Preface

The *global variational geometry* as introduced in this book is a branch of mathematics, devoted to extremal problems on the frontiers of differential geometry, global analysis, the calculus of variations, and mathematical physics. Its subject is, generally speaking, a geometric structure consisting of a smooth manifold endowed with a differential form.

More specifically, by a *variational structure*, or a *Lagrange structure*, we mean in this book a pair (Y,ρ), where Y is a *smooth fibred manifold* over an *n-dimensional* base manifold X and ρ a *differential n-form*, defined on the r-jet prolongation J^rY of Y. The forms ρ, satisfying a *horizontality condition*, are called the *Lagrangians*. The *variational functional*, associated with (Y,ρ), is the real-valued function $\Gamma_\Omega(\pi) \ni \gamma \to \rho_\Omega(\gamma) = \int J^r\gamma^*\rho \in \mathbf{R}$, where $\Gamma_\Omega(\pi)$ is the set of sections of Y over a compact set $\Omega \subset X$, $J^r\gamma$ is the r-jet prolongation of a section γ, and $J^r\gamma^*\rho$ is an n-form on X, the pull-back of ρ by $J^r\gamma$.

Over the past few decades the subject has developed to a self contained theory of extremals of *integral variational functionals* for sections of fibred manifolds, invariance theory under transformations of underlying geometric structures, and differential equations related to them. The *variational methods* for the study of these functionals extended the corresponding notions of global analysis such as differentiation and integration theory on manifolds. Innovations appeared in the developments of *topological methods* needed for a deeper understanding of the global character of variational concepts such as equations for extremals and conservation laws. It has also become clear that the higher-order variational functionals could hardly be studied without innovations in the multi-linear algebra, namely, in the decomposition theory of tensors and differential forms by the trace operation.

The resulting theory differs in many aspects from the classical approach to variational problems: The underlying *Euclidean spaces*, are replaced by smooth manifolds and fibred spaces, the classical Lagrange functions and their variations are replaced by Lagrange differential forms and their Lie derivatives, etc. Within the classical setting, a (first order) variational structure is a pair (Y,λ), where $Y = J^1(\mathbf{R}^n \times \mathbf{R}^m)$ is the 1-jet prolongation of the product $\mathbf{R}^n \times \mathbf{R}^m$ of Euclidean spaces, and in the canonical coordinates, $\lambda = L dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, where $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{nm} \to \mathbf{R}$ is a Lagrange function, depending on n independent variables, m dependent variables, and nm partial derivatives of dependent variables.

Basic geometric ideas allowing us to globalize the classical calculus of variations come from the concepts of E. Cartan [C] in the calculus of variations of simple integrals, and especially from the work of Lepage (see e.g. [Le]). Further developments after Cartan and Lepage have led to a deeper understanding of the structure and geometric nature of general variational procedures and their compatibility with manifold structures. Main contributors to the global theory are Dedecker [D1] (geometric approach to the calculus of variations, regularity), Garcia [G] (Poincare-Cartan form, invariant geometric operations, connections), Goldschmidt and Sternberg [GS] (Cartan form, vector-valued Euler-Lagrange form, Hamilton theory, Hamilton-Jacobi equation), Krupka [K13], [K1] (Lepage forms, higher-order variational functionals, infinitesimal first variation formula, Euler-Lagrange form, invariance), and Trautman [Tr1], [Tr2] (invariance of Lagrange systems, Noether's theory).

This book covers the subjects that are considered as basic in the classical monographs on the (local) calculus of variations on Euclidean spaces: variational functionals and their variations, the (first) variation formula, extremals and the Euler-Lagrange equations, invariance and conservation laws. We study these topics within the framework of much broader underlying structures, smooth manifolds. This requires, in particular, a systematic use of analysis and topology of manifolds. In addition, new questions appear in this framework such as for instance *global existence* of the notions, constructed in charts. We also study global properties of the Euler-Lagrange mapping; to this purpose two chapters devoted to sheaves and the variational sequence theory are included. It is however obvious that these themes do not reflect the foundations of the global variational theory completely. Further comprehensive expositions including applications, based on modern geometric methods in the calculus of variations on manifolds, can be found in the monographs Giachetta, Mangiarotti and Sardanashvily [GMS1], [GMS2], De Leon and Rodrigues [LR], Mangiarotti and Modugno [MM], and Mei Fengxiang and Wu Huibin [MW]. For orientation in recent research in these fields we refer to Krupka and Saunders [KS].

The text of the book requires a solid background in topology, multilinear algebra, and differential and integral calculus on manifolds; to this purpose we recommend the monograph Lee [L]. Essentials of the classical and modern calculus of variations can be found e.g. in Gelfand and Fomin [GF], Jost and Li-Jost [JL], and in the handbook Krupka and Saunders [KS], where differential forms are considered. For the theory of jets, natural bundles and applications we refer to original works of Ehresmann [E] and to the books Kolar, Michor and Slovak [KMS], Krupka and Janyska [KJ], and

Saunders [S]. We also need an elementary sheaf theory; our exposition extends a chapter of the book Wells [We]. For reference, some theorems and formulas are collected in the Appendix. We should especially mention the section devoted to the *trace decomposition theory* on real vector spaces, which is needed for the decomposition of differential forms on jet manifolds (Krupka [K15]); although the trace decomposition is an elementary topic, it is difficult to find an adequate reference in classical and contemporary algebraic literature.

Chapter 1 covers fundamentals of fibred manifolds and their jet prolongations. The usual topics related to the jet structure, such as the horizontalization morphism, jet prolongations of sections and morphism of fibred manifolds, and prolongations of vector fields are introduced. It should be pointed out that the vector fields and their jet prolongations represent a geometric, coordinate-free construction, replacing in the global variational theory the classical "variations of functions", and "induced variations" of their derivatives.

Chapter 2 studies differential forms on the jet prolongations of fibred manifolds. The *contact forms* are introduced, generating a *differential ideal* of the exterior algebra, and the corresponding decompositions of forms are studied. It is also shown that the *trace operation*, acting on the components of forms, leads to a decomposition related to the exterior derivative of forms. The meaning of the structure theorems for the global variational theory, explained in the subsequent chapters, consists in their variational interpretation; in different situations the decompositions lead to the Lagrangian forms, the source forms, the Helmholtz forms, etc.

Chapter 3 is devoted to the *formal divergence equations* on jet manifolds, a specific topic that needs independent exposition. It is proved that the integrability of these equations is equivalent with the vanishing of the Euler-Lagrange operator.

The objective of Chapters 4 – 6 is to study the behaviour of the variational functional $\Gamma_\Omega(\pi) \ni \gamma \to \rho_\Omega(\gamma) = \int J^r\gamma^* \rho \in \mathbf{R}$ with respect to the variable γ. But in general, the domain of definition $\Gamma_\Omega(\pi)$ has *no* natural algebraic and topological structures; this fact prevents an immediate application of the methods of the differentiation theory in topological vector spaces, based on the concept of the derivative of a mapping. However, even when *no* topology on $\Gamma_\Omega(\pi)$ has been introduced, the *geometric*, or *variational* method to investigate the functional ρ_Ω can still be used: we can always vary (deform) each section $\gamma \in \Gamma_\Omega(\pi)$ within the set $\Gamma_\Omega(\pi)$, and study the induced variations (deformations) of the value $\rho_\Omega(\gamma)$.

The key notions in **Chapter 4** are the *variational derivative*, *Lepage form*, the *first variation formula*, *Euler-Lagrange form*, *trivial Lagrangian*, *source form, Vainberg-Tonti Lagrangian*, and the *inverse problem of the calculus of variations* and the *Helmholtz expressions*.

The exposition begins with the description of variations of sections of the fibred manifold *Y*, considered as vector fields, and the *induced variations* of the variational functional $\int J^r\gamma^* \rho$. It turns out in this geometric setting that the induced variations are naturally characterized by the *Lie derivative* of ρ. An immediate consequence of this observation is that one can study the functional ρ_Ω by means of the differential calculus of forms and vector fields on the underlying jet manifold.

Next we introduce the fundamental concept of the global variational theory on fibred manifolds, a *Lepage form*. We prove that to any variational structure (Y,ρ) there always exists an *n*-form Θ_ρ with the following two properties: first, the form Θ_ρ defines the same integral variational functional as the form ρ, that is, the identity $J^r\gamma^*\rho = J^r\gamma^*\Theta_\rho$ holds for all sections γ of the fibred manifold Y, and second, the exterior derivative $d\Theta_\rho$ defines *equations for the extremals*, thus, γ is an extremal if and only if $d\Theta_\rho$ vanishes along $J^r\gamma$. Any form Θ_ρ is called a *Lepage equivalent* of the form ρ.

As a basic consequence of the existence of Lepage equivalents we derive a geometric, coordinate-free analogue of the classical (integral) first variation formula – the *infinitesimal first variation formula*, which is essentially the Lie derivative formula for the form Θ_ρ with respect to the vector fields defining the induced variations. The *infinitesimal first variation formula* becomes a main tool for further investigation of extremals and symmetries of the functional. It should also be noted that the geometric structure of the formula admits immediate extensions to *second* and *higher* variations.

We may say that these two properties defining Θ_ρ explain the meaning of the *first* and *second Lepage congruences*, considered by Lepage and Dedecker in their study of the classical variational calculus for submanifolds (cf. Dedecker [D1]).

The exterior derivative $d\Theta_\rho$ splits in two terms, one of them, characterizing extremals, is a (globally well-defined) differential form, the *Euler-Lagrange form*; its components in a fibred chart are the well-known *Euler-Lagrange expressions*. The corresponding system of partial differential equations, *Euler-Lagrange equations*, are then related to each fibred chart. Solving these equations requires their analysis in any concrete case from the local and global viewpoints.

Next we study in Chapter 4 the structure of the *Euler-Lagrange mapping*, assigning to a Lagrangian its Euler-Lagrange form. Since the Euler-Lagrange mapping is a morphism of Abelian groups of differential forms on the underlying jet spaces, its basic characteristics include descriptions of its *kernel* and its *image*. We describe these spaces by their *local* properties.

The *kernel* consists of *variationally trivial* Lagrangians – the Lagrangians whose Euler-Lagrange forms vanish identically. These Lagrangians are characterized in terms of the exterior derivative operator *d*; their local structure corresponds with the classical *divergence expressions*. The *global structure* depends on the topology of the underlying fibred manifold *Y*, and is studied in Chapter 8.

The problem of how to characterize the *image* of the Euler-Lagrange mapping is known as the *inverse problem of the calculus of variations*. Its simple coordinate version for systems of partial differential equations consists in searching for conditions when the given equations coincide with the

Euler-Lagrange equations of some Lagrangian. On a fibred manifold, the inverse problem is formulated for a *source form*, defined on $J^r Y$; it is required that the *components* of the source form coincide with the Euler-Lagrange expressions of a Lagrangian. We find the obstructions for variationality of source forms by means of the Lagrangians of Vainberg-Tonti type, constructed by a fibred homotopy operator, and used for the first time by Vainberg [87]. The resulting theorem gives the necessary and sufficient local variationality conditions in terms of the *Helmholtz expression* (cf. Anderson and Duchamp [AD] and Krupka [K8], [K11]).

Chapter 5 is devoted to variational structures whose Lagrangians, or Euler-Lagrange forms, admit some invariance transformations. The *invariance transformations* are defined naturally as the transformations preserving a given differential form; this immediately leads to criteria for a vector field to be a *generator* of these transformations. Then we prove a generalization of the *Noether's theorem* for a given variational structure (Y,ρ), relating the generators of invariance transformations of ρ with the existence of *conservation laws* for the solutions of the system of Euler-Lagrange equations. The theory extends the well-known classical results on invariance and conservation laws originally formulated for multiple-integral variational problems in Euclidean spaces (Noether [N]).

It should be noted that the invariance theorems for variational structures as stated in this book become comparatively simple (compare with Olver [O1], where a complete classical approach is given). The reason can be found in the fundamental concepts of the theory of variational structures – differential forms, for which invariance theorems are formulated. To explain the basic ideas, consider a manifold *Y* of dimension *p* endowed with a differential *n*-form ρ . Then for any vector field ξ on *Y*, the *Lie derivative* $\partial_\xi \rho$ can be expressed by the *Cartan's formula* $\partial_\xi \rho = i_\xi d\rho + di_\xi \rho$, where i_ξ is the *contraction* of ρ by the vector field by ξ and *d* is the *exterior derivative*. Then for any mapping $f : X \to Y$, where *X* is a manifold of dimension *n*, the Lie derivative satisfies $f^*\partial_\xi \rho = f^* i_\xi d\rho + df^* i_\xi \rho$. Thus, if ρ is *invariant* with respect to ξ , that is, $\partial_\xi \rho = 0$, we have $f^* i_\xi d\rho + df^* i_\xi \rho = 0$. If in addition *f* satisfies the equation $f^* i_\xi d\rho = 0$, then *f* necessarily satisfies the *conservation law equation* $df^* i_\xi \rho = 0$ (Noether's theorem). Similar conservation law theorems for variational structures on jet manifolds are proved along the same lines.

In **Chapter 6** we consider a few examples of *natural* variational structures as introduced in Krupka [K10] (for natural variational principles on Riemannian manifolds see Anderson [A1]). Main purpose is to establish basic (global) structures and find the corresponding Lepage forms. The *Hilbert variational functional* for the metric fields on a manifold (Hilbert [H]) and a variational functional for connections are briefly discussed. The approach should be compared with the standard formulation of the variational principles of the general relativity and other field theories. Clearly, these examples as well as many others whose role are variational principles of physics need a more complex and more detailed study.

As mentioned above, the theory of variational structures gives rise to the *Euler-Lagrange mapping*, which assigns to an *n*-form λ , a *Lagrangian*, an $(n+1)$ -form E_λ , the *Euler-Lagrange form* associated with λ . Its definition results from the properties of the exterior derivative operator *d*, an appropriate canonical decomposition of underlying spaces of forms, and from the concept of a Lepage form (cf. Krupka [K1]). On this basis we easily come to the basic observation that the Euler-Lagrange mapping can be included in a differential sequence of Abelian sheaves as one of its arrows. We proceed to introduce the sequence and the associated complex of global sections, and to study on this basis *global properties* of the Euler-Lagrange mapping.

To this purpose we first explain in **Chapter 7** elements of the sheaf theory (see e.g. Wells [We]). Attension is paid to those theorems, which are needed for the variational structures; complete proofs of these theorems are included. In particular, the formulation and proof of the *abstract De Rham theorem* is given.

The variational geometry is devoted to geometric, coordinate-independent properties of ρ_Ω . In particular, the geometric problems include the study of *critical points* (or *extremals*) of the variational functionals; their *maxima* and *minima*, where a topology on $\Gamma_\Omega(\pi)$ is needed, are not considered. Many other typical geometric problems are connected with various kinds of symmetries of the variational functionals and the corresponding equations for the extremals. The problem of restricting a given functional defined, say, on a Euclidean space, to a submanifold (the *constraint submanifold*) is obviously included in this framework.

It should be pointed out that the geometric variational theory completely covers the problems, related with the variational principles in *physical field theory* and *geometric mechanics*, where concrete underlying geometric structures and variational functionals are considered.

Chapter 8 is devoted to the *variational sequence* of order *r* for a fibred manifold *Y*. Its construction has *no a priori* relations with the theory of variational structures. The sequence is established on the observation that the *De Rham sequence* of differential forms on the *r*-jet prolongation J^rY has a remarkable *subsequence*, defined by the *contact forms*; the variational sequence is then defined to be the *quotient sheaf sequence* of the De Rham sheaf sequence (see Krupka [K19]).

With the obvious definition of the quotient groups, we denote the variational sequence as $0 \to \mathbf{R}_Y \to \Omega_0^r \to \Omega_1^r / \Theta_1^r \to \Omega_2^r / \Theta_2^r \to \Omega_3^r / \Theta_3^r \to \dots$. Its properties relevant to the calculus of variations can be divided into two parts:

(a) *Local properties*, represented by theorems on the structure of the classes of forms in the quotient sequence and morphisms between these quotient groups:

- the *classes* $[\rho]$ *of n-forms* $\rho \in \Omega_n^r$, where *n* is the dimension of the base of the base *X* of the fibred manifold *Y*, can canonically be identified with *Lagrangians* for the fibred manifold *Y*,

- the classes $[\rho]$ of $(n+1)$ -forms $\rho \in \Omega_{n+1}^r$ can canonically be identified with the *source forms*,

- the quotient morphism $E_n : \Omega_n^r / \Theta_n^r \to \Omega_{n+1}^r / \Theta_{n+1}^r$ is exactly the *Euler-Lagrange mapping* of the calculus of variations,

- the quotient morphism $E_{n+1} : \Omega_{n+1}^r / \Theta_{n+1}^r \to \Omega_{n+2}^r / \Theta_{n+2}^r$ is exactly the *Helmholtz mapping* of the calculus of variations.

All these classes and morphisms are described *explicitly* in fibred charts; their expressions coincide with the corresponding expressions given in Chapter 4. Thus, the variational sequence allows us to *rediscover* basic variational concepts from abstract structure constructions on the jet manifolds J^rY .

(b) *Global properties*, represented by the theorem on the cohomology of the *complex of global sections* of the variational sequence; this implies, on the basis of the De Rham theorem that:

- there exists an isomorphism between the cohomology groups of the complex of global sections and the De Rham cohomology groups,

- the obstructions for global variational triviality of Lagrangians lie in the cohomology group H^nY , where $n = \dim X$,

- the obstructions for global variationality of source forms lie in the cohomology group $H^{n+1}Y$.

We also provide a list of manifolds *Y* and its cohomology groups, which allows us to decide whether local variational triviality of a Lagrangian, resp. local variationality of a source form, necessarily implies its global triviality, resp. global variationality.

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