets *U*, *V* such that $V \subset U$ an Abelian group morphism $s_{VU} : \mathcal{S}V \to \mathcal{S}U$. $\mathcal{S}$ is said to be an *Abelian presheaf*, or just a *presheaf*, if the following conditions are satisfied:

(1) $\mathcal{S}\varnothing = 0$.

(2) $s_{UU} = \mathrm{id}_U$ for every open set $U \subset X$.

(3) $s_{WU} = s_{VU} \circ s_{WV}$ for all open sets $U, V, W \subset X$ such that $U \subset V \subset W$.

The topological space *X* is called the *base* of the Abelian presheaf $\mathcal{S}$. Elements of the Abelian groups $\mathcal{S}U$ are called *sections* of $\mathcal{S}$ over *U*, and the Abelian group morphisms s_{VU} are *restriction morphisms*, or just *restrictions* of $\mathcal{S}$. If $\gamma \in \mathcal{S}V$ and $U \subset V$, then the section $s_{VU}(\gamma)$ is called the *restriction* of the section γ to *U*.

Let $\mathcal{S}$ be an Abelian presheaf with base *X* and restrictions $\{s_{VU}\}$. Let *U* be an open subset of *X*. We say that two sections $\gamma, \delta \in \mathcal{S}U$ *coincide locally*, if there exists an open covering $\{U_\iota\}_{\iota \in I}$ of *U* such that for every $\iota \in I$

(1) $$s_{UU_\iota}(\gamma) = s_{UU_\iota}(\delta).$$

A family $\{\gamma_\iota\}_{\iota \in I}$ of sections of $\mathcal{S}$, where $\gamma_\iota \in \mathcal{S}U_\iota$, is said to be *compatible*, if the condition

(2) $$s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) = s_{U_\kappa, U_\iota \cap U_\kappa}(\gamma_\kappa)$$

holds for all $\iota, \kappa \in I$. We say that a family $\{\gamma_\iota\}_{\iota \in I}$ *locally generates* a section $\gamma \in \mathcal{S}U$, where $U = \bigcup U_\iota$, if

(3) $$s_{UU_\iota}(\gamma) = \gamma_\iota$$

for all $\iota \in I$. A family of sections, locally generating a section, is always compatible.

A *complete Abelian presheaf*, or an *Abelian sheaf*, is a presheaf $\mathcal{S}$ satisfying, in addition to conditions (1), (2) and (3) from the definition of an Abelian presheaf, the following two conditions:

(4) Any two sections of $\mathcal{S}$ which coincide locally, coincide.

(5) Every compatible family of sections of $\mathcal{S}$ locally generates a section of $\mathcal{S}$.

If an Abelian presheaf $\mathcal{S}$ is complete, then any section, locally generated by a compatible family of sections, is unique. Indeed, if γ_1, γ_2 are two sections locally generated by a compatible family $\{\gamma_\iota\}_{\iota \in I}$, then according to (5), $s_{UU_\iota}(\gamma_1) = \gamma_\iota = s_{UU_\iota}(\gamma_2)$, and property (4) implies $\gamma_1 = \gamma_2$.

Let $\mathcal{S}$ (resp. $\mathcal{T}$) be an Abelian presheaf over *X*, $\{s_{UV}\}$ (resp. $\{t_{UV}\}$) the family of restrictions of $\mathcal{S}$ (resp. $\mathcal{T}$). Let $f = \{f_U\}$ be a family of Abelian group morphisms $f_U : \mathcal{S}U \to \mathcal{T}U$. f is said to be a *morphism of Abelian presheaves*, or simply a *presheaf morphism*, if for every pair of open sets *U* and V in X such that $U \subset V$, the diagram

$$\begin{array}{ccc} SV & \xrightarrow{f_V} & TV \\ \downarrow s_{VU} & & \downarrow t_{VU} \\ SU & \xrightarrow{f_U} & TU \end{array} \tag{4}$$

commutes. We also denote this presheaf morphism by $f:S \to T$.

A *subpresheaf* S of an Abelian presheaf T is a presheaf such that SU is a subgroup of TU for every open set U in X. If ι_U are the corresponding inclusions, then the presheaf morphism $\iota:S \to T$, is called the *inclusion* of the subpresheaf S into T.

The *composition* of presheaf morphisms is defined in an obvious way. If $g:R \to S$ and $f:S \to T$ are two presheaf morphism, where $g=\{g_U\}$ and $f=\{f_U\}$, then we define $g \circ f:R \to T$ to be the family $\{g_U \circ f_U\}$.

If S is an Abelian presheaf, then the family $\mathrm{id}_S=\{\mathrm{id}_{SU}\}$ is a presheaf morphism, called the *identity morphism* of id_S. If $f:S \to T$ and $g:T \to S$ (resp. $h:T \to S$) are two Abelian presheaf morphisms and $g \circ f=\mathrm{id}_S$ (resp. $f \circ h=\mathrm{id}_T$), we call g (resp. h) a *left inverse* (resp. *right inverse*) for f. If f has a left inverse g and a right inverse h, then $h=(g \circ f)\circ h=g \circ (f \circ h)=g$ hence the presheaf morphism $h=g$ is unique. It is called the *inverse* of f and is denoted f^{-1}. f is called a *presheaf isomorphism*, if it has the inverse.

A Abelian presheaf morphism $f=\{f_U\}$ is called *injective* (resp. *surjective*), if the group morphisms f_U are injective (resp. surjective).

Let $f:S \to T$ be an Abelian presheaf morphism, $f=\{f_U\}$. We define a presheaf $\mathrm{Ker} f$ (resp. $\mathrm{Im} f$) as the correspondence, assigning to every open set $U \subset X$ the Abelian group $\mathrm{Ker} f_U \subset SU$ (resp. $\mathrm{Im} f_U \subset TU$), and to every two open sets $U,V \subset X$, where $U \subset V$, the restriction $s_{VU}|_{\mathrm{Ker} f_V}:\mathrm{Ker} f_V \to SU$ (resp. $t_{VU}|_{\mathrm{Im} f_V}:\mathrm{Im} f_V \to TU$). $\mathrm{Ker} f$ (resp. $\mathrm{Im} f$) is a subpresheaf of S (resp. T) called the *kernel* (resp. *image*) of f.

Remark 4 If the family $\{U_\iota\}_{\iota\in I}$ consists of two disjoint sets U_1, U_2, then condition (2) $s_{U_1,\emptyset}(\gamma_\iota)=s_{U_2,\emptyset}(\gamma_\kappa)$ reduces to the identity $0=0$. Thus, property (5), used for the definition of a complete presheaf, implies that there should always exist an extension of γ_1 and γ_2 to $U_1 \cup U_2$. This observation can sometimes be used to easily check that a presheaf is *not* complete: it is sufficient to verify that in the considered Abelian presheaf such an extension does not exist.

Examples 14. By definition the *sheaf of continuous sections* of an Abelian sheaf space, introduced in Section 7.3, is a sheaf.

15. Let S and T be Abelian sheaves with base X and let $f:S \to T$ be an Abelian presheaf morphism. It is easily seen that $\mathrm{Ker} f$ is a complete presheaf of S. Indeed, $\mathrm{Ker} f$ satisfies condition (4) from the definition of a sheaf. To investigate condition (5), denote by $\{s_{VU}\}$ (resp. $\{t_{VU}\}$) the family of restrictions of S (resp. T). Let $\{U_\iota\}_{\iota\in I}$ be a family of open sets in X,

$U = \bigcup U_\iota$. Let $\{\gamma_\iota\}_{\iota\in I}$ be a family of sections such that $\gamma_\iota \in (\mathrm{Ker} f)U_\iota$ and $s_{U_\iota,U_\iota\cap U_\kappa}(\gamma_\iota) = s_{U_\kappa,U_\iota\cap U_\kappa}(\gamma_\kappa)$ for all $\iota,\kappa \in I$. Then by condition (5), there exists $\gamma \in SU$ such that $s_{UU_\iota}(\gamma) = \gamma_\iota$. Using this condition and the commutative diagram (4), we get $t_{UU_\iota}(f_U(\gamma)) = f_{U_\iota}(s_{UU_\iota}(\gamma)) = f_{U_\iota}(\gamma_\iota) = 0$. Since T is complete, condition (5) implies $f_U(\gamma) = 0$.

16. The *trivial sheaf* over a topological space *X* is a complete presheaf, assigning to each open set $U \subset X$ the Abelian group id_U, with the restrictions $s_{UV}(\mathrm{id}_U) = \mathrm{id}_V$. The trivial sheaf over *X* is denoted by 0_X.

17. Assume that we have an Abelian sheaf space *S* with base *X* and projection σ. Consider the correspondence $\mathrm{Sec}\,S$, assigning to an open set $U \subset X$ the Abelian group $(\mathrm{Sec}\,S)U$ of *all*, not necessarily continuous, sections of the local homeomorphism σ, defined on *U*. To any open sets $U,V \subset X$ such that $U \subset V$ we assign the restriction mapping s_{VU} in a standard way; we get Abelian group morphisms $s_{VU} : (\mathrm{Sec}\,S)V \to (\mathrm{Sec}\,S)U$. In this way we get an Abelian sheaf $\mathrm{Sec}\,S$, called the sheaf of (*discontinuous*) *sections*, associated with the sheaf space *S*.

18. Let *X* be a topological space. Assign to every open set $U \subset X$ the Abelian group $C_{X,\mathbf{R}}U$ of continuous real-valued functions, defined on *U*, and to any open sets $U,V \subset X$ such that $U \subset V$, the restriction mapping defined as $C_{X,\mathbf{R}}V \ni f \to s_{VU}(f) = f|_U \in C_{X,\mathbf{R}}U$. This correspondence obviously satisfies the axioms (1) – (5) of a complete Abelian presheaf (Abelian sheaf). Indeed, axioms (1), (2) and (3) are satisfied trivially. To formally verify (4), suppose we have two continuous functions $f,g \in C_{X,\mathbf{R}}U$ such that

(5) $$s_{UU_\iota}(f) = f|_{U_\iota} = s_{UU_\iota}(g) = g|_{U_\iota}$$

for some open covering $\{U_\iota\}_{\iota\in I}$ of *U*. Clearly, then for every point $x \in U$, $f(x) = g(x)$, so *f* and *g* coincide on *U*. To verify axiom (5), consider a compatible family of continuous functions $\{f_\iota\}_{\iota\in I}$, where f_ι is defined on U_ι. Setting $f(x) = f_\iota(x)$ whenever $x \in U_\iota$, we get a continuous function *f*, defined on $U = \bigcup U_\iota$. Thus, the presheaf $C_{X,\mathbf{R}}$, defined in this way, is complete. This complete Abelian presheaf is referred to as the *sheaf of continuous functions* on the topological space *X*.

19. Let *X* be a smooth manifold. Assign to every open set $U \subset X$ the Abelian group $C^r_{X,\mathbf{R}}U$ of real-valued functions of class C^r, defined on *U*, where $r = 0,1,2,\ldots,\infty$, and to any open sets $U,V \subset X$ such that $U \subset V$, the restriction mapping $C^r_{X,\mathbf{R}}V \ni f \to s_{VU}(f) = f|_U \in C^r_{X,\mathbf{R}}U$. This correspondence obviously satisfies the axioms (1) – (5) of a complete presheaf; we get a complete Abelian presheaf called the *sheaf of functions of class* C^r on *X*.

20. Let *E* be a smooth vector bundle over a manifold *X* with projection π. For any $r = 0,1,2,\ldots,\infty$, assign to every open set $U \subset X$ the Abelian group $\Gamma^r_U(\pi)$ of C^r-sections of *E*, defined on *U*, and to any open sets $U,V \subset X$, where $U \subset V$, the restrictions $\Gamma_V(\pi) \ni \gamma \to s_{VU}(\gamma) = \gamma|_U \in \Gamma_U(\pi)$. This correspondence obviously satisfies the axioms (1) – (5) of a complete Abelian presheaf, the *sheaf of sections of class* C^r of the vector bundle *E*.

21. We show in this example that the image of a complete Abelian

presheaf by an Abelian presheaf morphism into a complete presheaf is not necessarily a complete subpresheaf. Consider the Abelian sheaf $C^{\infty}_{X,\mathbf{R}} = \Omega^0_X$ of smooth functions (0-forms) and the sheaf $T = \Omega^1_X$ of smooth 1-forms over the smooth manifold $X = \mathbf{R}^2 \setminus \{(0,0)\}$. The exterior derivative $d : \Omega^0_X \to \Omega^1_X$ defines, for every open set $U \subset X$, a morphism of Abelian groups $d : \Omega^0_X U \to \Omega^1_X U$, and a presheaf morphism $d : \Omega^0_X \to \Omega^1_X$. We show that the image presheaf $\operatorname{Im} d \subset \Omega^1_X$ does not satisfy condition (5) of a complete presheaf, so consequently, it is not complete. Consider in the canonical coordinates x, y in $\mathbf{R}^2$, the 1-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}. \tag{6}$$

Let $\{U_\iota\}_{\iota \in I}$ be a covering of *X* by open balls. Then by the Volterra-Poincare lemma, $\omega = d\varphi_\iota$ on U_ι, where $\varphi_\iota \in \Omega^0_X U_\iota$, but there is no function $\varphi \in \Omega^0_X$ satisfying $\omega = d\varphi$ (see e.g. Schwartz [Sc]). Thus ω is locally expressible as the exterior derivative, but there is *no* global function φ such that $\omega = d\varphi$.

7.5 Sheaf spaces associated with Abelian presheaves

We introduce in this section a correspondence, assigning to an Abelian presheaf an Abelian sheaf space, and to an Abelian presheaf morphism an Abelian sheaf space morphism, and study basic properties of this correspondence.

Let S be an Abelian presheaf with base *X*, $\{s_{VU}\}$ the family of its restriction mappings. For any point $x \in X$, consider the set of all pairs (U, γ), where *U* is a neighbourhood of *x* and γ a section of S, belonging to the Abelian group SU. There is an equivalence relation on this set "$\gamma \sim \delta$*, if there exists a neighbourhood W of x such that the restrictions of* γ *and* δ *to W coincide*". Indeed, the binary relation $\sim$ is obviously symmetric and reflexive. To show that it is transitive, consider three sections $\gamma_1 \in SU_1$, $\gamma_2 \in SU_2$, and $\gamma_3 \in SU_3$, such that $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$. Then by definition there exist two neighbourhoods *V* and *W* of the point *x* such that $V \subset U_1 \cap U_2$, $W \subset U_2 \cap U_3$ and $s_{U_1 V}(\gamma_1) = s_{U_2 V}(\gamma_2)$ and $s_{U_2 W}(\gamma_2) = s_{U_2 W}(\gamma_3)$. Then on $V \cap W$

$$\begin{aligned} s_{U_1, V\cap W}(\gamma_1) &= s_{V, V\cap W} \circ s_{U_1, V}(\gamma_1) = s_{V, V\cap W} \circ s_{U_2, V}(\gamma_2) = s_{U_2, V\cap W}(\gamma_2) \\ &= s_{W, V\cap W} \circ s_{U_2 W}(\gamma_2) = s_{W, V\cap W} \circ s_{U_3 W}(\gamma_3) = s_{U_3, V\cap W}(\gamma_3). \end{aligned} \tag{1}$$

The equivalence class of a section γ is called the *germ* of γ at the point *x* and is denoted by $[\gamma]_x$. Denote by S_x the quotient set and consider the set

$$\operatorname{Germ} S = \bigcup_{x \in X} S_x \tag{2}$$

Define a mapping $\sigma : \mathrm{Germ}\,\mathcal{S} \to X$ by the equation

(3) $\sigma([\gamma]_x) = x.$

We need a topology on the set $\mathrm{germ}\,\mathcal{S}$ and an Abelian group structure on each of the sets $\mathcal{S}_x$ defining on $\mathrm{Germ}\,\mathcal{S}$ the structure of a sheaf space of Abelian groups with base X and projection σ. Let U be an open set in X, $\gamma \in \mathcal{S}U$ a section. We define a mapping $\tilde{\gamma} : U \to \mathrm{Germ}\,\mathcal{S}$ by

(4) $\tilde{\gamma}(x) = [\gamma]_x.$

The set $\mathrm{germ}\,\mathcal{S}$ will be considered with the *final topology*, associated with the family $\{\tilde{\gamma}\}$, where γ runs through the set of sections of the presheaf $\mathcal{S}$; this is the strongest topology on the set $\mathrm{Germ}\,\mathcal{S}$ in which all the mappings $\tilde{\gamma}$ are continuous.

Note that if $\gamma \in \mathcal{S}U$ is a section then the set $\tilde{\gamma}(U)$ is open in $\mathrm{Germ}\,\mathcal{S}$. Clearly, if $\delta \in \mathcal{S}V$ is another section, we have

(5) $\tilde{\delta}^{-1}\tilde{\gamma}(U) = \{x \in V \mid \tilde{\delta}(x) = \tilde{\gamma}(x)\} = \{x \in U \cap V \mid \tilde{\delta}(x) = \tilde{\gamma}(x)\},$

which is an open subset of $U \cap V$ formed by all points x such that $\delta = \gamma$ on a neighbourhood of x. Now we apply the definition of the final topology to observe that $\tilde{\gamma}(U)$ is open.

It is easy to see that the mapping $\sigma : \mathrm{Germ}\,\mathcal{S} \to X$ defined by (4) is a local homeomorphism. If $y \in \mathrm{Germ}\,\mathcal{S}$ is any germ at $x \in X$ and $\gamma \in \mathcal{S}U$ any representative of y, then $W = \tilde{\gamma}(U)$ is a neighbourhood of y and

(6) $\sigma|_W \circ \tilde{\gamma} = \mathrm{id}_U, \quad \tilde{\gamma} \circ \sigma|_W = \mathrm{id}_W.$

Every fibre $\mathcal{S}_x$ of σ has the structure of an Abelian group defined by

(7) $[\gamma]_x + [\delta]_x = [s_{UW}(\gamma) + s_{VW}(\delta)]_x,$

where $\gamma \in \mathcal{S}U$, $\delta \in \mathcal{S}V$, and $W = U \cap V$. Clearly, this definition is correct, because the germ on the right-hand side is independent of the choice of the representatives γ and δ. Indeed, with obvious notation

$$\begin{aligned} &[s_{U'W'}(\gamma') + s_{U'W'}(\delta')]_x = [s_{W'W''}(s_{U'V'}(\gamma') + s_{U'V'}(\delta'))]_x \\ (8) \qquad &\quad = [s_{U'W''}(\gamma') + s_{U'W''}(\delta')]_x, \\ &[s_{UW}(\gamma) + s_{VW}(\delta)]_x = [s_{UW''}(\gamma) + s_{VW''}(\delta)]_x, \end{aligned}$$

where $W' = U' \cap V'$. Since one may choose the set W'' in such a way that $s_{UW''}(\gamma) = s_{U'W''}(\gamma)$ and $s_{VW''}(\delta) = s_{V'W''}(\delta')$, we have

(9) $[s_{UW}(\gamma) + s_{VW}(\delta)]_x = [s_{U'W''}(\gamma') + s_{V'W''}(\delta')]_x.$

It remains to check that the mapping $(p,q) \to (p-q)$ of the fibre product $\mathrm{Germ}\,S \times_X \mathrm{Germ}\,S$ into $\mathrm{Germ}\,S$ is continuous. Let (p_0, q_0) be an arbitrary point of the set $\mathrm{Germ}\,S \times_X \mathrm{Germ}\,S$, where $p_0 = [\gamma]_x$, $q_0 = [\delta]_x$. We may assume without loss of generality that $\gamma, \delta \in SW$, where W is a neighbourhood of x. Then $p_0 - q_0 = [\gamma - \delta]_x$. If $\eta = \gamma - \delta$, then $\tilde{\eta}(W)$ is a neighbourhood of the point $p_0 - q_0$. The set $\tilde{\gamma}(W) + \tilde{\delta}(W) \subset \mathrm{Germ}\,S \times \mathrm{Germ}\,S$ is open, and the set $(\tilde{\gamma}(W) + \tilde{\delta}(W)) \cap (\mathrm{Germ}\,S \times_S \mathrm{Germ}\,S)$ is open in the set $\mathrm{Germ}\,S \times_S \mathrm{Germ}\,S$. Since the image of $(\tilde{\gamma}(W) + \tilde{\delta}(W)) \cap (\mathrm{Germ}\,S \times_S \mathrm{Germ}\,S)$ under the mapping $(p,q) \to (p - q)$ coincides with $\tilde{\eta}(W)$, this mapping is continuous at (p_0, q_0). This completes the construction of the Abelian sheaf space $\mathrm{Germ}\,S$ from a given presheaf S.

We call $\mathrm{Germ}\,S$ the *Abelian sheaf space*, *associated* with the Abelian presheaf S. The continuous section $\tilde{\gamma} : U \to \mathrm{Germ}\,S$ is said to be *associated with* the section $\gamma \in SU$.

Let S (resp. T) be an Abelian presheaf over a topological space X, $\{s_{UV}\}$ (resp. $\{t_{UV}\}$) the family of restrictions of S (resp. T). Let $f = \{f_U\}$ be a *presheaf morphism* of the presheaf S into T. Denote by $\sigma : \mathrm{Germ}\,S \to X$ and $\tau : \mathrm{Germ}\,T \to X$ the corresponding sheaf spaces. We define a mapping $f : \mathrm{Germ}\,S \to \mathrm{Germ}\,T$ by the equation

$$f([\gamma]_x) = [f_U(\gamma)]_x, \tag{10}$$

where $[\gamma]_x \in \mathrm{Germ}\,S$ and $\gamma \in SU$ is any representative of the germ $[\gamma]_x$. It can be readily verified that the germ $[f_U(\gamma)]_x$ is defined independently of the choice of the representative γ. Indeed, let $\delta \in SV$ be such that $[\delta]_x = [\gamma]_x$. Then $s_{UW}(\gamma) = s_{VW}(\delta)$ for some neighbourhood W of the point x. Applying the definition of the presheaf morphism, we obtain

$$t_{UW} \circ f_U(\gamma) = f_W \circ s_{UW}(\gamma) = f_W \circ s_{VW}(\delta) = t_{VW} \circ f_V(\delta), \tag{11}$$

hence $[f_U(\gamma)]_x = [f_V(\delta)]_x$.

We assert that the mapping f, defined by (10), is a sheaf space morphism. f obviously satisfies $\tau \circ f = \sigma$. Note that if $\gamma \in SU$, then $f_U(\gamma)$ is a section of T; in particular, the mapping $x \to f([\gamma]_x) = f \circ \tilde{\gamma}(x) = [f_U(\gamma)]_x$ of U into the set $\mathrm{germ}\,T$ is continuous (with respect to the final topology on $\mathrm{germ}\,T$). This means, however, that $f \circ \tilde{\gamma}$ is continuous, and using the properties of the topology of the set $\mathrm{Germ}\,S$, we conclude that the mapping f is continuous. Finally, the restriction f_x of f to each fibre $(\mathrm{Germ}\,S)_x$ is an Abelian group morphism. Summarizing, we see that all conditions for f to be an Abelian sheaf space morphism hold. f is said to be *associated* with the Abelian presheaf morphism f.

Consider a sheaf space of Abelian groups S with base X and projection σ, the associated sheaf of Abelian groups $\mathrm{Sec}^{(c)} S$, and the sheaf space $\mathrm{germ}\,\mathrm{Sec}^{(c)} S$, associated with the sheaf $\mathrm{Sec}^{(c)} S$. Let $\sigma' : \mathrm{Germ}\,\mathrm{Sec}^{(c)} S \to X$ be sheaf space projection. Let $s \in S$ be a point and V a neighbourhood if s such that $\sigma|_V$ is a homeomorphism. Put $x = \sigma(s)$, $\gamma_s = (\sigma|_V)^{-1}$, and

(12) $\quad \nu_S(s) = [\gamma_s(x)].$

This defines a mapping $\nu_S : S \to \operatorname{Germ}\operatorname{Sec}^{(c)} S$ such that $\sigma' \circ \nu_S = \sigma$.

Lemma 8 (a) *Let S and T be two Abelian presheaves with base X, $\mathsf{f} : \mathsf{S} \to \mathsf{T}$ an Abelian presheaf morphism, and let $f : \operatorname{Germ}\mathsf{S} \to \operatorname{Germ}\mathsf{T}$ be the sheaf space morphism associated with f . Then for every point $x \in X$*

(13) $\quad (\operatorname{Germ}\operatorname{Ker}\mathsf{f})_x = \operatorname{Ker} f_x, \quad (\operatorname{Germ}\operatorname{Im}\mathsf{f})_x = \operatorname{Im} f_x.$

(b) *Let $f : \operatorname{Germ}\mathsf{R} \to \operatorname{Germ}\mathsf{S}$ (resp. $g : \operatorname{Germ}\mathsf{S} \to \operatorname{Germ}\mathsf{T}$) be the Abelian sheaf space morphism associated with an Abelian presheaf morphism $\mathsf{f} : \mathsf{R} \to \mathsf{S}$ (resp. $\mathsf{g} : \mathsf{S} \to \mathsf{T}$), and $h : \operatorname{Germ}\mathsf{R} \to \operatorname{Germ}\mathsf{T}$ the Abelian sheaf space morphism associated with the Abelian presheaf morphism $\mathsf{h} = \mathsf{g} \circ \mathsf{f}$. Then $h = g \circ f$.*

(c) *The mapping $\nu_S : S \to \operatorname{Germ}\operatorname{Sec}^{(c)} S$ is an Abelian sheaf space isomorphism.*

Proof (a) Let $[\gamma]_x \in \operatorname{Germ}\operatorname{Ker} f$. Then $\gamma \in (\operatorname{Ker} f)U$, where U is a neighbourhood of *x*. Thus the representative γ satisfies $\mathsf{f}_U(\gamma) = 0$ hence by (10), $f([\gamma]_x) = 0$ and $[\gamma]_x \in \operatorname{Ker} f$. Conversely, assume that $[\gamma]_x \in \operatorname{Ker} f$. Then by (10) $f([\gamma]_x) = [\mathsf{f}_V(\gamma)]_x = 0$. In particular, $\mathsf{f}_V(\gamma)$ is equivalent with the zero section, $t_{VU}(\mathsf{f}_V(\gamma)) = \mathsf{f}_V(s_{VU}(\gamma)) = 0$ for a neighbourhood *U* of *x* such that $U \subset V$. Thus $[\gamma]_x = [s_{VU}(\gamma)]_x$, where $s_{VU}(\gamma) \in \operatorname{Ker}\mathsf{f}_U$.

Let $[\delta]_x \in \operatorname{Germ}\operatorname{Im}\mathsf{f}$. Then for some neighbourhood *V* of *x*, $\delta = \mathsf{f}_V(\gamma)$, where $\gamma \in \mathsf{S}U$. Thus by (10), $f([\gamma]_x) = [\mathsf{f}_U(\gamma)]_x = [\delta]_x$ which means that $[\delta]_x \in \operatorname{Im} f_x$. Conversely, let $[\delta]_x \in \operatorname{Im} f_x$. Then there exists $[\gamma]_x$ such that $f_x([\gamma]_x) = [\delta]_x$. Assume that $\gamma \in \mathsf{S}V$, $\delta \in \mathsf{T}V$. Then on a neighbourhood *U* of *x*, $\mathsf{f}_U(s_{VU}(\gamma)) = t_{VU}(\delta)$ which implies $[\delta]_x = [t_{VU}(\delta)]_x = [\mathsf{f}_U s_{VU}(\gamma)]_x$, which is an element of the set $\operatorname{Germ}\operatorname{Im} f_x$.

(b) The proof is straightforward.

(c) We shall show that ν_S is an Abelian sheaf space isomorphism. Let $[\gamma]_x \in \operatorname{Germ}\operatorname{Sec}^{(c)} S$ be a germ represented by a section $\gamma \in (\operatorname{Sec}^{(c)} S)U$. Write $\tau_S([\gamma]_x) = \gamma(x)$. Clearly, the point $\gamma(x) \in S$ is defined independently of the choice of the representative γ . We have $\tau_S([\gamma]_x) = (\sigma_V)^{-1}(x)$, where *V* is a neighbourhood of the point $\gamma(x) \in S$ such that the restriction $\sigma|_U$ is a homeomorphism. Since $\nu_S \circ \tau_S([\gamma]_x) = \nu_S((\sigma_V)^{-1}(x)) = [(\sigma_V)^{-1}]_x = [\gamma]_x$ and

(14) $\quad \tau_S \circ \nu_S(s) = \tau_S([\gamma]_x) = \gamma_S(x) = s,$

τ_S is the inverse of ν_S .

We shall verify that ν_S is continuous. Let $s \in S$ be a point, $x = \sigma(s)$, *V* a neighbourhood of the point $\nu_S(s) \in \operatorname{Germ}\operatorname{Sec}^{(c)} S$. The point $\nu_S(s)$ has a neighbourhood $\tilde{\gamma}_s(U)$, where $\gamma_s : U \to S$ is a section, defined on a neighbourhood *U* of *x*, and $\tilde{\gamma}_s(y) = [\gamma_s]_y$. Since $\tilde{\gamma}_s$ is continuous, we may suppose that $\tilde{\gamma}_s(U) \subset V$. But the set $\gamma_s(U)$ is a neighbourhood of the point *s*, and $\nu_S(\gamma_s(U)) = \tilde{\gamma}_s(U) \subset V$, hence ν_S is continuous at *s*.

Now we shall show that for every point $x \in X$ and any two points $s_1, s_2 \in S_x$, $\nu_S(s_1 + s_2) = \nu_S(s_1) + \nu_S(s_2)$. Let V_1 (resp. V_2) be a neighbourhood of s_1 (resp. s_2) such that $\sigma|_{V_1}$ (resp. $\sigma|_{V_2}$) is a homeomorphism. One may suppose that $\sigma(V_1) = \sigma(V_2) = U$. Then $\gamma_{s_1}, \gamma_{s_2}, \gamma_{s_1+s_2} \in (\mathrm{Sec}^{(c)} S)U$ and by definition $[\gamma_{s_1}]_x + [\gamma_{s_2}]_x = [\gamma_{s_1} + \gamma_{s_2}]_x$, that is, $\nu_S(s_1) + \nu_S(s_2) = \nu_S(s_1 + s_2)$. This proves that the mapping ν_S is an Abelian sheaf space morphism.

The mapping ν_S is obviously injective and surjective hence bijective. The inverse mapping $(\nu_S)^{-1} : \mathrm{Germ}\,\mathrm{Sec}^{(c)} S \to S$ is continuous by the properties of the final topology, since for every section $\gamma \in \mathrm{Sec}^{(c)} S$ the composite $(\nu_S)^{-1} \circ \tilde{\gamma} = \gamma$ is continuous. Summarizing, this proves that ν_S is an Abelian sheaf space isomorphism.

We call the Abelian sheaf space isomorphism $\nu_S : S \to \mathrm{Germ}\,\mathrm{Sec}^{(c)} S$ the *canonical isomorphism*.

7.6 Sheaves associated with Abelian presheaves

The concepts of an *Abelian sheaf associated with an Abelian sheaf space* and the *Abelian sheaf space associated with an Abelian presheaf* allow to assign to any Abelian presheaf $\mathcal{S}$ the sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$, which is said to be *associated* with $\mathcal{S}$. We study properties of this correspondence.

Let $\mathcal{S}$ be an Abelian presheaf over a topological space X, $\{s_{VU}\}$ the family of its restrictions. For every open set $U \subset X$ define a morphism of Abelian groups $\vartheta_U : \mathcal{S}U \to (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S})U$ by

(1) $$\vartheta_U(\gamma) = \tilde{\gamma},$$

where $\tilde{\gamma}$ is a section of the sheaf $\mathrm{germ}\,\mathcal{S}$, associated with γ (Section 6.5, (4)). The Abelian presheaf morphism $\vartheta_{\mathcal{S}} = \{\vartheta_U\}$ of $\mathcal{S}$ into $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$ is said to be *canonical*.

Since for every open sets $U, V \subset X$ such that $U \subset V$, and every point $x \in U$, $\vartheta_U(s_{VU}(\gamma))(x) = [s_{VU}(\gamma)]_x = [\gamma]_x = \tilde{\gamma}(x) = \vartheta_V(\gamma)|_U(x)$, $\vartheta_{\mathcal{S}}$ commutes with the restrictions,

(2) $$\vartheta_U \circ s_{VU}(\gamma) = \vartheta_U(\gamma)|_U .$$

Note that any section δ of the sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$ is locally generated by a family of sections, generated by sections of $\mathcal{S}$. To prove it, consider a continuous section $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S})U$ and any point $x \in U$. By definition $\delta(x)$ is the germ of a section $\gamma_x \in \mathcal{S}U_x$, where U_x is a neighbourhood of the point x in U. That is, $\delta(x) = [\gamma_x]_x = \tilde{\gamma}_x(x)$. The projection $\sigma : \mathrm{Germ}\,\mathcal{S} \to X$ of the sheaf space $\mathrm{germ}\,\mathcal{S}$ is a local homeomorphism and $\sigma \circ \delta = \mathrm{id}_U$. On the other hand, $\sigma \circ \tilde{\gamma}_x = \mathrm{id}_{U_x}$, and since the inverse mapping is unique,

(3) $$\delta|_{U_x} = \tilde{\gamma}_x = \vartheta_{U_x}(\gamma_x).$$

Obviously, $U = \bigcup U_x$ and for any two points $x, y \in U$, $\delta|_{U_x} = \tilde{\gamma}_x$ hence

(4) $$\delta|_{U_x \cap U_y} = \tilde{\gamma}_x|_{U_x \cap U_y} = \tilde{\gamma}_y|_{U_x \cap U_y}.$$

Thus $[\gamma_x]_z = [\gamma_y]_z$ for every $z \in U_x \cap U_y$. Therefore, every point $z \in U_x \cap U_y$ has a neighbourhood W_z such that

(5) $$s_{U_x W_z}(\gamma_x) = s_{U_y W_z}(\gamma_y).$$

In view of (3) we say that the continuous section $\delta \in (\mathrm{Sec}^{(c)} \mathrm{Germ}\, S)U$ is *locally generated* by the family of sections $\{\gamma_x\}_{x \in U}$ of S.

Our aim now will be to find conditions ensuring that the canonical morphism $\vartheta_S : S \to \mathrm{Sec}^{(c)} \mathrm{Germ}\, S$ is a presheaf isomorphism.

Theorem 1 *Let S be an Abelian presheaf. The following conditions are equivalent:*

(1) *S is complete.*

(2) *The canonical presheaf morphism $\vartheta_S : S \to \mathrm{Sec}^{(c)} \mathrm{Germ}\, S$ is a presheaf isomorphism.*

Proof 1. Suppose that $\vartheta_S = \{\vartheta_U\}$ is a presheaf isomorphism. Let $\{s_{UV}\}$ be the restrictions of the presheaf S, $\{t_{UV}\}$ the restrictions of the sheaf $\mathrm{Sec}^{(c)} \mathrm{Germ}\, S$. Let $\{U_\iota\}_{\iota \in I}$ be a family of open sets in *X*, $U = \bigcup U_\iota$, and γ, δ two sections from SU such that $s_{UU_\iota}(\gamma) = s_{UU_\iota}(\delta)$. Then by the definition of the presheaf morphism, $\vartheta_{U_\iota} \circ s_{UU_\iota}(\gamma) = t_{UU_\iota} \circ \vartheta_U(\gamma) = t_{UU_\iota} \circ \vartheta_U(\delta)$. Hence $\vartheta_U(\gamma) = \vartheta_U(\delta)$ and, since ϑ_U is a group isomorphism, $\gamma = \delta$. This means that the presheaf S satisfies condition (4) of the definition of a complete presheaf. Now suppose that a family $\{\gamma_\iota\}_{\iota \in I}$, where $\gamma_\iota \in SU_\iota$, satisfies the condition $s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) = s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\kappa)$ for all $\iota, \kappa \in I$. Then

(7) $$\begin{aligned} \vartheta_{U_\iota \cap U_\kappa} \circ s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) &= t_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) \circ \vartheta_{U_\iota}(\gamma_\iota) \\ &= t_{U_\kappa, U_\iota \cap U_\kappa}(\gamma_\kappa) \circ \vartheta_{U_\kappa}(\gamma_\kappa), \end{aligned}$$

so there must exist a section $\delta \in (\mathrm{Sec}^{(c)} \mathrm{Germ}\, S)U$, where $U = \bigcup U_\iota$, such that $t_{UU_\iota}(\delta) = \vartheta_{U_\iota}(\gamma_\iota)$ for all indices $\iota \in I$. If $\gamma \in SU$ is such that $\delta = \vartheta_U(\gamma)$, we have $t_{UU_\iota} \circ \vartheta_U(\gamma) = \vartheta_{U_\iota} \circ s_{UU_\iota}(\gamma) = \vartheta_{U_\iota}(\gamma_\iota)$, hence $s_{UU_\iota}(\gamma) = \gamma_\iota$. Thus, condition (5) is also satisfied. This means, however, that S must be complete.

2. Conversely, suppose that the presheaf S is complete. We wish to show that there exists a presheaf morphism $f : \mathrm{Sec}^{(c)} \mathrm{Germ}\, S \to S$, $f = \{f_U\}$, such that $\vartheta_S \circ f = \mathrm{id}_{\mathrm{Sec}^{(c)} \mathrm{germ}\, S}$ and $f \circ \vartheta_S = \mathrm{id}_S$, that is,

(7) $$\vartheta_U \circ f_U = \mathrm{id}_{(\mathrm{Sec}^{(c)} \mathrm{germ}\, S)U}, \quad f_U \circ \vartheta_U = \mathrm{id}_{SU}$$

for all open sets $U \subset X$. Obviously, these equations have a solution f_U if and only if the mapping ϑ_U is bijective. Since we have already shown that ϑ_U is injective, it is sufficient to prove that it is surjective.

Let $\delta \in (\mathrm{Sec}^{(c)} \mathrm{Germ}\, S)U$ be a section, and let $x \in U$ be a point. Applying the definition of a presheaf (condition (3), Section 7.4) of to equation (3),

$$(8)\qquad s_{U_x\cap U_y,W_z}\circ s_{U_x,U_x\cap U_y}(\gamma_x)=s_{U_x\cap U_y,W_z}\circ s_{U_x,U_x\cap U_y}(\gamma_y).$$

Covering $U_x\cap U_y$ by the sets W_z we get from condition (4) of the definition of a presheaf

$$(9)\qquad s_{U_x,U_x\cap U_y}(\gamma_x)=s_{U_x,U_x\cap U_y}(\gamma_y).$$

Condition (5) now implies that there exists a section $\gamma\in SU$ such that

$$(10)\qquad s_{UU_x}(\gamma)=\gamma_x$$

for all $x\in U$. Therefore, the sections γ and γ_x belong to the same germ at every point of the set U_x. This means that $\tilde{\gamma}|_{U_x})=\tilde{\gamma}_x$ and

$$(11)\qquad \delta|_{U_x}=\tilde{\gamma}_x=\tilde{\gamma}|_{U_x}.$$

Since the presheaf of sections of the sheaf space $\mathrm{germ}\, S$ is a sheaf (Lemma 6), we get $\delta=\tilde{\gamma}$ proving that the mapping ϑ_U is surjective.

Consequently, the mapping f_U exists, and is given by the formula $f_U=(\vartheta_U)^{-1}$. It remains to show that $t_{VU}\circ f_V=f_U\circ s_{VU}$ for any two open sets $U,V\subset X$ such that $U\subset V$, where t_{VU} are restrictions of the presheaf $\mathrm{Sec}^{(c)}\,\mathrm{Germ}\, S$. Let $\delta\in(\mathrm{Sec}^{(c)}\,\mathrm{Germ}\, S)U$ be a section; then $\delta=\tilde{\gamma}=\vartheta_V(\gamma)$ for some section $\gamma\in SV$. We have

$$(12)\qquad s_{VU}\circ f_V(\tilde{\gamma})=s_{VU}\circ f_V\circ\vartheta_V(\gamma)=s_{VU}(\gamma),$$

and

$$(13)\qquad f_U\circ t_{VU}(\tilde{\gamma})=f_U\circ t_{VU}\circ\vartheta_V(\gamma)=f_U\circ\vartheta_U\circ s_{VU}(\gamma)=s_{VU}(\gamma),$$

proving the desired identity $t_{VU}\circ f_V=f_U\circ s_{VU}$. Now the proof is complete.

Theorem 2 *Let S (resp. T) be an Abelian presheaf with restrictions $\{s_{UV}\}$ (resp. $\{t_{UV}\}$), let $f:S\to T$ be an Abelian presheaf morphism. There exists a unique Abelian presheaf morphism $g:\mathrm{Sec}^{(c)}\,\mathrm{Germ}\, S\to\mathrm{Sec}^{(c)}\,\mathrm{Germ}\, T$ such that the diagram*

$$(14)\qquad \begin{array}{ccc} S & \xrightarrow{\quad f\quad} & T \\ \downarrow\vartheta_S & & \downarrow\vartheta_T \\ \mathrm{Sec}^{(c)}\,\mathrm{Germ}\, S & \xrightarrow{\quad g\quad} & \mathrm{Sec}^{(c)}\,\mathrm{Germ}\, T \end{array}$$

commutes.

Proof f generates a sheaf space morphism $f:\mathrm{Germ}\, S\to\mathrm{Germ}\, T$ by the formula $f([\gamma]_x)=[f_U(\gamma)]_x$, where U is a neighbourhood of x and $\gamma\in SU$

is a representative of the germ $[\gamma]_x$. f defines a sheaf morphism $g:\mathrm{Sec}^{(c)}\,\mathrm{Germ}\,S \to \mathrm{Sec}^{(c)}\,\mathrm{Germ}\,T$, $g=\{g_U\}$ by

(15) $$g_U(\delta) = f \circ \delta,$$

where $\delta \in (\mathrm{Sec}^{(c)}\,\mathrm{Germ}\,S)U$. Note that condition (10), Section 7.5 can be expressed in the form $f(\vartheta_{S,U}(\gamma)(x)) = \vartheta_{T,U}(f_U(\gamma))(x))$ or, equivalently, $f \circ \vartheta_{S,U}(\gamma) = \vartheta_{T,U} \circ f_U(\gamma)$, which implies

(16) $$g_U(\vartheta_{S,U}(\gamma)) = f \circ \vartheta_{S,U}(\gamma) = \vartheta_{T,U} \circ f_U(\gamma).$$

This proves existence and uniqueness of g.

To describe the morphism $g:\mathrm{Sec}^{(c)}\,\mathrm{Germ}\,S \to \mathrm{Sec}^{(c)}\,\mathrm{Germ}\,T$ explicitly, choose a continuous section $\delta \in (\mathrm{Sec}^{(c)}\,\mathrm{Germ}\,S)U$. We have already seen that there exists a family $\{\gamma_x\}_{x\in U}$ of sections $\gamma_x \in TU_x$, where U_x is a neighbourhood of *x* in *U*, such that

(17) $$\delta|_{U_x} = \vartheta_{S,U_x}(\gamma_x).$$

If $z \in U_x \cap U_y$, then $s_{U_x W_z}(\gamma_x) = s_{U_x W_z}(\gamma_y)$ on some neighbourhood W_z of the point *z* in $U_x \cap U_y$. Obviously, on U_x

(18) $$g_U(\delta)|_{U_x} = \vartheta_{T,U_x}(f_{U_x}(\gamma_x)),$$

because for every $y \in U_x$

(19) $$\begin{aligned} g_U(\delta)|_{U_x}(y) &= f(\delta(y)) = f(\vartheta_{S,U_x}(\gamma_x)(y)) = f([\gamma_x]_y) \\ &= [f_{U_x}(\gamma_x)]_y = \vartheta_{T,U_x}(f_{U_x}(\gamma_x))(y). \end{aligned}$$

Thus, if δ is locally generated by the family $\{\gamma_x\}_{x\in U}$, then $g_U(\delta)$ is locally generated by the family $\{f_{U_x}(\gamma_x)\}_{x\in U}$.

Note that if in diagram (14), T is a complete Abelian presheaf, then by Theorem 1, ϑ_T is an Abelian presheaf isomorphism, so we have, with obvious conventions,

(20) $$f = \vartheta_T^{-1} \circ g \circ \vartheta_S.$$

If S is a complete presheaf, then

(21) $$g = \vartheta_T \circ f \circ \vartheta_S^{-1}.$$

Corollary 2 *If S is a subpresheaf of an Abelian presheaf T, then the sheaf* $\mathrm{Sec}^{(c)}\,\mathrm{Germ}\,S$ *is a subsheaf of* $\mathrm{Sec}^{(c)}\,\mathrm{Germ}\,T$.

Corollary 3 (a) *Every complete Abelian presheaf is isomorphic with an Abelian sheaf, associated with an Abelian sheaf space.*

(b) *Every presheaf morphism of complete Abelian presheaves is expressible as a sheaf morphism, associated with a sheaf space morphism.*

Proof (a) This follows from Theorem 1.

(b) If both S and T in Theorem 2 are complete presheaves, then formulas (20) and (21) establish a one-to-one correspondence between presheaf morphisms f of complete presheaves and sheaf morphisms g associated with sheaf space morphisms.

Let $f:S\to T$ be an Abelian presheaf morphism, and suppose that the Abelian presheaf T is complete. Let $f:\mathrm{Germ}\,S\to\mathrm{Germ}\,T$ be the associated morphism of sheaf spaces. Note that we have defined the image $\mathrm{Im} f$ as a subpresheaf of T. On the other hand, we have also defined the image of the sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,S$ by the sheaf morphism induced by f, which is equal to the subsheaf $\mathrm{Sec}^{(c)}\mathrm{Im} f$ of the Abelian sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,T$. Obviously, $\mathrm{Im} f\subset\vartheta_T^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im} f)$, and $\vartheta_T^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im} f)$ is a complete subpresheaf of T. To distinguish between $\mathrm{Im} f$ and $\vartheta_T^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im} f)$, we sometimes call $\vartheta_T^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im} f)$ the *complete image* of S by the presheaf morphism f, or the *complete subpresheaf*, generated by S.

If S is a subpresheaf of the presheaf T, then the canonical inclusion $\iota_S:S\to T$ defines the image $\mathrm{Im}\,\iota_S$ and the complete image $\vartheta_T^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im}\,\iota_S)$. If the presheaf S is complete, then the following three subpresheaves S, $\mathrm{Im}\,\iota_S$ and $\vartheta_T^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im}\,\iota_S)$ coincide.

Examples 22. Let X be a topological space, G a group. We set for each non-void open set $U\subset X$, $GU=G$, and $G\emptyset=0$ (the neutral element of G). For any two open sets $U,V\subset X$ such that $U\subset V$, we set $s_{UV}:GU\to GV$ to be the restriction of the identity mapping id_G. Then the family $G=\{GU\}$ is a presheaf over X, called the *constant presheaf*. G is *not* complete, because it does not satisfy condition (5), Section 7.4 of the definition of a complete presheaf. Indeed, if U and V are *disjoint* open sets in X, and $g\in GU=G$, $h\in GV=G$ are two *different* points, then there is *no* element in G equal to both g and h (cf. Section 7.4, Remark 4). It is easily seen that the sheaf space, associated with the presheaf G, $\mathrm{Germ}\,G$, coincides with the *constant sheaf space* G_X (Section 7.2, Example 11).

Remark 5 One can define sheaves with *different* algebraic structure on the fibres than the Abelian group structure. Let $\sigma:S\to X$ be a local homeomorphism of topological spaces. Assume that for every point $x\in X$ the fibre S_x is a commutative ring with unity such that the subtraction $S\times_X S\ni(s_1,s_2)\to s_1-s_2\in S$ and multiplication $S\times_X S\ni(s_1,s_2)\to s_1\cdot s_2\in S$ are continuous. Then S is called the *sheaf space of commutative rings with unity*. If $\tau:T\to X$ is another local homeomorphism, such that the fibres T_x are modules over S_x and the mappings $T\times_X T\ni(t_1,t_2)\to t_1-t_2\in T$ and $S\times_X T\ni(s,t)\to s\cdot t\in S$ are continuous, then T is called a *sheaf space of S-modules*.