8 Variational sequences

We introduced in Chapter 4 the *Euler-Lagrange mapping* of the calculus of variations as an $\mathbf{R}$-linear mapping, assigning to a Lagrangian λ, defined on the *r*-jet prolongation J^rY of a fibred manifold *Y*, its Euler-Lagrange form E_λ. Local properties of this mapping are determined by the *components* of the Euler-Lagrange form, the Euler-Lagrange expressions of the Lagrangian λ. In this chapter we construct an exact sequence of Abelian sheaves, the *variational sequence*, such that one of its sheaf morphisms coincides with the Euler-Lagrange mapping. Existence of the sequence provides a possibility to study basic global characteristics of the Euler-Lagrange mapping in terms of the cohomology groups of the corresponding *complex of global sections*. Especially, for variational purposes the structure of the *kernel* and the *image* of the Euler-Lagrange mapping $\lambda \to E_\lambda$ is considered.

The variational sequence is defined by means of the exterior derivative operator, acting on differential forms on jet spaces. Recall that for *any* smooth, paracompact, Hausdorff manifold *X* the following facts have already been stated in Chapter 7:

(a) The set of real-valued functions, defined on open subsets of *X*, with standard restrictions, is a sheaf; the sets of *continuous*, C^k-*differentiable*, and *smooth* functions are also sheaves.

(b) More generally, the set of *differentiable* *k*-forms on open subsets of *X*, with standard restrictions, is a sheaf.

(c) The set of *closed* differentiable *k*-forms, defined on open subsets of *X*, with standard restrictions, is a sheaf.

(d) An *exact* form ρ on an open set $U \subset X$ is a form such that there exists a form η, defined on *U*, such that $\rho = d\eta$; the exact forms constitute a presheaf but *not* a sheaf: if $\{U_\iota\}_{\iota \in I}$ is an open covering of an open set $U \subset X$, such that $\rho|_{U_\iota} = d\eta_\iota$ for each $\iota \in I$, then in general there is no η such that $\rho = d\eta$.

This chapter treats the foundations of the variational sequence theory. The approach, which we have followed, is due to the original paper Krupka [K19]. Main innovations consist in the use of *variational projectors* (also called the *interior Euler-Lagrange operators*, see Anderson [A2], Krupka and Sedenková – Volná [KSe], Volná and Urban [VU]). The idea to apply sheaves comes from Takens [T].

A number of important topics have necessarily been omitted. For recent research in the structure of the variational sequence, its relations with topology, symmetries and differential equations, and possible extensions to Grassmann fibrations and submanifold theory we refer to Bloch, Krupka, Urban, Voicu, Volna and Zenkov [Bl], Brajercik and Krupka [BK], Francaviglia, Palese and Winterroth [FPW], Grigore [Gr], Krupka [K16], [K17], Krbek and Musilova [KM], Pommaret [Po], Urban and Krupka [UK1], Vitolo [Vit] (see also the handbook Krupka and Saunders [KS], where further references can be found).

Note that the variational sequence theory does *not* follow the approach to the “formal calculus of variations” based on a *variational bicomplex* *theory* on *infinite* jet prolongations of fibred manifolds, although some technical aspects of these two theories appear to be parallel (Anderson [A2], Anderson and Duchamp [AD], Dedecker and Tulczyjew [DT], Olver [O1], Saunders [S], Takens [T], Urban and Krupka [UK1], Vinogradov, Krasilschik and Lychagin [VKL] and others). The results, however, seem to be essentially different, and require a deeper comparison. It seems for instance that the infinite jet structure of the bicomplex theory is a serious obstacle for obtaining local and global characteristics of the "variational" morphisms within this theory; although a main motivation was to study these morphisms, no *explicit* (or at least *effective*) formulas say for the inverse problem of the calculus of variations and Helmholtz morphism have been derived yet.

As before, Y denotes in this chapter a smooth fibred manifold with n-dimensional base X and projection π, and $n+m=\dim Y$. J^rY is its *r-jet prolongation* and $\pi^r:J^rY\to X$, $\pi^{r,s}:J^rY\to J^sY$ are the canonical jet projections. For any open set $W\subset Y$, Ω_q^rW is the module of q-forms on the set $W^r=(\pi^{r,0})^{-1}(W)$, and Ω^rW is the exterior algebra of forms on W^r. The horizontalization morphism of the exterior algebra Ω^rW into $\Omega^{r+1}W$ is denoted by h. If Ξ is a π-projectable vector field and $J^r\Xi$ its r-jet prolongation, then to simplify notation we sometimes denote the contraction $i_{J^r\Xi}\rho$, and the Lie derivative $\partial_{J^r\Xi}\rho$ of a form ρ, just by $i_\Xi\rho$, or $\partial_\Xi\rho$.

8.1 The contact sequence

We saw in Section 7.10, Remark 6 that the exterior differential forms on a finite-dimensional smooth manifold X together with the exterior derivative morphism constitute a resolution of the constant sheaf $\mathbf{R}$ over X, the *De Rham resolution*. In this section we provide analogous construction for differential forms on the r-jet prolongation J^rY of a fibred manifold Y over X. We use the fibred structure of Y to construct a slightly modified version of the De Rham resolution, in which the underlying topological space is the manifold Y itself instead of J^rY.

Following our previous notation (Section 4, Section 6), consider a smooth fibred manifold Y with base X and projection π. For any open set W in Y, denote by Ω_0^rW the Abelian group of real-valued functions of class C^r (0-forms), defined on the open set $W^r\subset J^rY$; one can also consider Ω_0^rW with its algebraic structure of a commutative ring with unity. Next let $q\geq 1$, and denote by Ω_q^rW the Abelian group of q-forms of class C^r, defined on $W^r\subset J^rY$. This way we get, for every non-negative integer q, a correspondence $W\to\Omega_q^rW$, assigning to an open set $W\subset Y$ the Abelian group of q-forms on W^r. One can easily verify that this correspondence defines a *sheaf structure* on the family $\{\Omega_q^rW\}$, labelled by the open sets W. Indeed, to any two open sets W_1 and W_2 in Y such that $W_2\subset W_1$, and any $\rho\in\Omega_q^rW_1$, the restrictions $\Omega_q^rW_1\ni\rho\to\rho|_{W_2}\in\Omega_q^rW_2$ define an *Abelian presheaf structure* on $\{\Omega_q^rW\}$. Since this presheaf is obviously *complete*, it has the *Abelian sheaf structure* (Section 7.4); with this sheaf structure, the family $\{\Omega_q^rW\}$ will be referred to as the *sheaf of q-forms of order r* over Y, and will be denoted by Ω_q^r.

The exterior derivative operator d defines, for each $W \subset Y$, a sequence of Abelian groups

(1)
$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega_0^r W \xrightarrow{d} \Omega_1^r W \xrightarrow{d} \Omega_2^r W \xrightarrow{d} \ldots \xrightarrow{d} \Omega_n^r W$$
$$\xrightarrow{d} \Omega_{n+1}^r W \xrightarrow{d} \ldots \xrightarrow{d} \Omega_M^r W \to 0,$$

and an exact sequence of Abelian sheaves

(2)
$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega_0^r \xrightarrow{d} \Omega_1^r \xrightarrow{d} \Omega_2^r \xrightarrow{d} \ldots \xrightarrow{d} \Omega_n^r$$
$$\xrightarrow{d} \Omega_{n+1}^r \xrightarrow{d} \ldots \xrightarrow{d} \Omega_M^r \to 0.$$

We call this sequence the *De Rham (sheaf) sequence* over $J^r Y$.

We now construct a subsequence of the De Rham sequence. First recall the notion of a *contact form*, and introduce the notion of a *strongly contact form*, a (higher-order) analogy of a similar concept introduced in Section 2.

Let W be an open set in the fibred manifold Y. Recall that the *horizontalisation* $h : \Omega^r W \to \Omega^{r+1} W$ is a morphism of exterior algebras, which assigns to a q-form $\rho \in \Omega_q^r W$, $q \geq 1$, a π^{r+1}-horizontal q-form $h\rho \in \Omega_q^{r+1} W$ by the formula

(3)
$$h\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q) = \rho(J_x^r\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_q),$$

where $J_x^{r+1}\gamma \in W^{r+1}$ is any point and ξ_1, ξ_2, $\ldots$, ξ_q are any tangent vectors of $J^{r+1}Y$ at this point. If f is a function, then

(4)
$$hf = (\pi^{r+1,r})^* f.$$

One can equivalently introduce h as a morphism, defined in a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, by the equations

(5)
$$hf = f \circ \pi^{r+1,r}, \quad hdx^i = dx^i, \quad hdy^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k i} dx^i,$$

where f is any function on V^r and $0 \leq k \leq r$. A form $\rho \in \Omega_q^r W$ such that

(6)
$$h\rho = 0$$

is said to be *contact*. Clearly, every q-form ρ such that $q \geq n+1$ is contact, and the 1-forms

(7)
$$\omega^\sigma_{j_1 j_2 \ldots j_l} = dy^\sigma_{j_1 j_2 \ldots j_l} - y^\sigma_{j_1 j_2 \ldots j_l i} dx^i, \quad 0 \leq l \leq r-1,$$

defined on the open set $V^r \subset J^r Y$ are examples of contact 1-forms. The 1-forms $\{dx^i, \omega^\sigma_{j_1 j_2 \ldots j_k}, dy^\sigma_{l_1 l_2 \ldots l_{r-1} l_r}\}$, where the $1 \leq i \leq n$, $1 \leq \sigma \leq m$, $1 \leq k \leq r-1$, $1 \leq j_1 \leq j_2 \leq \ldots \leq j_k \leq n$, and $1 \leq l_1 \leq l_2 \leq \ldots \leq l_r \leq n$, constitute a basis of linear forms on the set V^r, called the *contact basis* (Section 2.1, Theorem 1).

The exterior derivative df, or more precisely, $(\pi^{r+1,r})^* df$, can be decomposed as $(\pi^{r+1,r})^* df = hdf + pdf$, where pdf is a contact 1-form, called the *contact component* of *f*. Any form $\rho \in \Omega_q^r W$, of more precisely $(\pi^{r+1,r})^* \rho$, has the *canonical decomposition* $(\pi^{r+1,r})^* \rho = h\rho + p_1\rho + p_2\rho + \ldots + p_q\rho$, where $h\rho$ is π^{r+1}-horizontal and $p_k\rho$ is *k-contact*; this condition can equivalently be expressed by saying that the chart expression of $p_k\rho$ is generated by the product of *k* exterior factors $\omega^\sigma_{j_1 j_2 \ldots j_p}$, where $0 \le p \le r$.

The 1-forms $\omega^\sigma_{j_1 j_2 \ldots j_k}$ and 2-forms $d\omega^\sigma_{j_1 j_2 \ldots j_{r-1}}$ locally generate the *contact ideal* $\Theta^r W$ of the exterior algebra $\Omega^r W$, which is *closed* under the exterior derivative operator *d*; its elements are called *contact forms*. The *contact q-forms* are elements of the *contact submodules* $\Omega_q^r W \cap \Theta^r W$. We need these submodules for $q \le n$; denote

(8) $$\Theta_q^r W = \Omega_q^r W \cap \Theta^r W, \quad q \le n.$$

The 1-forms $\omega^\sigma_{j_1 j_2 \ldots j_k}$, where $0 \le k \le r-1$, determined by a fibred atlas on *Y*, locally generate a (global) module of 1-forms, and an ideal $\Theta_0^r W$ of the exterior algebra $\Omega^r W$ (for definitions see Appendix 7). Clearly, the contact ideal contains $\Theta_0^r W$ as a subset.

Since the contact ideal is closed under the exterior derivative, we have the sequence of Abelian groups

(9) $$0 \longrightarrow \Theta_1^r W \xrightarrow{\ d\ } \Theta_2^r W \xrightarrow{\ d\ } \ldots \xrightarrow{\ d\ } \Theta_n^r W.$$

If $\rho \in \Theta_q^r W$ is a contact form and *f* is a function on W^r, then the formula

(10) $$d(f\rho) = df \wedge \rho + fd\rho$$

shows that the form $d(f\rho)$ is again a contact form. Thus, the mapping $\rho \to d(f\rho)$ is a morphism of Abelian groups; however, the exterior derivative in the sequence (9) is *not* a homomorphism of modules. Restricting the multiplication to *constant* functions *f*, that is, to *real numbers*, (9) can be considered as a sequence of real vector spaces.

Consider now the sets of *q*-forms $\Omega_q^r W$ such $n+1 \le q \le \dim J^r Y$. Denote $q = n+k$. If $\rho \in \Omega_{n+k}^r W$, then $h\rho = 0$, and also $p_1\rho = 0$, $p_2\rho = 0$, ..., $p_{k-1}\rho = 0$ identically (cf. Section 2.4, Theorem 8), thus ρ is always contact, and its canonical decomposition has the form

(11) $$(\pi^{r+1,r})^* \rho = p_k\rho + p_{k+1}\rho + \ldots + p_{k+n}\rho.$$

To introduce the notion of a strongly contact form, it is convenient to proceed in two steps.

First we slightly modify the definition given in Section 2.6 and introduce the class of strongly contact forms as follows. We say that an $(n+1)$-form $\rho \in \Omega_{n+1}^r W$ is *strongly contact*, if for every point $J_x^r \gamma \in V^r$ there exist an integer $s \ge r$, a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, at $\gamma(x) \in V$ and a contact n-form $\eta \in \Theta_n^s V$ such that

$$p_1((\pi^{s,r})^* \rho - d\eta) = 0. \tag{12}$$

Second, if $\rho \in \Omega_{n+k}^r W$ where $k \geq 2$, we say that ρ is *strongly contact*, if for every point $J_x^r \gamma \in V^r$ there exists $s \geq r$, a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, at $\gamma(x) \in V$ and a strongly contact $(n+k-1)$-form $\eta \in \Omega_{n+k}^s V$ such that

$$p_k((\pi^{s,r})^* \rho - d\eta) = 0. \tag{13}$$

Lemma 1 *Let* $\rho \in \Omega_{n+k}^r W$. *The following conditions are equivalent:*

(a) ρ *is strongly contact.*

(b) *There exists an integer* $s \geq r$ *and an* $(n+k-1)$*-form* $\eta \in \Omega_{n+k}^s V$ *such that*

$$(\pi^{s,r})^* \rho = \mu + d\eta, \quad p_k \mu = 0, \quad p_{k-1}\eta = 0. \tag{14}$$

Proof If ρ is strongly contact, then $(\pi^{s,r})^* \rho - d\eta = \mu$ for some form μ on V^s such that $p_k \mu = 0$. Then $(\pi^{s,r})^* \rho = \mu + d\eta$ proving (14). The converse is obvious.

Lemma 2 (a) *Every form* $\rho \in \Omega_{n+k}^r W$ *such that* $p_k \rho = 0$, *is strongly contact.*

(b) *Exterior derivative of a contact n-form is strongly contact. Exterior derivative of a strongly contact form is strongly contact.*

(c) *Let* Ξ *be a* π*-vertical vector field,* $\rho \in \Omega_{n+k}^r W$ *a strongly contact form. If* $k \geq 2$, *then the* $(n+k-1)$*-form* $i_\Xi \rho$ *is strongly contact.*

Proof (a) Obvious.

(b) We use the identity $p_{k+1}(d\rho - d\rho) = 0$.

(c) This follows from Lemma 9 and Section 2.5, Theorem 9. Indeed, for every π-vertical vector field Ξ

$$\begin{aligned} &i_\Xi p_k((\pi^{s,r})^* \rho - d\eta) \\ &\quad = p_{k-1}(i_\Xi (\pi^{s,r})^* \rho - i_\Xi d\eta) \\ &\quad = p_{k-1}(i_\Xi (\pi^{s,r})^* \rho - \partial_\Xi \eta) \\ &\quad = p_{k-1}(i_\Xi (\pi^{s,r})^* \rho + d i_\Xi \eta) = 0. \end{aligned} \tag{15}$$

But $p_{k-2} i_\Xi \eta = i_\Xi p_{k-1} \eta = 0$ proving (c).

Remark 1 It follows from Lemma 1 that the canonical decomposition of a strongly contact form $\rho \in \Theta_{n+k}^r W$ is

$$\begin{aligned} (\pi^{s,r})^* \rho &= p_k d\tau + p_{k+1}\rho + p_{k+2}\rho + \ldots + p_{n+k}\rho \\ &= d\tau + p_{k+1}(\rho - d\tau) + p_{k+2}(\rho - d\tau) + \ldots + p_{n+k}(\rho - d\tau), \end{aligned} \tag{16}$$

where the forms on the right-hand side are considered as canonically lifted to the set $V^s \subset J^s Y$.

Remark 2 One can formally extend the definition of a strongly contact form to the q-forms $\rho \in \Omega_q^r W$ such that $1 \le q \le n$. Indeed, we have for any contact form $\rho' \in \Theta_{q-1}^r W$, $h(\rho - d\rho') = h\rho$; thus if $h\rho = 0$ then we have $h(\rho - d\rho') = 0$ for any $\rho' \in \Theta_{q-1}^r W$.

Remark 3 The definition of a strongly contact form, given above, has its natural origin in the theory of systems of partial differential equations for mappings of n independent variables, defined by *differential forms of degree* $n+k > n$: such differential equations can equivalently be described by systems of *n-forms* arising by contraction of $(n+k)$-forms with k vector fields. For an *ad hoc* construction in this context, similar to the concept of a strongly contact form, see the *differential systems with independence condition* in Bryant, Chern, Gardner, Goldschmidt, Griffiths [Bry].

Remark 4 The definition of a strongly contact form is closely related to the concept of a Lepage form (Section 4.3).

Strongly contact $(n+k)$-forms on W^r constitute a *subgroup* $\Theta_q^r W$ of the Abelian group $\Omega_q^r W$; they do not form a submodule of $\Omega_q^r W$. The Abelian groups $\Theta_q^r W$ together with the exterior derivative d form a sequence

$$(17) \qquad \Theta_n^r W \xrightarrow{d} \Theta_{n+1}^r W \xrightarrow{d} \dots \xrightarrow{d} \Theta_M^r W \longrightarrow 0.$$

The index M of the last non-zero term in this sequence is

$$(18) \qquad M = m\binom{n+r-1}{n} + 2n - 1.$$

If η is a contact n-form, then η is automatically a strongly contact form. Thus, sequences (9) and (17) can be glued together. We get a sequence

$$(19) \qquad \begin{aligned} &0 \longrightarrow \Theta_1^r W \xrightarrow{d} \Theta_2^r W \xrightarrow{d} \dots \xrightarrow{d} \Theta_n^r W \\ &\xrightarrow{d} \Theta_{n+1}^r W \xrightarrow{d} \dots \xrightarrow{d} \Theta_M^r W \longrightarrow 0. \end{aligned}$$

The families of Abelian groups $\{\Theta_q^r W\}$, where W runs through open subsets of the fibred manifold Y, induce Abelian sheaves, and the sequences (18) induce a sequence of Abelian sheaves. Indeed, consider for any integer q such that $1 \le q \le M$ the family of Abelian groups $\Theta_q^r = \{\Theta_q^r W\}$. Any two open sets $W_1, W_2 \subset Y$ such that $W_2 \subset W_1$ define a morphism of Abelian groups $\Theta_q^r W_1 \ni \rho \to \rho|_{W_2} \in \Theta_q^r W_2$, the *restriction* of a form, defined on the open set $W_1^r \subset J^r Y$, to the open set $W_2^r \subset W_1^r$. Clearly, Θ_q^r with these restriction morphisms form an Abelian presheaf over Y. The restriction morphisms obviously satisfy the axioms of an Abelian sheaf (Section 7.4). Thus, the presheaf Θ_q^r has the structure of an Abelian sheaf.

If $1 \le q \le n$ (resp. $n+1 \le q \le M$) this sheaf is called the *sheaf of contact* (resp. *strongly contact*) *q-forms of order r* on *Y*.

Remark 5 The sheaf Θ^r_q, defined over the fibred manifold *Y*, *differs* from the sheaf of *q*-forms over the *r*-jet prolongation J^rY of *Y*; Θ^r_q can be characterized as the *direct image* of the sheaf of *q*-forms of order *r* over J^rY by the jet projection $\pi^{r,0}: J^rY \to Y$. Our construction, for the forms of degree $q \le n$, is the same as an analogous construction in Anderson and Duchamp [AD].

The sequences (18) induce the *sequence of Abelian sheaves*

$$\begin{aligned} (20) \qquad & 0 \longrightarrow \Theta^r_1 \xrightarrow{d} \Theta^r_2 \xrightarrow{d} \ldots \xrightarrow{d} \Theta^r_n \xrightarrow{d} \Theta^r_{n+1} \\ & \xrightarrow{d} \ldots \xrightarrow{d} \Theta^r_M \longrightarrow 0. \end{aligned}$$

The following basic observation shows that the De Rham sequence can be factored through the sequence (20).

Lemma 3 *The sequence of Abelian sheaves* (20) *is an exact subsequence of the De Rham sequence* (2).

Proof 1. To prove exactness of the sequence (20) at the term Θ^r_q, where $1 \le q \le n$, it is sufficient to consider differential forms defined on the chart neighbourhood of a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, on *Y*. However, for these differential forms the statement already follows from Section 2.7, Theorem 13.

2. Exactness at the terms Θ^r_q, where $n+1 \le q \le M$, follows from Section 2.7, Theorem 14.

The sequence (19) will be referred to as the *contact sequence*, or the *contact subsequence* of the De Rham sequence,

We show that the sheaves Θ^r_q in the contact subsequence are all soft. To describe the structure of these sheaves Θ^r_q such that $n+1 \le q \le M$, note that any *q*-form ρ on the *r*-jet prolongation J^rY identically satisfies

$$(21) \qquad h\rho = 0, \quad p_1\rho = 0, \quad p_2\rho = 0, \quad \ldots, \quad p_{q-n-1}\rho = 0$$

(Section 2.4, Theorem 8). We denote by $\Omega^r_{q\,(c)}W$ the submodule of the module of *q*-form Ω^r_qW defined by the condition

$$(22) \qquad p_{q-n}\rho = 0.$$

This condition states that the submodule $\Omega^r_{q\,(c)}W$ consists of the forms whose order of contactness is $\ge q-n+1$. The family of the modules $\Omega^r_{q\,(c)}W$ define the sheaf of modules

$$(23) \qquad \Omega^r_{q\,(c)} = \{\Omega^r_{q\,(c)}W\}.$$

Clearly $\Omega^r_{q\,(c)}$ is a soft sheaf.

Lemma 4 *For every integer q such that* $1 \le q \le M$ *the sheaf* Θ^r_q *is soft.*

Proof 1. If $1 \le q \le n$, then the sheaf Θ^r_q admits multiplication by functions so it is fine; then, however, according to Section 7.1, Theorem 4, the sheaf Θ^r_q is soft.

2. Consider the contact subsequence (20) and the short exact sequence

$$(24)\qquad 0 \longrightarrow \Theta^r_1 \xrightarrow{\ d\ } \Theta^r_2 \xrightarrow{\ d\ } d\Theta^r_2 \longrightarrow 0,$$

where $d\Theta^r_2$ denotes the image sheaf, $d\Theta^r_2 = \operatorname{Im} d$. Since the sheaves Θ^r_1 and Θ^r_2 are soft, the sheaf $d\Theta^r_2$ is also soft (Section 7.10, Corollary 1). Similarly, assign to the sequence

$$(25)\qquad 0 \longrightarrow \Theta^r_1 \xrightarrow{\ d\ } \Theta^r_2 \xrightarrow{\ d\ } \Theta^r_3 \xrightarrow{\ d\ } d\Theta^r_3 \longrightarrow 0$$

the short exact sequence

$$(26)\qquad 0 \longrightarrow \operatorname{Ker} d \longrightarrow \Theta^r_3 \xrightarrow{\ d\ } d\Theta^r_3 \longrightarrow 0.$$

Using exactness of (25) at Θ^r_3, we have $\operatorname{Ker} d = d\Theta^r_2$, so the sheaf $\operatorname{Ker} d$ in (26) is soft. Consequently, the sheaf $d\Theta^r_3$ is also soft. Continuing this way, we assign to the sequence

$$(27)\qquad 0 \longrightarrow \Theta^r_1 \xrightarrow{\ d\ } \Theta^r_2 \xrightarrow{\ d\ } \ldots \xrightarrow{\ d\ } \Theta^r_n \xrightarrow{\ d\ } d\Theta^r_n \longrightarrow 0$$

the short exact sequence

$$(28)\qquad 0 \longrightarrow \operatorname{Ker} d \longrightarrow \Theta^r_n \xrightarrow{\ d\ } d\Theta^r_n \longrightarrow 0$$

and since $\operatorname{Ker} d = d\Theta^r_{n-1}$ and this sheaf is soft, the sheaf $d\Theta^r_n$ is also soft.

Now consider the sheaf Θ^r_{n+1}. Note that by definition, we have a sheaf morphism, expressed (by means of representatives of the germs) as

$$(29)\qquad \Theta^r_n \times_Y \Omega^r_{n+1(c)} \ni (\tau,\mu) \to \mu + d\tau \in \Omega^r_{n+2},$$

where $\Theta^r_n \times_Y \Omega^r_{n+1(c)}$ is the fibre product of the sheaves Θ^r_n and $\Omega^r_{n+1(c)}$. The sheaf Θ^r_{n+1} can be regarded as the *image sheaf* of this morphism; its *kernel* consists of the pairs $(\tau,-d\tau) \in \Theta^r_n \times_Y d\Theta^r_n$. We get a short exact sequence

$$(30)\qquad 0 \longrightarrow \Theta^r_n \times_Y d\Theta^r_n \longrightarrow \Theta^r_n \times_X \Omega^r_{n+1(c)} \xrightarrow{\ d\ } \Theta^r_{n+1} \longrightarrow 0.$$

The sheaves $\Theta^r_n \times_Y d\Theta^r_n$ and $\Theta^r_n \times_X \Omega^r_{n+1(c)}$ in this sequence are fibre products of soft sheaves Θ^r_n, $d\Theta^r_n$, and $\Omega^r_{n+1(c)}$, and are therefore soft; hence the sheaf Θ^r_{n+1} is also soft.

Extending this construction to any of the sheaves Θ_q^r in the variational sequence (20), where $q \geq n+1$, we complete the proof.

8.2 The variational sequence

Consider the De Rham sequence (2), and its contact subsequence (19), Section 8.1. Using Section 8.1, Lemma 3 we get a commutative diagram

(1)
$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & \Theta_1^r \arrow[r, "d"] \arrow[d] & \Theta_2^r \arrow[r, "d"] \arrow[d] & \Theta_3^r \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & \mathbf{R}_Y \arrow[r] & \Omega_0^r \arrow[r, "d"] & \Omega_1^r \arrow[r, "d"] & \Omega_2^r \arrow[r, "d"] & \Omega_3^r \arrow[r, "d"] & \dots
\end{tikzcd}$$

in which $\mathbf{R}_Y \to \Omega_0^r$ is the canonical inclusion and the vertical arrows represent canonical inclusions of subsheaves. Passing to the quotient sheaves and quotient sheaf morphism, this diagram induces a commutative diagram, written in two parts as

(2)
$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & \Theta_1^r \arrow[r, "d"] \arrow[d] & \Theta_2^r \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & \mathbf{R}_Y \arrow[r] & \Omega_0^r \arrow[r, "d"] \arrow[dr] & \Omega_1^r \arrow[r, "d"] \arrow[d] & \Omega_2^r \arrow[r, "d"] \arrow[d] & \dots \\
 & & & \Omega_1^r/\Theta_1^r \arrow[r] \arrow[d] & \Omega_2^r/\Theta_2^r \arrow[r] \arrow[d] & \dots \\
 & & & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & \\
\dots \arrow[r] & \Theta_M^r \arrow[r] \arrow[d] & 0 \arrow[d] & & & \\
\dots \arrow[r, "d"] & \Omega_M^r \arrow[r, "d"] \arrow[d] & \Omega_M^r \arrow[r, "d"] & \dots \arrow[r, "d"] & \Omega_N^r \arrow[r] & 0 \\
\dots \arrow[r] & \Omega_M^r/\Theta_M^r \arrow[ur] \arrow[d] & & & & \\
 & 0 & & & &
\end{tikzcd}$$

The quotient sequence of Abelian sheaves, defined by this diagram,

(3) $$0 \longrightarrow \mathbf{R}_Y \longrightarrow \Omega_0^r \longrightarrow \Omega_1^r / \Theta_1^r \longrightarrow \Omega_2^r / \Theta_2^r \longrightarrow \Omega_3^r / \Theta_3^r \longrightarrow$$

is called the (*r-th* order) *variational sequence* over the fibred manifold *Y*. Since the De Rham sequence and its contact subsequence are exact, it can be easily verified that the quotient sequence is also exact (see also Section 7.7, Corollary 2). Thus, the variational sequence is a *resolution* of the constant sheaf $\mathbf{R}_Y$ over *Y*. We call the Abelian group morphisms in the (3) the *Euler-Lagrange morphisms* and denote them by $E_j : \Omega_j^r / \Theta_j^r \longrightarrow \Omega_{j+1}^r / \Theta_{j+1}^r$, or just by *E*. The variational sequence is also denoted by

(4) $$0 \longrightarrow \mathbf{R}_Y \longrightarrow \mathit{Var}_Y^r .$$

Consider the complex of global sections

(5) $$0 \longrightarrow \Omega_0^r Y \longrightarrow (\Omega_1^r / \Theta_1^r) Y \longrightarrow (\Omega_2^r / \Theta_2^r) Y \longrightarrow (\Omega_3^r / \Theta_3^r) Y \longrightarrow$$

associated with the variational sequence (4), its cohomology groups $H^k(\mathit{Var}_Y^r Y)$, and the cohomology groups of the fibred manifold *Y* with coefficients in the constant sheaf $\mathbf{R}_Y$; by the De Rham theorem, we identify these cohomology groups with the *De Rham cohomology gtoups*; thus $H^k Y = H^k(Y, \mathbf{R}_Y)$ (Section 7.10, Remark 6). We are now going to establish two theorems, representing central results of this chapter, namely the tools for the study of the global variational functionals, considered in Chapter 4 and Chapter 5 of this book.

Theorem 1 *The variational sequence* $0 \to \mathbf{R}_Y \to \mathit{Var}_Y^r$ *is an acyclic resolution of the constant sheaf* $\mathbf{R}_Y$.

Proof Since the sheaves Ω_k^r and Θ_k^r are soft (Section 8.1, Lemma 4), the quotient sheaves Ω_k^r / Θ_k^r are also soft (Section 7.9, Corollary 1). Then, however, the sheaves Ω_k^r / Θ_k^r are acyclic, so the resolution $0 \to \mathbf{R}_Y \to \mathit{Var}_Y^r$ is acyclic (Section 7.10, Lemma 18).

Theorem 2 *The cohomology groups* $H^k(\mathit{Var}_Y^r Y)$ *of the complex of global sections and the De Rham cohomology groups* $H^k Y$ *of the manifold Y are isomorphic*.

Proof This follows from Section 7.10, Theorem 5 (see also Corollary 13 and Remark 6).

Remark 6 The cohomology groups $H^k(Y, \mathbf{R}_Y)$ have been constructed by means of the topology of the underlying fibred manifold *Y*. On the other hand, it follows from Theorem 2 that the same cohomology groups characterize properties of the complex of global sections of the associated with the variational sequence. In this sense Theorem 2 clarifies the relationship between existence of global sections of the quotient Abelian groups and topological properties of *Y*.

8.3 Variational projectors

In this section we consider the columns of the diagram (3), Section 8.2, defining the variational sequence of order r over the fibred manifold Y. The main goal is to show that the *classes of forms* – elements of the quotient groups Ω_k^r/Θ_k^r, can be represented as *global differential forms*, defined on the s-jet prolongation J^sY for some s. Basic idea for constructing this representation leans on the definition of the quotient space, which is defined up to a canonical isomorphism. We shall construct an Abelian group of forms Φ_k^r and a group morphism $\mathscr{I}_k^r:\Omega_k^r \to \Phi_k^r$ such that $\operatorname{Ker}\mathscr{I}_k^r = \Theta_k^r$; then the quotient sheaf Ω_k^r/Θ_k^r becomes canonically isomorphic with the image $\operatorname{Im}\mathscr{I}_k^r \subset \Phi_k^r$, according to the diagram

$$
\begin{array}{ccccc}
 & & \Theta_k^r & & \\
 & & \downarrow & & \\
(1) & & \Omega_k^r & & \\
 & \swarrow & & \searrow & \\
\Omega_k^r/\Theta_k^r & & \longleftrightarrow & & \operatorname{Im}\mathscr{I}_k^r
\end{array}
$$

Let $k \geq 1$, let W be an open set in Y, and let η be a k-contact $(n+k)$-form η, defined on the open set W^{r+1} in J^rY. In a fibred chart (V,ψ), $\psi=(x^i,y^\sigma)$, on Y, η has an expression

(2) $$\eta = \sum_{0\leq k\leq r} \Phi_\sigma^{j_1 j_2 \dots j_k} \wedge \omega_{j_1 j_2 \dots j_k}^\sigma \wedge \omega_0,$$

where $\Phi_\sigma^{j_1 j_2 \dots j_k}$ are some $(k-1)$-contact $(k-1)$-forms. In this section we construct a decomposition of the canonical lift $(\pi^{2r+1,r+1})^*\eta$ of η to W^{2r+1}; to this purpose se use the property

(3) $$\omega_{j_1 j_2 \dots j_k}^\sigma \wedge \omega_0 = -d(\omega_{j_1 j_2 \dots j_{k-1}}^\sigma \wedge \omega_{j_k})$$

of the contact 1-forms $\omega_{j_1 j_2 \dots j_k}^\sigma$. Although the decomposition we introduce will be constructed by means of fibred charts, it will be independent of the chosen chars.

First consider the decomposition of $(n+1)$-forms, defined on the set W^{r+1}; the idea will be to identify in a form a summand, which is an *exact* form. The proof of the following theorem is based on the algebraic trace decomposition theory explained in Appendix 9.

Theorem 3 *Let* η *be a* 1*-contact* $\pi^{r+1,r}$*-horizontal* $(n+1)$*-form on* W^{r+1}*, expressed in a fibred chart* (V,ψ)*,* $\psi=(x^i,y^\sigma)$*, by*

(4) $$\eta = \sum_{0 \le |J| \le r} A_\sigma^J \omega_J^\sigma \wedge \omega_0.$$

(a) *There exist a* 1*-contact* ω^σ *-generated* $(n+1)$ *-form* $I_1\eta$ *on* V^{2r+1}*, a* 1*-contact n-form* $J_1\eta$ *and a* 2*-contact* $(n+1)$ *-form* $K_1\eta$ *, defined on* V^{2r+1}*, such that*

(5) $$(\pi^{2r+1,r+1})^* \eta = I_1\eta - dJ_1\eta + K_1\eta,$$

where

(6) $$\begin{aligned} I_1\eta &= \left(A_\sigma + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \dots d_{j_s} A_\sigma^{j_1 j_2 \dots j_s} \right) \omega^\sigma \wedge \omega_0, \\ J_1\eta &= \sum_{1 \le s \le r} \sum_{0 \le k \le r-1} (-1)^k d_{j_{s-k+1}} d_{j_{s-k+2}} \dots d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-k} j_{s-k+1} j_{s-k+2} \dots j_s} \omega^\sigma_{i_1 i_2 \dots i_{s-k-1}} \wedge \omega_{i_{s-k}}, \\ K_1\eta &= \sum_{1 \le s \le r} \sum_{0 \le k \le s-1} (-1)^{k+1} pd(d_{j_{s-k+1}} d_{j_{s-k+2}} \dots d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-k} j_{s-k+1} j_{s-k+2} \dots j_s}) \\ &\quad \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-k-1}} \wedge \omega_{i_{s-k}}. \end{aligned}$$

(b) *Suppose that we have a decomposition*

(7) $$(\pi^{2r+1,r+1})^* \eta = \eta_0 - d\eta_1 + \eta_2$$

such that η_0 *is* 1*-contact and* ω^σ *-generated,* η_1 *is* 1*-contact, and* η_2 *is a* 2*-contact form. Then*

(8) $$\eta_0 = I_1\eta, \quad d\eta_1 = dJ_1\eta, \quad \eta_2 = K_1\eta.$$

Proof (a) Write expression (4) as

(9) $$\begin{aligned} \eta &= \sum_{0 \le |J| \le r} A_\sigma^J \omega_J^\sigma \wedge \omega_0 \\ &= A_\sigma \omega^\sigma \wedge \omega_0 + \sum_{1 \le |J| \le r} A_\sigma^J \omega_J^\sigma \wedge \omega_0, \end{aligned}$$

and consider a summand $A_\sigma^J \omega_J^\sigma \wedge \omega_0$, where $|J| = s \ge 1$. Then in the standard index notation

(10) $$\begin{aligned} A_\sigma^J \omega_J^\sigma \wedge \omega_0 &= A_\sigma^{i_1 i_2 \dots i_s} \omega^\sigma_{i_1 i_2 \dots i_s} \wedge \omega_0 = -A_\sigma^{i_1 i_2 \dots i_s} d(\omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s}) \\ &= -d(A_\sigma^{i_1 i_2 \dots i_s} \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s}) + dA_\sigma^{i_1 i_2 \dots i_s} \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s} \\ &= hdA_\sigma^{i_1 i_2 \dots i_s} \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s} + pdA_\sigma^{i_1 i_2 \dots i_s} \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s} \\ &\quad - d(A_\sigma^{i_1 i_2 \dots i_s} \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s}) \end{aligned}$$

$$
\begin{aligned}
&= d(d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-1} j_s} \omega^\sigma_{i_1 i_2 \dots i_{s-2}} \wedge \omega_{i_{s-1}}) + d_{j_{s-1}} d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-2} j_{s-1} j_s} \omega^\sigma_{i_1 i_2 \dots i_{s-2}} \wedge \omega_0 \\
&- pd(d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-1} j_s}) \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-2}} \wedge \omega_{i_{s-1}} \\
&+ pdA_\sigma^{i_1 i_2 \dots i_s} \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s} - d(A_\sigma^{i_1 i_2 \dots i_s} \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s}) \\
&= d_{j_{s-1}} d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-2} j_{s-1} j_s} \omega^\sigma_{i_1 i_2 \dots i_{s-2}} \wedge \omega_0 \\
&- pd(d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-1} j_s}) \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-2}} \wedge \omega_{i_{s-1}} + pdA_\sigma^{i_1 i_2 \dots i_s} \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s} \\
&+ d(d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-1} j_s} \omega^\sigma_{i_1 i_2 \dots i_{s-2}} \wedge \omega_{i_{s-1}} - A_\sigma^{i_1 i_2 \dots i_s} \omega^\sigma_{i_1 i_2 \dots i_{s-1}} \wedge \omega_{i_s}).
\end{aligned}
$$

Further calculations yield

$$
\begin{aligned}
&A_\sigma^{i_1 i_2 \dots i_s} \omega^\sigma_{i_1 i_2 \dots i_s} \wedge \omega_0 = (-1)^s d_{j_1} d_{j_2} \dots d_{j_s} A_\sigma^{j_1 j_2 \dots j_s} \omega^\sigma \wedge \omega_0 \\
&\quad - \sum_{0 \le k \le s-1} (-1)^k pd(d_{j_{s-k+1}} d_{j_{s-k+2}} \dots d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-k} j_{s-k+1} j_{s-k+2} \dots j_s}) \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-k-1}} \wedge \omega_{i_{s-k}} \\
&\quad - d\left(\sum_{0 \le k \le s-1} (-1)^k d_{j_{s-k+1}} d_{j_{s-k+2}} \dots d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-k} j_{s-k+1} j_{s-k+2} \dots j_s} \omega^\sigma_{i_1 i_2 \dots i_{s-k-1}} \wedge \omega_{i_{s-k}} \right).
\end{aligned} \tag{11}
$$

These formulas prove statement (a).

(b) To prove (b), suppose that $\eta_0 - d\eta_1 + \eta_2 = 0$, where η_0 is 1-contact and ω^σ-generated, η_1 is 1-contact, and η_2 is a 2-contact form; we want to show that this condition implies $\eta_0 = 0$, $\eta_2 = 0$; indeed these conditions will also prove that $d\eta_1 = 0$. The forms η_0 and η_1 can be expressed in the form

$$
\eta_0 = A_\sigma \omega^\sigma \wedge \omega_0, \quad \eta_1 = B_\sigma{}^i \omega^\sigma \wedge \omega_i + \sum_{1 \le k \le 2r} B_\sigma^{j_1 j_2 \dots j_k \, i} \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i. \tag{12}
$$

If $k \ge 1$, then $B_\sigma^{j_1 j_2 \dots j_k \, i}$ can be decomposed as

$$
\begin{aligned}
&B_\sigma^{j_1 j_2 \dots j_k \, i} = \tilde{B}_\sigma^{j_1 j_2 \dots j_k \, i} \\
&\quad + \frac{1}{k+1}(B_\sigma^{j_1 j_2 \dots j_k \, i} - B_\sigma^{i j_2 j_3 \dots j_k \, j_1}) + \frac{1}{k+1}(B_\sigma^{j_1 j_2 \dots j_k \, i} - B_\sigma^{j_1 i j_3 j_4 \dots j_k \, j_2}) \\
&\quad + \dots + \frac{1}{k+1}(B_\sigma^{j_1 j_2 \dots j_k \, i} \dots B_\sigma^{j_1 j_2 \dots j_{k-1} i \, j_k}),
\end{aligned} \tag{13}
$$

where $\tilde{B}_\sigma^{j_1 j_2 \dots j_k \, i}$ is the symmetric component,

$$
\tilde{B}_\sigma^{j_1 j_2 \dots j_k \, i} = \frac{1}{k+1}(B_\sigma^{j_1 j_2 \dots j_k \, i} + B_\sigma^{i j_2 j_3 \dots j_k \, j_1} + B_\sigma^{j_1 i j_3 j_4 \dots j_k \, j_2} + \dots + B_\sigma^{j_1 j_2 \dots j_{k-1} i \, j_k}). \tag{14}
$$

Now calculating $p_1 d\eta_1$, we have

$$
\begin{aligned}
&p_1 d\eta_1 = hdB_\sigma{}^i \wedge \omega^\sigma \wedge \omega_i - B_\sigma{}^i \omega^\sigma_i \wedge \omega_0 \\
&\quad + \sum_{1 \le k \le 2r} hdB_\sigma^{j_1 j_2 \dots j_k \, i} \wedge \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i - \sum_{1 \le k \le 2r} B_\sigma^{j_1 j_2 \dots j_k \, i} \omega^\sigma_{j_1 j_2 \dots j_k i} \wedge \omega_0
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&= -d_i B_\sigma{}^i \omega^\sigma \wedge \omega_0 - B_\sigma{}^i \omega_i^\sigma \wedge \omega_0 \\
&- \sum_{1\le k\le 2r} d_i B_\sigma^{j_1 j_2 \dots j_k\ i} \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_0 - \sum_{1\le k\le 2r} B_\sigma^{j_1 j_2 \dots j_k\ i} \omega^\sigma_{j_1 j_2 \dots j_k i} \wedge \omega_0 \\
&= -d_i B_\sigma{}^i \omega^\sigma \wedge \omega_0 - (B_\sigma{}^{j_1} + d_i B_\sigma^{j_1\ i}) \omega^\sigma_{j_1} \wedge \omega_0 \\
&- \sum_{2\le k\le 2r} (d_i B_\sigma^{j_1 j_2 \dots j_k\ i} + B_\sigma^{j_1 j_2 \dots j_{k-1}\ j_k}) \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_0 \\
&- B_\sigma^{j_1 j_2 \dots j_{2r}\ i} \omega^\sigma_{j_1 j_2 \dots j_{2r} i} \wedge \omega_0 .
\end{aligned}
$$

Equation $\eta_0 - d\eta_1 + \eta_2 = 0$ implies $\eta_0 - p_1 d\eta_1 = 0$ hence

$$
\begin{aligned}
&(A_\sigma - d_{j_1} B_\sigma{}^{j_1}) \omega^\sigma \wedge \omega_0 - (d_i B_\sigma^{j_1\ i} + B_\sigma{}^{j_1}) \omega^\sigma_{j_1} \wedge \omega_0 \\
&\quad - (d_i B_\sigma^{j_1 j_2\ i} + B_\sigma^{j_1\ j_2}) \omega^\sigma_{j_1 j_2} \wedge \omega_0 - (d_i B_\sigma^{j_1 j_2\ i} + B_\sigma^{j_1\ j_2}) \omega^\sigma_{j_1 j_2} \wedge \omega_0 \\
&\quad - \dots - (d_i B_\sigma^{j_1 j_2 \dots j_{2r-1}\ i} + B_\sigma^{j_1 j_2 \dots j_{2r-2}\ j_{2r-1}}) \omega^\sigma_{j_1 j_2 \dots j_{2r-1}} \wedge \omega_0 \\
&\quad - (d_i B_\sigma^{j_1 j_2 \dots j_{2r}\ i} + B_\sigma^{j_1 j_2 \dots j_{2r-1}\ j_{2r}}) \omega^\sigma_{j_1 j_2 \dots j_{2r}} \wedge \omega_0 \\
&\quad - B_\sigma^{j_1 j_2 \dots j_{2r}\ j_{2r+1}} \omega^\sigma_{j_1 j_2 \dots j_{2r} j_{2r+1}} \wedge \omega_0 = 0,
\end{aligned} \tag{16}
$$

therefore the components $B_\sigma^{j_1 j_2 \dots j_k\ i}$ satisfy

$$
\begin{aligned}
&\tilde{B}_\sigma^{j_1 j_2 \dots j_{2r}\ j_{2r+1}} = 0, \\
&\tilde{B}_\sigma^{j_1 j_2 \dots j_{2r-1}\ j_{2r}} = -d_i B_\sigma^{j_1 j_2 \dots j_{2r}\ i}, \\
&\dots \\
&B_\sigma{}^{j_1} = -d_i B_\sigma^{j_1\ i}, \\
&B_\sigma{}^{j_1} = -d_i B_\sigma^{j_1\ i},
\end{aligned} \tag{17}
$$

and $A_\sigma = d_{j_1} B_\sigma{}^{j_1}$. Consequently

$$
\begin{aligned}
A_\sigma &= d_{j_1} B_\sigma{}^{j_1} = -d_{j_1} d_{j_2} B_\sigma^{j_1\ j_2} = -d_{j_1} d_{j_2} \tilde{B}_\sigma^{j_1\ j_2} = d_{j_1} d_{j_2} d_{j_3} B_\sigma^{j_1 j_2\ j_3} \\
&= d_{j_1} d_{j_2} d_{j_3} \tilde{B}_\sigma^{j_1 j_2\ j_3} = \dots = (-1)^{k-1} d_{j_1} d_{j_2} \dots d_{j_{k-1}} d_{j_k} B_\sigma^{j_1 j_2 \dots j_{k-1}\ j_k} \\
&= (-1)^{k-1} d_{j_1} d_{j_2} \dots d_{j_{k-1}} d_{j_k} \tilde{B}_\sigma^{j_1 j_2 \dots j_{k-1}\ j_k} \\
&= \dots = (-1)^{2r-1} d_{j_1} d_{j_2} \dots d_{j_{2r-1}} d_{j_{2r}} B_\sigma^{j_1 j_2 \dots j_{2r-1}\ j_{2r}} \\
&= (-1)^{2r-1} d_{j_1} d_{j_2} \dots d_{j_{2r-1}} d_{j_{2r}} \tilde{B}_\sigma^{j_1 j_2 \dots j_{2r-1}\ j_{2r}} \\
&= (-1)^{2r} d_{j_1} d_{j_2} \dots d_{j_{2r}} d_{j_{2r+1}} B_\sigma^{j_1 j_2 \dots j_{2r}\ j_{2r+1}} \\
&= (-1)^{2r} d_{j_1} d_{j_2} \dots d_{j_{2r}} d_{j_{2r+1}} \tilde{B}_\sigma^{j_1 j_2 \dots j_{2r}\ j_{2r+1}} \\
&= 0,
\end{aligned} \tag{18}
$$

proving that $A_\sigma = 0$ hence $\eta_0 = 0$.

Substituting from this identity to equations (17),

$$\begin{aligned}&\tilde{B}_{\sigma}^{j_1 j_2 \cdots j_{2r}\ j_{2r+1}} = 0, \quad d_i B_{\sigma}^{j_1 j_2 \cdots j_{2r}\ i} = -\tilde{B}_{\sigma}^{j_1 j_2 \cdots j_{2r-1}\ j_{2r}},\\ &d_i B_{\sigma}^{j_1 j_2 \cdots j_{2r-1}\ i} = -\tilde{B}_{\sigma}^{j_1 j_2 \cdots j_{2r-2}\ j_{2r-1}}, \quad \dots, \quad d_i B_{\sigma}^{j_1 j_2\ i} = -\tilde{B}_{\sigma}^{j_1\ j_2},\\ &d_i B_{\sigma}^{j_1\ i} = -B_{\sigma}^{\ j_1}, \quad d_{j_1} B_{\sigma}^{\ j_1} = 0.\end{aligned} \tag{19}$$

Then by Section 3.1, Remark 2 and Section 3.2, Theorem 1, the functions $B_{\sigma}^{\ j_1}, B_{\sigma}^{j_1\ i}, B_{\sigma}^{j_1 j_2\ i}, \dots, B_{\sigma}^{j_1 j_2 \cdots j_{2r-1}\ i}, B_{\sigma}^{j_1 j_2 \cdots j_{2r}\ i}$ depend on the variable x^i only. Then formula (12) implies $p_2 d\eta_1 = 0$, hence, from equation $\eta_0 - d\eta_1 + \eta_2 = 0$, $\eta_2 = 0$. This proves (b).

Note that for any n-form ρ on W^r, the 1-contact component $p_1\rho$ is an n-form on the set W^{r+1}, and since $p_1 d\rho = p_1 dh\rho + p_1 dp_1\rho = dh\rho + p_1 dp_1\rho$, the 1-contact $(n+1)$-form $p_1 dp_1\rho$ is also defined on W^{r+1}. Therefore, the form $I_1 p_1 dp_1\rho$ is defined and is an $(n+1)$-form on W^{2r+1}.

Corollary 1 *The form $I_1 p_1 dp_1\rho$ vanishes identically,*

$$I_1 p_1 dp_1 \rho = 0. \tag{20}$$

Proof We have the identity

$$\begin{aligned}&(\pi^{2r+1,r+1})^* p_1 dp_1\rho\\ &= (\pi^{2r+1,r+1})^*(dp_1\rho - p_2 dp_1\rho - p_3 dp_1\rho - \dots - p_{n+1} dp_1\rho)\\ &= d(\pi^{2r+1,r+1})^* p_1\rho - p_2(\pi^{2r,r+1})^* dp_1\rho\end{aligned} \tag{21}$$

because $p_3 dp_1\rho = 0$, $p_4 dp_1\rho = 0$, $\dots$, $p_{n+1} dp_1\rho = 0$. Comparing this formula with decomposition (5) and using the uniqueness of the component $I_1 p_1 dp_1\rho$ (Theorem 3, (b)) we get identity (20).

Remark 7 If $p_2 d\eta_1$ is ω^{σ}-generated, then $p_2 d\eta_1 = 0$ (see the proof of Theorem 3).

Remark 8 Part (b) of Theorem 3 can alternatively be proved by means of the properties of Lepage forms. Note that the uniqueness condition $\eta_0 - d\eta_1 + \eta_2 = 0$ implies that $\eta_0 = p_1 d\eta_1$; this means, however, that η_1 is a *Lepage form* whose Lagrangian $h\eta_1 = 0$ is the *zero Lagrangian*. Using Section 4.3, Theorem 3 we get $\eta_1 = d\kappa + \mu$, where the form κ is 1-contact and the form μ is of order of contactness ≥ 2. Then, however, $d\eta_1 = d\mu$, which is a form of order or contactness ≥ 2. Equation $\eta_0 - d\eta_1 + \eta_2 = 0$ now implies that $\eta_0 = 0$ because η_0 is 1-contact (and $-d\mu + \eta_2$ is of order of contactness ≥ 2).

Next consider $(n+k)$-forms on W^{r+1} for arbitrary $k \geq 1$. The following result generalizes Theorem 3.

Theorem 4 *Let* $k \geq 1$, *let* η *be a k-contact,* $\pi^{r+1,r}$*-horizontal* $(n+k)$*-form on* W^{r+1}, *expressed in a fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *by*

$$\eta = \sum_{0 \leq k \leq r} \Phi_\sigma^{j_1 j_2 \dots j_k} \wedge \omega_{j_1 j_2 \dots j_k}^\sigma \wedge \omega_0. \tag{22}$$

There exist k-contact ω^σ*-generated* k*-form* $I_k \eta$ *on* V^{2r+1}, *a* $(k-1)$*-contact* $(n+k-1)$*-form* $J_k \eta$ *and an* $(k+1)$*-contact* $(n+k)$*-form* $K_k \eta$, *defined on* V^{2r+1}, *such that*

$$(\pi^{2r+1,r+1})^* \eta = I_k \eta - dJ_k \eta + K_k \eta. \tag{23}$$

(b) *Suppose that we have a decomposition*

$$(\pi^{2r+1,r+1})^* \eta = \eta_0 - d\eta_1 + \eta_2 \tag{24}$$

such that η_0 *is* 1*-contact and* ω^σ*-generated,* η_1 *is* 1*-contact, and* η_2 *is a* 2*-contact form. Then*

$$\eta_0 = I_k \eta. \tag{25}$$

Proof (a) Let $k \geq 1$, let *W* be an open set in *Y*, and let η be a *k*-contact, $(n+k)$-form, defined on some open set W^{r+1}. In a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, on *Y*, η has a unique decomposition

$$\eta = \eta_0 + \eta_1 + \eta_2 + \ldots + \eta_r, \tag{26}$$

where η_0 is the ω^σ-generated component, η_1 includes all $\omega_{j_1}^\sigma$-generated terms, which do not contain any factor ω^σ, η_2 includes all $\omega_{j_1 j_2}^\sigma$-generated terms, which do not contain any factors ω^σ, $\omega_{j_1}^\sigma$, etc.; finally, η_r consists of $\omega_{j_1 j_2 \dots j_r}^\sigma$-generated terms which do not include any factors ω^σ, $\omega_{j_1}^\sigma$, $\omega_{j_1 j_2}^\sigma$, $\dots$, $\omega_{j_1 j_2 \dots j_{r-1}}^\sigma$.

η_r has an expression

$$\eta_r = \Psi_\sigma^{j_1 j_2 \dots j_r} \wedge \omega_{j_1 j_2 \dots j_r}^\sigma \wedge \omega_0 \tag{27}$$

for some $(k-1)$-contact $(k-1)$-forms $\Psi_\sigma^{j_1 j_2 \dots j_r}$, which do not include any factors ω^σ, $\omega_{j_1}^\sigma$, $\omega_{j_1 j_2}^\sigma$, $\dots$, $\omega_{j_1 j_2 \dots j_{r-1}}^\sigma$. Then by (3)

$$\begin{aligned} \eta_r &= -\Psi_\sigma^{j_1 j_2 \dots j_r} \wedge d(\omega_{j_1 j_2 \dots j_{r-1}}^\sigma \wedge \omega_{j_r}) \\ &= (-1)^k d(\Psi_\sigma^{j_1 j_2 \dots j_r} \wedge \omega_{j_1 j_2 \dots j_{r-1}}^\sigma \wedge \omega_{j_r}) - (-1)^k d\Psi_\sigma^{j_1 j_2 \dots j_r} \wedge \omega_{j_1 j_2 \dots j_{r-1}}^\sigma \wedge \omega_{j_r} \\ &= -(-1)^k p_{k-1} d\Psi_\sigma^{j_1 j_2 \dots j_r} \wedge \omega_{j_1 j_2 \dots j_{r-1}}^\sigma \wedge \omega_{j_r} \\ &\quad + (-1)^k d(\Psi_\sigma^{j_1 j_2 \dots j_r} \wedge \omega_{j_1 j_2 \dots j_{r-1}}^\sigma \wedge \omega_{j_r}) \\ &\quad - (-1)^k p_k d\Psi_\sigma^{j_1 j_2 \dots j_r} \wedge \omega_{j_1 j_2 \dots j_{r-1}}^\sigma \wedge \omega_{j_r}. \end{aligned} \tag{28}$$

The term $p_{k-1}d\Psi_\sigma^{j_1j_2\dots j_r}\wedge\omega^\sigma_{j_1j_2\dots j_{r-1}}\wedge\omega_{j_r}$ in this expression is k-contact (and therefore contains the factor $dx^1\wedge dx^2\wedge\ldots\wedge dx^n$), and is generated by the forms $\omega^\sigma_{j_1j_2\dots j_{r-1}}$. Thus, from the definition of the $(k-1)$-component $p_{k-1}d\Psi_\sigma^{j_1j_2\dots j_r}$ it follows that the form $p_{k-1}d\Psi_\sigma^{j_1j_2\dots j_r}\wedge\omega^\sigma_{j_1j_2\dots j_{r-1}}\wedge\omega_{j_r}$ contains the exterior factors $\omega^\nu_{l_1l_2\dots l_{r-1}}$, $\omega^\nu_{l_1l_2\dots l_r}$ and $\omega^\nu_{l_1l_2\dots l_ri_1}$ only. Decomposition (26) now reads

$$
(29)\qquad \begin{aligned}
(\pi^{2r+1,r+1})^*\eta &= \eta_0+\eta_1+\eta_2+\ldots+\eta_{r-2}+\tilde\eta_{r-1}\\
&+(-1)^k d(\Psi_\sigma^{j_1j_2\dots j_r}\wedge\omega^\sigma_{j_1j_2\dots j_{r-1}}\wedge\omega_{j_r})\\
&-(-1)^k p_k d\Psi_\sigma^{j_1j_2\dots j_r}\wedge\omega^\sigma_{j_1j_2\dots j_{r-1}}\wedge\omega_{j_r},
\end{aligned}
$$

where $\tilde\eta_{r-1}$ can be written as

$$
(30)\qquad \begin{aligned}
\tilde\eta_{r-1} &= \eta_{r-1}-(-1)^k p_{k-1}d\Psi_\sigma^{j_1j_2\dots j_r}\wedge\omega^\sigma_{j_1j_2\dots j_{r-1}}\wedge\omega_{j_r}\\
&=\Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge\omega^\sigma_{j_1j_2\dots j_{r-1}}\wedge\omega_0.
\end{aligned}
$$

Then, however,

$$
(31)\qquad \begin{aligned}
\tilde\eta_{r-1} &= \Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge d(\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_{j_{r-1}})\\
&=-(-1)^k p_{k-1}d\Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_{j_{r-1}}\\
&+(-1)^k d(\Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_{j_{r-1}})\\
&-(-1)^k p_k d\Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_{j_{r-1}}.
\end{aligned}
$$

The term $p_{k-1}d\Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_{j_{r-1}}$ in this expression is k-contact, contains the factor ω_0, and is generated by the forms $\omega^\sigma_{j_1j_2\dots j_{r-1}}$. From the definition of the $(k-1)$-component $p_{k-1}d\Psi_\sigma^{j_1j_2\dots j_r}$ it follows that this term contains the exterior factors $\omega^\nu_{l_1l_2\dots l_{r-2}}$, $\omega^\nu_{l_1l_2\dots l_{r-1}}$, $\omega^\nu_{l_1l_2\dots l_r}$, $\omega^\nu_{l_1l_2\dots l_ri_1i_2}$ only. The decomposition (26) (or (29)) now reads

$$
(32)\qquad \begin{aligned}
(\pi^{2r+1,r+1})^*\eta &= \eta_0+\eta_2+\eta_3+\ldots+\eta_{r-3}+\tilde\eta_{r-2}\\
&+(-1)^k d(\Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_{j_{r-1}})\\
&-(-1)^k p_k d\Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_{j_{r-1}}\\
&+(-1)^k(d(\Psi_\sigma^{j_1j_2\dots j_r}\wedge\omega^\sigma_{j_1j_2\dots j_{r-1}}\wedge\omega_{j_r})\\
&-p_k d\Psi_\sigma^{j_1j_2\dots j_r}\wedge\omega^\sigma_{j_1j_2\dots j_{r-1}}\wedge\omega_{j_r}),
\end{aligned}
$$

where

$$
(33)\qquad \begin{aligned}
\tilde\eta_{r-2} &= \eta_{r-2}-(-1)^k p_{k-1}d\Psi_\sigma^{j_1j_2\dots j_{r-1}}\wedge\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_{j_{r-1}}\\
&=\Psi_\sigma^{j_1j_2\dots j_2}\wedge\omega^\sigma_{j_1j_2\dots j_{r-2}}\wedge\omega_0.
\end{aligned}
$$

Continuing in the same way we get after $r-1$ steps

$$(34)\qquad \begin{aligned}(\pi^{2r+1,r+1})^*\eta &= \eta_0+\tilde{\eta}_1+(-1)^k d(\Psi_\sigma^{j_1j_2}\wedge\omega_{j_1}^\sigma\wedge\omega_{j_2})\\ &-(-1)^k p_k d\Psi_\sigma^{j_1j_2}\wedge\omega_{j_1j_2}^\sigma\wedge\omega_{j_2}\\ &+\ldots+(-1)^k d(\Psi_\sigma^{j_1j_2\ldots j_{r-1}}\wedge\omega_{j_1j_2\ldots j_{r-2}}^\sigma\wedge\omega_{j_{r-1}})\\ &-(-1)^k p_k d\Psi_\sigma^{j_1j_2\ldots j_{r-1}}\wedge\omega_{j_1j_2\ldots j_{r-2}}^\sigma\wedge\omega_{j_{r-1}}\\ &+(-1)^k d(\Psi_\sigma^{j_1j_2\ldots j_r}\wedge\omega_{j_1j_2\ldots j_{r-1}}^\sigma\wedge\omega_{j_r})\\ &-(-1)^k p_k d\Psi_\sigma^{j_1j_2\ldots j_r}\wedge\omega_{j_1j_2\ldots j_{r-1}}^\sigma\wedge\omega_{j_r},\end{aligned}$$

where

$$(35)\qquad \begin{aligned}\tilde{\eta}_1 &= \eta_1-(-1)^k p_{k-1}d\Psi_\sigma^{j_1j_2}\wedge\omega_{j_1}^\sigma\wedge\omega_{j_2}\\ &=\Psi_\sigma^{j_1}\wedge\omega_{j_1}^\sigma\wedge\omega_0.\end{aligned}$$

The form $\tilde{\eta}_1$ contains $\omega_{j_1}^\sigma$, $\omega_{j_1j_2}^\sigma$, $\ldots$, $\omega_{j_1j_2\ldots j_r}^\sigma$, $\omega_{j_1j_2\ldots j_ri_1}^\sigma$, $\ldots$, $\omega_{j_1j_2\ldots j_ri_1i_2\ldots i_{r-1}}^\sigma$ but no factor ω^σ. Then

$$(36)\qquad \begin{aligned}\tilde{\eta}_1 &= -\Psi_\sigma^{j_1}\wedge d(\omega^\sigma\wedge\omega_{j_1})\\ &=(-1)^k d(\Psi_\sigma^{j_1}\wedge\omega^\sigma\wedge\omega_{j_1})-(-1)^k d\Psi_\sigma^{j_1}\wedge\omega^\sigma\wedge\omega_{j_1}\\ &=-(-1)^k p_{k-1}d\Psi_\sigma^{j_1}\wedge\omega^\sigma\wedge\omega_{j_1}\\ &+(-1)^k d(\Psi_\sigma^{j_1}\wedge\omega^\sigma\wedge\omega_{j_1})-(-1)^k p_k d\Psi_\sigma^{j_1}\wedge\omega^\sigma\wedge\omega_{j_1},\end{aligned}$$

and

$$(37)\qquad \begin{aligned}&(\pi^{2r+1,r+1})^*\eta\\ &=\eta_0-(-1)^k p_{k-1}d\Psi_\sigma^{j_1}\wedge\omega^\sigma\wedge\omega_{j_1}\\ &-(-1)^{k-1}d(\Psi_\sigma^{j_1}\wedge\omega^\sigma\wedge\omega_{j_1}+\Psi_\sigma^{j_1j_2}\wedge\omega_{j_1}^\sigma\wedge\omega_{j_2}\\ &+\ldots+\Psi_\sigma^{j_1j_2\ldots j_{r-1}}\wedge\omega_{j_1j_2\ldots j_{r-2}}^\sigma\wedge\omega_{j_{r-1}}+\Psi_\sigma^{j_1j_2\ldots j_r}\wedge\omega_{j_1j_2\ldots j_{r-1}}^\sigma\wedge\omega_{j_r})\\ &-(-1)^k(p_k d\Psi_\sigma^{j_1}\wedge\omega^\sigma\wedge\omega_{j_1}+p_k d\Psi_\sigma^{j_1j_2}\wedge\omega_{j_1j_2}^\sigma\wedge\omega_{j_2}\\ &+\ldots+p_k d\Psi_\sigma^{j_1j_2\ldots j_{r-1}}\wedge\omega_{j_1j_2\ldots j_{r-2}}^\sigma\wedge\omega_{j_{r-1}}\\ &+p_k d\Psi_\sigma^{j_1j_2\ldots j_r}\wedge\omega_{j_1j_2\ldots j_{r-1}}^\sigma\wedge\omega_{j_r}).\end{aligned}$$

Summarizing

$$(38)\qquad (\pi^{2r+1,r+1})^*\eta = I_k\eta - dJ_k\eta + K_k\eta,$$

where

$$
\begin{aligned}
&I_k\eta = \eta_0 - (-1)^k p_{k-1} d\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1},\\
&J_k\eta = (-1)^{k-1}(\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1} + \Psi_\sigma^{j_1 j_2} \wedge \omega_{j_1}^\sigma \wedge \omega_{j_2}\\
(39)\qquad &\quad + \ldots + \Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}} + \Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r}),\\
&K_k\eta = (-1)^{k-1} p_{k+1}(d\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1} + d\Psi_\sigma^{j_1 j_2} \wedge \omega_{j_1 j_2}^\sigma \wedge \omega_{j_2}\\
&\quad + \ldots + d\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}} + d\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r}).
\end{aligned}
$$

(b) To prove uniqueness of the component $I_k\eta$, we adapt to the decomposition (38) a classical integration approach. It is sufficient to consider the case when

$$(40)\qquad I_k\eta - dJ_k\eta + K_k\eta = 0,$$

and to prove that $I_k\eta = 0$. Choose π-vertical vector fields Ξ_1, Ξ_2, …, Ξ_k on Y and consider the pull-back of this n-form by the r-jet prolongation of a section γ of Y, $J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k\eta$. Clearly, the pull-back $J^{2r+1}\gamma^*$ annihilates contact n-forms. Since the Lie derivative of a contact form by a π-vertical vector field is a contact form (Section 2.5, Theorem 9, (d)), hence

$$
\begin{aligned}
&J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k\eta\\
&= J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} dJ_k\eta + J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} K_k\eta\\
(41)\qquad &= J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_2} (\partial_{\Xi_1} J_k\eta - di_{\Xi_1} J_k\eta) + J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} K_k\eta\\
&= -J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_3} i_{\Xi_2} di_{\Xi_1} J_k\eta
\end{aligned}
$$

because the forms $i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_2} \partial_{\Xi_1} J_k\eta$ and $i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} K_k\eta$ are contact. Repeating this step,

$$
\begin{aligned}
&J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k\eta\\
&= -J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_4} i_{\Xi_3} \partial_{\Xi_2} i_{\Xi_1} J_k\eta + J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_4} i_{\Xi_3} di_{\Xi_2} i_{\Xi_1} J_k\eta\\
&= J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_4} i_{\Xi_3} di_{\Xi_2} i_{\Xi_1} J_k\eta\\
(42)\qquad &= \ldots = (-1)^p J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_{p+2}} i_{\Xi_{p+1}} di_{\Xi_p} i_{\Xi_{p-1}} \ldots i_{\Xi_2} i_{\Xi_1} J_k\eta\\
&= \ldots = (-1)^k J^{2r+1}\gamma^* di_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_2} i_{\Xi_1} J_k\eta\\
&= (-1)^k dJ^{2r+1}\gamma^* i_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_2} i_{\Xi_1} J_k\eta.
\end{aligned}
$$

Thus, integrating over an arbitrary piece $\Omega \subset X$ with boundary $\partial\Omega$,

$$
\begin{aligned}
&\int_\Omega J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k\eta = (-1)^k \int_\Omega dJ^{2r+1}\gamma^* i_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_2} i_{\Xi_1} J_k\eta\\
(43)\qquad &= (-1)^k \int_{\partial\Omega} J^{2r+1}\gamma^* i_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_2} i_{\Xi_1} J_k\eta.
\end{aligned}
$$

This identity holds for all π-vertical vector fields Ξ_1, Ξ_2, $\ldots$, Ξ_k, but on the other hand, the right-hand side depends on their values along the boundary $\partial\Omega$ only. Replace the vector field Ξ_1 with $f\Xi_1$, where *f* is a function, defined on a neighbourhood of Ω, vanishing along $\partial\Omega$. Then we get

$$(44) \qquad \int_\Omega J^{2r+1}\gamma^* i_{\Xi_k}\ldots i_{\Xi_2} i_{f\Xi_1} I_k\eta = (-1)^k \int_\Omega f J^{2r+1}\gamma^* i_{\Xi_k}\ldots i_{\Xi_2} i_{\Xi_1} I_k\eta = 0.$$

Since the function *f* is arbitrary in the interior of the pieace Ω, this is only possible when the integrand satisfies $J^{2r+1}\gamma^* i_{\Xi_k}\ldots i_{\Xi_2} i_{\Xi_1} I_k\eta = 0$. Finally, the section γ is also arbitrary; since through every point of the domain of definition of the form $i_{\Xi_k}\ldots i_{\Xi_2} i_{\Xi_1} I_k\eta$ passes the $(2r+1)$ -jet prolongation $J^{2r+1}\gamma$ of γ, therefore

$$(45) \qquad I_k\eta = 0.$$

This proves that the form $I_k\eta$ in formula (38) is defined uniquely by the assumptions of Theorem 4.

Corollary 1 extends to arbitrary forms as follows.

Corollary 2 *For any integer* $k \ge 1$ *and any* $(n+k-1)$*-form* ρ *on* W^r *the form* $I_k p_k dp_k \rho$ *vanishes,*

$$(46) \qquad I_k p_k dp_k \rho = 0$$

Proof Using the canonical decomposition of the form $dp_1\rho$ we get the identity

$$(47) \qquad \begin{aligned} &(\pi^{2r+1,r+1})^* p_k dp_k \rho \\ &= (\pi^{2r+1,r+1})^*(dp_k\rho - p_{k+1}dp_k\rho - p_{k+2}dp_k\rho - \ldots - p_{k+n}dp_k\rho) \\ &= d(\pi^{2r+1,r+1})^* p_k\rho - p_{k+1}(\pi^{2r,r+1})^* dp_k\rho \end{aligned}$$

because the components satisfy the conditions $p_{k+1}dp_k\rho = 0$, $p_{k+2}dp_k\rho = 0$, $\ldots$, $p_{k+n}dp_k\rho = 0$. Comparing this formula with decomposition (23) and using the uniqueness of the component $I_1 p_1 dp_1 \rho$ (Theorem 4, (b)) we get identity (46).

Our next aim is to determine an explicit formula for the component $I_k\eta$ of a form η by a geometric construction; the result will be proved on a successive application of Theorem 3.

Theorem 5 (a) *Let* η *be a* 2*-contact,* $\pi^{r+1,r}$*-horizontal* $(n+2)$*-form on the set* W^{r+1}. *Then for any* π*-vertical vector fields* Ξ_1 *and* Ξ_2

$$(48) \qquad i_{\Xi_2} i_{\Xi_1} I_2\eta = \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1}\eta - i_{\Xi_1} I_1 i_{\Xi_2}\eta).$$

(b) *Let* $k \geq 2$ *and let* η *be a k-contact,* $\pi^{r+1,r}$*-horizontal* $(n+k)$*-form defined on* W^{r+1}. *Then for any* π*-vertical vector fields* Ξ_1, Ξ_2. …, Ξ_k

(49)
$$\begin{aligned} i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_1} I_k \eta = \frac{1}{k}(& i_{\Xi_k} i_{\Xi_{k-1}} \dots i_{\Xi_2} I_{k-1} i_{\Xi_1} \eta - i_{\Xi_k} i_{\Xi_{k-1}} \dots i_{\Xi_3} i_{\Xi_1} I_{k-1} i_{\Xi_2} \eta \\ & - i_{\Xi_k} i_{\Xi_{k-1}} \dots i_{\Xi_4} i_{\Xi_2} i_{\Xi_1} I_{k-1} i_{\Xi_3} \eta - \dots - i_{\Xi_{k-1}} \dots i_{\Xi_2} i_{\Xi_1} I_{k-1} i_{\Xi_k} \eta). \end{aligned}$$

Proof (a) From the decompositions

(50)
$$(\pi^{2r+1,r+1})^* i_{\Xi_1} \eta = \begin{cases} i_{\Xi_1} I_2 \eta - i_{\Xi_1} dJ_2 \eta + i_{\Xi_1} K_2 \eta \\ I_1 i_{\Xi_1} \eta - dJ_1 i_{\Xi_1} \eta + K_1 i_{\Xi_1} \eta \end{cases}$$

it follows that

(51)
$$\begin{aligned} & i_{\Xi_2} i_{\Xi_1} I_2 \eta - \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1} \eta - i_{\Xi_1} I_1 i_{\Xi_2} \eta) = i_{\Xi_2} i_{\Xi_1} dJ_2 \eta - i_{\Xi_2} i_{\Xi_1} K_2 \eta \\ & \quad + \frac{1}{2}(-i_{\Xi_2} dJ_1 i_{\Xi_1} \eta + i_{\Xi_2} K_1 i_{\Xi_1} \eta - i_{\Xi_1} dJ_2 i_{\Xi_1} \eta + i_{\Xi_1} K_1 i_{\Xi_2} \eta). \end{aligned}$$

Using the properties of the Lie derivative operator (see Appendix 5), we can write

(52)
$$\begin{aligned} & i_{\Xi_2} i_{\Xi_1} dJ_2 \eta + \frac{1}{2}(-i_{\Xi_2} dJ_1 i_{\Xi_1} \eta - i_{\Xi_1} dJ_2 i_{\Xi_1} \eta) \\ & \quad = i_{\Xi_2} \partial_{\Xi_1} J_2 \eta - i_{\Xi_2} d i_{\Xi_1} J_2 \eta \\ & \quad + \frac{1}{2}(-\partial_{\Xi_2} J_1 i_{\Xi_1} \eta + d i_{\Xi_2} J_1 i_{\Xi_1} \eta - \partial_{\Xi_1} J_2 i_{\Xi_1} \eta + d i_{\Xi_1} J_2 i_{\Xi_1} \eta) \\ & \quad = i_{\Xi_2} \partial_{\Xi_1} J_2 \eta - \partial_{\Xi_2} i_{\Xi_1} J_2 \eta - d i_{\Xi_2} i_{\Xi_1} J_2 \eta \\ & \quad + \frac{1}{2}(-\partial_{\Xi_2} J_1 i_{\Xi_1} \eta + d i_{\Xi_2} J_1 i_{\Xi_1} \eta - \partial_{\Xi_1} J_2 i_{\Xi_1} \eta + d i_{\Xi_1} J_2 i_{\Xi_1} \eta), \end{aligned}$$

thus

(53)
$$\begin{aligned} & i_{\Xi_2} i_{\Xi_1} I_2 \eta - \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1} \eta - i_{\Xi_1} I_1 i_{\Xi_2} \eta) \\ & \quad = i_{\Xi_2} \partial_{\Xi_1} J_2 \eta - \partial_{\Xi_2} i_{\Xi_1} J_2 \eta - d i_{\Xi_2} i_{\Xi_1} J_2 \eta \\ & \quad + \frac{1}{2}(-\partial_{\Xi_2} J_1 i_{\Xi_1} \eta + d i_{\Xi_2} J_1 i_{\Xi_1} \eta - \partial_{\Xi_1} J_2 i_{\Xi_1} \eta + d i_{\Xi_1} J_2 i_{\Xi_1} \eta) \\ & \quad - i_{\Xi_2} i_{\Xi_1} K_2 \eta + \frac{1}{2}(i_{\Xi_2} K_1 i_{\Xi_1} \eta + i_{\Xi_1} K_1 i_{\Xi_2} \eta). \end{aligned}$$

Now integrating

$$\int_{\Omega} J^{2r+1}\gamma^*\left(i_{\Xi_2}i_{\Xi_1}I_2\eta - \frac{1}{2}(i_{\Xi_2}I_1 i_{\Xi_1}\eta - i_{\Xi_1}I_1 i_{\Xi_2}\eta\right)$$

(54)

$$= \int_{\partial\Omega} J^{2r+1}\gamma^*\left(-i_{\Xi_2}i_{\Xi_1}J_2\eta + \frac{1}{2}(i_{\Xi_2}J_1 i_{\Xi_1}\eta + i_{\Xi_1}J_2 i_{\Xi_1}\eta)\right).$$

To conclude that this condition implies

(55) $$i_{\Xi_2}i_{\Xi_1}I_2\eta - \frac{1}{2}(i_{\Xi_2}I_1 i_{\Xi_1}\eta - i_{\Xi_1}I_1 i_{\Xi_2}\eta = 0$$

we proceed as in the proof of Theorem 4.

(b) To complete the proof we apply elementary induction.

According to Theorem 5, formula (24) defines a mapping I_k from the Abelian group of k-contact $(n+k)$-forms on W^{r+1} to $\pi^{2r+1,0}$-horizontal $(n+k)$-forms on W^{r+1}. I_k is clearly a morphism of Abelian groups.

Theorem 6 (a) *Condition* $I_k\eta = 0$ *is satisfied if and only if* η *is a strongly contact form.*

(b) *The mapping* I_k *satisfies*

(56) $$I_k \circ I_k = I_k$$

Proof (a) This follows from Theorem 4, (b).

(b) To prove (b), write $(\pi^{2r+1,r+1})^*\eta = I_k\eta - dJ_k\eta + K_k\eta$. Then

(57)
$$\begin{aligned}&(\pi^{2(2r+1+,2r+2)})^*(\pi^{2r+1,r+1})^*\eta\\ &= I_k(\pi^{2r+1,r+1})^*\eta - dJ_k(\pi^{2r+1,r+1})^*\eta + K_k(\pi^{2r+1,r+1})^*\eta\end{aligned}$$

and from the properties of the pull-back operation

(58)
$$\begin{aligned}&(\pi^{2(2r+1+,2r+2)})^*(\pi^{2r+1,r+1})^*\eta\\ &= (\pi^{2(2r+1+,2r+2)})^*(I_k\eta - dJ_k\eta + K_k\eta)\\ &= (\pi^{2(2r+1+,2r+2)})^*I_k\eta - d(\pi^{2(2r+1+,2r+2)})^*J_k\eta + (\pi^{2(2r+1+,2r+2)})^*K_k\eta\\ &= I_kI_k\eta - dJ_kI_k\eta + K_kI_k\eta\\ &- d(\pi^{2(2r+1+,2r+2)})^*J_k\eta + (\pi^{2(2r+1+,2r+2)})^*K_k\eta.\end{aligned}$$

Comparing (57) with (58) and using the uniqueness of these decompositions (Theorem 4 (b)) we get formula (56).

Remark 9 Property (a) characterizes the *kernel* of the mapping I_k. Its *image* consists of the k-contact, ω^σ-generated $(n+k)$-forms ε on W^{2r+1} for which the equation

(59) $$\varepsilon = I_k\eta$$

has a solution η. The corresponding *integrability conditions*, which should be satisfied by ε, are determined by the structure of the mapping I_k, and can be studied by means of the formal divergence equations (Chapter 3).

Remark 10 The uniqueness of the component $I_k\eta$ in the decomposition (24) means that the pull-back of the vector space of *k*-contact $(n+k)$-forms on W^{r+1} is isomorphic with the direct sum of two subspaces of the vector space of *k*-contact $(n+k)$-forms on W^{2r+1}, one of which is the subspace of strongly contact forms.

We conclude this section by extending the decomposition (24), defined for *k*-contact $(n+k)$-forms on W^{r+1}, to any forms $\rho\in\Omega^r_{n+k}W$. Substituting in formula (21) $\eta=p_k\rho$ we get

$$\begin{aligned}(60)\qquad &(\pi^{2r+1,r})^*\rho\\ &=(\pi^{2r+1,r+1})^*p_k\rho+(\pi^{2r+1,r+1})^*(p_{k+1}\rho+p_{k+2}\rho+\ldots+p_{k+n}\rho)\\ &=(\pi^{2r+1,r+1})^*(I_kp_k\rho-dJ_kp_k\rho+K_kp_k\rho)\\ &+(\pi^{2r+1,r+1})^*(p_{k+1}\rho+p_{k+2}\rho+\ldots+p_{k+n}\rho)\\ &=(\pi^{2r+1,r+1})^*I_kp_k\rho-d(\pi^{2r+1,r+1})^*J_kp_k\rho\\ &+(\pi^{2r+1,r+1})^*K_kp_k\rho+(\pi^{2r+1,r+1})^*(p_{k+1}\rho+p_{k+2}\rho+\ldots+p_{k+n}\rho).\end{aligned}$$

Therefore, setting

$$\begin{aligned}(61)\qquad &\mathcal{I}_k\rho=(\pi^{2r+1,r+1})^*I_kp_k\rho,\\ &\mathcal{J}_k\rho=(\pi^{2r+1,r+1})^*J_kp_k\rho,\\ &\mathcal{K}_k\rho=(\pi^{2r+1,r+1})^*K_kp_k\rho+(\pi^{2r+1,r+1})^*(p_{k+1}\rho+p_{k+2}\rho+\ldots+p_{k+n}\rho)\end{aligned}$$

we get the decomposition

$$(62)\qquad (\pi^{2r+1,r})^*\rho=\mathcal{I}_k\rho-d\mathcal{J}_k\rho+\mathcal{K}_k\rho.$$

According to Theorem 4, this formula defines a mapping $\rho\to\mathcal{I}_k\rho$ of the Abelian group of $\Omega^r_{n+k}W$ of $(n+k)$-forms, defined on W^r, into the Abelian group $\Omega^{2r+1}_{n+k}W$ of $(n+k)$-forms on W^{2r+1}.

The following lemma summarizes elementary properties of the mapping $\Omega^r_{n+k}W\ni\rho\to\mathcal{I}_k\rho\in\Omega^{2r+1}_{n+k}W$. As before, to simplify notation, we omit obvious pull-back operations on differential forms with respect to the canonical jet projections $\pi^{r,s}:J^rY\to J^sY$.

Theorem 7 (a) *The mapping* $\rho\to\mathcal{I}_k\rho$ *of the Abelian group* $\Omega^r_{n+k}W$ *into* $\Omega^{2r+1}_{n+k}W$ *is a morphism of Abelian groups.*

(b) *The kernel of the mapping* $\mathcal{I}$ *is the Abelian group of strongly contact forms* $\Theta^r_{n+k}W$, *and its image is isomorphic with the quotient group* $\Omega^r_{n+k}W/\Theta^r_{n+k}W$.

(c) *For every* $\rho \in \Omega^r_{n+k} W$ *the mapping* $\mathscr{I}$ *satisfies*

(63) $$\mathscr{I}_k \mathscr{I}_k \rho = \mathscr{I}_k \rho.$$

Proof (a) Obvious.

(b) If $\mathscr{I}_k \rho = 0$, then by Lemma 3, ρ is *strongly contact*, thus ρ belongs to the Abelian group $\Theta^r_{n+k} W$.

(c) Applying the pull-back operation to both sides of formula (62) and using the properties $\mathscr{J}_k \mathscr{I}_k \rho = 0$ and $\mathscr{K}_k \mathscr{I}_k \rho = 0$ of the mappings $\mathscr{I}_k$, $\mathscr{J}_k$ and $\mathscr{K}_k$,

(64) $$\begin{aligned} &(\pi^{2(2r+1)+1,2r+1})^* (\pi^{2r+1,r})^* \rho \\ &= \mathscr{I}_k (\pi^{2r+1,r})^* \rho - d\mathscr{J}_k (\pi^{2r+1,r})^* \rho + \mathscr{K}_k (\pi^{2r+1,r})^* \rho, \end{aligned}$$

and

(65) $$\begin{aligned} &(\pi^{2(2r+1)+1,2r+1})^* (\pi^{2r+1,r})^* \rho \\ &= (\pi^{2(2r+1)+1,2r+1})^* (\mathscr{I}_k \rho - d\mathscr{J}_k \rho + \mathscr{K}_k \rho) \\ &= (\pi^{2(2r+1)+1,2r+1})^* \mathscr{I}_k \rho - d(\pi^{2(2r+1)+1,2r+1})^* \mathscr{J}_k \rho \\ &+ (\pi^{2(2r+1)+1,2r+1})^* \mathscr{K}_k \rho \\ &= \mathscr{I}_k \mathscr{I}_k \rho - d\mathscr{J}_k \mathscr{I}_k \rho + \mathscr{K}_k \mathscr{I}_k \rho \\ &- d(\pi^{2(2r+1)+1,2r+1})^* \mathscr{J}_k \rho + (\pi^{2(2r+1)+1,2r+1})^* \mathscr{K}_k \rho \\ &= \mathscr{I}_k \mathscr{I}_k \rho - d(\pi^{2(2r+1)+1,2r+1})^* \mathscr{J}_k \rho + (\pi^{2(2r+1)+1,2r+1})^* \mathscr{K}_k \rho. \end{aligned}$$

Comparing these formulas and using the uniqueness of the decompositions we get assertion (c).

We call the Abelian group morphism $\Omega^r_{n+k} W \ni \rho \to \mathscr{I}_k \rho \in \Omega^{2r+1}_{n+k} W$ the k-th *variational projector*. To simplify notation we sometimes write just $\mathscr{I}$ instead of $\mathscr{I}_k$.

8.4 The Euler-Lagrange morphisms

Consider the variational sequence (3), Section 8.2

(1) $$0 \longrightarrow \mathbf{R}_Y \longrightarrow \Omega^r_0 \longrightarrow \Omega^r_1 / \Theta^r_1 \longrightarrow \Omega^r_2 / \Theta^r_2 \longrightarrow \Omega^r_3 / \Theta^r_3 \longrightarrow$$

Note that by definition of the horizontalization morphism $h : \Omega^r_p W \to \Omega^{r+1}_p W$ the equivalence relation on the Abelian group $\Omega^r_p W$ associated with the subgroup of contact forms $\Theta^r_p W \subset \Omega^r_p W$ coincides with the equivalence relation defined by h. Similarly, Part Theorem 7, Section 8.3 shows that for each $k \geq 1$ the equivalence relation on the Abelian group $\Omega^r_{n+k} W$, associated with the subgroup $\Theta^r_{n+k}W \subset \Omega^r_{n+k}W$ of strongly contact forms, coincides with the equivalence relation induced by the *variational projectors* $\mathscr{I}_k$. Thus the diagram, defining the variational sequence, can be expressed as

(2)
$$
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
 & & 0 \arrow[r] \arrow[d] & \Theta^r_1 \arrow[r, "d"] \arrow[d] & \Theta^r_2 \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \Theta^r_n \arrow[d] \\
0 \arrow[r] & \mathbf{R}_Y \arrow[r] & \Omega^r_0 \arrow[r, "d"] \arrow[dr] & \Omega^r_1 \arrow[r, "d"] \arrow[d, "h"] & \Omega^r_2 \arrow[r, "d"] \arrow[d, "h"] & \dots \arrow[r, "d"] & \Omega^r_n \arrow[d, "h"] \\
 & & & h^r_1 \arrow[r] \arrow[d] & h^r_2 \arrow[r] \arrow[d] & \dots \arrow[r] & h^r_n \arrow[d] \\
 & & & 0 & 0 & & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & & \\
{} \arrow[r] & \Theta^r_{n+1} \arrow[r] \arrow[d] & \Theta^r_{n+2} \arrow[r] \arrow[d] & \dots \arrow[r] & \Theta^r_M \arrow[r] \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & \Omega^r_{n+1} \arrow[r, "d"] \arrow[d, "h"] & \Omega^r_{n+2} \arrow[r, "d"] \arrow[d, "h"] & \dots \arrow[r, "d"] & \Omega^r_M \arrow[r, "d"] \arrow[d, "h"] & \Omega^r_{M+1} \arrow[r, "d"] & {} \\
{} \arrow[r] & \mathscr{I}^r_{n+1} \arrow[r] \arrow[d] & \mathscr{I}^r_{n+2} \arrow[r] \arrow[d] & \dots \arrow[r] & \mathscr{I}^r_M \arrow[d] \arrow[ur] & & \\
 & 0 & 0 & & 0 & &
\end{tikzcd}
$$

The corresponding representation of the variational sequence (1) is

(3)
$$
\begin{aligned}
&0 \longrightarrow \mathbf{R}_Y \longrightarrow \Omega^r_0 \xrightarrow{E_0} h^r_1 \xrightarrow{E_1} h^r_2 \xrightarrow{E_2} \dots \xrightarrow{E_{n-1}} h^r_n \\
&\xrightarrow{E_n} \mathscr{I}^r_1 \xrightarrow{E_{n+1}} \mathscr{I}^r_2 \xrightarrow{E_{n+2}} \dots
\end{aligned}
$$

The Abelian group morphisms E_k in this sequence will be called the *Euler-Lagrange morphisms*. Our task in this section will be to determine the structure of the morphisms E_k. The formulas we derive establish *explicit* correspondence between the morphisms E_k and basic concepts of the calculus of variations on fibred manifolds such as the Euler-Lagrange mapping and the Helmholtz mappings, etc. The following two theorems give us a way to calculate the chart expressions of these morphisms E_k.

Theorem 8 *The Euler-Lagrange morphisms in the variational sequence* (3) *can be expressed as*

(4) $$E_k h\rho = \begin{cases} hd\rho, & \rho \in \Omega_k^r W, \quad 0 \le k \le n-1, \\ I_1 dh\rho, & \rho \in \Omega_n^r W, \quad k = n, \end{cases}$$

and

(5) $$E_{n+k}\mathscr{I}_k \rho = I_{k+1} dp_k \rho, \quad \rho \in \Omega_{n+k}^r W, \quad k \ge 1.$$

Proof If $\rho \in \Omega_n^r W$, then $E_n h\rho = \mathscr{I}_1 d\rho = I_1 p_1 d\rho = I_1 p_1 dh\rho + I_1 p_1 dp_1 \rho$. Thus, by Section 8.3, Corollary 1,

(6) $$E_n h\rho = I_1 dh\rho.$$

If $\rho \in \Omega_{n+k}^r W$, where $k \ge 1$, then

(7) $$\begin{aligned} E_{n+k}\mathscr{I}_k \rho &= \mathscr{I}_{k+1} d\rho = I_{k+1} p_{k+1} d\rho \\ &= I_{k+1} p_{k+1} dp_k \rho + I_{k+1} p_{k+1} dp_{k+1} \rho \end{aligned}$$

hence, by Corollary 2, (46)

(8) $$E_{n+k}\mathscr{I}_k \rho = I_{k+1} dp_k \rho.$$

Theorem 9 *Let* (V,ψ), $\psi = (x^i, y^\sigma)$, *be a fibred chart on Y.*
(a) *If* $f \in \Omega_0^r V$, *then*

(9) $$E_0 f = d_i f \cdot dx^i.$$

(b) *Let* $1 \le k \le n-1$ *and let* $h\rho \in \mathscr{I}_j^r V$ *be a class. Then if* $h\rho$ *is expressed by*

(10) $$h\rho = \rho_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k},$$

then the image $E_k h\rho$ *is given by*

(11) $$E_k h\rho = d_{i_0} \rho_{i_1 i_2 \dots i_k} \wedge dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}.$$

Proof We prove assertion (b). According to the trace decomposition theorem (Section 2.2, Theorem 3), a form $\rho \in \Omega_k^r V$ has an expression

(12) $$\rho = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|I| = r-1} d(\omega_J^\sigma \wedge \Psi_\sigma^J) + \rho_0,$$

where ρ_0 is the traceless component of ρ and Φ_σ^J, Ψ_σ^J are some forms. Since the morphism *h* annihilates the contact forms ω_J^σ and $d\omega_I^\sigma$, ρ_0 has an expression

$$\begin{aligned}
\rho_0 &= A_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} \\
&+ A^{J_1}_{\sigma_1 \ i_2 i_3 \dots i_k} dy^{\sigma_1}_{J_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_k} \\
&+ A^{J_1 \ J_2}_{\sigma_1 \ \sigma_2 \ i_3 i_4 \dots i_k} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_k} \\
&+ \dots + A^{J_1 \ J_2 \ \dots \ J_{k-1}}_{\sigma_1 \ \sigma_2 \dots \sigma_{k-1} \ i_k} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \dots \wedge dy^{\sigma_{j-1}}_{J_{j-1}} \wedge dx^{i_k} \\
&+ A^{J_1 \ J_2 \ \dots \ J_k}_{\sigma_1 \ \sigma_2 \dots \sigma_k} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \dots \wedge dy^{\sigma_k}_{J_k},
\end{aligned} \tag{13}$$

where the coefficients $A^{J_1 \ J_2 \ \dots \ J_s}_{\sigma_1 \ \sigma_2 \dots \sigma_s \ i_{s+1} i_{s+2} \dots i_k}$ are *traceless*. Thus, any class $h\rho$ is expressed as the k-form

$$h\rho = \rho_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}, \tag{14}$$

where

$$\begin{aligned}
\rho_{i_1 i_2 \dots i_k} &= (A_{i_1 i_2 \dots i_k} + A^{J_1}_{\sigma_1 \ i_2 i_3 \dots i_k} y^{\sigma_1}_{J_1 i_1} + A^{J_1 \ J_2}_{\sigma_1 \ \sigma_2 \ i_3 i_4 \dots i_k} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \\
&+ \dots + A^{J_1 \ J_2 \ \dots \ J_{k-1}}_{\sigma_1 \ \sigma_2 \dots \sigma_{k-1} \ i_k} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \dots y^{\sigma_{k-1}}_{J_{k-1} i_{k-1}} + A^{J_1 \ J_2 \ \dots \ J_k}_{\sigma_1 \ \sigma_2 \dots \sigma_k} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \dots y^{\sigma_k}_{J_k i_k}) \\
&\operatorname{Alt}(i_1 i_2 \dots i_k).
\end{aligned} \tag{15}$$

The class $hd\rho$ of $d\rho$ is then given by

$$hd\rho = d_{i_0} \rho_{i_1 i_2 \dots i_k} \wedge dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}. \tag{16}$$

Clearly, $hd\rho$ is defined on V^{r+1}.

Remark 11 If $k = n-1$, then since $\varepsilon^{l i_1 i_2 \dots i_{n-1}} \omega_l = dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}}$, the class $h\rho = \rho_{i_1 i_2 \dots i_{n-1}} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}}$ (10) can be written as $h\rho = \rho^l \omega_l$. Then the image $E_{n-1} h\rho$ is expressed as

$$\begin{aligned}
E_{n-1} h\rho &= d_{i_0} \rho_{i_1 i_2 \dots i_{n-1}} \wedge dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}} \\
&= d_{i_0} \rho_{i_1 i_2 \dots i_{n-1}} \varepsilon^{i_0 i_1 i_2 \dots i_{n-1}} \wedge \omega_0 = d_i \rho^i \cdot \omega_0 \\
&= hdh\rho,
\end{aligned} \tag{17}$$

where $d_i \rho^i$ is the *formal divergence* of the family ρ^l. Thus, the Euler-Lagrange morphism E_{n-1} can also be expressed in short as $E_{n-1} = hd$.

Now we study the Euler-Lagrange morphisms E_{n+k} for $k \geq 0$. We derive explicit formulas for $k = 0,1$; in subsequent sections, these formulas will be compared with basic variational concepts, which appeared already in the previous sections devoted to the calculus of variations.

In order to study the morphism E_n, we find the chart expression of the class $h\rho$ of a form $\rho \in \Omega^r_n V$. According to the trace decomposition theorem (Section 2.2, Theorem 3), ρ has an expression

(18) $$\rho = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|I|=r-1} d\omega_I^\sigma \wedge \Psi_\sigma^I + \rho_0,$$

where ρ_0 is the traceless component of ρ. Clearly $h\rho = h\rho_0$. But ρ_0 has an expression

(19) $$\begin{aligned} \rho_0 &= A_{i_1 i_2 \dots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n} \\ &+ A_{\sigma_1}^{J_1}{}_{i_2 i_3 \dots i_n} dy_{J_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n} \\ &+ A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \dots i_n} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_n} \\ &+ \dots + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-1}}^{J_{n-1}}{}_{i_n} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge \dots \wedge dy_{J_{n-1}}^{\sigma_{n-1}} \wedge dx^{i_n} \\ &+ A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_n}^{J_n} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge \dots \wedge dy_{J_n}^{\sigma_n}, \end{aligned}$$

where the summation indices satisfy $|J_1| = |J_2| = \dots = |J_{n+1}| = r$, and the coefficients $A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_s}^{J_s}{}_{i_{s+1} i_{s+2} \dots i_n}$ are *traceless*. Thus, any class $h\rho$ can be expressed as the *n*-form

(20) $$h\rho = \rho_{i_1 i_2 \dots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n},$$

where

(21) $$\begin{aligned} \rho_{i_1 i_2 \dots i_n} &= A_{i_1 i_2 \dots i_n} + A_{\sigma_1}^{J_1}{}_{i_2 i_3 \dots i_n} y_{J_1 i_1}^{\sigma_1} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2}{}_{i_3 i_4 \dots i_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 i_2}^{\sigma_2} \\ &+ \dots + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_{n-1}}^{J_{n-1}}{}_{i_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 i_2}^{\sigma_2} \dots y_{J_{n-1} i_{n-1}}^{\sigma_{n-1}} + A_{\sigma_1}^{J_1}{}_{\sigma_2}^{J_2} \dots {}_{\sigma_n}^{J_n} y_{J_1 i_1}^{\sigma_1} y_{J_2 j_2}^{\sigma_2} \dots y_{J_n i_n}^{\sigma_n} \\ &\mathrm{Alt}(i_1 i_2 \dots i_n). \end{aligned}$$

Thus $h\rho$ can also be characterized as

(22) $$h\rho = \mathscr{L}\omega_0,$$

where $\mathscr{L} = \varepsilon^{i_1 i_2 \dots i_n} \rho_{i_1 i_2 \dots i_n}$ (Section 4.1, (12)).

Remark 12 In variational terminology, the class $\lambda = h\rho$ is the *Lagrangian*, associated with the *n*-form ρ, that is, an element of the module $\Omega_{n,X}^{r+1} V$ of π^{r+1}-horizontal forms, defined on $V^{r+1} \subset J^{r+1}Y$. The function $\mathscr{L}$, characterizing the class $h\rho$ locally, is the *Lagrange function*, associated with $h\rho$ (and with the given fibred chart, cf. Section 4.1).

We can now prove the following theorem.

Theorem 10 *If the class $h\rho$ of an n-form $\rho \in \Omega_n^r V$ is expressed as*

(23) $$h\rho = \mathscr{L}\omega_0,$$

then

(24) $$E_n h\rho = \left(\frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \dots d_{j_s} \frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2 \dots j_s}} \right) \omega^\sigma \wedge \omega_0 .$$

Proof The class $E_n h\rho$ is defined to be $E_n h\rho = \mathscr{I}_1 d\rho = I_1 dh\rho$ (Theorem 8, (4)). Since

(25) $$dh\rho = d\mathscr{L} \wedge \omega_0 = \sum_{0 \le |J| \le r+1} \frac{\partial \mathscr{L}}{\partial y^\nu_J} \omega^\nu_J \wedge \omega_0 ,$$

we have

(26) $$I_1 dh\rho = \left(\frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \dots d_{j_s} \frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2 \dots j_s}} \right) \omega^\sigma \wedge \omega_0$$

(Section 8.3, Theorem 3).

Now we find the chart expression of the class $\mathscr{I}_1\rho$ of a form $\rho \in \Omega^r_{n+1}V$. Writing $p_1\rho$ as

(27) $$p_1\rho = \sum_{0 \le s \le r} A_\sigma^{j_1 j_2 \dots j_s} \omega^\sigma_{j_1 j_2 \dots j_s} \wedge \omega_0 ,$$

we get, according to Section 8.3, Theorem 3,

(28) $$\mathscr{I}_1\rho = I_1 p_1 \rho = \varepsilon_\sigma \omega^\sigma \wedge \omega_0 ,$$

where

(29) $$\varepsilon_\sigma = A_\sigma + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \dots d_{j_s} A_\sigma^{j_1 j_2 \dots j_s} .$$

Remark 13 According to formula (28) the class $\varepsilon = \mathscr{I}_1\rho$ of a form $\rho \in \Omega^r_{n+1}V$ is an element of the Abelian group $\Omega^{2r+1}_{n+1,Y}V$ of $\pi^{2r+1,0}$-horizontal forms, defined on the set $V^{2r+1} \subset J^{2r+1}Y$; in the variational theory, elements of the Abelian groups $\Omega^{2r+1}_{n+1,Y}V$ are the *source forms* on the fibred manifold Y (cf. Section 4.9).

Theorem 11 *If the class* $\mathscr{I}_1\rho$ *of an* $(n+1)$*-form* $\rho \in \Omega^r_{n+1}V$ *is expressed as*

(30) $$\mathscr{I}_1\rho = \varepsilon_\sigma \omega^\sigma \wedge \omega_0 ,$$

then

(31) $$E_{n+1}\mathscr{I}_1\rho = \frac{1}{2} \sum_{0 \le k \le r} H_{\sigma}{}^{j_1 j_2 \dots j_k}_{\nu}(\varepsilon) \omega^\nu_{j_1 j_2 \dots j_k} \wedge \omega^\sigma \wedge \omega_0 ,$$

where

$$(32)\qquad \begin{aligned} H_{\sigma\nu}^{j_1 j_2 \ldots j_k}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{j_1 j_2 \ldots j_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \\ &- \sum_{l=k+1}^{s} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \ldots j_k p_{k+1} p_{k+2} \ldots p_l}}. \end{aligned}$$

Proof The image $E_{n+1}\mathscr{I}_1\rho$ is defined by the equation $E_{n+1}\mathscr{I}_1\rho = \mathscr{I}_2 d\rho$. However, if $\mathscr{I}_1\rho$ is defined on V^s, then $\mathscr{I}_1\mathscr{I}_1\rho = (\pi^{2s+1,s})^*\mathscr{I}_1\rho$ (Section 8.3, Theorem 7), thus, the image can also be calculated from the equation

$$(33)\qquad \begin{aligned} E_{n+1}\mathscr{I}_1\mathscr{I}_1\rho &= E_{n+1}(\pi^{2s+1,s})^*\mathscr{I}_1\rho = (\pi^{2s+1,s})^* E_{n+1}\mathscr{I}_1\rho \\ &= (\pi^{2s+1,s})^*\mathscr{I}_2 d\rho. \end{aligned}$$

We apply this formula to the representation (30) of the class of ρ. Setting $\mathscr{I}_1\rho = \varepsilon$ we have

$$(34)\qquad E_{n+1}\mathscr{I}_1\varepsilon = \mathscr{I}_2 d\varepsilon = I_2 p_2 d\varepsilon = I_2 d\varepsilon .$$

This expression can be easily determined by means of the mapping I_2, defined by the condition

$$(35)\qquad i_{\Xi_2} i_{\Xi_1} I_2 d\varepsilon = \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1} d\varepsilon - i_{\Xi_1} I_1 i_{\Xi_2} d\varepsilon),$$

where Ξ_1 and Ξ_2 are any π-vertical vector fields (Section 8.3, Theorem 5). From this expression we conclude that

$$(36)\qquad I_2 d\varepsilon = \frac{1}{2}\sum_{0\le k\le r} H_{\sigma\ \nu}^{\ j_1 j_2 \ldots j_k}(\varepsilon)\omega^\nu_{j_1 j_2 \ldots j_k} \wedge \omega^\sigma \wedge \omega_0,$$

where the components $H_{\sigma\ \nu}^{\ j_1 j_2 \ldots j_k}(\varepsilon)$ are given by (32).

Consider the variational sequence (3). Theorem 10 shows that the morphism E_n in this Abelian sheaf sequence is exactly the *Euler-Lagrange mapping* of the calculus of variations (cf. Section 4.5). The mappings E_{n-1} and E_{n+1} also admit a direct variational interpretation (Remark 11, Theorem 11). In the subsequent sections we consider the part of the variational sequence Var^r_Y including E_n,

$$(37)\qquad \ldots \longrightarrow h^r_{n-1} \xrightarrow{E_{n-1}} h^r_n \xrightarrow{E_n} \mathscr{I}^r_1 \xrightarrow{E_{n+1}} \mathscr{I}^r_2 \longrightarrow \ldots$$

and the corresponding part of the associated *complex of global sections* $\mathsf{Var}^r_Y Y$ (Section 8.2, (5)),

$$(38)\qquad \ldots \longrightarrow h^r_{n-1}Y \xrightarrow{E_{n-1}} h^r_n Y \xrightarrow{E_n} \mathscr{I}^r_1 Y \xrightarrow{E_{n+1}} \mathscr{I}^r_2 Y \longrightarrow \ldots$$

Since by Section 8.2, Theorem 2, the cohomology groups $H^k(\mathit{Var}^r_Y Y)$ of (38) and the cohomology groups $H^k(Y, \mathbf{R}_Y)$ are isomorphic, this fact allows us to complete the properties of the *kernel* and the *image* of the Euler-Lagrange mapping by their *global* characteristics. The results bind together properties of the *variationally trivial Lagrangians*, and *variational source forms* with the *topology* of the underlying fibred manifold *Y* in terms of its (De Rham) cohomology groups.

Remark 14 In general, to determine the De Rham cohomology groups of a smooth manifold or a smooth fibred manifold is a hard problem; for basic theory of the De Rham cohomology we refer to Lee [L] and Warner [W]; in simple cases one can apply the *Künneth theorem* (Bott and Tu [BT]).

The following are well-known standard examples of manifolds and their cohomology groups:

(a) *Euclidean spaces* $\mathbf{R}^n$: $H^k\mathbf{R}^n = 0$ for all $k \geq 1$.

(b) *Spheres* S^n:

$$(38)\qquad H^k S^n = \begin{cases} \mathbf{R}, & k = 0, n \\ 0, & 0 < k < n \end{cases}$$

(c) *Punctured Euclidean spaces* (complements of one-point sets $\{x\}$ in $\mathbf{R}^n$), *complements of closed balls* $B \subset \mathbf{R}^n$:

$$(39)\qquad H^k(\mathbf{R}^n \setminus \{x\}) = H^k(\mathbf{R}^n \setminus B) = H^k S^{n-1}.$$

(d) Tori $T^k = S^1 \times S^1 \times \ldots \times S^1$ (k factors S^1):

$$(40)\qquad H^k T^n = \mathbf{R}^{\binom{n}{k}}.$$

(e) *Möbius band*:

$$(41)\qquad H^k M = H^k S^1.$$

(f) $H^0 X \times Y) = \mathbf{R}$

(h) *Cartesian products* (*Künneth theorem*), $k \geq 0$:

$$(42)\qquad H^k(X \times Y) = \bigoplus_{r+s=k} H^r X \otimes H^s Y$$

(h) *Disjoint unions* (M_1, M_2 disjoint):

$$(43)\qquad H^k(M_1 \cup M_2) = H^k M_1 \oplus H^k M_2.$$

8.5 Variationally trivial Lagrangians

Let W be an open set in Y. Recall that a Lagrangian $\lambda \in h_n^r W$ is called *variationally trivial*, if its Euler-Lagrange form vanishes,

(1) $\qquad E_n \lambda = 0.$

This condition can be considered as an *equation* for the unknown n-form λ. Our main objective in this section is to summarize previous local results on the solutions of this equation and to complete these results by a theorem on global solutions.

The mapping E_n is the Euler-Lagrange morphism in the complex of global sections

(2) $\qquad \ldots \longrightarrow h_{n-1}^r W \xrightarrow{E_{n-1}} h_n^r W \xrightarrow{E_n} \mathscr{I}_1^r W \xrightarrow{E_{n+1}} \mathscr{I}_2^r W \longrightarrow \ldots$

and equation (1) has the meaning of the *integrability condition* for the corresponding equation for an unknown $(n-1)$-form η,

(3) $\qquad \lambda = E_{n-1}\eta.$

Thus, since $E_{n-1}\eta$ is defined to be $hd\eta$, equation (3) can also be written as

(4) $\qquad \lambda = hd\eta.$

Integrability condition (1), representing exactness of the sheaf variational sequence, ensures existence of *local* solutions, defined on chart neighbourhoods in the set W. According to Theorem 9, Section 4.8, the following conditions are equivalent:

(a) λ *is variationally trivial*.

(b) *For any fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *such that* $V \subset W$, *there exist functions* $g^i : V^r \to \mathbf{R}$, *such that on* V^r, λ *is expressible as* $\lambda = \mathscr{L}\omega_0$, *where*

(5) $\qquad \mathscr{L} = d_i g^i.$

(c) *For every fibred chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *such that* $V \subset W$, *there exists an* $(n-1)$*-form* $\mu \in \Omega_{n-1}^{r-1} V$ *such that on* V^r

(6) $\qquad \lambda = hd\mu.$

A question still remains open, namely, under what conditions there exists a solution μ, defined *globally* over W or, in other words, when a given Lagrangian, locally expressible as “divergence”, can be expressed as a “divergence” globally.

The following theorem is an immediate consequence of the properties of the complex of global sections (38), Section 8.4).

Theorem 12 *Let Y be a fibred manifold over an n-dimensional manifold X, such that* $H^nY = 0$. *Let* $\lambda \in h_n^r Y$ *be a Lagrangian. Then the following conditions are equivalent:*

(a) λ *is variationally trivial.*

(b) *There exists an* $(n-1)$*-form* $\mu \in \Omega_{n-1}^{r-1} Y$ *such that on* $J^r Y$

(7) $$\lambda = hd\mu.$$

Proof 1. We show that (a) implies (b). In view of Section 4.8, Theorem 9, only existence of μ, defined globally on J^rY, needs proof. But by Section 8.2, Theorem 2 the cohomology groups $H^k(\mathit{Var}_Y^r Y)$ are isomorphic with the De Rham cohomology groups $H^k(Y, \mathbf{R}_Y)$; thus, condition $H^nY = 0$ implies $H^n(\mathit{Var}_Y^r Y) = 0$ proving existence of μ.

2. The converse is obvious.

On analogy with the De Rham sequence, a variationally trivial Lagrangian can also be called *variationally closed*. A variationally closed Lagrangian $\lambda \in h_n^r W$ is called *variationally exact*, if $\lambda = hd\mu$ for some $\mu \in h_{n-1}^r W$. Theorem 12 then says that if $H^nY = 0$, then every variationally closed Lagrangian is variationally exact.

In the following examples we refer to the cohomology groups given in Section 8.4, Remark 13.

Examples (Obstructions for variational triviality) 1. If the fibred manifold *Y* is the Cartesian product $\mathbf{R}^n \times \mathbf{R}^m$, endowed with the first canonical projection, then every variationally trivial Lagrangian on *Y* is variationally exact.

2. Let $Y = S^3$, and consider S^3 as a fibred manifold over S^2 (the *Hopf fibration*). Then $H^3S^3 = \mathbf{R} \neq 0$, therefore, a variationally trivial Lagrangian on J^rS^3 need not be closed.

3. If $Y = \mathbf{R}^n \times Q$, then the Künneth theorem yields $H^n(\mathbf{R}^n \times Q) = H^nQ$. Thus, if $H^nQ = 0$, then variational triviality always implies variational exactness. If for example Q is an *n*-sphere S^n, punctured Euclidean space $\mathbf{R}^{n+1} \setminus \{0\}$, or the *k*-torus T^k, then variational triviality does not imply variational exactness.

8.6 Global inverse problem of the calculus of variations

Let W be an open set in *Y*. Recall that a source form $\varepsilon \in \mathscr{F}_1^r W$ is said to be *variational*, if there exists a Lagrangian $\lambda \in h_n^r W$ such that its Euler-Lagrange form $E_n\lambda$ coincides with ε,

(1) $$\varepsilon = E_n\lambda.$$

ε is said to be *locally variational*, if there exists an atlas on Y, consisting of fibred charts, such that for each chart (V,ψ), $\psi = (x^i, y^\sigma)$, from this atlas, the restriction of ε to V^s is variational.

The mapping E_n in formula (1) is the *Euler-Lagrange morphism* in the complex of global sections

(2) $$\ldots \longrightarrow h^r_{n-1}W \xrightarrow{E_{n-1}} h^r_n W \xrightarrow{E_n} \mathscr{I}^r_1 W \xrightarrow{E_{n+1}} \mathscr{I}^r_2 W \longrightarrow \ldots$$

which determines the *integrability condition* for equation (1)

(3) $$E_{n+1}\varepsilon = 0.$$

The problem to determine conditions ensuring existence of the Lagrangian λ, and to determine λ as a function of the source form ε, is the *inverse problem of the calculus of variations*.

If the source form ε is expressed in the form

(4) $$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0,$$

then equation (1) is expressed as a system of partial differential equations

(5) $$\varepsilon_\sigma = \frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \ldots d_{j_s} \frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2 \ldots j_s}}, \quad 1 \le \sigma \le m,$$

for an unknown function $\mathscr{L} = \mathscr{L}(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$. Integrability condition (3) is then of the form

(6) $$E_{n+1}\varepsilon = \frac{1}{2} \sum_{0 \le k \le r} H_{\sigma\ \nu}^{\ j_1 j_2 \ldots j_k}(\varepsilon)\omega^\nu_{j_1 j_2 \ldots j_k} \wedge \omega^\sigma \wedge \omega_0 = 0,$$

where $H_{\sigma\ \nu}^{\ j_1 j_2 \ldots j_k}(\varepsilon)$ are the *Helmholtz expressions* (Section 8.4, Theorem 11); thus if s is the *order* of the functions ε_σ, the integrability condition reads

(7) $$\begin{gathered} \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{j_1 j_2 \ldots j_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \\ - \sum_{l=k+1}^{s} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \ldots j_k p_{k+1} p_{k+2} \ldots p_l}} = 0, \\ 1 \le \sigma, \nu \le m, \quad 0 \le k \le s, \quad 1 \le j_1, j_2, \ldots, j_k \le n. \end{gathered}$$

Integrability condition (7) ensures existence of local solutions λ of equation (1), or, which is the same, solutions $\mathscr{L}$ of the system (5); solutions are given explicitly by the *Vainberg-Tonti Lagrangians*

(8) $$\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0,$$

where

$$
\begin{aligned}
(9)\qquad & \mathscr{L}_{\varepsilon}(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_s}) \\
& = y^\sigma \int_0^1 \varepsilon_\sigma(x^i, ty^\nu, ty^\nu_{j_1}, ty^\nu_{j_1 j_2}, \dots, ty^\nu_{j_1 j_2 \dots j_s}) dt
\end{aligned}
$$

(Section 4.9, (3), Section 4.10, Theorem 12 and Theorem 13, Section 8.4, Theorem 11).

In this section we complete these results by a theorem ensuring existence of *global* solutions of equation (1), where the open set $W \subset Y$ coincides with the fibred manifold *Y*.

The following result completes properties of the source forms by establishing a topological condition ensuring that local variationality implies (global) variationality.

Theorem 13 *Let Y be a fibred manifold with n-dimensional base X, such that* $H^{n+1}Y = 0$. *Let* $\varepsilon \in \mathscr{S}^r_1 W$ *be a source form. Then the following conditions are equivalent:*

(a) ε *is locally variational.*

(b) ε *is variational.*

Proof This assertion is an immediate consequence of the existence of an isomorphism between the cohomology groups $H^k(\mathit{Var}^r_Y Y)$ and the De Rham cohomology groups $H^k(Y, \mathbf{R}_Y)$ (Section 8.2, Theorem 2); thus, condition $H^{n+1}Y = 0$ implies $H^{n+1}(\mathit{Var}^r_Y Y) = 0$ as required.

Remark 14 The meaning of Theorem 13 can be rephrased as follows. First, it states that in order to ensure that a given source form ε is *locally variational*, one should verify that its components satisfy the *Helmholtz conditions* (7); and second, if in addition the $(n+1)$-st cohomology group $H^{n+1}Y$ of the underlying fibred manifold vanishes, then ε is automatically variational.

Examples (Obstructions for global variationality) 4. If $Y = \mathbf{R} \times M$, where *M* is the Möbius band, then $H^2 Y = 0$ hence local variationality always implies variationality.

5. If $Y = S^1 \times M$, where S^1 is the circle and *M* is the Möbius band, then $H^2 Y = H^2(S^1 \times M) = H^1 S^1 \oplus H^1 M = \mathbf{R} \oplus \mathbf{R} = \mathbf{R}^2$. Thus in general, local variationality does not imply variationality.

6. If the 3-sphere S^3 is considered as a fibred manifold over S^2 (Hopf fibration), then since $H^3 S^3 = \mathbf{R} \neq 0$, local variationality does not necessarily imply global variationality.

7. If $k \geq l$ then the *k*-torus T^k can be fibred over the *l*-torus T^l by means of the Cartesian projection. Since $H^{l+1} T^k \neq 0$, we have obstructions agains global variationality.

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