

## Second order variational problem and 2–dimensional concircular geometry\*

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### Abstract

It is proved that the set of geodesic circles in two dimensions may be given a variational description and the explicit form of it is presented. In the limit case of the Euclidean geometry a certain claim of uniqueness of such description is proved. A formal notion of ‘spin’ force is discovered as a by-product of the variation procedure involving the acceleration.

## 1 Introduction

The concircular geometry deals with geodesic circles in (pseudo)-Riemannian space. Geodesic circles in two dimensions are those curves in 2-dimensional (pseudo)-Riemannian space who preserve the Frenet curvature along them. In relativity theory this coincides with the definition of the uniformly accelerated one-dimensional motion of a test particle. The ordinary differential equation to govern such curves has order three [1]. Thus the Lagrange function should involve second derivatives and, at the same time, it should depend linearly on them.

Aiming at simplification of the exposition and of the accompanying notations, let us agree not to be confused with such notions as vector or bivector norm in pseudo-Riemannian geometry. Although the outcome of present investigation lucidly concerns both the proper Riemannian and the pseudo-Riemannian geometries, for the sake of prudence one may restrict oneself to the case of proper Riemannian space, and it still will remain evident, wherein the results will be valid in actually the pseudo-Riemannian framework as well. Thus hereinafter we shall somewhat vaguely use the terms *Riemannian* and *Euclidean*, keeping in mind that strictly speaking, some details of pure technical developments can in fact apply only to proper Riemannian case.

Consider the following Lagrange function in 2–dimensional Euclidean space:

$$(1) \quad L = L_{II} + L_I = \frac{\epsilon_{ij} \dot{u}^i \dot{u}^j}{\|\mathbf{u}\|^3} - m \|\mathbf{u}\| ,$$

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with  $\epsilon_{ij}$  denoting the skew-symmetric covariant Levi–Civita symbol. The first addend,  $L_{II}$ , is the so-called signed first Frenet curvature of a path. Further in this contribution we show that the expression (1) as a candidate for the Lagrange function is very tightly defined by the conditions of the symmetry of corresponding equation of motion and by the request that the Frenet curvature be preserved along the extremal curves.

Formula (1) clearly suggests accepting same Lagrange function also for general Riemannian case,

$$(2) \quad L^R = k - m\|\mathbf{u}\| .$$

To prove the preservation of curvature  $k$  along the extremals of (2) we need some further tools as introduced below.

## 2 Means from higher order mechanics of Ostrohrads'kyj

### 2.1 Parametric homogeneity

Let  $T^q M = \{x^j, u^j, \dot{u}^j, \ddot{u}^j, \dots, u^{(q-1)j}\}$  denote the manifold of  $q^{\text{th}}$ -order Ehresmann velocities to the base manifold  $M$  of dimension  $n$ . The prolonged reparametrization group  $Gl_n^q = J_0^q(\mathbb{R}, \mathbb{R})_0$  acts on the manifold  $T^q M = J_0^q(\mathbb{R}, M)$  by composition of jets (in our case  $n = 2$ ). As far as the Lagrange function (2) depends on the derivatives of at most second order, it lives on the space  $T^2 M$ . The infinitesimal counterpart of the above mentioned parameter transformations of  $T^2 M$  (we put  $q = 2$ ) is given by so-called fundamental fields (for arbitrary order consult [2, 3]):

$$\zeta_1 = u^i \frac{\partial}{\partial u^i} + 2 \dot{u}^i \frac{\partial}{\partial \dot{u}^i}, \quad \zeta_2 = u^i \frac{\partial}{\partial \dot{u}^i} .$$

If a function  $F$  defined on  $T^2 M$  does not change under arbitrary parameter transformations discussed above, then it with necessity satisfies the following sufficient conditions:

$$(3) \quad \zeta_1 F = 0, \quad \zeta_2 F = 0 .$$

On the other hand, if a function  $L$  on  $T^2 M$  defines a parameter-independent autonomous variational problem with the action functional

$$\int L(x^j, u^j, \dot{u}^j) d\zeta ,$$

then it also with necessity satisfies the so-called Zermelo sufficient conditions [4, 5]:

$$(4) \quad \zeta_1 L = L, \quad \zeta_2 L = 0 .$$

The generalized momenta are being conventionally introduced by the next expressions:

$$p_i^{(2)} = \frac{\partial L}{\partial \dot{u}^i}, \quad p_i^{(1)} = \frac{\partial L}{\partial u^i} - \frac{d}{d\zeta} p_i^{(2)},$$

while the Hamilton function reads:

$$H = p_i^{(2)} \dot{u}^i + p_i^{(1)} u^i - L.$$

This Hamilton function may also be expressed in different way [3, 8]:

$$(5) \quad H = \zeta_1 L + \frac{d}{d\zeta} \zeta_2 L - L.$$

As the Hamilton function is a constant of motion, from (3), (4), and (5) we immediately obtain the following proposition:

**Proposition 2.1.** *Let a function  $L_{II}$  be parameter-independent, and let another function  $L_I$  define a parameter-independent variational problem on  $T^2M$ . Then  $L_{II}$  is constant along the extremals of the variational problem, defined by the Lagrange function*

$$(6) \quad L = L_{II} + L_I.$$

This holds because  $L_{II} = -H$  with  $H$  corresponding to (6).

Frenet curvature is constant along the extremals of (2), so by the Proposition 2.1 we have right to state:

**Claim 2.1** ([6, 7]). *The Lagrange function(2) constitutes the variational principle for the geodesic circles.*

Now we wish to provide evidence that in the limit case of Euclidean space the corresponding Euler-Poisson equation may be specified by means of symmetry considerations together with the curvature preservation requirement. This means that the inverse variational problem tools should be applied.

## 2.2 The generalized Helmholtz conditions and symmetry.

Following Tulczyjew (see [9, 3]), let us introduce some operators, acting in the graded algebra of differential forms who live on manifolds  $T^qM$  of varying order  $q$  of jets:

1. The total derivative:

$$d_T f = u^i \frac{\partial f}{\partial x^i} + \dot{u}^i \frac{\partial f}{\partial u^i} + \ddot{u}^i \frac{\partial f}{\partial \dot{u}^i} + \cdots + u^{q_i} \frac{\partial f}{\partial u^{q-1_i}}, \quad dd_T = d_T d;$$

2. For each of  $r \leq q$  the derivations of zero degree:

$$\begin{aligned} i_0(\omega) &= \text{deg}(\omega) \omega, & i_r(f) &= 0, & i_r(dx^i) &= 0, \\ i_r(d^k u^i) &= \frac{(k+1)!}{(k-r+1)!} d^{k-r} u^i, & i_r(d^k u^i) &= 0, & \text{if } k < r-1; \end{aligned}$$

3. The Lagrange derivative:

$$\delta = \left( i_0 - d_T i_1 + \frac{1}{2} d_T^2 i_2 - \frac{1}{6} d_T^3 i_3 + \cdots + \frac{(-1)^q}{q!} d_T^q i_q \right) d.$$

It is of common knowledge that the Euler–Poisson expressions constitute a covariant object.

**Lemma 2.1 ([9]).** *Let a system of some differential expressions of the third order form a covariant object—the differential one-form*

$$(7) \quad \varepsilon = E_i (x^j, u^j, \dot{u}^j, \ddot{u}^j) dx^i.$$

Then  $\varepsilon = \delta(L)$  for some (local)  $L$  if and only if

$$(8) \quad \delta(\varepsilon) = 0.$$

Developing the criterion (8) amounts to establishing a general pattern for the expression (7),

$$(9) \quad E_i = A_{ij}(x^l, u^l) \ddot{u}^j + \dot{u}^p \frac{\partial}{\partial u^p} A_{ij}(x^l, u^l) \dot{u}^j + B_{ij}(x^l, u^l) \dot{u}^j + q_i(x^l, u^l),$$

and to some generalized Helmholtz conditions [8, 10, 11], cast in the form of a system of partial differential equations, imposed on the coefficients  $A_{ij} = -A_{ji}$ ,  $B_{ij}$ , and  $q_i$ :

$$\begin{aligned} (10a) \quad & \partial_{u^{[i}} A_{j]l} = 0 \\ & 2 B_{[ij]} - 3 \mathbf{D}_1 A_{ij} = 0 \\ (10b) \quad & 2 \partial_{u^{[i}} B_{j]l} - 4 \partial_{x^{[i}} A_{j]l} + \partial_{x^l} A_{ij} + 2 \mathbf{D}_1 \partial_{u^l} A_{ij} = 0 \\ & \partial_{u^{(i}} q_{j)} - \mathbf{D}_1 B_{(ij)} = 0 \\ & 2 \partial_{u^l} \partial_{u^{[i}} q_{j]} - 4 \partial_{x^{[i}} B_{j]l} + \mathbf{D}_1^2 \partial_{u^l} A_{ij} + 6 \mathbf{D}_1 \partial_{x^{[i}} A_{j]l} = 0 \\ & 4 \partial_{x^{[i}} q_{j]} - 2 \mathbf{D}_1 \partial_{u^{[i}} q_{j]} - \mathbf{D}_1^3 A_{ij} = 0, \end{aligned}$$

where the notation  $\mathbf{D}_1 = u^p \partial_{x^p}$  was introduced.

The Euclidean symmetry means that everywhere on the submanifold  $E$  defined by the system of equations  $E_l = 0$  the shifted system  $X(E_l)$  vanishes too, where

$X$  denotes the prolonged generator of (pseudo)-orthogonal transformations. We denote this criterion as

$$(11) \quad X(E_l) |_E = 0.$$

That we tend to embrace nothing more but only the *geodesic circles* as extremals, falls into similar condition:

$$(12) \quad (d_T k) |_E = 0.$$

As far as in two-dimensional space ( $\dim M = 2$ ) the skew-symmetric matrix  $A_{ij}$  is invertible, it is not difficult to implement conditions (11) and (12).

If one wishes to include in the set of extremals all those Euclidean geodesics that refer to the natural parameter, one has to imply one more condition:

$$(13) \quad E_l |_{\dot{\mathbf{u}}=0}.$$

**Theorem 2.1.** *Let a third order autonomous dynamical equation  $\mathbf{E} = \mathbf{0}$  in two-dimensional space obey conditions:*

1.  $\delta(\varepsilon) = 0$ ;
2. *The system of ODEs  $\{E_j = 0\}$  possesses the Euclidean symmetry;*
3. *The system  $\{E_j = 0\}$  possesses the first integral — the Frenet curvature  $k$ , and includes all curves of constant curvature as its solutions;*
4. *It also includes the strait lines with natural parametrization,  $\dot{\mathbf{u}} = \mathbf{0}$ .*

Then

$$E_i = \frac{\epsilon_{ij}\ddot{u}^j}{\|\mathbf{u}\|^3} - 3 \frac{(\dot{\mathbf{u}} \cdot \mathbf{u})}{\|\mathbf{u}\|^5} \epsilon_{ij}\dot{u}^j + m \frac{\|\mathbf{u}\|^2 \dot{u}_i - (\dot{\mathbf{u}} \cdot \mathbf{u}) u_i}{\|\mathbf{u}\|^3}.$$

The Lagrange function is given by (1).

### Remarks.

- If, for instance, we took  $L = k\sqrt{u_i u^i}$ , then  $H = 0$  for this Lagrange function, and the Proposition 2.1 wouldn't work.
- Because of the non-degeneracy of the matrix  $A_{ij}$ , there cannot exist a parameter-invariant variational problem in two dimensions that would produce strictly the third order Euler–Poisson equation. But, if we omit the first addend  $k$  in (2), then what remains defines the conventional parameter-invariant problem for the Riemannian projective geodesic paths. So, what fixes the parameter along the extremal in our case, is the Frenet curvature  $k$  in (2).

One should confer with [12] and [13] on these remarks.

### 2.3 Proof of the Theorem 2.1

Before passing to the proof of the above Theorem let us notice two simplification formulæ which hold at specific occasion of two dimensions. Namely, for arbitrary vectors  $\mathbf{a}$ ,  $\mathbf{c}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  it is true that

$$(14) \quad \|\mathbf{a} \wedge \mathbf{c}\| = \sqrt{|\det[g_{ij}]| |\epsilon_{ij} a^i c^j|} \quad \text{and} \quad \|\mathbf{a} \wedge \mathbf{c}\| \|\mathbf{v} \wedge \mathbf{w}\| = |(\mathbf{a} \wedge \mathbf{c}) \cdot (\mathbf{v} \wedge \mathbf{w})| ,$$

where, as usual,  $(\mathbf{a} \wedge \mathbf{c}) \cdot (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{a} \cdot \mathbf{v})(\mathbf{c} \cdot \mathbf{w}) - (\mathbf{c} \cdot \mathbf{v})(\mathbf{a} \cdot \mathbf{w})$  [14]. Also, let us agree to postpone the proof of the second part of statement 3 of Theorem 2.1 until more general Riemannian case proved in Section 3.2.

*Proof of the necessity implication of Theorem 2.1 assumptions.* In order to meet the condition 4 of the Theorem 2.1 in the form(13), we have to remove the array  $\mathbf{q}$  from (9). Next we write down the first part of the statement 3 given by means of (12). Starting with the expression

$$(15) \quad k = \frac{\|\mathbf{u} \wedge \dot{\mathbf{u}}\|}{\|\mathbf{u}\|^3}$$

of the Frenet curvature we substitute  $\ddot{\mathbf{u}}$  in

$$d_T k = \frac{(\mathbf{u} \wedge \dot{\mathbf{u}}) \cdot (\mathbf{u} \wedge \ddot{\mathbf{u}})}{\|\mathbf{u}\|^3 \|\mathbf{u} \wedge \dot{\mathbf{u}}\|} - 3 \frac{\|\mathbf{u} \wedge \dot{\mathbf{u}}\| (\mathbf{u} \cdot \dot{\mathbf{u}})}{\|\mathbf{u}\|^5}$$

by  $\ddot{\mathbf{u}} = -A^{-1}(\dot{\mathbf{u}} \cdot \partial_u) A \dot{\mathbf{u}} - A^{-1} B \dot{\mathbf{u}}$  of (9) and then split the expression (12) by the powers of  $\dot{\mathbf{u}}$  to obtain separately

$$(16a) \quad (\mathbf{u} \cdot \mathbf{u})(\mathbf{u} \wedge \dot{\mathbf{u}}) \cdot (\mathbf{u} \wedge (A^{-1}(\dot{\mathbf{u}} \cdot \partial_u) A \dot{\mathbf{u}})) + 3 (\mathbf{u} \cdot \dot{\mathbf{u}}) \|\mathbf{u} \wedge \dot{\mathbf{u}}\|^2 = 0$$

$$(16b) \quad (\mathbf{u} \wedge \dot{\mathbf{u}}) \cdot (\mathbf{u} \wedge (A^{-1} B \dot{\mathbf{u}})) = 0 .$$

Let us recall that the covariant and the contravariant Levi–Civita symbols are related by  $\epsilon_{ij} e^{jl} = -\delta_i^l$  and also let matrix  $A$  be expressed as  $A_{ij} = A_{12} \epsilon_{ij}$ . With these agreements the first addend in (16a) becomes

$$\frac{1}{A_{12}} \|\mathbf{u}\|^2 \|\mathbf{u} \wedge \dot{\mathbf{u}}\|^2 (\dot{\mathbf{u}} \cdot \partial_u) A_{12} ,$$

thus reducing (16a) by means of (14) to the partial differential equation

$$\|\mathbf{u}\|^2 (\dot{\mathbf{u}} \cdot \partial_u) A_{12} + 3 A_{12} (\mathbf{u} \cdot \dot{\mathbf{u}}) = 0$$

that in turn yields the solution

$$A_{12} = \alpha \|\mathbf{u}\|^{-3} .$$

Now we see that matrix  $A$  satisfies the relations

$$(17) \quad \dot{\mathbf{u}} \cdot \partial_u A = -3 \frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{\|\mathbf{u}\|^2} A,$$

and, evidently,

$$(18) \quad e^{ij} u_i \frac{\partial}{\partial u^j} A = 0,$$

with the help of which the Euler–Poisson expression (9) becomes

$$(19) \quad \mathbf{E} = A\ddot{\mathbf{u}} - 3 \frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{\|\mathbf{u}\|^2} A\dot{\mathbf{u}} + B\dot{\mathbf{u}},$$

so that the submanifold  $\mathbf{E} = \mathbf{0}$  is now defined by the equation

$$(20) \quad \ddot{\mathbf{u}} = 3 \frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{\|\mathbf{u}\|^2} A^{-1} B\dot{\mathbf{u}}.$$

Again with the help of (14) the equation (16b) takes the shape

$$\|\mathbf{u} \wedge (A^{-1} B\dot{\mathbf{u}})\| = 0, \quad \text{or} \quad \epsilon_{ij} e^{jp} B_{pl} u^i \dot{u}^l = 0,$$

from where it follows that

$$(21) \quad u^p B_{pl} = 0.$$

The generators of the Euclidean symmetry are enumerated by an arbitrary constant  $\varpi$  and an arbitrary constant array  $\boldsymbol{\chi} = \{\chi^i\}$  and they read:

$$(22a) \quad \boldsymbol{\chi} \cdot \partial_x \left( \equiv \chi^i \frac{\partial}{\partial x^i} \right);$$

$$(22b) \quad \varpi e^{ij} \left( x_i \frac{\partial}{\partial x^j} + u_i \frac{\partial}{\partial u^j} + \dot{u}_i \frac{\partial}{\partial \dot{u}^j} + \ddot{u}_i \frac{\partial}{\partial \ddot{u}^j} \right).$$

Applying criterion (11) with  $X = \boldsymbol{\chi} \cdot \partial_x$  and taking into account the substitution (20) ends in

$$(23) \quad -\frac{\boldsymbol{\chi} \cdot \partial_x \alpha}{\alpha} B\dot{\mathbf{u}} + \boldsymbol{\chi} \cdot \partial_x B\dot{\mathbf{u}} = \mathbf{0}.$$

Applying criterion (11) with  $X$  equal to (22b) and again calling to mind the substitution (20) with the help of

$$A_{lj} e^{ij} A^{-1}{}_i{}^p = \frac{1}{A_{12}} A^{-1}{}_l{}^p = -g_{il} e^{ip}$$

ends in

$$g_{ij}e^{il}B_{lp}\dot{u}^p + e^{il}u_i\frac{\partial}{\partial u^l}B_{jp}\dot{u}^p + e^{il}B_{jl}\dot{u}_i = 0, \quad \text{identically with respect to } \dot{u}^p,$$

from where we conclude:

$$(24) \quad e^{il}u_i\frac{\partial}{\partial u^l}B_{jp} + g_{ij}e^{il}B_{lp} + g_{ip}e^{il}B_{jl} = 0.$$

We may deduce from (24) that the skew-symmetric part of  $B$  should satisfy the equation:

$$(25) \quad e^{ij}u_i\frac{\partial}{\partial u^j}B_{[lp]} + g_{il}e^{ij}B_{[jp]} + g_{ip}e^{ij}B_{[lj]} = 0.$$

Let the skew-symmetric part of matrix  $B$  be presented as  $\beta\epsilon_{ij}$ . Then equation (25) confirms that  $\beta$  should be a differential invariant:

$$(26) \quad e^{ij}u_i\frac{\partial}{\partial u_j}\beta = 0.$$

But the variationality condition (10a) now says:

$$(27) \quad 2\beta = 3\mathbf{u}\cdot\boldsymbol{\partial}_x\alpha.$$

Applying the left hand side operator of (26) to (27) along with equation (18) produces

$$\epsilon_{ji}e^{ip}\frac{\partial}{\partial x^p}\alpha = 0.$$

Thus  $\alpha$  does not depend on  $x^i$ . Looking back at (27) immediately implies  $\beta = 0$ , matrix  $B$  being symmetric thus. In addition, we see that matrix  $B$  also should not depend on  $x^i$  by the reason of relation (23).

Now it is time to turn back to the constraint (21). Of course, we could have used it much earlier, but we prefer to unleash it now. So, the two equations, contained there, allow us to prescribe the shape to the matrix  $B$  as follows (independent of its virtual symmetry). Let  $B_{12} = b_1u_2$ ,  $B_{21} = b_2u_1$ . Then from (21) one has:

$$B_{ij} = b_iu_j - (\mathbf{b}\cdot\mathbf{u})g_{ij}.$$

But we already know that  $B_{[ij]} = 0$ . This immediately implies that  $\mathbf{b}$  and  $\mathbf{u}$  must be collinear,  $\mathbf{b} = \mu\mathbf{u}$ , thus suggesting the following form of matrix  $B$ :

$$(28) \quad B_{ij} = \mu(u_iu_j - (\mathbf{u}\cdot\mathbf{u})g_{ij}).$$

Let us again act on (28) with the operator  $e^{ij}u_i\frac{\partial}{\partial u^j}$  and make use of (24). After some simplifications we get:

$$e^{ij}u_i\frac{\partial}{\partial u^j}\mu = 0,$$



what suggests that  $\mu$  depends on  $u^i$  exclusively via  $\mathbf{u} \cdot \mathbf{u}$ .

The definite step consists in applying the second valid variational criterion, that of (10b). It is efficient to make contraction of (10b) with  $u^i$  on the left and in meanwhile not to forget about the constraint (21). One obtains:

$$u^i \frac{\partial}{\partial u^i} B_{jp} = -B_{jp}.$$

Together with the guise (28) this produces

$$\left( 2(\mathbf{u} \cdot \mathbf{u}) \frac{\partial \mu}{\partial \|\mathbf{u}\|^2} + 3\mu \right) (u_i u_j - (\mathbf{u} \cdot \mathbf{u}) g_{ij}) = 0,$$

what clearly has the solution  $\mu = \frac{m}{(\mathbf{u} \cdot \mathbf{u})^{3/2}}$  and so says the finite appearance of  $B$ :

$$B_{ij} = \frac{m}{(\mathbf{u} \cdot \mathbf{u})^{3/2}} (u_i u_j - (\mathbf{u} \cdot \mathbf{u}) g_{ij}).$$

□

### 3 The variational description of geodesic circles

#### 3.1 The variational equation

Before calculating the variation of the integrand in the functional expression  $\int k d\zeta$  let us agree on some basic formulæ. If  $v$  denotes the infinitesimal shift of the path  $x^i(\zeta)$  and if  $\tilde{D}$  stands for the covariant differentiation operator according to that shift, then the covariant variation of any vector field  $\xi$  along this path is given by

$$(29) \quad \langle v, \tilde{D}\xi \rangle^i = \langle v, d\xi^i \rangle + \Gamma_{lj}^i \xi^j v^l.$$

Let the covariant derivative of a vector field be notated by prime. And let us introduce a special designation for the evaluation of Riemannian curvature on velocities as follows:

$$\sigma_j^l = R_{ji,p}{}^l u^i u^p.$$

The vector differential one-form  $\boldsymbol{\sigma} = [\sigma_j^l]$  is semi-basic when the projection  $TM \rightarrow M$  is considered:  $\langle v, \boldsymbol{\sigma} \rangle^l = \sigma_j^l v^j$ . Let  $\boldsymbol{\theta}$  denote the vector one-form representing the identity:  $\boldsymbol{\theta} = [\delta_j^l]$ . Next formulæ replace then the usual interchange rule between infinitesimal variation and ordinary differentiation:

$$(30) \quad \begin{aligned} \tilde{D}\mathbf{u} &= \boldsymbol{\theta}' \text{ [this recapitulates definition (29)],} \\ \tilde{D}(\mathbf{u}') &= (\tilde{D}\mathbf{u})' - \boldsymbol{\sigma} \text{ [this recapitulates the definition of the tensor } R_{ji,p}{}^l \text{].} \end{aligned}$$

Further on we shall find escape from highly tangled and tedious calculations in the truth of the following relation (valid in two dimensions only):

$$(31) \quad (\mathbf{a} \cdot \mathbf{a})(\mathbf{v} \wedge \mathbf{c}) \cdot (\mathbf{v} \wedge \mathbf{c}) - (\mathbf{a} \cdot \mathbf{v})(\mathbf{v} \wedge \mathbf{c}) \cdot (\mathbf{a} \wedge \mathbf{c}) + (\mathbf{a} \cdot \mathbf{c})(\mathbf{v} \wedge \mathbf{c}) \cdot (\mathbf{a} \wedge \mathbf{v}) = 0,$$

along with the simplification formulæ (14).

The above formal and highly symbolic notations save place and time and help to avoid unessential calculative details, whereas keeping the skeleton of the variational procedure untouched and faithfully tracing the logical outlines of our development as well as producing the correct final result.

With these prerequisites we calculate the covariant variation of the Frenet curvature (15), discarding terms which present total covariant derivatives:

$$\begin{aligned}
\tilde{D}k &= \frac{(\mathbf{u} \wedge \mathbf{u}') \cdot (\tilde{\mathbf{D}}\mathbf{u} \wedge \mathbf{u}')}{\|\mathbf{u}\|^3 \|\mathbf{u} \wedge \mathbf{u}'\|} - 3 \frac{\|\mathbf{u} \wedge \mathbf{u}'\|}{\|\mathbf{u}\|^5} (\mathbf{u} \cdot \tilde{\mathbf{D}}\mathbf{u}) + \frac{(\mathbf{u} \wedge \mathbf{u}') \cdot (\mathbf{u} \wedge \tilde{\mathbf{D}}\mathbf{u}')}{\|\mathbf{u}\|^3 \|\mathbf{u} \wedge \mathbf{u}'\|} = \\
& \text{[by (14), (30), and Leibniz rule]} \\
&= 2 \frac{\|\tilde{\mathbf{D}}\mathbf{u} \wedge \mathbf{u}'\|}{\|\mathbf{u}\|^3} - 3 \frac{\|\mathbf{u} \wedge \mathbf{u}'\|}{\|\mathbf{u}\|^5} (\mathbf{u} \cdot \tilde{\mathbf{D}}\mathbf{u}) - 3 \frac{\|\tilde{\mathbf{D}}\mathbf{u} \wedge \mathbf{u}\|}{\|\mathbf{u}\|^5} (\mathbf{u} \cdot \mathbf{u}') \\
& \quad - \frac{(\mathbf{u} \wedge \mathbf{u}') \cdot (\mathbf{u} \wedge \boldsymbol{\sigma})}{\|\mathbf{u}\|^3 \|\mathbf{u} \wedge \mathbf{u}'\|} \\
&= - \frac{\|\tilde{\mathbf{D}}\mathbf{u} \wedge \mathbf{u}'\|}{\|\mathbf{u}\|^3} - \frac{(\mathbf{u} \wedge \mathbf{u}') \cdot (\mathbf{u} \wedge \boldsymbol{\sigma})}{\|\mathbf{u}\|^3 \|\mathbf{u} \wedge \mathbf{u}'\|} \quad \text{[by (31)]} \\
&= \frac{\|\boldsymbol{\theta} \wedge \mathbf{u}'\|}{\|\mathbf{u}\|^3} - 3 \frac{\|\boldsymbol{\theta} \wedge \mathbf{u}'\|}{\|\mathbf{u}\|^5} (\mathbf{u} \cdot \mathbf{u}') - \frac{(\mathbf{u} \wedge \mathbf{u}') \cdot (\mathbf{u} \wedge \boldsymbol{\sigma})}{\|\mathbf{u}\|^3 \|\mathbf{u} \wedge \mathbf{u}'\|} \quad \text{[by Leibniz rule again]}.
\end{aligned}$$

Let us introduce one more succinct notation:

$$\mathcal{R}_j = \frac{\sqrt{|\det[g_{ij}]|}}{\|\mathbf{u}\|^3} \epsilon_{il} R_{jn,p}{}^l u^i u^p u^n$$

The relation between this scalar semi-basic one form  $\mathcal{R}_j dx^j$  and previously introduced vector semi-basic one form  $\sigma^i_j dx^j$  is obvious:

$$\sqrt{|\det[g_{ij}]|} \frac{\epsilon_{il} u^i \sigma^l_j}{\|\mathbf{u}\|^3} = \mathcal{R}_j.$$

Both quantities satisfy the constraint imposed on the contraction with velocity:

$$(32) \quad \mathcal{R}_j u^j = 0,$$

along with

$$(33) \quad u_i \sigma^i_j = 0.$$

Now the Euler–Poisson equation for the complete Lagrange function (2) may be expressed in the form, valid in each case of different signature of metric tensor  $g_{ij}$  with the help of Hodge star operator:

$$(34) \quad \mathbf{E}^R = - \frac{* \mathbf{u}''}{\|\mathbf{u}\|^3} + 3 \frac{(\mathbf{u} \cdot \mathbf{u}')}{\|\mathbf{u}\|^5} * \mathbf{u}' + m \frac{(\mathbf{u} \cdot \mathbf{u}) \mathbf{u}' - (\mathbf{u}' \cdot \mathbf{u}) \mathbf{u}}{\|\mathbf{u}\|^3} - \mathcal{R} = 0$$

**Remark.** The force  $\mathcal{R}$  may be given another shape thanks to the relation (33):

$$\mathcal{R}_l dx^l = \frac{(\mathbf{u} \wedge \mathbf{u}') \cdot (\mathbf{u} \wedge \boldsymbol{\sigma})}{\|\mathbf{u}\|^3 \|\mathbf{u} \wedge \mathbf{u}'\|} = \frac{\boldsymbol{\sigma} \cdot \mathbf{u}'}{\|\mathbf{u}\| \|\mathbf{u} \wedge \mathbf{u}'\|} = \frac{1}{2} R_{lj,pi} u^j S^{pi} dx^l,$$

where  $S^{pi} = \frac{(\mathbf{u} \wedge \mathbf{u}')^{pi}}{\|\mathbf{u}\| \|\mathbf{u} \wedge \mathbf{u}'\|}$  is a formally introduced ‘spin’ tensor.

### 3.2 Completeness of variational description of geodesic circles

It remains to prove that every geodesic circle may be given a consistent parametrization, which makes it an extremal of the variational problem with the Lagrange function (2).

*The governing equation for the geodesic circles.* With the intention to derive a dynamical differential equation, governing the motion along a geodesic path, we put equal to zero the derivative of the Frenet curvature function  $k$  in terms of natural parametrization by  $ds = \sqrt{u_i u^i} d\zeta$ :

$$(35) \quad \mathbf{u}'_s \cdot \mathbf{u}''_s = 0.$$

To it we add the obvious constraint

$$(36) \quad \mathbf{u}'_s \cdot \mathbf{u}'_s + \mathbf{u}_s \cdot \mathbf{u}''_s = 0,$$

which merely presents the differential consequence of

$$(37) \quad \mathbf{u}_s \cdot \mathbf{u}'_s = 0.$$

Next we solve the system of equations (35) and (36) for  $\mathbf{u}''_s$  to obtain

$$(38) \quad (u''_s)_l = \frac{\epsilon_{li} (u'_s)^i}{\epsilon_{ij} (u'_s)^i (u_s)^j} \mathbf{u}'_s \cdot \mathbf{u}'_s.$$

We leave it to the Reader to check with the help of (37) and of  $\mathbf{u}_s \cdot \mathbf{u}_s = 1$  that in two-dimensional space the late (38) by means of the relation  $\epsilon_{il} (u'_s)^i = (u'_s)_l \epsilon_{ij} (u'_s)^i (u_s)^j$  reduces to the well known governing equation of geodesic circles

$$(39) \quad \mathbf{u}''_s + (\mathbf{u}'_s \cdot \mathbf{u}'_s) \mathbf{u} = 0.$$

In order to dispense with the constraint  $\mathbf{u}_s \cdot \mathbf{u}_s = 1$  we recalculate the derivatives in (39) by the reparametrization from  $s$  to an arbitrary elapsed parameter  $\zeta$  along the path of a geodesic circle to see at last that geodesic circles accept characterization as the integral curves of the following parameter-homogeneous differential equation:

$$(40) \quad \frac{\mathbf{u}''}{\|\mathbf{u}\|^3} = \frac{\mathbf{u} \cdot \mathbf{u}''}{\|\mathbf{u}\|^5} \mathbf{u} + 3 \frac{\mathbf{u} \cdot \mathbf{u}'}{\|\mathbf{u}\|^5} \mathbf{u}' - 3 \frac{(\mathbf{u} \cdot \mathbf{u}')^2}{\|\mathbf{u}\|^7} \mathbf{u}.$$

*Proof of the exhaustiveness of extremal set.* Let us complement equation (40) by the following additional one, which is consistent with the equation (34) (as its consequence) and will play the role of the means to fix the way of parametrization along the extremal curve:

$$(41) \quad \frac{\mathbf{u} \cdot \mathbf{u}''}{\|\mathbf{u}\|^3} - 3 \frac{(\mathbf{u} \cdot \mathbf{u}')^2}{\|\mathbf{u}\|^5} = \left( \frac{m}{\|\mathbf{u}\|} \mathbf{u} \wedge \mathbf{u}' - \mathbf{u} \wedge \mathcal{R} \right).$$

For the sake of efficiency, let us evaluate the Euler–Poisson expression (34) on some arbitrary vector  $\mathbf{v}$ :

$$\mathbf{E}^R \cdot \mathbf{v} = \frac{* (\mathbf{v} \wedge \mathbf{u}'')}{\|\mathbf{u}\|^3} - 3 \frac{\mathbf{u} \cdot \mathbf{u}'}{\|\mathbf{u}\|^5} * (\mathbf{v} \wedge \mathbf{u}') + \frac{m}{\|\mathbf{u}\|^3} (\mathbf{u} \wedge \mathbf{u}') \cdot (\mathbf{u} \wedge \mathbf{v}) - \mathcal{R} \cdot \mathbf{v}.$$

If now we substitute  $\mathbf{u}''$  in this equation with the expression from (40) and simultaneously take into account the additional equation (41), we will get:

$$\begin{aligned} \mathbf{E}^R \cdot \mathbf{v} &= - \frac{* (\mathbf{v} \wedge \mathbf{u}) * (\mathbf{v} \wedge \mathcal{R})}{\|\mathbf{u}\|^2} + \frac{m}{\|\mathbf{u}\|^3} * (\mathbf{v} \wedge \mathbf{u}) * (\mathbf{u} \wedge \mathbf{u}') \\ &\quad + \frac{m}{\|\mathbf{u}\|^3} (\mathbf{u} \wedge \mathbf{u}') \cdot (\mathbf{u} \wedge \mathbf{v}) - \mathcal{R} \cdot \mathbf{v} \\ &= - \frac{(\mathbf{v} \cdot \mathbf{u})(\mathbf{u} \cdot \mathcal{R})}{\|\mathbf{u}\|^2} + \mathbf{v} \cdot \mathcal{R} - \mathcal{R} \cdot \mathbf{v} \equiv 0 \end{aligned}$$

because of (32) □

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