Differential Geometry and Its Applications
Proc. Conf., Opava (Czech Republic), August 27–31, 2001
Silesian University, Opava, 2001, 447–459

Autoparallel variational description of the free relativistic top third order dynamics[1]

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Abstract. A second order variational description of the autoparallel curves of some differential-geometric connection for the third order Mathisson's "new mechanics" of a relativistic free spinning particle is suggested starting from general requirements of invariance and "variationality".

Keywords. Lagrangian, higher-order connection, variationality, invariance, Ostrohrads'kyj mechanics, classical spin, relativistic top.

MS classification. 49N45, 58E30, 70G45, 70G65, 70H50 83A05.

1. Introduction

In 1937 M. Mathisson, in the article named "Neue Mechanik materieller Systeme", see [5], introduced a third order differential equation to describe the motion of quasi-classical relativistic particle with inner angular momentum given by a skew-symmetric tensor $S^{\alpha\beta}$:

$$m_0 \frac{Du^\alpha}{d\tau} = S^{\alpha\beta} \frac{D^2 u_\beta}{d\tau^2} - \frac{1}{2} R^\alpha_{\beta\gamma\delta} u^\beta S^{\gamma\delta}, \tag{1}$$

where the velocity four-vector $u^\alpha, \alpha \in \overline{(0,3)}$ is subject to the usual constraint $u^\alpha u_\alpha = 1$. Equation (1) in fact was considered by Mathisson under the assumption of later well-known "Pirani auxiliary condition"

$$u_\beta S^{\alpha\beta} = 0, \tag{2}$$

[1] This paper is in final form and no version of it will be submitted for publication elsewhere. Research supported by Grants MSM:J10/98:192400002 of the Ministry of Education, Youth and Sports, and GAČR 201/00/0724 of the Grant Agency of the Czech Republic.

which fixes one possible way of choosing the point of reference within the tube of world lines followed by different points of extended object with dipole angular momentum $S^{\alpha\beta}$. This way or that, one may pose the following question: what geometry is best suited for the description of physical particles with complicated internal structure? In presence of the gravitational field such geometry of course will incorporate the curvature tensor, but the other question arises then to invent a local model for such a future geometry. And with this approach in mind, we start with the pseudo-Euclidean space, endowed not only with the usual structure of geodesic straight lines, but also with some other structure, the autoparallel curves of which would satisfy also the unparametrized version of Mathisson's equation (1) with zero curvature tensor $R^{\alpha}_{\beta\gamma\delta}$.

The constraint (2) suggests the idea to introduce the spin four-vector

$$s_\delta = \frac{1}{2\|\boldsymbol{u}\|}\varepsilon_{\alpha\beta\gamma\delta}u^\alpha S^{\beta\gamma},$$

and it was proved in [6] and published in [7] that in terms of this spin vector the Mathisson equation (1) is equivalent to the following one (we put $R^{\alpha}_{\beta\gamma\delta} = 0$):

$$\varepsilon_{\alpha\beta\gamma\delta}\ddot{u}^\beta u^\gamma s^\delta - 3\frac{\dot{u}_\beta u^\beta}{\|\boldsymbol{u}\|^2}\,\varepsilon_{\alpha\beta\gamma\delta}\dot{u}^\beta u^\gamma s^\delta - m_0\big(\|\boldsymbol{u}\|^2\dot{u}_\alpha - \dot{u}_\beta u^\beta u_\alpha\big) = 0, \tag{3}$$

subject to the constraint

$$s_\alpha u^\alpha = 0. \tag{4}$$

And we recall that spin four-vector $\boldsymbol{s}$ is a constant vector along the word line of the particle as long as no gravitational field is considered.

The equation (3) does not change under arbitrary reparametrizations of the world line, i.e., under arbitrary local transformations of the independent variable τ, the parameter, and because of that it is often said that the equation is presented in homogeneous form, or that it is parameter-independent.

Now, we set the following two-fold task:

1. invent a variational description for the equation (3);
2. try to add some parametrization to equation (3) in such a way that the (parametried) autoparallel curves of the corresponding second order connection would also satisfy (3) everywhere on the constraint submanifold (4).

2. Variationality

As far as we are interested in the parameter-homogeneous form of a variational third-order equation that should be equivalent to the equation (3), and also as far as we intend to impose pseudo-Euclidean symmetry, it is convenient to work in the variables $u^0 = 1$, $v^i = u^i$, $i \in \overline{(1,3)}$, $x^0 = t$, that is to pass to the manifold of r-th order contact elements in the manifold $M = \{t, x^i\}$. We note that pseudo-Euclidean transformations permute the variables t and x^i.

Let, in general, $T^r_p M$ denote the bundle of Ehresmann p-velocities of order r to manifold M and let $C^r(p, M)$ denote the manifold of r-th order contact elements of p-dimensional submanifolds in $p+q$ dimensional manifold M. The group $Gl^r(p)$ of invertible jets from $\mathbb{R}^p$ to $\mathbb{R}^p$ which both start and terminate at $0 \in \mathbb{R}^p$, acts on the right upon the manifold $T^r_p M$ by jet composition rule. This action, as we shall see, is in charge of parameter (independent variable) transformations of the velocities from $T^r_p M$ and hence governs the transformations of a variational equation in parametric form. The generators are the generalized Liouville fields

$$(5) \qquad \zeta^{\mathrm{M}}_n = \sum_{|\mathrm{N}|=0}^{r-|\mathrm{M}|} \binom{|\mathrm{M}|+|\mathrm{N}|}{|\mathrm{M}|} u^\alpha_{\mathrm{N}+1_n} \frac{\partial}{\partial u^\alpha_{\mathrm{N}+\mathrm{M}}}, \quad 1 \le |\mathrm{M}| \le r,$$

where, as common, multi-indexes $\mathrm{M} = (\mu_1, \ldots, \mu_p)$ and $\mathrm{N} = (\nu_1, \ldots, \nu_p)$ both belong to $\mathbb{N}^p$ with the length defined by $|\mathrm{N}| = \nu_1 + \cdots + \nu_p$, and the multi-index 1_n corresponds to partial differentiation along the direction of the n-th independent variable τ^n, $n \in \overline{(1, p)}$. In future we shall abuse the notation u^α_0 in place of x^α, $x^\alpha = t^\alpha$ if $\alpha \le p$. The zero section of $T^r_p M$ is well defined and we have the quotient projection with respect to the above mentioned action

$$(6) \qquad \wp : T^r_p M \backslash \{0\} \to C^r(p, M).$$

On the manifold M we shall define a variational problem, invariant under the action of pseudo-Euclidean group on M. A Lagrangian will mean a semi-basic with respect to M local p-form defined on $C^r(p, M)$, and two such forms will be recognized equivalent if in common domain their difference belongs to the ideal, generated by contact forms. As our considerations on $C^r(p, M)$ are local and infinitesimal, we shall profit from the local isomorphism

$$C^r(p, M) \approx J^r(\mathbb{R}^p, \mathbb{R}^q).$$

And further on, let us recall the isomorphism $J^r(\mathbb{R}^p, \mathbb{R}^q) \approx J^r(\mathbb{R}^p \times \mathbb{R}^q)$, where the right-hand side means the bundle of jets of cross sections of the fibration $\mathbb{R}^{p+q} \to \mathbb{R}^p$. From among the equivalent Lagrangians on $C^r(p, M)$ it is always possible to fix the unique representative, semi-basic with respect to $\mathbb{R}^p$ in this local representation.

Let us introduce the notation v^i_Ω, with $\Omega = (\omega_1, \ldots, \omega_p)$, for the canonical coordinates in $J^r(\mathbb{R}^p, \mathbb{R}^q)$ and let (t^w, x^i), $w \in \overline{(1, p)}$, $i \in \overline{(1, q)}$, be the corresponding local coordinates in M. One would like to pull the variational problem posed on $C^r(p, M)$, back to the manifold $T^r_p M$ in the temptation to obtain some variational equation in the parameter-homogeneous form on M, and in case $p = 1$ to construct then a kind of higher-order connection on some $T^k M$, $k < 2r$, in such a way, that the autoparallel curves of this connection would prescribe some parametrization to the unparametrized integral submanifolds of the initial parameter-independent variational problem. But the pull-back of a one-form is again one-form, and what we need is a local Lagrange *function* on $T^r_p M$, not a form. The way out is to consider the manifold $T^r_p M$ as a rudiment of the parameter-extended space $J^r(\mathbb{R}^p, M)$ in the following way.

First, recall the isomorphism $J^r(\mathbb{R}^p, M) \approx \mathbb{R}^p \times J^r(\mathbb{R}^p, M)(0)$, given by the correspondence $j^r\sigma(\tau) \to (\tau, j^r(\sigma \circ \delta_\tau)(0))$, where δ_τ is the translation by τ in $\mathbb{R}$. Then notice that $J^r(\mathbb{R}^p, M)(0)$ is exactly the definition of $T^r_p M$ and apply the projection onto the second factor,

$$(7) \qquad J^r(\mathbb{R}^p, M) \approx \mathbb{R}^p \times T^r_p M \xrightarrow{p_2(r)} T^r_p M.$$

Now, the idea is to pull a variational problem from the manifold $C^r(p, M)$ back to the manifold $J^r(\mathbb{R}^p, M)$ and then to find on $J^r(\mathbb{R}^p, M)$ an equivalent Lagrangian of the form $\mathcal{L}_0\, d\tau^1 \wedge \cdots \wedge d\tau^p$. The function $\mathcal{L}_0$ will then in fact be defined on the space $T^r_p M$. To make our consideration precise, let us recall some calculus on $J^r(\mathbb{R}^p, M)$.

2.1. Lagrange differential

Let us introduce an abridged notation $Y^r = J^r(\mathbb{R}^p, M)$ and, of course, Y will stand in place of $\mathbb{R}^p \times M$. Also let $\Omega_r^{\cdot,\cdot} = \sum \Omega_r^{h,v}$ denote the module of semi-basic with respect to $\mathbb{R}^p$ differential forms on Y^r with values in the dual $T^*(Y^r/\mathbb{R}^p)$ to the bundle $T(Y^r/\mathbb{R}^p)$ of $\mathbb{R}^p$-vertical tangent vectors to Y^r; h and v mean the corresponding degrees in the bigraded module

$$\Omega_r^{h,v} \approx \mathrm{Sec}(\textstyle\bigwedge^v T^*(Y^r/\mathbb{R}^p) \otimes_{Y^r} \bigwedge^h T^*\mathbb{R}^p).$$

It is not our goal here to present any definition of the Euler–Lagrange differential $\boldsymbol{\delta}$ (see [2] or [9]). We merely recall that it is possible to interpret the operator $\boldsymbol{\delta}$ as one acting from $\Omega_r^{h,v}$ to $\Omega_{2r}^{h,v+1}$ so that for any $\lambda \in \Omega_r^{p,0}$ the result of applying $\boldsymbol{\delta}$ belongs to $\Omega_{2r}^{p,1}$, and in fact $\boldsymbol{\delta}\lambda$ is a semi-basic p-form taking values in $T^*(Y/\mathbb{R}^p)$ alone. Its components in $T^*(Y/\mathbb{R}^p)$ along some local coordinates $\{x^\alpha\}$ in M are the classical Euler–Lagrange expressions. Let us identify the fibre bundle $\bigwedge T^*(Y^r/\mathbb{R}^p)$ with the reciprocal image of $\bigwedge T^*(T^r_p M)$ along the projection (7). We think of the algebra $\Omega(T^r_p M)$ of differential forms on $T^r_p M$ as of $\Omega^0(T^r_p M)$-subalgebra of $\Omega_r^{0,\cdot}$, the inclusion being defined by the reciprocal image construction along $p_2(r)$. The operator $\boldsymbol{\delta}$ takes $\Omega(T^r_p M)$ into the $\Omega^0(T^{2r}_p M)$-subalgebra $\Omega(T^{2r}_p M)$ of $\Omega_{2r}^{0,\cdot}$. We denote the restriction of the operator $\boldsymbol{\delta}$ to the algebra $\Omega(T^r_p M)$ by $\boldsymbol{\delta}^T$.

Now, consider some Lagrangian

$$(8) \qquad \lambda = \mathcal{L}_0 d^p\tau \in \Omega_r^{p,0},$$

where $d^p\tau$ stands for the p-fold exterior product $d\tau^1 \wedge \cdots \wedge d\tau^p$, and, in general, function $\mathcal{L}_0$ may depend on $\tau \in \mathbb{R}^p$. We say that such a Lagrangian defines a variational problem in extended parametric form. In this case,

$$(9) \qquad \boldsymbol{\delta}\lambda = \boldsymbol{\varepsilon}_0 \otimes d^p\tau \in \Omega_{2r}^{p,1}, \quad \text{where} \quad \boldsymbol{\varepsilon}_0 = \boldsymbol{\delta}\mathcal{L}_0.$$

Let

$$(10) \qquad p^r : T^r_p M \to M$$

denote the standard projection. We observe that essentially $\boldsymbol{\varepsilon}_0$ is a cross-section of the induced bundle $p_2(2r)^* p^{2r*} T^*M$.

Let υ be the graph of a local immersion $\sigma : \mathbb{R}^p \to M$. Recall that by the definition of the action of pull-backs on vector bundle valued differential forms,

$$j^{2r}\upsilon^*\boldsymbol{\delta}\lambda \in \mathrm{Sec}(\sigma^* T^* M \otimes \textstyle\bigwedge^p T^*\mathbb{R}^p),$$

$$j^{2r}\upsilon^*\boldsymbol{\delta}\lambda = (\boldsymbol{\delta}\mathcal{L}_0 \circ j^{2r}\sigma) \otimes d^p\tau,$$

where $j^{2r}\upsilon$ denotes the prolongation of the cross-section υ and $j^{2r}\sigma$ is the essential component of the cross-section $j^{2r}\upsilon$. Thus the Euler–Lagrange equations appear to have two equivalent guises:

$$j^{2r}\upsilon^*\boldsymbol{\delta}\lambda = 0 \quad \text{or} \quad \boldsymbol{\delta}\mathcal{L}_0 \circ j^{2r}(\sigma) = 0. \tag{11}$$

Let us assume that the Lagrange function $\mathcal{L}_0$ does not depend on parameter $\tau \in \mathbb{R}^p$: $\mathcal{L}_0 = p_2(r)^*\mathcal{L}$. Consider the second component $\partial_{2r}\sigma$ of the jet $j^{2r}\sigma$ under the projection $p_2(2r) : J^{2r}(\mathbb{R}^p, M) \to T_p^{2r}M$. The Euler–Lagrange equations take the shape of

$$(\boldsymbol{\delta}^T\mathcal{L}) \circ \partial_{2r}\sigma = 0. \tag{12}$$

2.2. Parametric invariance

Introduce an arbitrary local change of parameter $\mathbb{R}^p \to \mathbb{R}^p$ and let us see how it effects a variational problem in extended parametric form on the fibred manifold $\pi : J^r(\mathbb{R}^p, M) \to \mathbb{R}^p$, given by (8). The standard prolongation of the pair of morphisms $(f, \mathrm{id}) : \mathbb{R}^p \times M \to \mathbb{R}^p \times M$ is denoted by $J^r(f, \mathrm{id}) : f^*J^r(\mathbb{R}^p, M) \to J^r(\mathbb{R}^p, M)$ and is defined by the property

$$J^r(f, \mathrm{id}) \circ (\pi^* f)^{-1} \circ j^r\sigma \circ f = j^r(\sigma \circ f) \tag{13}$$

for arbitrary jet $j^r\sigma \in J^r(\mathbb{R}^p, M)$, and we mention that in standard functorial notations morphism $\pi^* f : f^*J^r(\mathbb{R}^p, M) \to J^r(\mathbb{R}^p, M)$ is a bijection as long as the mapping f is a diffeomorphism. Let (W, σ) be a pair consisting of a compact set W in $\mathbb{R}^p$ and of a mapping σ from W into M. Diffeomorphism f acts upon such pairs by means of the rule $f : (W, \sigma) \mapsto (f^{-1}W, \sigma \circ f)$. Let S be a function, defined for each pair (W, σ) by means of $S : (W, \sigma) \mapsto \int_W j^r\sigma^*\lambda$. We demand that the function S be equivariant with respect to the action of f, that is,

$$S \circ f = S, \tag{14}$$

and in this case the variational problem is called a parameter-invariant one. By (13) and by the well-known change of variables formula,

$$S(W, \sigma) = \int_{f^{-1}W} f^* j^r\sigma^*\lambda,$$

we obtain:

$$(S\circ f)(W,\sigma)=S(f^{-1}W,\sigma\circ f)=\int_{f^{-1}W}\big(j^r(\sigma\circ f)\big)^*\lambda$$
$$=\int_{f^{-1}W} f^* j^r\sigma^*\big((\pi^* f)^{-1}\big)^*\big(J^r(f,\mathrm{id})\big)^*\lambda$$
$$=S\Big[W,\big((\pi^* f)^{-1}\big)^*\big(J^r(f,\mathrm{id})\big)^*\lambda\Big].$$

Now the parametric invariance (14) means that

(15) $$J^r(f,\mathrm{id})^*\lambda=(\pi^* f)^*\lambda.$$

The identification (7) implies that $f^*J^r(\mathbb{R}^p,M)\approx\mathbb{R}^p\times T^r_pM$ and $\pi^*f=(f\times\mathrm{id})$, thus (15) takes the form

(16) $$\mathcal{L}_0\circ J^r(f,\mathrm{id})=\mathcal{L}_0\circ(f\times\mathrm{id}).\det\frac{\partial f}{\partial\tau}.$$

The infinitesimal analogue of (16) reads

$$\langle\boldsymbol{\zeta}^r,\mathbf{d}_\pi\mathcal{L}_0\rangle=\boldsymbol{\zeta}(\mathcal{L}_0)+\mathcal{L}_0.\mathrm{tr}\,\frac{\partial\boldsymbol{\zeta}}{\partial\tau},$$

where $\boldsymbol{\zeta}$ generates some local flow on $\mathbb{R}^p$, $\boldsymbol{\zeta}^r$ denotes the standard prolongation of $\boldsymbol{\zeta}$ to the space $J^r(\mathbb{R}^p,M)$, and $\mathbf{d}_\pi$ is the fibre differential along fibres of π. As far as $\partial\boldsymbol{\zeta}/\partial\tau$ is an arbitrary matrix, we conclude that $\mathcal{L}_0$ must not depend on τ (put $\partial\boldsymbol{\zeta}/\partial\tau=0$) and thus essentially is defined and may be thought of as some function on T^r_pM alone: $\mathcal{L}_0=p_2(r)^*\mathcal{L}$. The calculation of $\boldsymbol{\zeta}^r$ according to the standard procedure, [12], ultimates in Zermelo–Géhéniau's conditions

(17) $$\boldsymbol{\zeta}^{\mathrm{M}}_n(\mathcal{L})=\delta^{\mathrm{M}}_{1_n}\mathcal{L},$$

where fields $\boldsymbol{\zeta}^{\mathrm{M}}_n$ are given by (5).

It is well known, that each invariance of Lagrangian λ implies the invariance of the corresponding differential form $\boldsymbol{\delta}\lambda$ (see [3] for technical details). In our notations,

$$J^{2r}(f,\mathrm{id})^*\boldsymbol{\delta}\lambda=(f\times\mathrm{id})^*\boldsymbol{\delta}\lambda,$$

and in terms of the projection (7) it gives for $\boldsymbol{\varepsilon}_0=p_2(2r)^*\boldsymbol{\delta}^T\mathcal{L}$, as defined in (9),

$$J^{2r}(f,\mathrm{id})^*p_2(2r)^*\boldsymbol{\delta}^T\mathcal{L}=\big(p_2(2r)^*\boldsymbol{\delta}^T\mathcal{L}\big).\det\frac{\partial f}{\partial\tau}.$$

The infinitesimal analogue in terms of fibre derivative $\mathbf{d}_{p^{2r}}$ with respect to the fibration $p^{2r}:T^{2r}_pM\to M$ reads

$$\big\langle\boldsymbol{\zeta}^{2r},\,p_2(2r)^*\mathbf{d}_{p^{2r}}\boldsymbol{\delta}^T\mathcal{L}\big\rangle=\big(p_2(2r)^*\boldsymbol{\delta}^T\mathcal{L}\big).\mathrm{tr}\,\frac{\partial\boldsymbol{\zeta}}{\partial\tau},$$

and again in course of the arbitrariness of $\boldsymbol{\zeta}$ we come up to the following formulation of parametric invariance of the Euler–Lagrange form $\boldsymbol{\varepsilon}_0 = p_2(2r)^*\boldsymbol{\delta}^T\mathcal{L}$:

$$\langle \boldsymbol{\zeta}_n^{\mathrm{M}}, \mathbf{d}_{p^{2r}}\boldsymbol{\delta}^T\mathcal{L}\rangle = \delta_{1_n}^{\mathrm{M}}\boldsymbol{\delta}^T\mathcal{L}. \tag{18}$$

2.3. Transition from $C^r(p, M)$ to parameter-homogeneous form

The projection (6) in local coordinates is given by the following formula, which may be deduced from general reflections on the subject of transformation rules for derivatives, [1],

$$u^i_{n_1\cdots n_r} = \sum_{k=1}^{r} \mathbf{P}^{w_1\cdots w_k}_{n_1\cdots n_r}\wp^i_{w_1\cdots w_k},$$

where we put

$$\wp^i_{w_1\cdots w_k} = v^i_\Omega \circ \wp,$$

v^i_Ω being the coordinates in $C^r(p, M)$, which coincide with u^i_{N} for $\mathrm{N} = \Omega$ when $u^w_{\mathrm{N}} = \delta^{1_w}_{\mathrm{N}}$, and in the multi-index $\Omega = (\omega_1\cdots\omega_p)$, of length k, every ω_w denotes the number of repetitions of w in the sequence $(w_1\cdots w_k)$. The matrix $\mathbf{P}$ is calculated according to the formula:

$$\mathbf{P}^{w_1\cdots w_k}_{n_1\cdots n_r} = \sum_{\substack{1\le r_1\le\cdots\le r_k\le r\\ r_1+\cdots+r_k=r}} \frac{r!}{r_1!\cdots r_k!\,\rho_1!\cdots\rho_{r-k+1}!}$$
$$u^{(w_1}_{(n_1\cdots n_{r_1}}u^{w_2}_{n_{r_1+1}\cdots n_{r_1+r_2}}\cdots u^{w_k)}_{n_{r_1+r_2+\cdots+r_{k-1}+1}\cdots n_{r_1+r_2+\cdots+r_k})},$$

where each ρ_k means the number of repetitions of k in the sequence $(r_1\cdots r_k)$ and parentheses denote the symmetrization procedure.

We now proceed further in the realization of our main goal: to represent a variational problem, initially posed on the contact manifold $C^r(p, M)$, by means of some parameter-homogeneous form of an equivalent variational problem, this time on the manifold T^r_pM. Let be given in some local chart of $C^r(p, M)$ an $\mathbb{R}^p$-semi-basic representative

$$\Lambda = L\,d^pt, \quad d^pt = dt^1\wedge\cdots\wedge dt^p \tag{19}$$

of a class of equivalent Lagrangians (see [8]). The pull-back of Λ along the total projection $\mathfrak{p} = \wp\circ p_2(r)$ equals $L\circ\mathfrak{p}\;.\;d^pt$. Let us decompose the p-form d^pt with respect to the basis, constituted by the p-form $d^p\tau$ and by the forms

$$(d^p\tau)^{\alpha_1\cdots\alpha_l}_{n_1\cdots n_l} = \vartheta^{\alpha_1}\wedge\cdots\vartheta^{\alpha_l}\wedge\frac{\partial}{\partial\tau^{n_1}}\lrcorner\cdots\lrcorner\frac{\partial}{\partial\tau^{n_l}}\lrcorner\, d^p\tau,$$

$1\le l\le p$, $1\le n_1<\cdots<n_l\le p$, $1\le\alpha_1<\cdots<\alpha_l\le p+q$, where $\vartheta^\alpha = dx^\alpha - u^\alpha_n d\tau^n$ are the first order contact forms on the manifold $J^r(\mathbb{R}^p, M)$ and $\tau\in\mathbb{R}^p$. In fact, only terms with $d^p\tau$ and $(d^p\tau)^{w_1\cdots w_l}_{n_1\cdots n_l}$, $1\le w_1<\cdots<w_l\le p$

survive in this decomposition, and we obtain

$$d^p t = \det \mathbf{U} \,.\, d^p \tau + \sum_{l=1}^{p} \sum_{\substack{1 \le n_1 < \cdots < n_l \le p \\ 1 \le w_1 < \cdots < w_l \le p}} \bar{\mathbf{U}}^{n_1 \cdots n_l}_{w_1 \cdots w_l} (d^p \tau)^{w_1 \cdots w_l}_{n_1 \cdots n_l},$$

where $\bar{\mathbf{U}}^{n_1 \cdots n_l}_{w_1 \cdots w_l}$ denotes the algebraic adjunct of the minor $\mathbf{U}^{w_1 \cdots w_l}_{n_1 \cdots n_l}$ in the matrix $\mathbf{U} = (u^w_n)$.

Let us consider for a moment another local chart (u'^i, x'^i, t'^i) of the manifold $C^r(p, M)$, denote by ϕ_C the corresponding transition function and let $\Lambda' = L' \, d^p t'$ be such a representative, that $\phi^*_C \Lambda' - \Lambda$ belongs to the ideal, generated by differential forms

$$(d^p t)^{i_1 \cdots i_l}_{w_1 \ldots w_l} = \theta^{i_1} \wedge \cdots \theta^{i_l} \wedge \frac{\partial}{\partial t^{w_1}} \lrcorner \cdots \lrcorner \frac{\partial}{\partial t^{w_l}} \lrcorner \, d^p t,$$

$1 \le l \le p$, $1 \le w_1 < \cdots < w_l \le p$, $1 \le i_1 < \cdots < i_l \le q$, where $\theta^i = dx^i - v^i_w dt^w$ are the first order contact forms on the manifold $J^r(\mathbb{R}^p, \mathbb{R}^q)$ and $t \in \mathbb{R}^p$.

The pull-back operation preserves the corresponding contact ideal ([8]):

$$\mathfrak{p}^* \theta^i = \vartheta^i - \wp^i_w \vartheta^w$$

as well, as the coherent transition function φ_J in the manifold $J^r(\mathbb{R}^p, M)$ does, and it may be proved that the difference $\varphi^*_J(L' \circ \mathfrak{p} \,.\, d^p t') - L \circ \mathfrak{p} \,.\, d^p t$ belongs to the contact ideal on $J^r(\mathbb{R}^p, M)$. Hence our considerations are intrinsic.

Let us recall the notations (8, 9, 11, 12), and introduce the shortcut notation

$$\wp_\phi = \phi_C \circ \wp : T^r_p M \to J^r(\mathbb{R}^p, \mathbb{R}^q),$$

same for each r.

Proposition 1. *Let* $\mathcal{L} = (L \circ \wp_\phi) . \det \mathbf{U}$. *The equations*

$$(\boldsymbol{\delta} L) \circ \wp_\phi \circ \partial_{2r} \sigma = 0$$

and

$$\boldsymbol{\delta}^T (\mathcal{L}) \circ \partial_{2r} \sigma = 0$$

are equivalent.

The Lagrange function $\mathcal{L}$ and the corresponding differential form $\boldsymbol{\delta}^T \mathcal{L}$ obviously satisfy Zermelo–Géhéniau's conditions (17) and (18).

Remark 1. We strive to give an (in fact trivial) algorithm for building up a Lagrange function and the corresponding Euler–Lagrange equations in parameter-homogeneous form directly from solutions of an inverse variational problem on contact manifold $C^r(p, M)$. But treating this latter problem, especially in the aspects of equivalence and symmetry of differential equations, appears to be more convenient in terms of the *Lepagean equivalents*, [4] (cf. [8, 10, 11]). So it would be of interest to translate the reparametrization technique, presented in this section, directly into the language of Lepagean differential forms theory.

2.4. Third order equations with pseudo-Euclidean symmetry

In case of the system of ordinary differential Euler–Lagrange equations (we follow the tradition of calling them *Euler–Poisson equations*) the vector-valued differential form $\boldsymbol{\delta}\Lambda$ of Λ as in (19), takes the shape

$$\boldsymbol{\delta}\Lambda = E_i\, dx^i \otimes dt, \tag{20}$$

where E_i are the Euler–Poisson expressions. We call the problem of finding Euler–Lagrange equations with prescribed symmetry and of prescribed order, the invariant inverse problem of that order in the calculus of variations. In case of third order Euler–Poisson equations with pseudo-Euclidean symmetry in four-dimensional space, one solution was found in ([6]) and announced in ([7]). It is essential that a four-vector parameter $\boldsymbol{s} = (s^\alpha)$ should enter in variational equations of the third order to make them obey the pseudo-Euclidean symmetry. This parameter does not undergo any variations. Physically, it is responsible for an intrinsic dipole momentum of a relativistic test particle. As the problem was posed on contact manifold, we obtain the solution in terms of the coordinates on the contact manifold $C^3(1,M)$:

$$\begin{aligned}\mathbf{E} = {} & \frac{\mathbf{v}'' \times (\mathbf{s} - s_0\mathbf{v})}{[(1+\mathbf{v^2})(s_0^2+\mathbf{s^2}) - (s_0 + \mathbf{s}\cdot\mathbf{v})^2]^{\frac{3}{2}}} \\ & - 3\,\frac{(s_0^2+\mathbf{s^2})\mathbf{v}'\cdot\mathbf{v} - (s_0+\mathbf{s}\cdot\mathbf{v})\mathbf{s}\cdot\mathbf{v}'}{[(1+\mathbf{v^2})(s_0^2+\mathbf{s^2}) - (s_0 + \mathbf{s}\cdot\mathbf{v})^2]^{\frac{5}{2}}}\,\mathbf{v}' \times (\mathbf{s} - s_0\mathbf{v}) \\ & + m\,\frac{(1+\mathbf{v^2})\mathbf{v}' - (\mathbf{v}'\cdot\mathbf{v})\mathbf{v}}{(1+\mathbf{v^2})^{\frac{3}{2}}(s_0^2+\mathbf{s^2})^{\frac{3}{2}}},\end{aligned} \tag{21}$$

produced by any of the following Lagrange functions,

$$\begin{aligned}L_{(i)} = {} & \frac{s_0}{s_0^2+\mathbf{s^2}} \cdot \frac{(s_0^2+\mathbf{k}_{(i)}{}^{\mathbf{2}})(s_i - s_0 v_i) - s_i(\mathbf{k}_{(i)}\cdot\mathbf{z}_{(i)})}{(s_0^2+\mathbf{k}_{(i)}{}^{\mathbf{2}})\,\mathbf{z}_{(i)}{}^{\mathbf{2}} - (\mathbf{k}_{(i)}\cdot\mathbf{z}_{(i)})^2} \\ & \cdot \frac{[\mathbf{v}', (\mathbf{s}-s_0\mathbf{v}), \mathbf{e}_{(i)}]}{(\mathbf{s}-s_0\mathbf{v})^{\mathbf{2}} + (\mathbf{s}\times\mathbf{v})^{\mathbf{2}}} - \frac{m}{(s_0^2+\mathbf{s^2})^{\frac{3}{2}}}\sqrt{1+\mathbf{v^2}},\end{aligned}$$

where some shortcut notations were introduced:

$$\mathbf{k}_{(i)} = \mathbf{s} - s_i\,\mathbf{e}_{(i)}, \quad \mathbf{z}_{(i)} = (\mathbf{s} - s_0\mathbf{v}) - (s_i - s_0 v_i)\,\mathbf{e}_{(i)},$$

and vectors $\mathbf{e}_{(i)}$ form a basis in $\mathbb{R}^3$.

Remark 2. By virtue of a certain proposition of ([6]) it is not realistic to try to find any third order variational equation with pseudo-Euclidean symmetry in four-dimensional space without introducing into it some additional quantities, constructed from the representations of the pseudo-Euclidean group.

Proposition 1 immediately allows us to build the parameter-homogeneous form of the expression (21) by means of the following prescription: if

$$\boldsymbol{\delta}^T \mathcal{L} = \mathcal{E}_\alpha \, dx^\alpha \quad \text{and} \quad \boldsymbol{\delta}(L\, dt) = E_i \, dx^i \otimes dt,$$

then

$$\mathcal{E}_\alpha \, dx^\alpha = -\frac{dx^i}{d\tau} \cdot (E_i \circ \wp_\phi)\, dt + \frac{dt}{d\tau} \cdot (E_i \circ \wp_\phi)\, dx^i .$$

So for (21) we obtain:

$$\begin{aligned} \boldsymbol{\mathcal{E}} = {} & \frac{*\ddot{\boldsymbol{u}} \wedge \boldsymbol{u} \wedge \boldsymbol{s}}{\|\boldsymbol{s} \wedge \boldsymbol{u}\|^3} - 3\, \frac{*\dot{\boldsymbol{u}} \wedge \boldsymbol{u} \wedge \boldsymbol{s}}{\|\boldsymbol{s} \wedge \boldsymbol{u}\|^5} \, (\dot{\boldsymbol{u}} \wedge \boldsymbol{s}) \cdot (\boldsymbol{u} \wedge \boldsymbol{s}) \\ & + \frac{m}{\|\boldsymbol{s}\|^3} \left[\frac{\dot{\boldsymbol{u}}}{\|\boldsymbol{u}\|} - \frac{\dot{\boldsymbol{u}} \cdot \boldsymbol{u}}{\|\boldsymbol{u}\|^3} \boldsymbol{u} \right] \\ = {} & 0, \end{aligned} \tag{22}$$

and again Proposition 1 helps to guess the family of four Lagrange functions, each of which produces equation (22):

$$\mathcal{L}_{(\alpha)} = \frac{*\dot{\boldsymbol{u}} \wedge \boldsymbol{u} \wedge \boldsymbol{s} \wedge \boldsymbol{e}_{(\alpha)}}{\|\boldsymbol{s}\|^2 \|\boldsymbol{s} \wedge \boldsymbol{u}\|} \cdot \frac{\boldsymbol{s}^2 u_\alpha + (\boldsymbol{s} \cdot \boldsymbol{u})\, s_\alpha}{(u_\alpha \boldsymbol{s} - s_\alpha \boldsymbol{u})^2 - (\boldsymbol{s} \wedge \boldsymbol{u})^2} - \frac{m}{\|\boldsymbol{s}\|^3} \|\boldsymbol{u}\|, \tag{23}$$

with vectors $\boldsymbol{e}_{(\alpha)}$ constituting a basis in M. Equation (22) possesses the first integral

$$\frac{\boldsymbol{s} \cdot \boldsymbol{u}}{\|\boldsymbol{u}\|}, \tag{24}$$

and by comparison with (3) and (4) we calculate that every time we choose

$$\boldsymbol{s} \cdot \boldsymbol{u} = 0, \tag{25}$$

it describes the free motion of a relativistic top.

3. Autoparallel reparametrization of geodesic curves

It was argued, in [10], that an arbitrary third order equation $\boldsymbol{\xi} : T^2M \to T^3M$ of the local form

$$\ddot{u}^\alpha = \xi^\alpha(\dot{u}^\beta, u^\beta, x^\beta) \tag{26}$$

defines an autoparallel curve only in the case, when the functions ξ^α satisfy (in terms of the vector field $\boldsymbol{\xi}$) the following commutation relations with the Liouville fields (5):

$$(27)\qquad \begin{cases}(T\wp)[\boldsymbol{\zeta}^1,\boldsymbol{\xi}] = (T\wp)\boldsymbol{\xi}\\ (T\wp)[\boldsymbol{\zeta}^2,\boldsymbol{\xi}] = \mathbf{0},\end{cases}$$

which might be put into the local form by the following PDE system with constant Lagrange multipliers μ and κ

$$(28)\qquad \dot{u}^\alpha - \frac{1}{3}\frac{\partial\xi^\alpha}{\partial\dot{u}^\beta}u^\beta = \kappa u^\alpha$$

$$(29)\qquad \xi^\alpha - \frac{1}{3}\frac{\partial\xi^\alpha}{\partial u^\beta}u^\beta - \frac{2}{3}\frac{\partial\xi^\alpha}{\partial\dot{u}^\beta}\dot{u}^\beta = \mu u^\alpha.$$

It remains to solve the equations (28, 29), and to find the functions ξ^α for the representation (26) of the equation (22). In order to cast the equation (22) into the form (26), solved with respect to the highest order derivatives, we add to it one more equation of general type

$$(30)\qquad \ddot{\boldsymbol{u}}\cdot\boldsymbol{u} = \|\boldsymbol{u}\|^2\Psi(\dot{\boldsymbol{u}},\boldsymbol{u}),$$

and that will prescribe some kind of parametrization along the unparametrized curves – the solutions of (22). Next we also make use of the physical constraint (25). To proceed further, contract the vector equation (22) with the tensor $*\boldsymbol{u}\wedge\boldsymbol{s}$ and differentiate the first integral (24) twice. This helps to solve the equation (22) with respect to $\ddot{\boldsymbol{u}}$:

$$(31)\qquad \ddot{\boldsymbol{u}} = 3\frac{\dot{\boldsymbol{u}}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\dot{\boldsymbol{u}} - 3\frac{(\dot{\boldsymbol{u}}\cdot\boldsymbol{u})^2}{\|\boldsymbol{u}\|^4}\boldsymbol{u} - m\frac{\|\boldsymbol{u}\wedge\boldsymbol{s}\|}{\|\boldsymbol{s}\|^3\|\boldsymbol{u}\|} * \dot{\boldsymbol{u}}\wedge\boldsymbol{u}\wedge\boldsymbol{s} + u.\Psi.$$

Comparing (31) with (26), we rewrite (28, 29) in terms of Ψ, and then applying the compatibility conditions to the system of PDE {(28, 29)} shows that $\kappa = 0$. The Ansatz for Ψ is

$$\Psi = \frac{3}{\|\boldsymbol{u}\|^2}\left(\tfrac{1}{2}\|\dot{\boldsymbol{u}}\|^2 + \psi\right),$$

and from (28) there arises a constraint on possible functions ψ:

$$(32)\qquad \boldsymbol{u}.\frac{\partial\psi}{\partial\dot{\boldsymbol{u}}} = 0.$$

Let us apply symmetry concept to the equation (26). The group of transformations of M must not operate on the parameter τ. In case of pseudo-Euclidean group the generators read:

$$(33)\qquad X = \Omega^{\alpha\beta}u_\alpha\frac{\partial}{\partial u^\beta} + \Omega^{\alpha\beta}\dot{u}_\alpha\frac{\partial}{\partial\dot{u}^\beta} + \Omega^{\alpha\beta}\ddot{u}_\alpha\frac{\partial}{\partial\ddot{u}^\beta} + \Omega^{\alpha\beta}s_\alpha\frac{\partial}{\partial s^\beta},$$

with arbitrary skew-symmetric matrix parameter $\Omega^{\alpha\beta}$.

Now apply X to the equation (31) and observe that if $\boldsymbol{a}$ is a vector, then $Xa^\alpha = -\eta^{\alpha\beta}\Omega_{\beta\gamma}a^\gamma$, where $\eta^{\alpha\beta}$ is the constant canonical diagonal metric tensor of pseudo-Euclidean M.

This observation together with (32) and (29) suggests the solution

$$\Psi = \frac{3}{\|\boldsymbol{u}\|^2}\left(\tfrac{1}{2}\,\|\dot{\boldsymbol{u}}\|^2 + A\|\dot{\boldsymbol{u}} \wedge \boldsymbol{u}\|^{\frac{4}{3}}\right), \qquad \mu = 0,$$

with arbitrary scalar constant A

Proposition 2. *The autoparallel curves in four-dimensional pseudo-Euclidean space describe the motion of the free relativistic top and satisfy the equation*

$$\ddot{\boldsymbol{u}} = 3\,\frac{\dot{\boldsymbol{u}}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\,\dot{\boldsymbol{u}} - 3\left[\frac{(\dot{\boldsymbol{u}}\cdot\boldsymbol{u})^2}{\|\boldsymbol{u}\|^4} - \frac{1}{2}\,\frac{\|\dot{\boldsymbol{u}}\|^2}{\|\boldsymbol{u}\|^2} - A\frac{\|\dot{\boldsymbol{u}}\wedge\boldsymbol{u}\|^{\frac{4}{3}}}{\|\boldsymbol{u}\|^2}\right]\boldsymbol{u}$$
$$- m\,\frac{\|\boldsymbol{u}\wedge\boldsymbol{s}\|}{\|\boldsymbol{s}\|^3\|\boldsymbol{u}\|} * \dot{\boldsymbol{u}}\wedge\boldsymbol{u}\wedge\boldsymbol{s}.$$

The world lines are those among the extremal curves of the Lagrange function (23), *who agree with the physical constraint* (4).

The constant A corresponds to different ways of the parametrization of world lines. One may chose $A = 0$.

References

[1] M. Kawaguchi, An introduction to the theory of higher order spaces II. Higher order spaces in multiple parameters, *Research Association of Applied Geometry. Memoirs (Tokyo)* 4 (1968) 578–592. (Misc. 6, 86–100).

[2] I. Kolář, On the Euler–Lagrange differential in fibred manifolds, *Rep. Math. Phys.* 12 (1977) (3) 301–305.

[3] I. Kolář, Lie derivatives and higher-order Lagrangians, in: *Differential Geometry and Applications*, Proc. Conf. (ČSSR–GDR–Pol.) (Univ. Karlova, Praha, 1981) 117–123.

[4] O. Krupková, *The Geometry of Ordinary Variational Equations*, Lecture Notes in Math. 1678 (Springer, Berlin, 1997).

[5] M. Mathisson, Neue Mechanik materieller Systeme, *Acta Phys. Polon.* 6 (1937) (3) 163–200.

[6] R.Ya. Matsyuk, Poincaré-invariant equations of motion in Lagrangian mechanics with higher derivatives, Thesis (Institute for Applied Problems in Mechanics and Mathematics, L'viv, Ukraine 1984), in Russian.

[7] R.Ya. Matsyuk, Lagrangian analysis of invariant third-order equations of motion in relativistic mechanics of classical particles, *Sov. Phys.–Dokl. (USA)* 30 (1985) (11) 923–925.

[8] R.Ya. Matsyuk, Symmetries of vector exterior differential systems and the inverse problem in second-order Ostrohrads'kyj mechanics, *J. Nonlinear Math. Phys.* 4 (1997) (1/2) 89–97.

[9] R.Ya. Matsyuk, Integration by parts and vector differential forms in higher order variational calculus on fibred manifolds, *Mat. Studii, (L'viv, Ukraine)* 11 (1999) (1) 85–107.
[10] R.Ya. Matsyuk, A first order prolongation of the conventional space, in: *Differential Geometry and Applications*, Proc. Conf. 1995, Brno, Czech Republic (Masaryk Univ., Brno, 1996) 403–415.
[11] R.Ya. Matsyuk, Third order relativistic dynamics: classical spinning particle travelling in a plane, *Condensed Matter Physics (L'viv, Ukraine)* 1 (1998) (3) 453–462.
[12] L.V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, Orlando, 1982).

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