7.7 Sequences of Abelian groups, complexes

We summarize in this section elementary notions of the homological algebra of sequences of Abelian groups such as the complex, the connecting homorphism, and the long exact sequence.

A family $A^* = \{A^i, d^i\}_{i \in Z}$, of Abelian groups and their morphisms $d^i : A^i \to A^{i+1}$, indexed with the integers $i \in Z$, is called a *sequence of Abelian groups*. The family of the group morphisms in this sequence is denoted by $\{d^i\}_{i \in Z}$. We usually write A^* in the form

(1) $$\ldots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \ldots$$

Note that the asterisk in the symbol A^* of the sequence refers to the position of indices in the sequence.

A sequence of Abelian groups may begin or end with an infinite string of trivial, one-element Abelian groups 0, and their trivial group morphisms. If $A^i = 0$ for all $i < 0$, then the sequence A^* is said to be *non-negative*, and is written as $A^* = \{A^i, d^i\}_{i \in N}$, with indexing set the non-negative integers, or

(2) $$0 \longrightarrow A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} A^2 \xrightarrow{d^3} \ldots$$

In this notation the mapping $0 \to A^0$ is the *trivial* group morphism. If there exist the smallest and greatest integer *r* and *s*) such that $A^r \neq 0$ and $A^s \neq 0$, then the sequence A^* is said to be *finite*, and A^r (resp. A^s) is called its *first* (resp. *last*) element. In this case we write A^* as

(3) $$0 \longrightarrow A^r \xrightarrow{d^r} A^{r+1} \xrightarrow{d^{r+1}} \ldots \xrightarrow{d^{s-1}} A^s \xrightarrow{d^s} 0$$

with trivial group morphisms $0 \to A^r$ and $A^s \to 0$.

To simplify notation, we sometimes omit the indexing set and write just $A^* = \{A^i, d^i\}$, or $A^* = \{A^i, d\}$ for the sequence (3) when no misunderstanding may arise.

A sequence of Abelian groups $A^* = \{A^i, d^i\}$ is said to be *exact* at the term A^q, if $\operatorname{Ker} d^q = \operatorname{Im} d^{q-1}$. A^* is an *exact sequence*, if it is exact in *every* term. Exact sequence of the form

(4) $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence*.

The following are elementary properties of short exact sequences.

Lemma 9 (a) *The sequence* (4) *is exact at C if and only if the group morphism g is surjective.*

(b) *The sequence* (4) *is exact at A if and only if the f is injective.*

(c) *A sequence of Abelian groups*

(5) $$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} B/A \longrightarrow 0$$

in which $A \subset B$, $\iota : A \to B$ *is inclusion and* $\pi : B \to B/A$ *is the quotient projection, is a short exact sequence*.

(d) *Suppose we have a diagram*

(6) $$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0 & \xrightarrow{f^0} & A^1 & \xrightarrow{f^1} & A^2 & \longrightarrow & 0 \\ & & \downarrow\varphi^0 & & \downarrow\varphi^1 & & & & \\ 0 & \longrightarrow & B^0 & \xrightarrow{g^0} & B^1 & \xrightarrow{g^1} & B^2 & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are short exact sequences of Abelian groups, φ^0 *and* φ^1 *are morphisms of Abelian groups, and the first square commutes,*

(7) $$g^0 \circ \varphi^0 = \varphi^1 \circ g^1.$$

Then there exists a unique morphism of Abelian groups $\varphi^2 : A^2 \to B^2$ *such that the second square of the diagram*

(8) $$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0 & \xrightarrow{f^0} & A^1 & \xrightarrow{f^1} & A^2 & \longrightarrow & 0 \\ & & \downarrow\varphi^0 & & \downarrow\varphi^1 & & \downarrow\varphi^2 & & \\ 0 & \longrightarrow & B^0 & \xrightarrow{g^0} & B^1 & \xrightarrow{g^1} & B^2 & \longrightarrow & 0 \end{array}$$

commutes.

(e) *Consider the exact sequence of Abelian groups* (4) *and the quotient projection* $\pi : B \to B/f(A)$. *There exists a unique group isomorphism* $\varphi : C \to B/f(A)$ *such that the diagram*

(9) $$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow\mathrm{id}_A & & \downarrow\mathrm{id}_B & & \downarrow\varphi & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{\pi} & B/f(A) & \longrightarrow & 0 \end{array}$$

commutes.

Proof 1. Assertions (a), (b), and (c) are immediate consequences of definitions.

2. Consider the diagram (6). We first construct a morphism of Abelian groups $\varphi^2 : A^2 \to B^2$ and then prove its uniqueness. Let $a'' \in A^2$ be a point. We set

(10) $$\varphi^2(a'') = g^1\varphi^1(a'),$$

where $a' \in A^1$ is any element such that $f^1(a') = a''$. We shall show that this equation defines a point $\varphi^2(a'') \in B^2$ independently of the choice of a'. Let $a_1', a_2' \in A^1$ be any two points such that $f^1(a_1') = a''$ and $f^1(a_2') = a''$. Then $f^1(a_1' - a_2') = 0$ hence $a_1' - a_2' = f^0(a)$ for some $a \in A^1$ (exactness of the first row). Then, however, $g^1(\varphi^1(a_1')) = g^1(\varphi^1(a_2')) + g^1(\varphi^1(f^0(a))) = g^1(\varphi^1(a_2'))$ because $g^1(\varphi^1(f^0(a))) = g^1(g^0(\varphi^0(a))) = 0$ (exactness of the second row). Therefore, formula (10) defines a mapping $\varphi^2 : A^2 \to B^2$, and the same formula immediately implies that φ^2 satisfies the condition $\varphi^2 \circ f^1 = g^1 \circ \varphi^1$. This means that the second square of the diagram (6) commutes.

To show that the mapping φ^2 is a group morphism, take $a_1'', a_2'' \in A^2$ and $a_1', a_2' \in A^1$ such that $f^1(a_1') = a_1''$ and $f^1(a_2') = a_2''$. Then we have $f^1(a_1' + a_2') = a_1'' + a_2''$, therefore

$$(11) \qquad \varphi^2(a_1'' + a_2'') = g^1(\varphi^1(a_1' + a_2')) = \varphi^2(a_1'') + \varphi^2(a_2'')$$

since both g^1 and φ^1 are group morphisms. This proves existence of the group morphism φ^2. Its uniqueness follows from the surjectivity of f^1.

3. To prove (e) we combine (c) and (d).

A sequence of Abelian groups $A^* = \{A^i, d^i\}$ is called a *complex of Abelian groups*, or just a *complex*, if

$$(12) \qquad d^{i+1} \circ d^i = 0$$

for all i. The family of group morphisms $d^* = \{d^i\}$ is called the *differential* of the complex A^*. Condition (12) is equivalent to saying that the kernel $\operatorname{Ker} d^{i+1}$ and the image $\operatorname{Im} d^i$ satisfy $\operatorname{Im} d^i \subset \operatorname{Ker} d^{i+1}$. To simplify notation, we usually denote the Abelian group morphisms d^i by the same letter, d; condition (12) then reads $d \circ d = 0$.

Let $A^* = \{A^i, d\}$ be a complex. For every index i, the complex A^* defines an Abelian group $H^i A^*$, the i-th *cohomology group* of A^*, by

$$(13) \qquad H^i A^* = \operatorname{Ker} d^{i+1} / \operatorname{Im} d^i .$$

Elements of this group are called *i-th cohomology classes* of the complex A^*. Note that the complex is exact in the *i-th* term if and only if the *i-th* cohomology group $H^i A^*$ is trivial.

If A is an Abelian group, then any exact sequence Abelian groups of the form

$$(14) \qquad 0 \longrightarrow A \xrightarrow{\ \varepsilon\ } B^0 \xrightarrow{\ d\ } B^1 \xrightarrow{\ d\ } B^2 \xrightarrow{\ d\ } \dots$$

is called a *resolution* of A. A resolution (14) defines a non-negative complex $B^* = \{B^i, d\}$ as

$$(15) \qquad 0 \longrightarrow B^0 \xrightarrow{\ d\ } B^1 \xrightarrow{\ d\ } B^2 \xrightarrow{\ d\ } B^3 \xrightarrow{\ d\ } \dots$$

such that

(16) $H^0 B^* = A, \quad H^i B^* = 0, \quad i \geq 1.$

Using this complex the resolution can also be expressed in a shortened form

(17) $$0 \longrightarrow A \xrightarrow{\varepsilon} B^*$$

Let $A^* = \{A^i, d\}$ and $B^* = \{B^i, d'\}$ be two complexes, and let $\Phi = \{\varphi^i\}$ be a family of Abelian group morphisms $\varphi^i : A^i \to B^i$. These complexes and group morphisms can be expressed by the diagram

(18) $$\begin{array}{ccccccccc} \dots & \longrightarrow & A^{i-1} & \xrightarrow{d} & A^i & \xrightarrow{d} & A^{i+1} & \longrightarrow & \dots \\ & & \downarrow\varphi^{i-1} & & \downarrow\varphi^i & & \downarrow\varphi^{i+1} & & \\ \dots & \longrightarrow & B^{i-1} & \xrightarrow{d'} & B^i & \xrightarrow{d'} & B^{i+1} & \longrightarrow & \dots \end{array}$$

If all squares in this diagram commute, that is,

(19) $$\varphi^{i+1} \circ d = d' \circ \varphi^i,$$

then we say that Φ is a *morphism of the complex* A^* into B^*. Property (19) can also be expressed by writing $\Phi : A^* \to B^*$. The *composition* of two morphisms Φ and Ψ , defined in an obvious way, and is denoted by $\Psi \circ \Phi$.

As before, the asterisk in the following lemma denotes position of indices, labelling different elements of Abelian groups belonging to a complex.

Lemma 10 *Let* $A^* = \{A^i_j, d^i_j\}$ *and* $A_* = \{A^j_i, \delta^j_i\}$ *be two families of non-negative complexes. Suppose that we have a commutative diagram*

(20) $$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^0_0 & \xrightarrow{d^0_0} & A^1_0 & \xrightarrow{d^1_0} & A^2_0 & \xrightarrow{d^2_0} & A^3_0 & \longrightarrow & \dots \\ & & \downarrow\delta^0_0 & & \downarrow\delta^1_0 & & \downarrow\delta^2_0 & & \downarrow\delta^3_0 & & \\ 0 & \longrightarrow & A^0_1 & \xrightarrow{d^0_1} & A^1_1 & \xrightarrow{d^1_1} & A^2_1 & \xrightarrow{d^2_1} & A^3_1 & \longrightarrow & \dots \\ & & \downarrow\delta^0_1 & & \downarrow\delta^1_1 & & \downarrow\delta^2_1 & & \downarrow\delta^3_1 & & \\ 0 & \longrightarrow & A^0_2 & \xrightarrow{d^0_2} & A^1_2 & \xrightarrow{d^1_2} & A^2_2 & \xrightarrow{d^2_2} & A^3_2 & \longrightarrow & \dots \\ & & \downarrow\delta^0_2 & & \downarrow\delta^1_2 & & \downarrow\delta^2_2 & & \downarrow\delta^3_2 & & \\ 0 & \longrightarrow & A^0_3 & \xrightarrow{d^0_3} & A^1_3 & \xrightarrow{d^1_3} & A^2_3 & \xrightarrow{d^2_3} & A^3_3 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

such that that all its rows (resp. columns) except possibly the first row (resp. column) are exact sequences of Abelian groups. Then for each $q \geq 0$ *the cohomology groups* $H^q A^*_0$ *and* $H^q A^0_*$ *are isomorphic.*

Proof Let $q=0$ and let $[a]\in H^0A^0_* = \operatorname{Ker}\delta^0_0$. Then $[a]=a$, $\delta^0_0(a)=0$ hence $\delta^1_0 d^0_0(a)=d^0_1\delta^0_0(a)=0$ and injectivity of δ^1_0 implies $d^0_0(a)=0$, that is, $a\in\operatorname{Ker}d^0_0 = H^0A^*_0$. Thus, $H^0A^0_*\subset H^0A^*_0$. The opposite inclusion is obtained in the same way.

Consider the case $q\geq 1$. Let $[a]\in H^qA^0_* = \operatorname{Ker}\delta^0_q/\operatorname{Im}\delta^0_{q-1}$, and let a be a representative of $[a]$. Then $\delta^0_q(a)=0$ hence $\delta^1_q d^0_q(a)=d^0_{q-1}\delta^0_q(a)=0$, that is, $d^0_q(a)\in\operatorname{Ker}\delta^1_q=\operatorname{Im}\delta^1_{q-1}$, and for some $b_1\in A^1_{q-1}$,

$$d^0_q(a)=\delta^1_{q-1}(b_1). \tag{21}$$

But $\delta^2_{q-1}d^1_{q-1}(b_1)=d^1_q\delta^1_{q-1}(b_1)=d^1_q d^0_q(a)=0$ and $d^1_{q-1}(b_1)\in\operatorname{Ker}\delta^2_{q-1}=\operatorname{Im}\delta^2_{q-2}$. Thus, for some $b_2=A^1_{q-1}$ we have $d^1_{q-1}(b_1)=\delta^2_{q-2}(b_2)$.

Suppose that for some k, $1\leq k\leq q-2$, and A^k_{q-k}, there exists $b_k\in A^k_{q-k}$ such that $d^k_{q-1}(b_k)=\delta^{k+1}_{q-k-1}(b_{k+1})$. Then

$$\delta^{k+2}_{q-k-1}d^{k+1}_{q-k-1}(b_{k+1})=d^{k+1}_{q-k}\delta^{k+1}_{q-k-1}(b_{k+1})=d^{k+1}_{q-k}d^k_{q-k}(b_k)=0 \tag{22}$$

hence $d^{k+1}_{q-k-1}(b_{k+1})\in\operatorname{Ker}\delta^{k+2}_{q-k-1}=\operatorname{Im}\delta^{k+2}_{q-k-2}$. Thus for some $b_{k+2}\in A^{k+1}_{q-k-2}$,

$$d^{k+1}_{q-k-1}(b_{k+1})=\delta^{k+2}_{q-k-2}(b_{k+2}). \tag{23}$$

The construction is described by the following part of diagram (20):

$$\begin{tikzcd}
 & & b_{k+2} & A^{k+2}_{q-k-2} \arrow[d, "\delta^{k+2}_{q-k-2}"] \\
 & b_{k+1} & A^{k+1}_{q-k-1} \arrow[r, "d^{k+1}_{q-k-1}"] \arrow[d, "\delta^{k+1}_{q-k-1}"] & A^{k+2}_{q-k-1} \arrow[d, "\delta^{k+2}_{q-k-1}"] \\
b_k & A^k_{q-k} \arrow[r, "d^k_{q-k}"] \arrow[d, "\delta^k_{q-k}"] & A^{k+1}_{q-k} \arrow[r, "d^{k+1}_{q-k}"] \arrow[d, "\delta^{k+1}_{q-k}"] & A^{k+2}_{q-k} \\
 & A^k_{q-k+1} \arrow[r, "d^k_{q-k+1}"] & A^{k+1}_{q-k+1} &
\end{tikzcd} \tag{24}$$

For $k=q-2$, formula (23) gives $d^{q-1}_1(b_{q-1})=\delta^q_0(b_q)$ hence

$$\delta^{q+1}_0 d^q_0(b_q)=d^q_1\delta^q_0(b_q)=d^q_1 d^{q-1}_1(b_{q-1})=0, \tag{25}$$

and injectivity of δ^{q+1}_0 implies $d^q_0(b_q)=0$ hence $b_q\in\operatorname{Ker}d^q_0$.

Thus, to a representative a of a class $[a]\in H^qA^0_*$ we have constructed a sequence $(b_1,b_2,\ldots,b_q)$ such that $b_i\in A^i_{q-1}$ for each i, $b_q\in\operatorname{Ker}d^q_0$, and

$$d^0_q(a)=\delta^1_{q-1}(b_1),\quad d^k_{q-k}(b_k)=\delta^{k+1}_{q-k-1}(b_{k+1}). \tag{26}$$

Let a' be another representative of the class $[a]$, and let $(b'_1,b'_2,\ldots,b'_q)$

be another sequence satisfying condition (26),

$$d^0_q(a') = \delta^1_{q-1}(b'_1), \quad d^k_{q-k}(b'_k) = \delta^{k+1}_{q-k-1}(b'_{k+1}). \tag{27}$$

We set $a'' = a - a'$, $b''_i = b_i - b'_i$. We wish to show that $[b''_q] = 0$ hence $b''_q \in \operatorname{Im} d^{q-1}_0$. By definition $[a''] = 0$ hence $a'' \in \operatorname{Im} \delta^0_{q-1}$ and $a'' = \delta^0_{q-1}(c_1)$ for some $c_1 \in A^0_{q-1}$. But by (26) and (27), $\delta^1_{q-1}(b''_1) = d^0_q(\delta^0_{q-1}(c_1)) = \delta^1_{q-1} d^0_{q-1}(c_1)$, which implies $b''_1 - d^0_{q-1}(c_1) \in \operatorname{Ker} \delta^1_{q-1} = \operatorname{Im} \delta^1_{q-2}$, hence for some $c_2 \in A^1_{q-2}$,

$$b''_1 - d^0_{q-1}(c_1) = \delta^1_{q-2}(c_2). \tag{28}$$

Now suppose that for some $k \ge 1$ and some $c_k \in A^{k-1}_{q-k}$ there exists $c_{k+1} \in A^{k-1}_{q-k-1}$ such that $b''_k - d^{k-1}_{q-k}(c_k) = \delta^k_{q-k-1}(c_{k+1})$. Using (26), (27) and (28),

$$\begin{aligned} \delta^{k+1}_{q-k-1}(b''_{k+1}) &= d^k_{q-k}(b''_k) = d^k_{q-k}(\delta^k_{q-k-1}(c_{k+1}) + d^{k-1}_{q-k}(c_k)) \\ &= d^k_{q-k}\delta^k_{q-k-1}(c_{k+1}) = \delta^{k+1}_{q-k-1} d^k_{q-k-1}(c_{k+1}), \end{aligned} \tag{29}$$

so that $b''_{k+1} - d^k_{q-k-1}(c_{k+1}) \in \operatorname{Ker} \delta^{k+1}_{q-k-1} = \operatorname{Im} \delta^{k+1}_{q-k-2}$. Thus, for some $c_{k+2} \in A^{k+1}_{q-k-2}$

$$b''_{k+1} - d^k_{q-k-1}(c_{k+1}) = \delta^{k+1}_{q-k-2}(c_{k+2}). \tag{30}$$

The derivation of this formula includes the following part of diagram (20) of Lemma 10:

$$\begin{array}{cccccccc} & & & & & & c_{k+2} & A^{k+1}_{q-k-2} \\ & & & & & & & \downarrow \delta^{k+1}_{q-k-2} \\ & & & c_{k+1} & A^k_{q-k-1} & \xrightarrow{d^k_{q-k}} & & A^{k+1}_{q-k-1} \\ & & & & \downarrow \delta^k_{q-k-1} & & & \downarrow \delta^{k+1}_{q-k-1} \\ c_{k+1} & A^{k-1}_{q-k} & \xrightarrow{d^{k-1}_{q-k}} & & A^k_{q-k} & \xrightarrow{d^k_{q-k}} & & A^{k+1}_{q-k} \\ & \downarrow \delta^{k-1}_{q-k} & & & \downarrow \delta^k_{q-k} & & & \\ & A^{k-1}_{q-k+1} & \xrightarrow{d^{k-1}_{q-k+1}} & & A^k_{q-k-1} & & & \end{array} \tag{31}$$

If $k = q-2$, formula (30) gives for some $c_q \in A^{q-1}_0$

$$b''_{q-1} - d^k_1(c_{q-1}) = \delta^{q-1}_0(c_q). \tag{32}$$

Then by (26), (27), and (32)

$$\begin{aligned} \delta^q_0(b''_q) &= d^{q-1}_1(b''_{q-1}) = d^{q-1}_1(\delta^{q-1}_0(c_q) + d^{q-2}_1(c_{q-1})) \\ &= d^{q-1}_1\delta^{q-1}_0(c_q) = \delta^q_0 d^{q-1}_0(c_q), \end{aligned} \tag{33}$$

that is, $b''_q - d_0^{q-1}(c_q) = 0$ because δ_0^q is injective. Therefore, $b''_q \in \operatorname{Im} d_0^{q-1}$.

Consequently, equation

(34) $\quad f^q([a]) = [b_q]$

defines a mapping $f^q : H^q A_*^0 \to H^q A_0^*$ which is a morphism of Abelian groups. In the same way we define a morphism of Abelian groups $f_q : H^q A_0^* \to H^q A_*^0$, and it remains to verify that the morphism f_q is the inverse of f^q.

Let $[b] \in H^q A_0^*$ be a class, represented by an element b. There exists a sequence $(a_1, a_2, \ldots, a_q)$, where $a_i \in A_i^{q-1}$, such that

(35) $\quad \delta_0^q(b) = d_1^{q-1}(a_1), \quad \delta_k^{q-k}(a_k) = d_{k+1}^{q-k-1}(a_{k+1}),$

where $k = 1, 2, \ldots, q-1$. By definition,

(36) $\quad f_q([b]) = [a_q].$

Let $[b] = [b_q]$, where $[b_q]$ is determined by (34). Taking $a_1 = b_{q-1}$, $a_2 = b_{q-2}$, $\ldots$, $a_{q-1} = b_1$, $a_q = a$ we get from (21) and (23) that (35) is satisfied. Consequently, $[a_q] = [a]$ proving that f_q is the inverse of f^q.

This completes the proof of Lemma 10.

Now we consider three complexes $A^* = \{A^i, d^i\}$, $B^* = \{B^i, \delta^i\}$ and $C^* = \{C^i, \Delta^i\}$ and two morphisms of complexes $\Phi : A^* \to B^*$, $\Phi = \{\varphi^i\}$, and $\Psi : B^* \to C^*$, $\Psi = \{\psi^i\}$ between them. The composition of these morphisms yields a morphism of complexes $\Psi \circ \Phi : A^* \to C^*$, defined by

(37) $\quad (\Psi \circ \Phi)^q = \psi^q \circ \varphi^q.$

We show that under some exactness hypothesis these morphisms induce an exact sequence of Abelian groups, formed by cohomology groups of these complexes.

Note that the morphism Φ induces the diagrams

(38)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im} d^{i-1} & \longrightarrow & \operatorname{Ker} d^i & \longrightarrow & H^i A^* & \longrightarrow & 0 \\
 & & \downarrow\varphi^i & & \downarrow\varphi^i & & \downarrow\varphi_i & & \\
0 & \longrightarrow & \operatorname{Im} \delta^{i-1} & \longrightarrow & \operatorname{Ker} \delta^i & \longrightarrow & H^i B^* & \longrightarrow & 0
\end{array}$$

where the first two vertical arrows are the restrictions of the morphism φ^i to the subgroups of A^i, the mappings $\operatorname{Im} d^{i-1} \to \operatorname{Ker} d^i$ and $\operatorname{Im} \delta^{i-1} \to \operatorname{Ker} \delta^i$ are the canonical inclusions, and φ_i is the unique morphism of Abelian groups for which the second square in the diagram (38) commutes (Lemma 10, (e)).

The following statement is sometimes referred to as the *zig-zag lemma*. Its proof is based on the technique known as the *diagram chasing*.

Lemma 11 *Let* $A^* = \{A^i, d^i\}$, $B^* = \{B^i, \delta^i\}$ *and* $C^* = \{C^i, \Delta^i\}$ *be three non-negative complexes,* $\Phi : A^* \to B^*$, $\Phi = \{\varphi^i\}$, *and* $\Psi : B^* \to C^*$, $\Psi = \{\psi^i\}$ *morphisms of complexes. Suppose that we have a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A^0 \arrow[r, "d^0"] \arrow[d, "\varphi^0"] & A^1 \arrow[r, "d^1"] \arrow[d, "\varphi^1"] & A^2 \arrow[r, "d^2"] \arrow[d, "\varphi^2"] & \dots \\
0 \arrow[r] & B^0 \arrow[r, "\delta^0"] \arrow[d, "\psi^0"] & B^1 \arrow[r, "\delta^1"] \arrow[d, "\psi^1"] & B^2 \arrow[r, "\delta^2"] \arrow[d, "\psi^2"] & \dots \\
0 \arrow[r] & C^0 \arrow[r, "\Delta^0"] \arrow[d] & C^1 \arrow[r, "\Delta^1"] \arrow[d] & C^2 \arrow[r, "\Delta^2"] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{39}$$

with exavt columns. Then for every $q \ge 0$ *there exists a morphism of sequences of Abelian groups* $\partial = \{\partial^q\}$, $\partial^q : H^q C^* \to H^{q+1} A^*$ *such that the sequence of Abelian groups*

$$\begin{aligned} 0 &\longrightarrow H^0 A^* \xrightarrow{\ \varphi^0\ } H^0 B^* \xrightarrow{\ \psi^0\ } H^0 C^* \xrightarrow{\ \partial^0\ } H^1 A^* \\ &\xrightarrow{\ \varphi_1\ } H^1 B^* \xrightarrow{\ \psi_1\ } H^1 C^* \xrightarrow{\ \partial^1\ } H^2 A^* \xrightarrow{\ \varphi^2\ } \end{aligned} \tag{40}$$

is exact.

Proof 1. First we construct the group morphisms $\partial^q : H^q C^* \to H^{q+1} A^*$. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A^{q-1} \arrow[r, "d^{q-1}"] \arrow[d, "\varphi^{q-1}"] & A^q \arrow[r, "d^q"] \arrow[d, "\varphi^q"] & A^{q+1} \arrow[r, "d^{q+1}"] \arrow[d, "\varphi^{q+1}"] & A^{q+2} \arrow[d, "\varphi^{q+2}"] \\
B^{q-1} \arrow[r, "\delta^{q-1}"] \arrow[d, "\psi^{q-1}"] & B^q \arrow[r, "\delta^q"] \arrow[d, "\psi^q"] & B^{q+1} \arrow[r, "\delta^{q+1}"] \arrow[d, "\psi^{q+1}"] & B^{q+2} \\
C^{q-1} \arrow[r, "\Delta^{q-1}"] \arrow[d] & C^q \arrow[r, "\Delta^q"] \arrow[d] & C^{q+1} \arrow[d] & \\
0 & 0 & 0 &
\end{tikzcd} \tag{41}$$

Let $[c] \in H^q C^* = \operatorname{Ker} \Delta^q / \operatorname{Im} \Delta^q$ be a class, represented by an element $c \in \operatorname{Ker} \Delta^q$. Since ψ^q is surjective, there exists an element $b \in B^q$ such that $\psi^q(b) = c$. But $\psi^{q+1} \delta^q(b) = \Delta^q \psi^q(b) = 0$ so that $\delta^q(b) \in \operatorname{Ker} \psi^{q+1}$ and by exactness of the third column, there exists an element $a \in A^{q+1}$ such that $\delta^q(b) = \varphi^{q+1}(a)$. Since $\varphi^{q+2} d^{q+1}(a) = \delta^{q+1}\varphi^{q+1}(a) = \delta^{q+1}\delta^q(a) = 0$, and since φ^{q+2} is injective, $d^{q+1}(a) = 0$ and $a \in \operatorname{Ker} d^{q+1}$. Thus, given $c \in \operatorname{Ker} \Delta^q$, there exists $b \in B^q$ and $a \in \operatorname{Ker} d^{q+1}$ such that

(42) $$c = \psi^q(b), \quad \delta^q(b) = \varphi^{q+1}(a).$$

If c' is some other representative of the class $[c]$, then there exist $b' \in B^q$, $a' \in \operatorname{Ker} d^{q+1}$ and $d \in C^{q-1}$ such that

(43) $$c' = \psi^q(b'), \quad \delta^q(b') = \varphi^{q+1}(a'), \quad c' = c - \Delta^{q-1}(d).$$

We show that $[a] = [a']$. We have $d = \psi^{q-1}(b_0)$ for some $b_0 \in B^{q-1}$ (by surjectivity of ψ^{q-1}). Thus, $\psi^q \delta^{q-1}(b_0) = \Delta^{q-1}\psi^{q-1}(b_0) = \Delta^{q-1}(d)$, and the third formula (41) gives $\psi^q(b' - b + \delta^{q-1}(b_0)) = 0$, that is, by exactness of the column, $b' - b + \delta^{q-1}(b_0) \in \operatorname{Im}\varphi^q$. Thus, $b' - b + \delta^{q-1}(b_0) = \varphi^q(a_0)$ for some $a_0 \in A^q$. But $\delta^q(b' - b + \delta^{q-1}(b_0)) = \delta^q\varphi^q(a_0) = \varphi^{q+1}d^q(a_0)$ by commutativity of the diagram (41). Applying (42) and (43) and the property $\delta^{q+1}\delta^q = 0$ of the complex B^* one obtains $\varphi^{q+1}(a') - \varphi^{q+1}(a) = \varphi^{q+1}d^q(a_0)$. Finally, injectivity of φ^{q+1} yields $a' - a = d^q(a_0)$. This proves that $[a] = [a']$.

Now since the class $[a]$ is defined independently of the choice of the representative c of the class $[c]$, we may define a mapping ∂^q of H^qC^* into $H^{q+1}A^*$ by the formula

(44) $$\partial^q([c]) = [a].$$

It is easily verified that this mapping is an Abelian group morphism. Let c_1 be a representative of a class $[c_1]$ in H^qC^*. There exists $b_1 \in B^q$ and $a_1 \in \operatorname{Ker} d^{q+1}$ such that $c_1 = \psi^q(b_1)$, $\delta^q(b_1) = \varphi^{q+1}(a_1)$. Similarly, let c_2 be a representative of a class $[c_2]$ in H^qC^*. There exist elements $b_2 \in B^q$ and $a_2 \in \operatorname{Ker} d^{q+1}$ such that $c_2 = \psi^q(b_2)$, $\delta^q(b_2) = \varphi^{q+1}(a_2)$. Then

(45) $$c_1 + c_2 = \psi^q(b_1 + b_2), \quad \delta^q(b_1 + b_2) = \varphi^{q+1}(a_1 + a_2),$$

proving that ∂^q is a group morphism.

2. Now we prove exactness of the sequence of Abelian groups (40). We proceed in several steps.

(a) Exactness at $H^0A^* = \operatorname{Ker} d^0$ is obvious: Since $H^0B^* = \operatorname{Ker}\delta^0$ and the commutativity of the left upper square in the diagram (39) implies $\varphi^0(\operatorname{Ker} d^0) \subset \operatorname{Ker}\delta^0$, exactness at H^0A^* follows from injectivity of φ^0.

(b) We verify exactness at the term H^0B^*. Let $b \in H^0B^* = \operatorname{Ker}\delta^0$ and $b \in \operatorname{Ker}\psi^0$. Then $b = \varphi^0(a)$ for some $a \in A_0 = H^0A^*$, and we want to show that $a \in \operatorname{Ker} d^0$. But $\varphi^1 d^0(a) = \delta^0\varphi^0(a) = \delta^0(b) = 0$ hence $d^0(a) = 0$ (injectivity of φ^1). and $a \in \operatorname{Ker} d^0 = H^0A^*$. Thus $\operatorname{Ker}\psi^0 = \operatorname{Im}\varphi^0$.

(c) We prove exactness at H^0C^*. Consider an element $c \in H^0C^*$ such that $c \in \operatorname{Ker}\Delta^0$, that is, $\partial^0 c = 0$. We want to show that $c = \psi^0(b)$ for some $b \in H^0B^* = \operatorname{Ker}\delta^0$. By definition, $\partial^0 c = [a]$, where $a \in \operatorname{Ker} d^1$ is an arbitrary point such that for some $b' \in B^0$, $c = \psi^0(b')$ and $\delta^0(b') = \varphi^1(a)$ (42). But $[a] = 0$ hence $a \in \operatorname{Im} d^0$ and $a = d^0(a')$ for some $a' \in A^0$. Consequently, $\delta^0(b') = \varphi^1 d^0(a') = \delta^0 \varphi^0(a')$. We set $b = b' - \varphi^0(a')$. Then

(46) $$\delta^0(b) = \delta^0(b') - \delta^0\varphi^0(a') = 0,$$

that is, $b \in \operatorname{Ker}\delta^0$. Moreover,

(47) $$\psi^0(b) = \psi^0(b') - \psi^0\varphi^0(a') = \psi^0(b') = c,$$

thus $\operatorname{Ker}\delta^0 \subset \operatorname{Im}\psi^0$.

Conversely, if $c \in \operatorname{Im}\psi^0$, then $c = \psi^0(b)$ for some $b \in H^0B^* = \operatorname{Ker}\delta^0$, and $\partial^0(c) = [a]$, where $c = \psi^0(b')$ and $\delta^0(b') = \varphi^1(a)$ for some $b' \in B^0$, $a \in \operatorname{Ker} d^1$ (42). But $\psi^0(b - b') = 0$ hence $b - b' = \varphi^0(a')$, where $a' \in A^0$. Now $\varphi^1 d^0(a') = \delta^0\varphi^0(a') = \delta^0(b - b') = -\delta^0(b') = -\varphi^1(a)$ that is, by injectivity, $d^0(a') = -a$. Hence $[a] = -[d^0(a')] = 0$ and we get $\operatorname{Im}\psi^0 \subset \operatorname{Ker}\partial^0$.

Summarizing, $\operatorname{Im}\psi^0 = \operatorname{Ker}\partial^0$ as required.

(d) We check exactness at H^qA^*, where $q > 0$. Let $[a] \in H^qA^*$ and $\varphi_q([a]) = 0$. Since $\varphi_q([a]) = [\varphi^q(a)] = 0$, we have $\varphi^q(a) \in \operatorname{Im}\delta^{q-1}$. Thus, there exists $b \in B^{q-1}$ such that $\delta^{q-1}(b) = \varphi^q(a)$. We set $c = \psi^{q-1}(b)$. Then by definition, $\partial([c]) = [a]$, therefore $\operatorname{Ker}\varphi_q \subset \operatorname{Im}\partial^{q-1}$.

Conversely, consider a class $[c] \in H^{q-1}C^*$. Then $\varphi_q\delta^{q-1}([c]) = \varphi_q([a])$, where $c = \psi^{q-1}(b)$, $\delta^{q-1}(b) = \varphi^q(a)$ for some $b \in B^{q-1}$, $a \in \operatorname{Ker} d^q$. But then $\varphi_q\delta^{q-1}([c]) = [\varphi^q(a)] = [\delta^{q-1}(b)] = 0$ since $H^qB^* = \operatorname{Ker}\delta^q / \operatorname{Im}\delta^{q-1}$.

(e) We prove exactness at H^qB^*, $q > 0$. Let $[b] \in H^qB^*$ be a class such that $\psi_q([b]) = [\psi^q(b)] = 0$. Then $\psi^q(b) \in \operatorname{Im}\Delta^{q-1}$ hence there exists $c \in C^{q-1}$ such that $\psi^q(b) = \Delta^{q-1}(c)$. But $c = \psi^{q-1}(b')$ for some $b' \in B^{q-1}$; applying Δ^{q-1} we have $\Delta^{q-1}\psi^{q-1}(b') = \psi^q\delta^{q-1}(b')$, that is, $\psi^q(b) = \psi^q\delta^{q-1}(b')$ hence $\psi^q(b - \delta^{q-1}(b')) = 0$ and $b - \delta^{q-1}(b') = \varphi^q(a)$ for some $a \in A^q$. Now $\varphi^{q+1}d^q(a) = \delta^q\varphi^q(a) = \delta^q(b - \psi^q\delta^{q-1}(b')) = 0$ because $\delta^q(a) = 0$, $\delta^q\delta^{q-1} = 0$. Hence $d^q(a) = 0$ and $a \in \operatorname{Ker} d^q$. Now

(48) $$\varphi_q([a]) = [\varphi^q(a)] = [b - \delta^{q-1}(b')] = [b],$$

so we get the inclusion $\operatorname{Ker}\psi_q \subset \operatorname{Im}\varphi_q$.

The inverse inclusion follows from the equality $\psi_q \circ \varphi^q = 0$ and from the diagram (38), which implies

(49) $$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im} d^{q-1} & \longrightarrow & \operatorname{Ker} d^q & \longrightarrow & H^qA^* & \longrightarrow & 0 \\
 & & \downarrow\varphi^q & & \downarrow\varphi^q & & \downarrow\varphi_q & & \\
0 & \longrightarrow & \operatorname{Im}\delta^{q-1} & \longrightarrow & \operatorname{Ker}\delta^q & \longrightarrow & H^qB^* & \longrightarrow & 0 \\
 & & \downarrow\psi^q & & \downarrow\psi^q & & \downarrow\psi_q & & \\
0 & \longrightarrow & \operatorname{Im}\Delta^{q-1} & \longrightarrow & \operatorname{Ker}\delta^q & \longrightarrow & H^qC^* & \longrightarrow & 0
\end{array}$$

in which the group morphisms φ_q and ψ_q are unique, and the composition law $(\Psi\circ\Phi)^q=\psi^q\circ\varphi^q$ (37) holds.

(f) We prove exactness at H^qC^*, where $q>0$. Let $[c]\in H^qC^*$ be a class such that $\partial^q([c])=0$. We want to show that there exists $[b]\in H^qB^*$ such that $[c]=\psi_q([b])$. Let c be a representative of $[c]$. By (42) there exist an element $b\in B^q$ and $a\in\operatorname{Ker}d^{q+1}$ such that $c=\psi^q(b)$, $\delta^q(b)=\varphi^{q+1}(a)$. From the condition $\partial^q([c])=0$ it follows that $[a]=0$ hence $a\in\operatorname{Im}d^q$ and $a=d^q(a')$ for some $a'\in A^q$. Then $\delta^q(b)=\varphi^{q+1}d^q(a')=\delta^q\varphi^q(a')$ hence $b-\varphi^q(a')\in\operatorname{Ker}\delta^q$. Setting $b'=b-\varphi^q(a')$ we have $\delta^q(b')=0$, $b'\in\operatorname{Ker}\delta^q$. Moreover, $\psi^q(b')=\psi^q(b-\varphi^q(a'))=\psi^q(b)=c$, therefore

$$\psi_q([b'])=[\psi^q(b')]=[c]. \tag{50}$$

This implies that $\operatorname{Ker}\partial^q\subset\operatorname{Im}\psi_q$.

Conversely, let $[c]\in\operatorname{Im}\psi_q$. Then $[c]=\psi_q([b])=[\psi^q(b)]$ for some element $[b]\in H^qB^*$. Thus $\partial^q([c])=[a]$, where $c=\psi^q(b')$, $\delta^q(b')=\varphi^{q+1}(a)$ for some $b'\in B^q$. But $\psi^q(b-b')=0$ so that $b-b'=\varphi^q(a')$, where $a'\in A^q$. Now

$$\varphi^{q+1}d^q(a')=\delta^q\varphi^q(a')=\delta^q(b-b')=-\delta^q(b') \tag{51}$$

hence $\varphi^{q+1}(a)=-\varphi^{q+1}d^q(a')$, $\varphi^{q+1}(a+d^q(a'))=0$, and $a+d^q(a')=0$. Hence $[a]=-[d^q(a')]=0$, therefore $\operatorname{Im}\psi_q\subset\operatorname{Ker}\partial^q$. This completes the proof.

The exact sequence of Abelian groups (40) is referred to as the *long exact sequence*, associated with the morphisms of complexes $\Phi:A^*\to B^*$ and $\Psi:B^*\to C^*$. The family of Abelian group morphisms $\partial=\{\partial^q\}$, where $\partial^q:H^qC^*\to H^{q+1}A^*$, is called the *connecting morphism*, associated to the morphisms Φ and Ψ.

The following two corollaries follow from the long exact sequence (40).

Corollary 4 *Suppose that in the commutative diagram of morphisms of Abelian groups*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{52}$$

all columns are exact. Then if two rows are exact, the third row is also exact.

Corollary 5 *Let* A^*, B^* *and* C^* *be three non-negative complexes,* $\Phi : A^* \to B^*$ *and* $\Psi : B^* \to C^*$ *morphisms of complexes. Suppose that the diagram* (39) *commutes and all its columns are exact. Then if any two of the complexes* A^*, B^*, *and* C^* *are exact, the third is also exact.*

Proof This follows from the long exact sequence (40).

7.8 Exact sequences of Abelian sheaves

The concepts we have introduced for sequences of Abelian groups apply to sequences of Abelian sheaves. First we briefly formulate the definitions, and describe basic properties of exact sequences. Then we study the *canonical resolution* of an Abelian sheaf, an exact sequence, relating properties of a sheaf with topological properties of its base space.

A family $S^* = \{S^i, f^i\}_{i \in \mathrm{Z}}$ of Abelian sheaves S^i over the same base, and their morphisms $f^i : S^i \to S^{i+1}$, indexed with the integers $i \in \mathrm{Z}$, is called a *sequence of Abelian sheaves*. The family of sheaf morphisms in this sequence is denoted by $\{f^i\}_{i \in \mathrm{Z}}$. The sequence S^* is called a *non-negative*, if $S^i = 0$ for all $i < 0$. Then the sequence S^* is usually written as $S^* = \{S^i, f^i\}_{i \in \mathrm{N}}$, with indexing set the non-negative integers N, or just as

$$(1) \qquad 0 \longrightarrow S^0 \xrightarrow{\ f^0\ } S^1 \xrightarrow{\ f^1\ } S^2 \xrightarrow{\ f^2\ } \dots$$

In this notation the mapping $0 \to S^0$ is the *trivial* sheaf morphism. If there exist the smallest and greatest integers *r* and *s* such that $S^r \neq 0$ and $S^s \neq 0$, then the sequence S^* is said to be *finite*, and S^r (resp. S^s) is called its *first* (resp. *last*) element. In this case we write S^* as

$$(2) \qquad 0 \longrightarrow S^r \xrightarrow{\ f^r\ } S^{r+1} \xrightarrow{\ f^{r+1}\ } \dots \xrightarrow{\ f^{s-1}\ } S^s \longrightarrow 0$$

with trivial sheaf morphisms $0 \to S^r$ and $S^s \to 0$. To further simplify notation, we sometimes omit the indexing set and write just $S^* = \{S^i, f^i\}$, or $S^* = \{S^i, f\}$ instead of $S^* = \{S^i, f^i\}_{i \in \mathrm{N}}$.

Let $S^* = \{S^i, f^i\}$ be a family of sheaves of Abelian groups over a topological space *X*, $x \in X$ a point. Denote by $S^p_x = (\mathrm{Germ}\, S^p)_x$ the *fibre* of the sheaf space $\mathrm{Germ}\, S^p$ over *x*, and by $f^p_x : S^p_x \to S^{p+1}_x$ the restriction to the fibre of the morphism $f^i : S^i \to S^{i+1}$. Restricting all the sheaf morphisms to the fibres S^p_x we get a sequence of Abelian groups

$$(3) \qquad 0 \longrightarrow S^0_x \xrightarrow{\ f^0_x\ } S^1_x \xrightarrow{\ f^1_x\ } S^2_x \xrightarrow{\ f^2_x\ } \dots$$

This sequence is called the *restriction* of the sequence (1) to the point *x*.

The sequence S^* (1) is said to be *exact* at the term S^q *over x*, if the restricted sequence (3) is exact as the sequence of Abelian groups, that is, if

$\operatorname{Ker} f_x^q = \operatorname{Im} f_x^{q-1}$. $S*$ is said to be *exact* at the term S^q if it is exact at *x* for every $x \in X$. We say that $S*$ is an *exact sequence*, if it is exact in every term S^q.

A sequence of Abelian sheaves $S* = \{S^i, f^i\}$, such that

(4) $$f^{q+1} \circ f^q = 0$$

for all *q* is called a *differential sequence*. An exact sequence is a differential sequence.

Let S be an Abelian sheaf. An exact sequence of the form

(5) $$0 \longrightarrow S \xrightarrow{\varepsilon} T^0 \xrightarrow{f^0} T^1 \xrightarrow{f^1} T^2 \xrightarrow{f^2} \ldots$$

is called a *resolution* of S. The resolution defines a non-negative differential sequence $T* = \{T^i, f^i\}$. To shorten notation we sometimes write the sequence (5) as

(6) $$0 \longrightarrow S \xrightarrow{f} T*$$

the mappings being understood.

An exact sequence of the form

(7) $$0 \longrightarrow R \xrightarrow{f} S \xrightarrow{g} T \longrightarrow 0$$

where $0 \to R$ and $T \to 0$ are trivial sheaf morphisms, is called a *short exact sequence*.

Let *Y* be a subspace of the topological space *X*. Denote by S_Y the restriction of the Abelian sheaf S to *Y* and by f_Y^i the restriction of the sheaf morphism $f^i : S^i \to S^{i+1}$ to *Y*. We obtain a sequence of sheaves

(8) $$0 \longrightarrow S_Y^0 \xrightarrow{f_Y^0} S_Y^1 \xrightarrow{f_Y^1} S_Y^2 \xrightarrow{f_Y^2} \ldots$$

called the *restriction* of the sequence $S* = \{S^i, f^i\}$ to the subspace *Y*.

The following are elementary properties of exact sequences.

Lemma 12 (a) *A sequence of Abelian sheaves* $S* = \{S^i, f^i\}$ *is exact at* S^q *if and only if* $\operatorname{Ker} f^q = \operatorname{Im} f^{q-1}$.

(b) *If a sequence of Abelian sheaves* $S* = \{S^i, f^i\}$ *over a topological space X is exact at the term* S^q, *then its restriction to a subspace* $Y \subset X$ *is exact at* S_Y^q.

(c) *A sequence of sheaves of the form* (7) *is exact at* T *if and only if the sheaf morphism* g *is surjective*.

(d) *A sequence of sheaves of the form* (7) *is exact at* R *if and only if the sheaf morphism* f *is injective*.

(e) *A sequence of Abelian sheaves*

(9) $$0 \longrightarrow R \xrightarrow{\iota} S \xrightarrow{\pi} S/\iota(R) \longrightarrow 0$$

where $R \subset S$ *is a subsheaf,* $\iota : R \to S$ *its inclusion,* $S/\iota(R)$ *the quotient sheaf and* $\pi : S \to S/\iota(R)$ *the quotient projection, is a short exact sequence.*

(f) *Suppose we have a diagram*

(10) $$\begin{array}{ccccccccc} 0 & \longrightarrow & R_0 & \xrightarrow{f^0} & R_1 & \xrightarrow{f^1} & R_2 & \longrightarrow & 0 \\ & & \downarrow \varphi_0 & & \downarrow \varphi_1 & & & & \\ 0 & \longrightarrow & S_0 & \xrightarrow{g^0} & S_1 & \xrightarrow{g^0} & S_2 & \longrightarrow & 0 \end{array}$$

such that the horizontal sequences are short exact sequences of sheaves, φ_0 *and* φ_1 *are sheaf morphisms and*

(11) $$g^0 \circ \varphi^0 = \varphi^1 \circ f^0.$$

Then there exists a unique Abelian sheaf morphism $\varphi_2 : R^2 \to S^2$ *such that the second square of the diagram*

(12) $$\begin{array}{ccccccccc} 0 & \longrightarrow & R_0 & \xrightarrow{f^0} & R_1 & \xrightarrow{f^1} & R_2 & \longrightarrow & 0 \\ & & \downarrow \varphi^0 & & \downarrow \varphi^1 & & \downarrow \varphi^2 & & \\ 0 & \longrightarrow & S_0 & \xrightarrow{g^0} & S_1 & \xrightarrow{g^1} & S_2 & \longrightarrow & 0 \end{array}$$

commutes.

(g) *Consider the exact sequence of Abelian sheaves* (7)*, the quotient sheaf* $S/f(R)$ *and the quotient projection* $\pi : S \to S/f(R)$*. There exists a unique sheaf isomorphism* $\varphi : T \to S/f(R)$ *such that*

(13) $$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{f^0} & S & \xrightarrow{f^1} & T & \longrightarrow & 0 \\ & & \downarrow \mathrm{id}_R & & \downarrow \mathrm{id}_S & & \downarrow \varphi & & \\ 0 & \longrightarrow & R & \xrightarrow{g^0} & S & \xrightarrow{\pi} & S/f(R) & \longrightarrow & 0 \end{array}$$

commutes.

Proof 1. We prove assertion (a). Suppose that S^* is exact at S^q. Then by definition $\mathrm{Ker} f_x^q = \mathrm{Im} f_x^{q-1}$ for every x, where f_x^q is the restriction of the sheaf space morphism $f^q : \mathrm{Germ} S^q \to \mathrm{Germ} S^{q+1}$, associated with f^q, to x. Thus

(14) $$\mathrm{Ker}\, f^q = \bigcup_{x \in X} \mathrm{Ker}\, f_x^q = \bigcup_{x \in X} \mathrm{Im}\, f_x^{q-1} = \mathrm{Im}\, f^q.$$

Then $\operatorname{Ker} f^q = \operatorname{Sec}^{(c)} \operatorname{Ker} f^q = \operatorname{Sec}^{(c)} \operatorname{Im} f^{q-1} = \operatorname{Im} f^{q-1}$ as required. The converse is obvious.

2. Assertions (b), (c), (d) and (e) of Lemma 12 are immediate consequences of definitions.

3. To prove (f) we apply (b) and Lemma 9, (d).

4. To prove (g) we apply (b) and Lemma 9, (e).

A sequence of Abelian sheaves (1) over a topological space X induces, for every open set U in X, the Abelian groups $S^i U$ of continuous sections and their morphisms $f_U^i : S^i U \to S^{i+1} U$. We usually denote these morphisms by the same letters, f^i. The sequence of Abelian groups is then denoted by

(15) $$0 \longrightarrow S^0 U \xrightarrow{\ f^0\ } S^1 U \xrightarrow{\ f^1\ } S^2 U \xrightarrow{\ f^2\ } \dots$$

and is said to be *induced* by the sequence of sheaves (1). In particular, if $U = X$, the sequence of Abelian groups

(16) $$0 \longrightarrow S^0 X \xrightarrow{\ f^0\ } S^1 X \xrightarrow{\ f^1\ } S^2 X \xrightarrow{\ f^2\ } \dots$$

is referred to as the *sequence of global sections*, associated with the sequence of Abelian sheaves (1).

Exactness of the sequence (1) does not imply exactness of (15). This is demonstrated by the following example.

7.9 Cohomology groups of a sheaf

In this section we construct a resolution of an Abelian sheaf, known as the *canonical*, or *Godement resolution* (Godement [G]). We also introduce canonical morphisms of the canonical resolutions, and study properties of the corresponding diagrams.

Consider the sheaf space $\operatorname{Germ} S$, associated with S and the sheaf of (not necessarily continuous) sections of the sheaf space $\operatorname{Germ} S$, denoted by

(1) $$C^0 S = \operatorname{Sec} \operatorname{Germ} S$$

(cf. Section 7.4, Example 17). We have the *canonical injective sheaf morphism* $\iota : \operatorname{Sec}^{(c)} \operatorname{Germ} S \to C^0 S$. Since $\operatorname{Sec}^{(c)} \operatorname{Germ} S$ is canonically isomorphic with the Abelian sheaf S, setting

(2) $$D^1 S = C^0 S / \operatorname{Im} \varepsilon$$

we get an exact sequence of sheaves

(3) $$0 \longrightarrow S \xrightarrow{\ \iota\ } C^0 S \longrightarrow D^1 S \longrightarrow 0.$$

The same construction can be repeated for the sheaf D^1S. Replacing S with D^1S, we have the Abelian sheaf of (*discontinuous*) sections of the sheaf space $\mathrm{Germ}\, D^1S$, $C^0D^1S = \mathrm{Sec}\,\mathrm{Germ}\, D^1S$, the Abelian sheaf of *continuous sections* $\mathrm{Sec}^{(c)}\mathrm{Germ}\, D^1S$, canonically isomorphic with the sheaf D^1S, and the canonical sheaf morphism of continuous sections into discontinuous sections, $\iota^1 : \mathrm{Sec}^{(c)}\mathrm{Germ}\, D^1S \to \mathrm{Sec}\,\mathrm{Germ}\, D^1S$. Setting $D^1(D^1S) = C^0(D^1S)/\operatorname{Im}\varepsilon^1$ we get an exact sequence

$$0 \longrightarrow D^1S \xrightarrow{\ \iota^1\ } C^0D^1S \longrightarrow D^1D^1S \longrightarrow 0. \tag{4}$$

Combining these two constructions

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & S \arrow[d] & & & \\
0 \arrow[r] & S \arrow[r] & C^0S \arrow[r] \arrow[d] & D^1S \arrow[r] & 0 & \\
 & 0 \arrow[r] & D^1S \arrow[r] \arrow[d] & C^1S \arrow[r] & D^1D^1S \arrow[r] & 0 \\
 & & 0 & & &
\end{tikzcd} \tag{5}$$

Similarly we get, with obvious notation, the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] & C^0S \arrow[d] & & \\
 & 0 \arrow[r] & D^1S \arrow[r] \arrow[d] & C^1S \arrow[d] & \\
 & & 0 \arrow[r] & D^2S \arrow[r] \arrow[d] & C^2S \arrow[d] \\
 & & & 0 \arrow[r] & D^3S \arrow[r] & C^2S
\end{tikzcd} \tag{6}$$

etc. This diagram gives rise to the sheaf morphisms $c^p : C^pS \to C^{p+1}S$, for every $p \geq 0$. We get a sequence of sheaves of Abelian groups

$$0 \longrightarrow S \xrightarrow{\ \iota\ } C^0S \xrightarrow{\ c^0\ } C^1S \xrightarrow{\ c^1\ } C^2S \xrightarrow{\ c^2\ } \dots \tag{7}$$

Lemma 13 *The sequence of sheaves of Abelian group* (7) *is a resolution of the sheaf* S.

Proof We want to verify exactness. Since ε is injective, the sequence is exact at S. To check exactness at the term C^0S, we use the diagram (7), where the sheaf morphism $g: C^0S \to D^1S$ is the quotient morphism and $h: D^1S \to C^1S$ is an inclusion. Let $a \in \operatorname{Im} \varepsilon$. Evidently $a \in \operatorname{Ker} c^0$ since $c^0 = h \circ g$ and $a \in \operatorname{Ker} g$. Conversely, let $a \in \operatorname{Ker} c^0$. Then $h(g(a)) = 0$ and since h is injective, $g(a) = 0$ and $a \in \operatorname{Ker} h$ hence $a \in \operatorname{Im} \varepsilon$. Exactness at C^qS can be proved in the same way.

The resolution (7) of the Abelian sheaf S is called the *canonical resolution*. Setting $C*S = \{C^iS, c^i\}$, we can write the sequence (7) as

(8) $$0 \longrightarrow S \xrightarrow{\iota} C*S$$

The Abelian sheaves C^pS, where $p \geq 0$, in the sequence (8), have some specific properties, namely, they belong to the class of soft sheaves. A sheaf of Abelian groups S over a topological space *X* is said to be *soft* if any section of the associated sheaf space $\operatorname{Germ} S$, defined on a closed subset $Y \subset X$, can be prolonged to a global section of S.

Lemma 14 *The sheaves* C^pS*, where* $p \geq 0$*, are soft.*

Proof It is sufficient to show that the sheaf $C^0S = \operatorname{Sec} \operatorname{Germ} S.$ is soft; the same proof applies to C^pS, where $p > 0$. Let $Y \subset X$ be a closed subset, $\delta \in C^0S$ any section of $\operatorname{Germ} S$, defined on *Y*. By definition, $\delta(x)$, where *x* is a point of *Y*, is the germ of a (not necessarily continuous) section $\gamma \in SU$, where *U* is a neighbourhood of *x* in *X*; thus $\delta(x) = [\gamma]_x$. Consider a family of (not necessarily continuous) sections $\gamma_x \in SU_x$ such that $\delta(x) = [\gamma_x]_x$ for all points $x \in Y$, and set

(9) $$\tilde{\delta}(x) = \begin{cases} [\gamma_x]_x, & x \in Y \\ 0, & x \notin Y \end{cases}$$

Then $\tilde{\delta}$ is a global section of the sheaf space $\operatorname{Germ} S$. Here 0 is the germ of the zero section, defined on the open set $X \setminus Y \subset X$.

Let $\nu: S \to T$ be a morphism of Abelian sheaves over a topological space *X*. We shall construct a family of sheaf morphisms $\nu^p: C^pS \to C^pT$, $p \geq 0$, between the canonical resolutions $0 \to S \to C*S$ and $0 \to T \to C*T$ of these sheaves, such that the diagram

(10) $$\begin{array}{ccccccccccc} 0 & \longrightarrow & S & \xrightarrow{\iota_S} & C^0S & \longrightarrow & C^1S & \longrightarrow & C^2S & \longrightarrow \\ & & \downarrow \nu & & \downarrow \nu^0 & & \downarrow \nu^1 & & \downarrow \nu^2 & \\ 0 & \longrightarrow & T & \xrightarrow{\iota_T} & C^0T & \longrightarrow & C^1T & \longrightarrow & C^2T & \longrightarrow \end{array}$$

commutes.

Let $S = \operatorname{Germ} S$ and $T = \operatorname{Germ} T$ be the associated sheaf spaces, σ and τ the corresponding sheaf space projections, and let $\tilde{\nu}: S \to T$ be the associated sheaf space morphism. Recall that $\tilde{\nu}$ is defined as the mapping $S \ni [\gamma]_x \to \tilde{\nu}([\gamma]_x) = [\nu_U(\gamma)]_x \in T$, where $\gamma \in SU$ is a representative of the germ $[\gamma]_x$ (Section 7.5, (10)). We shall consider the Abelian sheaves S and T as the sheaves of continuous sections of the sheaf spaces S and T.

Then $C^0 S$ and $C^0 T$ are the corresponding Abelian sheaves of *discontinuous* sections. We set for any section $\delta : U \to S$

$$\nu^0(\delta) = \tilde{\nu} \circ \delta. \tag{11}$$

This formula defines the first square in the diagram (10). If δ is a continuous section of $C^0 S$, we have $\nu^0(\iota_S(\gamma)) = \tilde{\nu} \circ \iota_S(\gamma)$

$$\begin{aligned}(\nu^0 \iota_S(\gamma))(x) &= \tilde{\nu}(\iota_S(\gamma)(x)) = \tilde{\nu}(\gamma(x)) = \tilde{\nu}([\gamma]_x) \\ &= [\nu_U(\gamma)]_x = \nu \circ \gamma(x) = \iota_T(\nu \circ \gamma)(x),\end{aligned} \tag{12}$$

proving the commutativity.

Consider the next squares in the diagram (10)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & C^0 S & \longrightarrow & D^1 S & \longrightarrow & 0 \\ & & \downarrow \nu & & \downarrow \nu^0 & & \downarrow \bar{\nu}^1 & & \\ 0 & \longrightarrow & T & \longrightarrow & C^0 T & \longrightarrow & D^1 T & \longrightarrow & 0 \end{array} \tag{13}$$

defining $\bar{\nu}^1$ (Lemma12, (f)). If we replace S (resp. T) with $D^i S$ (resp. $D^i T$), where $i \geq 1$, we get the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^i S & \longrightarrow & C^i S & \longrightarrow & D^{i+1} S & \longrightarrow & 0 \\ & & \downarrow \bar{\nu}^i & & \downarrow \nu^i & & \downarrow \bar{\nu}^{i+1} & & \\ 0 & \longrightarrow & D^i T & \longrightarrow & C^i T & \longrightarrow & D^{i+1} T & \longrightarrow & 0 \end{array} \tag{14}$$

We show that the i-th square also commutes. Combining (6) and (14) and using a suitable temporary notation we get the commutative diagrams

$$\begin{array}{ccc} C^{i-1} S & \xrightarrow{a} & D^i S \\ \downarrow \nu^{i-1} & & \downarrow \bar{\nu}^i \\ C^{i-1} T & \xrightarrow{b} & D^i T \end{array} \qquad \begin{array}{ccc} C^{i-1} S & \xrightarrow{g} & C^i S \\ \downarrow \nu^{i-1} & & \downarrow \bar{\nu}^i \\ C^{i-1} T & \xrightarrow{h} & C^i T \end{array} \qquad \begin{array}{ccc} D^i S & \xrightarrow{b} & C^i S \\ \downarrow \bar{\nu}^{i-1} & & \downarrow \nu^i \\ D^i T & \xrightarrow{d} & C^i T \end{array} \tag{15}$$

Combining these diagrams with (2) we obtain

$$g = b \circ a, \quad d \circ \bar{\nu}^i = \nu^i \circ b, \quad \bar{\nu}^i \circ a = c \circ \nu^{i-1}, \quad h = d \circ c, \tag{16}$$

which implies $\nu^i \circ g = \nu^i \circ b \circ a = d \circ \bar{\nu}^i \circ a = d \circ c \circ \nu^{i-1} = h \circ \nu^{i-1}$. Since $i \geq 1$, this proves commutativity of all squares in the diagram (10).

The family of sheaf morphisms $\{\nu, \nu^0, \nu^1, \nu^2, \ldots\}$ is called the *canonical morphism* of the canonical resolutions $0 \to S \to C^* S$ and $0 \to T \to C^* T$, associated with the Abelian sheaf morphism $\nu : S \to T$.

Elementary properties of the canonical resolutions are formulated in the following lemma.

Lemma 15 (a) *The canonical resolution of a trivial Abelian sheaf* 0_X *over a topological space X consists of the trivial sheaves* $C^p 0_X = 0_X$.

(b) *The canonical resolution associated with the identity sheaf morphism* id_S *is the identity morphism* $\{\mathrm{id}_S, \mathrm{id}_{C^0S}, \mathrm{id}_{C^1S}, \mathrm{id}_{C^2S}, \ldots\}$.

(c) *If the Abelian sheaf morphism* $\nu : S \to T$ *is injective (resp. surjective), then each* $\nu^p : S^p \to T^p$ *is injective (resp. surjective).*

(d) *Let* R, S, *and* T *be three Abelian sheaves with base X,* $\mu : R \to S$, $\nu : S \to T$ *two Abelian sheaf morphisms, and* $\eta = \nu \circ \mu$. *Then the diagram*

(17)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & R & \longrightarrow & C^0R & \longrightarrow & C^1R & \longrightarrow & C^2R & \longrightarrow \\
 & & \downarrow \mu & & \downarrow \mu^0 & & \downarrow \mu^1 & & \downarrow \mu^2 & \\
0 & \longrightarrow & S & \longrightarrow & C^0S & \longrightarrow & C^1S & \longrightarrow & C^2S & \longrightarrow \\
 & & \downarrow \nu & & \downarrow \nu^0 & & \downarrow \nu^1 & & \downarrow \nu^2 & \\
0 & \longrightarrow & T & \longrightarrow & C^0T & \longrightarrow & C^1T & \longrightarrow & C^2T & \longrightarrow
\end{array}$$

satisfies, for every $p \geq 0$,

(18) $$\eta^p = \nu^p \circ \mu^p.$$

(e) *Suppose that the first column of the diagram*

(19)
$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & S & \longrightarrow & C^0S & \longrightarrow & C^1S & \longrightarrow & C^2S & \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & S & \longrightarrow & C^0S & \longrightarrow & C^1S & \longrightarrow & C^2S & \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & T & \longrightarrow & C^0T & \longrightarrow & C^1T & \longrightarrow & C^2T & \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & S^2 & \longrightarrow & C^0S^2 & \longrightarrow & C^1S^2 & \longrightarrow & C^2S^2 & \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

consists of the resolution

(20) $$0 \longrightarrow S \xrightarrow{\ \varepsilon\ } S^0 \xrightarrow{\ f^0\ } S^1 \xrightarrow{\ f^1\ } S^2 \xrightarrow{\ f^2\ }$$

of the sheaf S, *the rows are formed by the canonical resolutions, and the columns are the canonical morphisms of the canonical resolutions. Then this* *diagram commutes, and all its columns are exact.*

Proof (a) This follows from formulas (2) – (4).

(b) We set in (10) $S=T$, $\nu=\mathrm{id}_S$. Then $\nu^0 : C^0S \to C^0S$ satisfies

$$\nu^0 = \mathrm{id}_{C^0S} \tag{21}$$

and (13) implies

$$\bar{\nu}^1 = \mathrm{id}_{D^1S}. \tag{22}$$

hence $\nu^1 = \mathrm{id}_{C^1S}$ and by induction $\nu^i = \mathrm{id}_{C^iS}$ for all $i \ge 1$.

(c) This follows from (11).

(d) Denote by $\tilde{\mu}$ ($\tilde{\nu}$, resp. $\tilde{\eta}$) the sheaf space morphism associated with μ (ν, resp. η). Since $\eta = \nu \circ \mu$, we have $\tilde{\eta} = \tilde{\nu} \circ \tilde{\mu}$ (Section 7.7, Lemma 9, (b)). Thus, using (11) we get for every section $\delta : U \to \mathrm{Germ}\, S$, $\eta_0(\delta) = \tilde{\eta} \circ \delta = \tilde{\nu} \circ \tilde{\mu} \circ \delta = \tilde{\nu} \circ \mu_0(\delta) = \nu_0(\mu_0(\delta))$ proving (d) for $p = 0$. Repeating this procedure we get $\eta^i = \nu^i \circ \mu^i$ for all $i \ge 1$.

(e) Commutativity is ensured by diagram (10). We want to prove exactness of the *p*-th column of the diagram (19). Consider the second column

$$0 \longrightarrow C^0S \xrightarrow{\ \varepsilon^0\ } C^0S^0 \xrightarrow{\ f^{00}\ } C^0S^1 \xrightarrow{\ f^{10}\ } C^0S^2 \xrightarrow{\ f^{20}\ } \tag{23}$$

Exactness at the term C^0S follows from the injectivity of ε^0 (see (c)). Now let $\delta : U \to \mathrm{Germ}\, C^0S^0$ be a section such that $f^{00}(\delta) = \tilde{f}^0 \circ \delta = 0$. Then if $\delta(x) = [\gamma_x]_x$ for some continuous section $\gamma_x : U_x \to \mathrm{Germ}\, C^0S^0$, we have $\tilde{f}^0([\gamma_x]_x) = 0$ and $[\gamma_x]_x \in \mathrm{Ker}\, \tilde{f}^0 = \mathrm{Im}\, \tilde{\varepsilon}_x$. Therefore, δ is a section of $\mathrm{Im}\, \varepsilon$, proving exactness at C^0S^0. Continuing in the same way we get exactness of the first column. Exactness in the next columns can be proved by induction.

Corollary 6 *Suppose that we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & S & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R' & \longrightarrow & S' & \longrightarrow & T' & \longrightarrow & 0 \end{array} \tag{24}$$

with exact rows. Then for every $i \ge 0$ *the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^iR & \longrightarrow & C^iS & \longrightarrow & C^iT & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^iR' & \longrightarrow & C^iS' & \longrightarrow & C^iT' & \longrightarrow & 0 \end{array} \tag{25}$$

commutes, and has exact rows.

Proof To prove commutativity of the diagram (25) we use commutativity of the square

(26)
$$\begin{array}{ccc} R & \xrightarrow{h} & S \\ \downarrow\mu & & \downarrow\nu \\ R' & \xrightarrow{k} & S' \end{array}$$

in (24) and formulas (2) – (4). Exactness of the rows follows from Lemma 15, (e).

Corollary 7 *For any isomorphism of Abelian sheaves $f:R\to S$ the sheaf morphisms $f^p:C^pR\to C^pS$ are isomorphisms.*

Proof This follows from Lemma 15, (b) and(d).

Let S be an Abelian sheaf over a topological space *X*. Consider the canonical resolution of S

(27) $$0 \longrightarrow S \xrightarrow{\iota} C^0S \xrightarrow{c^0} C^1S \xrightarrow{c^1} C^2S \xrightarrow{c^2}$$

Taking *global sections* of every term we obtain a complex of Abelian groups

(28)
$$\begin{gathered} 0 \longrightarrow SX \xrightarrow{\iota} (C^0S)X \xrightarrow{c^0} (C^1S)X \\ \xrightarrow{c^1} (C^2S)X \xrightarrow{c^2} \dots \end{gathered}$$

where the induced Abelian group morphisms in this diagram are denoted by the same letters as in the sequence (27). Denote by $(C*S)X$ the non-negative complex

(29) $$0 \longrightarrow C^0S \xrightarrow{c^0} C^1S \xrightarrow{c^1} C^2S \xrightarrow{c^2} \dots$$

Then (28) can also be written as

(30) $$0 \longrightarrow SX \xrightarrow{\iota} (C*S)X \ .$$

We set for every $p\geq 0$

(31) $$H^p(X,S) = H^p((C*S)X).$$

The Abelian group $H^p(X,S)=H^p((C*S)X)$. is called the *p-th cohomology group* of the topological space *X with coefficients* in the sheaf S .

Lemma 16 *Let S be an Abelian sheaf over a topological space X. The complex of Abelian groups* (28) *is exact at the terms SX and $(C^0S)X$.*

Proof Let $\gamma\in SX$ and let $\iota(\gamma)=0$. Then by definition $\iota(\gamma(x))=0$ for all $x\in X$. Since the canonical resolution (27) is exact at S we have $\gamma(x)=0$ for every *x* hence $\gamma=0$. Thus, the complex (28) is exact at SX .

We prove exactness at $(\mathcal{C}^0\mathcal{S})X$. Only inclusion $\operatorname{Ker} c^0 \subset \operatorname{Im} \iota$ needs proof. Let $\gamma \in (\mathcal{C}^0\mathcal{S})X$ and let $c^0(\gamma) = 0$. Then $c^0(\gamma)(x) = 0$ for every point $x \in X$. But (27) is exact at the term $\mathcal{C}^0\mathcal{S}$ hence to each $x \in X$ there exists a unique germ $s_x \in \mathcal{S}_x$ such that $\iota(s_x) = \gamma(x) = 0$, and we have a mapping $X \ni x \to \delta(x) = s_x \in \mathcal{S}$ satisfying $\iota \circ \delta = \gamma$. We want to show that this mapping is continuous. Let $x_0 \in X$ be a point. There exists a neighbourhood V (resp. W, resp. U) of the point $\delta(x_0)$ (resp. $\iota(\delta(x_0))$, resp. x_0) such that $\iota|_V : V \to W$ (resp. $\gamma|_U : U \to W$) is a homeomorphism. Then the composition $(\iota|_V)^{-1} \circ \gamma|_U : U \to V$ satisfies, for each $x \in U$,

$$\iota((\iota|_V)^{-1} \circ \gamma|_U(x)) = \gamma(x) = \iota(\delta(x)). \tag{32}$$

Since $\delta(x), (\iota|_V)^{-1} \circ \gamma|_U(x) \in \mathcal{S}_x$ and the restriction of ι to the fibre $\mathcal{S}_x$ is injective, we have $\delta(x) = (\iota|_V)^{-1} \circ \gamma|_U(x)$, which shows that the mapping δ is continuous at x_0. Consequently $\operatorname{Ker} c^0 \subset \operatorname{Im} \iota$.

Corollary 8 *For any Abelian sheaf $\mathcal{S}$ with base X, $H^0(X,\mathcal{S}) = \mathcal{S}X$.*

Let $\mathcal{S}$ and $\mathcal{T}$ be Abelian sheaves over a topological space X, $\nu : \mathcal{S} \to \mathcal{T}$ a morphism of Abelian sheaves, and let $\{\nu, \nu^0, \nu^1, \nu^2, \dots\}$ be the canonical morphism of the canonical resolutions of these sheaves. This morphism induces a comutative diagram of Abelian groups of global sections

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{S}X & \xrightarrow{\iota_{\mathcal{S}}} & (\mathcal{C}^0\mathcal{S})X & \longrightarrow & (\mathcal{C}^1\mathcal{S})X & \longrightarrow & (\mathcal{C}^2\mathcal{S})X & \longrightarrow \\
 & & \downarrow \nu & & \downarrow \nu^0 & & \downarrow \nu^1 & & \downarrow \nu^2 & \\
0 & \longrightarrow & \mathcal{T}X & \xrightarrow{\iota_{\mathcal{T}}} & (\mathcal{C}^0\mathcal{T})X & \longrightarrow & (\mathcal{C}^1\mathcal{T})X & \longrightarrow & (\mathcal{C}^2\mathcal{T})X & \longrightarrow
\end{array} \tag{33}$$

and a commutative diagram of non-negative complexes of global sections

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{C}^0\mathcal{S})X & \longrightarrow & (\mathcal{C}^1\mathcal{S})X & \longrightarrow & (\mathcal{C}^2\mathcal{S})X & \longrightarrow & \dots \\
 & & \downarrow \nu^0 & & \downarrow \nu^1 & & \downarrow \nu^2 & & \\
0 & \longrightarrow & (\mathcal{C}^0\mathcal{T})X & \longrightarrow & (\mathcal{C}^1\mathcal{T})X & \longrightarrow & (\mathcal{C}^2\mathcal{T})X & \longrightarrow & \dots
\end{array} \tag{34}$$

with obvious notation for the morphisms. Applying standard definitions we obtain, passing to the quotiens, the induced group morphisms of cohomology groups $\nu_q : H^q(X,\mathcal{S}) \to H^q(X,\mathcal{T})$, $q \ge 0$.

If $\mu : \mathcal{T} \to \mathcal{P}$ is some other Abelian sheaf morphism and the family $\{\mu, \mu^0, \mu^1, \mu^2, \dots\}$ is the morphism of the corresponding canonical resolutions, $\mu_q : H^q(X,\mathcal{T}) \to H^q(X,\mathcal{P})$, we have for every $q \ge 0$, an Abelian group morphism $(\mu \circ \nu)_q : H^q(X,\mathcal{S}) \to H^q(X,\mathcal{P})$. Using Lemma 12, (f) and Lemma 15, (d)

$$\mu^q \circ \nu^q = (\mu \circ \nu)^q. \tag{35}$$

Corollary 9 *If* $\nu : \mathcal{S} \to \mathcal{T}$ *is an isomorphism of Abelian sheaves, then* $\nu_q : H^q(X,\mathcal{S}) \to H^q(X,\mathcal{T})$ *is an Abelian group isomorphism for every* $q \geq 0$.

7.10 Sheaves over paracompact Hausdorff spaces

All sheaves considered in this section are *Abelian sheaves* over topological spaces whose topology is *Hausdorff* and *paracompact*.

Recall that an Abelian sheaf $\mathcal{S}$ with base *X* can be considered as the sheaf of continuous sections of the corresponding Abelian sheaf space $S = \mathrm{Germ}\,\mathcal{S}$, defined on open subsets of *X*. Every morphism $f : \mathcal{S} \to \mathcal{T}$ of Abelian sheaves can be considered as a morphism of Abelian sheaf spaces $f : S \to T$.

A *soft sheaf* is by definition a sheaf $\mathcal{S}$ with base *X* such that every continuous section of $\mathcal{S}$, defined on a closed subset of *X* can be prolonged to a global section. The proof of the following theorem on short exact sequences of soft sheaves is based on the *Zorn's lemma*.

Theorem 3 *Let X be a paracompact Hausdorff space, and let*

(1) $$0 \longrightarrow \mathcal{R} \xrightarrow{\ f\ } \mathcal{S} \xrightarrow{\ g\ } \mathcal{T} \longrightarrow 0$$

be a short exact sequence of sheaves over X. If $\mathcal{R}$ *is a soft sheaf, then the sequence of Abelian groups of global sections*

(2) $$0 \longrightarrow \mathcal{R}X \xrightarrow{\ f_X\ } \mathcal{S}X \xrightarrow{\ g_X\ } \mathcal{T}X \longrightarrow 0$$

is exact.

Proof 1. We prove exactness at $\mathcal{R}X$. If $\gamma \in \mathcal{R}X$ and $f_X(\gamma) = 0$, then $f(\tilde{\gamma}(x)) = 0$, then for every point $x \in X$ we get, by injectivity of *f*, $\tilde{\gamma}(x) = 0$. Thus the germ $\tilde{\gamma}(x)$ can be represented at every point by the zero section hence $\gamma = 0$.

2. We prove exactness of the sequence (2) at $\mathcal{S}X$. Let $\gamma \in \mathrm{Ker}\, g_X$. Then $\mathrm{Ker}\, g_X(\gamma) = 0$ hence $g \circ \tilde{\gamma}(x) = 0$ for all $x \in X$. Since the sequence (1) is exact at $\mathcal{S}$, to every point $x \in X$ there exists an element $\delta(x) \in R$ such that $f(\delta(x)) = \gamma(x)$ and, since the morphism *f* is injective, this point is unique. Since $\sigma \circ f = \rho$, where σ (resp. σ) is the projection of *S* (resp. *T*), we have $\rho \circ \delta = \sigma \circ f \circ \delta = \sigma \circ \gamma = \mathrm{id}_X$ showing that δ is a global section of R . To show that δ is continuous, observe that $f \circ \delta = \gamma$ is continuous; then the continuity of δ follows from the property of *f* to be a local homeomorphism.

3. We show that the mapping g_X is surjective. Let $\gamma \in \mathcal{T}X$ be a global section of $\mathcal{T}$. Since the sequence of Abelian sheaves (1) is exact at $\mathcal{T}$, to each point $x \in X$ there exists a neighbourhood U_x and a continuous section $\beta_x \in \mathcal{S}U_x$ such that $g_{U_x}(\beta_x) = \gamma|_{U_x}$. Thus, in a different notation, there exists an open covering $\{U_\iota\}_{\iota \in I}$ of *X*, such that for each $\iota \in I$ there exists $\beta_\iota \in \mathcal{S}U_\iota$

with the property

(3) $g_{U_\iota}(\beta_\iota) = \gamma|_{U_\iota}$.

Since X is paracompact and Hausdorff, there exists a locally finite open covering $\{V_\iota\}_{\iota\in I}$ of X such that $\mathrm{Cl} V_\iota \subset U_\iota$ (Cl denotes the *closure*). The sets $K_\iota = \mathrm{Cl} V_\iota$ are closed, and form a closed covering $\{K_\iota\}_{\iota\in I}$ of X. Thus, to every $\iota \in I$ we have assigned a pair (K_ι, β_ι), where $\beta_\iota \in SU_\iota$. Consider the non-empty set $\mathcal{K}$ of pairs (K, β), where $K = \bigcup K_\kappa$ is the union of some sets belonging to the family $\{K_\iota\}_{\iota\in I}$, and β is a section of S defined on the open set $U = \bigcup U_\kappa$. $\mathcal{K}$ becomes a *partially ordered set*, defined by the order relation "$(K,\beta) \le (K',\beta')$ *if* $K \subset K'$ *and* $\beta'|_U = \beta$".

We show that any linearly ordered family of subsets of the set $\mathcal{K}$ has an upper bound. Let $\{(K_\lambda, \beta_\lambda)\}_{\lambda\in L}$ be a linearly ordered family of subsets of $\mathcal{K}$, $K_\lambda \subset U_\lambda$. Denote $K = \bigcup K_\lambda$; then $K \subset U = \bigcup U_\lambda$. The family $\{\beta_\lambda\}_{\lambda\in L}$ is a compatible family of sections of the sheaf S. But every compatible family of sections of S locally generates a section of S (Section 7.4, condition (5)), thus, there exists a section $\beta \in SU$ such that $\beta|_{U_\lambda} = \beta$ for each $\lambda \in L$. Then the pair (K, β) is the *upper bound* of the linearly ordered family $\{(K_\lambda, \beta_\lambda)\}_{\lambda\in L}$.

This shows that the set $\mathcal{K}$ satisfies the assumptions of the *Zorn's lemma*, therefore, it has a maximal element (K_0, β_0). It remains to show that $K_0 = X$. Suppose the opposite; then there exists a point $x \in X$ such that $x \notin K_0$, and since $K = \bigcup K_\iota = X$, there must exist an index $\iota \in I$ such that $K_\iota \not\subset K_0$. On $K_\iota \cap K_0$, $g \circ (\beta_0 - \beta_\iota) = \gamma_0 - \gamma_\iota = 0$. But the sequence (2) is exact at SX hence $f(\delta) = \beta_0 - \beta_\iota$ for some $\delta \in R(K_\iota \cap K_0)$. Since R is soft, δ can be prolonged to a section $\bar{\delta}$ over X; then $\delta = \bar{\delta}|_{K_\iota \cap K_0}$. We define a section $\bar{\beta}$ over $K_\iota \cup K_0$ by the conditions

(4) $\bar{\beta}|_{K_0} = \beta_0, \quad \bar{\beta}|_{K_\iota} = \beta_\iota + f(\delta).$

Clearly, the $\bar{\beta}$ is defined correctly since on $K_\iota \cap K_0$

(5) $\beta_0|_{K_\iota \cap K_0} = (\beta_\iota + f(\delta))|_{K_\iota \cap K_0} = (\beta_\iota + \beta_0 - \beta_\iota)|_{K_\iota \cap K_0}$.

Consequently, the pair $(K_\iota \cup K_0, \bar{\beta})$ belongs to the set $\mathcal{K}$. But this pair satisfies $(K_0, \beta_0) \le (K_\iota \cup K_0, \bar{\beta})$, which contradicts maximality of the pair (K_0, β_0) unless $K_0 = X$.

Corollary 10 *If the Abelian sheaves* R *and* S *in the short exact sequence* (1) *are soft, then also the Abelian sheaf* T *is soft.*

Proof Let K be a closed set in the base X, and consider the restriction of the exact sequence (1) to K. The restricted sequence is also exact. Then by Theorem 3, the corresponding sequence of Abelian group (2) over K is exact. Choose a section $\gamma \in TK$. There exists $\delta \in SK$ such that $g_K(\delta) = \gamma$. If $\tilde{\delta}$ is an extension of δ to X, then $g_X(\tilde{\delta}) = g \circ \tilde{\delta}$ is the extension of γ to X.

Corollary 11 *Let X be a paracompact Hausdorff space and let*

(6) $$0 \longrightarrow \mathsf{S}_0 \xrightarrow{f_0} \mathsf{S}_1 \xrightarrow{f_1} \mathsf{S}_2 \xrightarrow{f_2} \dots$$

be an exact sequence of Abelian sheaves over X. If each of the sheaves S_0, S_1, S_2, … *is soft, then the induced sequence of Abelian groups*

(7) $$0 \longrightarrow \mathsf{S}_0 X \longrightarrow \mathsf{S}_1 X \longrightarrow \mathsf{S}_2 X \longrightarrow \dots$$

is exact.

Proof The sequence (6) is exact if and only if for each $i = 1,2,3,\dots$ the sequence

(8) $$0 \longrightarrow \operatorname{Ker} f_i \longrightarrow \mathsf{S}_i \xrightarrow{f_i} \operatorname{Ker} f_{i+1} \longrightarrow 0$$

is exact. Since by hypothesis $\operatorname{Ker} f_1 = \mathsf{S}_0$ and S_1 are soft sheaves, the sheaf $\operatorname{Ker} f_2$ is also soft (Corollary 10). Since the sheaf S_1 is soft, the sheaf $\operatorname{Ker} f_3$ must also be soft, according to Corollary 10, etc. Therefore, for all *i*, the sequence of global sections

(9) $$0 \longrightarrow (\operatorname{Ker} f_i)X \longrightarrow \mathsf{S}_i X \xrightarrow{f_i} (\operatorname{Ker} f_{i+1})X \longrightarrow 0$$

is exact, by Theorem 3. Now it is immediate that the sequence (7) must be exact.

Corollary 12 *If* S *is a soft sheaf over a paracompact Hausdorff space X, then* $H^q(X,\mathsf{S}) = 0$ *for all* $q \geq 1$.

Proof Consider the canonical resolution of S,

(10) $$0 \longrightarrow \mathsf{S} \xrightarrow{\iota} \mathsf{C}^0\mathsf{S} \xrightarrow{\mathsf{c}^0} \mathsf{C}^1\mathsf{S} \xrightarrow{\mathsf{c}^1} \mathsf{C}^2\mathsf{S} \xrightarrow{\mathsf{c}^2}$$

(Section 7.6, (7)). Since all the sheaves $\mathsf{C}^i\mathsf{S}$ are soft (Section 7.8, Lemma 14), the associated sequence of global sections

(11) $$0 \longrightarrow (\mathsf{C}^9\mathsf{S})X \xrightarrow{\mathsf{c}^0} (\mathsf{C}^1\mathsf{S})X \xrightarrow{\mathsf{c}^1} (\mathsf{C}^2\mathsf{S})X \xrightarrow{\mathsf{c}^2} \dots$$

is exact (Corollary 11). Now Corollary 12 follows from the definition of a cohomology group.

Examples 22. Let G be an Abelian group, X *connected Hausdorff space*, and $S = X \times G$ the constant sheaf space (Section 7.2, Example 11). We show that the constant sheaf $\operatorname{Sec}^{(c)} S$ is *not* soft. Let *x* and *y* be two different points of the base *X*. Consider the closed subset $Y = \{x\} \cup \{y\}$ of X and the section γ of *S* defined on *Y* by $\gamma(x) = g$, $\gamma(y) = h$, where *g* and *h* are two distinct point of *G*. If *U* is a neighbourhood of *x* and *V* is a neighbourhood of y such that $U \cap V = \emptyset$, then we have a section $\tilde{\gamma}: U \cup V \to S$, equal to g on U and h on V. The restriction of $\tilde{\gamma}$ to Y is equal to γ; in particular, $\tilde{\gamma}$ is continuous. But since X is connected, $\tilde{\gamma}$ cannot be prolonged to a global continuous section of S.

23. If X is a *normal space*, then every continuous, real-valued function defined on a closed subspace of X, can be prolonged to a globally defined continuous function (*Tietze theorem*). Consequently, the sheaf $C_{X,\mathbf{R}}$ is soft (cf. Section 7.4, Example 18).

24. We shall show that the sheaf of modules $\mathcal{S}$ over a soft sheaf of commutative rings with unity $\mathcal{R}$ is soft. Let X be the base of $\mathcal{R}$ (and $\mathcal{S}$), K a closed subset of X, and let $\gamma \in \mathrm{Sec}^{(c)} \mathcal{S}$ be a continuous section, defined on K. Then by definition γ can be prolonged to a continuous section, also denoted by γ, defined on a neighbourhood U of K. Define a continuous section $\rho \in \mathrm{Sec}^{(c)}(K \cup (X \setminus U)))$ by

$$\rho(x) = \begin{cases} 1, & x \in K, \\ 0 & x \in X \setminus U. \end{cases} \tag{12}$$

Since $\mathcal{R}$ is soft, there exists a section $\tilde{\rho} \in \mathrm{Sec}^{(c)} X$ prolonging ρ to X. We define $\tilde{\gamma}(x) = \tilde{\rho}(x) \cdot \gamma(x)$; $\tilde{\gamma}$ is the desired prolongation of γ.

25. The sum of two soft subsheaves of a sheaf is a soft subsheaf (cf. Section 7.2, Example 13).

Let $\mathcal{S}$ be an Abelian sheaf over a topological space X, $\eta: \mathcal{S} \to \mathcal{S}$ a sheaf morphism. We define the *support* of η to be a closed subspace of X

$$\operatorname{supp} \eta = \operatorname{cl}\{x \in X \mid \eta(x) \neq 0\}. \tag{13}$$

Let $\{U_\iota\}_{\iota \in I}$ be a locally finite open covering of the paracompact Hausdorff space X, $\mathcal{S}$ an Abelian sheaf with base X. By a *sheaf partition of unity* for $\mathcal{S}$, subordinate to $\{U_\iota\}_{\iota \in I}$ we mean any family $\{\chi_\iota\}_{\iota \in I}$ of sheaf morphisms $\chi_\iota: \mathcal{S} \to \mathcal{S}$ over X with the following two properties:

(1) $\operatorname{supp} \chi_\iota \subset U_\iota$ for every $\iota \in I$.

(2) For every point $x \in X$

$$\sum_{\iota \in I} \chi_\iota(s) = s. \tag{14}$$

Note that the sum on the left-hand side of formula (14) is well-defined, because for every fixed point s the summation is taking place through only a *finitely many* indices ι from the indexing set I.

An Abelian sheaf $\mathcal{S}$ is said to be *fine*, if to every locally finite open covering $\{U_\iota\}_{\iota \in I}$ of X there exists a sheaf partition of unity $\{\chi_\iota\}_{\iota \in I}$ subordinate to $\{U_\iota\}_{\iota \in I}$.

Theorem 4 *Every fine Abelian sheaf over a paracompact Hausdorff space is soft.*

Proof Let $\mathcal{S}$ be an Abelian sheaf over a paracompact Hausdorff space X, $S = \operatorname{Germ}\mathcal{S}$, and let σ be the projection of S. Let Y be a closed subspace of X, γ a continuous section, defined on Y. To every point $x \in Y$ there exists a neighbourhood U_x of x and a continuous section $\gamma_x : U_x \to S$ such that $\gamma(x) = \gamma_x$. Shrinking γ_x to $U_x \cap Y$ we get a continuous section of the restriction of S to $U_x \cap Y$. Shrinking U_x if necessary we may assume without loss of generality that $\gamma_x|_{U_x \cap Y} = \gamma|_{U_x \cap Y}$. The sets U_x together with the set $X \setminus Y$ cover X. Since X is paracompact, there exists a locally finite refinement $\{V_\iota\}_{\iota \in I}$ of this covering. If for some $\iota \in I$, $V_\iota \cap Y \neq \emptyset$, then there exists a continuous section $\gamma_\iota : V_\iota \to S$ such that $\gamma_\iota|_{V_\iota \cap Y} = \gamma|_{V_\iota \cap Y}$; if $V_\iota \cap Y = \emptyset$, we set $\gamma_\iota = 0$. In this way we assign to each of the sets V_ι a continuous section $\gamma_\iota : V_\iota \to S$.

Let $\{\eta_\iota\}_{\iota \in I}$ be a partition of unity subordinate to the covering $\{V_\iota\}_{\iota \in I}$. Set for all $\iota \in I$

(15) $$\delta_\iota(x) = \begin{cases} \eta_\iota(\gamma_\iota(x)), & x \in V_\iota, \\ 0, & x \in X \setminus V_\iota, \end{cases}$$

where 0 denotes the neutral element of the Abelian group S_x. We get a mapping $\delta_\iota : X \to S$ satisfying the condition $\sigma \circ \delta_\iota = \mathrm{id}_X$. This mapping is obviously continuous on the set V_ι, and also on a neighbourhood $X \setminus \operatorname{supp} \eta_\iota$ of the closed set $X \setminus V_\iota$. We set $\delta = \sum \delta_\iota$. Then δ is a global continuous section of the sheaf space S. Then for every point $x \in X$,

(16) $$\delta(x) = \sum_{V_\kappa \ni x} \eta_\kappa(\gamma_\kappa(x)) = \sum_\kappa \eta_\kappa \gamma(x) = \left(\sum_\kappa \eta_\kappa \right) \gamma(x) = \gamma(x).$$

Therefore, $\delta|_Y = \gamma$.

Examples 26. The Abelian sheaf $C_{X,\mathbf{R}}$ of continuous real-valued functions on a paracompact Hausdorff space X is fine. Indeed, any locally finite open covering $\{U_\iota\}_{\iota \in I}$ of X, and any subordinate partition of unity $\{\chi_\iota\}_{\iota \in I}$, define a sheaf partition of unity as the family of sheaf morphisms $f \to \chi_\iota f$. The Abelian sheaf $C_{X,\mathbf{R}}$ can also be considered as a *sheaf of commutative rings with unity*.

27. Let $\mathcal{S}$ be a sheaf of $C_{X,\mathbf{R}}$-modules over a paracompact Hausdorff space X, let S be the associated sheaf space, with projection $\sigma : S \to X$. Every continuous function $f : X \to \mathbf{R}$ defines an Abelian sheaf morphism of the sheaf space S by

(17) $$f_S(s) = f(\sigma(s)) \cdot s.$$

If $\{U_\iota\}_{\iota \in I}$ is an open covering of X, and $\{\chi_\iota\}_{\iota \in I}$ a partition of unity on X, subordinate to $\{U_\iota\}_{\iota \in I}$, then formula (17) applies to the functions from the family of functions $\{\chi_\iota\}_{\iota \in I}$; the corresponding family of sheaf morphisms $\{\chi_{\iota,S}\}_{\iota \in I}$ is then a sheaf partition of unity on S. Consequently, the Abelian sheaf S is fine.

28. The Abelian sheaves $C^r_{X,\mathbf{R}}$ of r times continuously differentiable functions on a smooth manifold X, where $r = 0,1,2,\ldots,\infty$, are fine (cf. Example 26), and can also be considered as *sheaves of commutative rings with unity*.

29. Every sheaf of modules over a fine sheaf of commutative rings with unity is fine.

Let us consider a short exact sequence of Abelian sheaves over a paracompact Hausdorff manifold X

$$0 \longrightarrow R \xrightarrow{\ f\ } S \xrightarrow{\ g\ } T \longrightarrow 0 , \tag{18}$$

and the related commutative diagram of the canonical resolutions

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & C^0R & \longrightarrow & C^1R & \longrightarrow & C^2R & \longrightarrow & \\
 & & \downarrow f & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & C^0S & \longrightarrow & C^1S & \longrightarrow & C^2S & \longrightarrow & \\
 & & \downarrow g & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & C^0T & \longrightarrow & C^1T & \longrightarrow & C^2T & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{19}$$

This diagram induces the commutative diagram of global sections

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & RX & \longrightarrow & (C^0R)X & \longrightarrow & (C^1R)X & \longrightarrow & (C^2R)X & \longrightarrow & \\
 & & \downarrow f & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & SX & \longrightarrow & (C^0S)X & \longrightarrow & (C^1S)X & \longrightarrow & (C^2S)X & \longrightarrow & \\
 & & \downarrow g & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TX & \longrightarrow & (C^0T)X & \longrightarrow & (C^1T)X & \longrightarrow & (C^2T)X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{20}$$

All the sheaves C^iR, C^iS, and C^iT in (19) are soft (Section 7.9, Lemma 14). Applying Corollary 11, we see that the columns are exact. Therefore, by Lemma 11, we get the *long exact sequence*

(21)
$$\begin{array}{l} 0 \longrightarrow H^0(X,R) \xrightarrow{f} H^0(X,S) \xrightarrow{g} H^0(X,T) \xrightarrow{\partial^0} \\ H^1(X,R) \longrightarrow H^1(X,S) \longrightarrow H^1(X,T) \xrightarrow{\partial^1} H^2(X,R) \end{array}$$

where the family $(\partial^0, \partial^1, \partial^2, \ldots)$ is the *connected morphism*.

The long exact sequence can be applied to commutative diagrams of short exact sequences.

Lemma 17 *Let X be a paracompact Hausdorff space. Suppose that the commutative diagram of Abelian sheaves over X*

(22)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{f} & S & \xrightarrow{g} & T & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow k & & \downarrow j & & \\ 0 & \longrightarrow & \bar{R} & \xrightarrow{\bar{f}} & \bar{S} & \xrightarrow{\bar{g}} & \bar{T} & \longrightarrow & 0 \end{array}$$

whose rows are exact. Then the diagram

(23)
$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(X,R) & \xrightarrow{f} & H^0(X,S) & \xrightarrow{g} & H^0(X,T) & \xrightarrow{\partial^0} \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & H^0(X,\bar{R}) & \xrightarrow{\bar{f}} & H^0(X,\bar{S}) & \xrightarrow{\bar{g}} & H^0(X,\bar{T}) & \xrightarrow{\partial^0} \end{array}$$
$$\begin{array}{ccccccc} H^1(X,R) & \longrightarrow & H^1(X,S) & \longrightarrow & H^1(X,T) & \xrightarrow{\partial^1} & H^2(X,R) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(X,\bar{R}) & \longrightarrow & H^1(X,\bar{S}) & \longrightarrow & H^1(X,\bar{T}) & \xrightarrow{\partial^1} & H^2(X,\bar{R}) \end{array}$$

where the first (resp. the second) row is the long exact sequence associated with the first (resp. the second) row in (24)*, commutes.*

Proof It is enough to prove commutativity of the squares in (23) containing the group morphisms ∂^i. Commutativity of the other squares is an immediate consequence of the diagrams (22) and Section 7.9, (10).

Consider the square

(24)
$$\begin{array}{ccc} H^0(X,T) & \xrightarrow{\partial^0} & H^1(X,R) \\ \downarrow & & \downarrow \\ H^0(X,\bar{T}) & \xrightarrow{\partial^0} & H^1(X,\bar{R}) \end{array}$$

For the purpose of this proof denote by $\varepsilon_R : R \to C^0R$ and $c_R^i : C^iR \to C^{i+1}R$

the corresponding sheaf morphisms in the canonical resolution of the sheaf R, $0 \to R \to C^0 R \to C^1 R \to C^2 R \to \dots$, and introduce analogous notation for the sheaves S and T. Let $c \in H^0(X,T) = \operatorname{Ker} c_T^0$. There exist an element $b \in (C^0 S)X$ and $a \in \operatorname{Ker} c_R^1$ such that $c \in g^0(b)$, $c_S^0(b) = f^1(a)$, and by definition

$$(25) \qquad \begin{aligned} &\partial^0(c) = [a], \\ &h^1 \partial^0(c) = h^1([a]) = [h^1(a)]. \end{aligned}$$

We set

$$(26) \qquad \bar{b} = k_0(b), \quad \bar{a} = h^1(a).$$

Then we get by immediate calculations $\bar{g}^0(b') = \bar{g}^0 k^0(b) = j^0 g^0(b) = j^0(c)$, $\bar{f}^1(\bar{a}) = \bar{f}^1(h^1(a)) = k^1 f^1(a)$, and $c_{S'}^0(\bar{b}) = c_{S'}^0 k^0(b) = k^0 c_S^0(b) = k^1 f^1(a)$. Hence $\bar{b}$ and $\bar{a}$ satisfy

$$(27) \qquad j^0(c) = \bar{g}(\bar{b}), \quad c_S^0(\bar{b}) = \bar{f}^1(\bar{a}).$$

Consequently,

$$(28) \qquad \partial^0 \bar{j}^0(c) = a' = h^1 \partial^0(c)$$

proving commutativity of (24).

Commutativity of the square

$$(29) \qquad \begin{array}{ccc} H^q(X,T) & \xrightarrow{\partial^q} & H^{q+1}(X,R) \\ \downarrow & & \downarrow \\ H^q(X,\bar{T}) & \xrightarrow{\partial^q} & H^{q+1}(X,\bar{R}) \end{array}$$

can be proved in the same way. Let $[c] \in H^q(X,T) = \operatorname{Ker} c_T^q / \operatorname{Im} c_T^{q-1}$. There exist elements $b \in (C^q S)X$ and $a \in \operatorname{Ker} c_R^{q+1}$ such that

$$(30) \qquad c = g^q(b), \quad c_S^q(b) = f^{q+1}(a),$$

and by definition

$$(31) \qquad \begin{aligned} &\partial^q([c]) = [a], \\ &h^{q+1} \partial^q([c]) = h^{q+1}([a]) = [h^{q+1}(a)]. \end{aligned}$$

We denote

$$(32) \qquad \bar{b} = k^k(b), \quad \bar{a} = h^{q+1}(a).$$

Then

$$\begin{aligned}(33)\qquad &\overline{g}^q(\overline{b}) = \overline{g}^q k^q(b) = j^q g^q(b) = j^q(c),\\ &\overline{f}^{\,q+1}(\overline{a}) = \overline{f}^{\,q+1} h^{q+1}(a) = k^{q+1} f^{q+1}(a),\\ &c_{\overline{S}}^q(\overline{b}) = c_{\overline{S}}^q k^q(b) = k^{q+1} c_S^q(b) = k^{q+1} f^{q+1}(a),\end{aligned}$$

so that

$$(34)\qquad c_{\overline{S}}^q(\overline{b}) = \overline{f}^{\,q+1}(\overline{a}).$$

Now using the definition of ∂^q we get

$$\begin{aligned}(35)\qquad &\partial^q j^q([c]) = \partial^q([j^q(c)]) = [a']\\ &\quad = [h^{q+1}(a)] = h^{q+1}\partial^q([c]),\end{aligned}$$

which proves commutativity of the square (29).

An Abelian sheaf S over a topological space X is said to be *acyclic*, if $H^q(X,S) = 0$ for all $q \geq 1$. A resolution of S

$$(36)\qquad 0 \longrightarrow S \longrightarrow S^0 \longrightarrow S^1 \longrightarrow S^2 \longrightarrow \dots$$

is said to be *acyclic*, if each of the sheaves S^i, where $i \geq 1$, is acyclic.

Lemma 18 *Let S be an Abelian sheaf over a paracompact Hausdorff space X.*
(a) *If S is soft, it is acyclic.*
(b) *The canonical resolution of S is acyclic.*

Proof (a) This follows from Corollary 12.
(b) We want to show that each of the sheaves C^pS, where $p \geq 0$, is acyclic. But we have already shown that these sheaves are soft (Section 7.9, Lemma 14)); since by hypothesis the base X of S is paracompact and Hausdorff, they are acyclic by part (a) of this lemma.

Denote by T^*X the complex $0 \to T^0X \to T^1X \to T^2X \to \dots$, and let $H^q(T^*X)$ be the q-th cohomology group of this complex.

Theorem 5 (Abstract De Rham theorem) *Let S be an Abelian sheaf over a paracompact Hausdorff manifold X, let*

$$(37)\qquad 0 \longrightarrow S \longrightarrow T^0 \longrightarrow T^1 \longrightarrow T^2 \longrightarrow \dots$$

be a resolution of S. If this resolution is acyclic, then for every $q \geq 0$ the cohomology groups $H^q(X,S)$ and $H^q(T^*X)$ are isomorphic.

Proof Let us consider the following commutative diagram of Abelian sheaves

(38)
$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & C^0S & \longrightarrow & C^1S & \longrightarrow & C^2S & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^0 & \longrightarrow & C^0T^0 & \longrightarrow & C^1T^0 & \longrightarrow & C^2T^0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^1 & \longrightarrow & C^0T^1 & \longrightarrow & C^1T^1 & \longrightarrow & C^2T^1 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^2 & \longrightarrow & C^0T^1 & \longrightarrow & C^1T^1 & \longrightarrow & C^2T^1 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

with exact rows and columns, and the associated diagram of global sections

(39)
$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & SX & \longrightarrow & (C^0S)X & \longrightarrow & (C^1S)X & \longrightarrow & (C^2S)X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^0X & \longrightarrow & (C^0T^0)X & \longrightarrow & (C^1T^0)X & \longrightarrow & (C^2T^0)X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^1X & \longrightarrow & (C^0T^1)X & \longrightarrow & (C^1T^1)X & \longrightarrow & (C^2T^1)X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^2X & \longrightarrow & (C^0T^1)X & \longrightarrow & (C^1T^1)X & \longrightarrow & (C^2T^1)X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

By Section 7.9, Corollary 6 and Corollary 7, every column in this diagram except possibly the first one, is exact. We shall show that each row, except possibly the first row, is exact.

Consider the *k*-the row

(40) $$0 \longrightarrow T^kX \longrightarrow (C^0T^k)X \longrightarrow (C^1T^k)X \longrightarrow (C^2T^k)X \longrightarrow$$

This sequence is exact at the first and the second terms (Section 7.9, Lemma 16). Since the sheaf T^k is acyclic, we have for each $q \geq 1$,

(41) $$H^q(X,T^k) = 0,$$

which means that the sequence (40) is exact everywhere. In particular, the diagram (40) is exact everywhere except possibly the first column and the first row. Now we apply (Section 7.7, Lemma 10).

Corollary 13. *For any two acyclic resolutions of an Abelian sheaf* S *over a paracompact Hausdorff space X, expressed by the diagram*

$$
(42)\qquad \begin{tikzcd}[row sep=small]
 & & \mathsf{R}^0 \arrow[r] & \mathsf{R}^1 \arrow[r] & \mathsf{R}^2 \arrow[r] & \ldots \\
0 \arrow[r] & \mathsf{S} \arrow[ur] \arrow[dr] & & & & \\
 & & \mathsf{T}^0 \arrow[r] & \mathsf{T}^1 \arrow[r] & \mathsf{T}^2 \arrow[r] & \ldots
\end{tikzcd}
$$

the cohomology groups of the complexes of global sections $H^q(\mathsf{R}*X)$ *and* $H^q(\mathsf{T}*X)$ *are isomorphic*.

Proof Indeed, according to Theorem 5, $H^q(\mathsf{R}*X)$ and $H^q(\mathsf{T}*X)$ are isomorphic with the cohomology group $H^q(X,\mathsf{S})$.

Examples 30. *Any sheaf* S *of* C^r*-sections of a smooth vector bundle over a smooth paracompact Hausdorff manifold X admits multiplication by functions of class* C^r *and is therefore fine*. *Consequently,* S *is soft* (Theorem 4) *and acyclic* (Lemma 18).

Remark 6 Consider an *n*-dimensional smooth manifold *X*, the constant sheaf $\mathbf{R}$ and the sheaves of *p*-forms Ω^p of class C^∞ on *X*. The exterior derivative of differential forms $d:\Omega^p \to \Omega^{p+1}$ defines a differential sequence

$$
(43)\qquad 0 \longrightarrow \mathbf{R} \longrightarrow \Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d\ } \Omega^2 \longrightarrow
$$

where the mapping $\mathbf{R} \to \Omega^0$ is the canonical inclusion. It follows from the Volterra-Poincaré lemma that this sequence is *exaxt*, therefore, it is a resolution of the constant sheaf $\mathbf{R}$. Since the sheaves Ω^p are fine they are soft (Example 29, Example 30) and acyclic (Lemma 18). Thus, the resolution (43) is acyclic; in particular, according to the abstract De Rham theorem, the cohomology groups $H^q(\Omega*X)$ of the complex of global sections

$$
(44)\qquad 0 \longrightarrow \Omega^0 X \xrightarrow{\ d\ } \Omega^1 X \xrightarrow{\ d\ } \Omega^2 X \xrightarrow{\ d\ } \ldots
$$

coincide with the cohomology groups $H^q(X,\mathbf{R})$. The sequence (43) is called the *De Rham sequence* (of sheaves); (44) is the *De Rham sequence* of differential forms on *X*, and the groups $H^q(\Omega*X)$, usually denoted just by H^qX, are the *De Rham cohomology groups* of *X*. Note that according to Corollary 13, for *any* acyclic resolution of the constant sheaf $\mathbf{R}$ on *X*,

$$
(45)\qquad 0 \longrightarrow \mathbf{R} \longrightarrow \mathsf{S}*,
$$

the cohomology groups $H^q(\mathsf{S}*X)$ *coincide* (that is, are isomorphic) with the De Rham cohomology groups H^qX,

$$
(46)\qquad H^q(\mathsf{S}*X) = H^qX.
$$

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