# TOWARDS THE PHYSICAL SIGNIFICANCE OF THE $(k^2 + A)\|u\|$ METRIC.*

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**Abstract**

We offer an example of the second order Kawaguchi metric function the extremal flow of which generalizes the flat space-time model of the semi-classical spinning particle to the framework of the pseudo-Riemannian space-time. The general shape of the variational Euler–Poisson equation of the fourth order in the (pseudo-)Riemannian space is being developed too.

**Introduction.** In 1946 Fritz Bopp in an attempt to describe the relativistic motion of the charged particle influenced by self-radiation in flat space-time considered a Lagrange function [1][1)], which, in the absence of the external electromagnetic field, may be expressed in terms of the particle's world line Frenet curvature as follows:

$$L^k = (k^2 + A)\|u\| \,, \tag{1}$$

where $u$ denotes the derivative $\dot{x}$ of the configuration space variable $x$ with respect to the evolution parameter $\xi$ along the particle's world line $x^n(\xi)$. Later different modifications of Bopp Lagrangian were introduced, among them a more general expression was investigated by Lovelock in 1963 [2][2)]. Then, in 1972, Riewe, still staying in the framework of flat space-time, proposed an equation of the fourth order with the purpose to give a description of the semi-classical "Zitterbewegung" of test particle with an internal degree of freedom:

$$\frac{d^4 x^n}{ds^4} + \omega^2 \frac{d^2 x^n}{ds^2} = 0 \,, \tag{2}$$

where the derivatives are calculated with respect to the natural parameter. Recently in papers [4] and [5] I showed that the Riewe equation follows from the Bopp Lagrangian under the a posteriori imposed constraint $k^2 = \frac{1}{3} A + \frac{2}{3}\omega^2$.

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1) Numbers in brackets refer to the references at the end of the paper.

2) We even do not try to present the exhaustive bibliography on the subject here.

The goal of the present communication is to obtain a generalization of the equation (2) from the variational principle with the fundamental function (1) in the (pseudo-)Riemannian case. The space, endowed with the metric function (1), may be considered as an example of a Kawaguchi space, because this function $L^k$ satisfies Zermelo conditions.

**§ 1. The covariant momenta.** Let us introduce the following change of local coordinates in the second-order velocities space:

$$\{x^n, u^n, \dot{u}^n\} \mapsto \{x^n, u^n, u'^n\}\,,$$

where the prime stands for the covariant derivative. Let us also denote the local expression of the Lagrange function in terms of the new coordinates by $\tilde{L}$. The following formulæ produce then the receipt of the recalculation of partial derivatives:

$$\text{(3)}\qquad \frac{\partial L}{\partial u^n} = \frac{\partial \tilde{L}}{\partial u^n} + 2\,\frac{\partial \tilde{L}}{\partial u'^q}\Gamma^q{}_{mn}u^m, \qquad \frac{\partial L}{\partial x^n} = \frac{\partial \tilde{L}}{\partial x^n} + \frac{\partial \tilde{L}}{\partial u'^q}\,\frac{\partial \Gamma^q{}_{ml}}{\partial x^n}\,u^l u^m\,.$$

For further use we recall the familia conventions from the Riemannian geometry

$$\text{(4)}\qquad a'^{\,n} = \frac{da^n}{d\xi} + \Gamma^n{}_{lm}a^m u^l, \qquad a'_{\,n} = \frac{da_n}{d\xi} - \Gamma^m{}_{ln}a_m u^l\,,$$

$$\text{(5)}\qquad \frac{\partial g_{mn}}{\partial x^k} = g_{ml}\Gamma^l{}_{kn} + g_{nl}\Gamma^l{}_{km}\,,$$

$$\text{(6)}\qquad R_{kmn}{}^l = \frac{\partial \Gamma^l{}_{kn}}{\partial x^m} - \frac{\partial \Gamma^l{}_{mn}}{\partial x^k} + \Gamma^l{}_{mq}\Gamma^q{}_{kn} - \Gamma^l{}_{kq}\Gamma^q{}_{mn}\,.$$

Let us introduce the covariant momenta

$$\text{(7)}\qquad \pi^{(1)} = \frac{\partial \tilde{L}}{\partial u'}\,, \qquad \pi = \frac{\partial \tilde{L}}{\partial u} - \pi^{(1)\prime}\,.$$

**Proposition 1** *Let some Lagrange function $L$ depend on all the variables exclusively through the differential invariants $\gamma = u{\cdot}u$, $\beta = u{\cdot}u'$, and $\alpha = u'{\cdot}u'$ only. In this case the Euler–Poisson expression is:*

$$\text{(8)}\qquad \boxed{\mathcal{E}_n = -\pi'{}_n - \pi^{(1)}{}_l R_{nkm}{}^l u^m u^k}$$

*The proof is given in steps:*

*Step 1.* In second order Ostrohrads'kyj mechanics the Euler–Poisson expression $\mathcal{E}$, that constitutes the system of variational Euler–Poisson equations $\{\mathcal{E}_n = 0\}$ is known to be conveniently put down in terms of the momenta

$$\text{(9)}\qquad p^{(1)}_n = \frac{\partial L}{\partial \dot{u}^n}\,, \qquad p_n = \frac{\partial L}{\partial u^n} - \frac{dp^{(1)}_n}{d\xi}\,,$$

as follows

$$\text{(10)}\qquad \mathcal{E}_n = \frac{\partial L}{\partial x^n} - \frac{dp_n}{d\xi} = 0\,.$$

*Step 2.* The covariant momentum $\pi$, profiting from the first of the formulæ (3) together with the covariant derivative pattern (4), is presented as:

$$\pi_n = \frac{\partial L}{\partial u^n} - 2\,\Gamma^q{}_{mn} u^m \pi^{(1)}{}_q - \pi^{(1)\prime}{}_n \,. \tag{11}$$

The covariant derivative of the momentum $\pi^{(1)}$, again profiting from the pattern (4), writes down as

$$\pi^{(1)\prime}{}_n = \frac{d}{d\xi}\pi^{(1)}{}_n - \Gamma^m{}_{ln}\pi^{(1)}{}_m u^l \,. \tag{12}$$

*Step 3.* in terms of the covariant quantities above, the non-covariant quantity $p_n$ from the expression (9) is given by the following calculation:

$$\begin{aligned} p_n &= \pi_n + 2\,\Gamma^q{}_{mn} u^m \pi^{(1)}{}_q + \pi^{(1)\prime}{}_n - && \text{(by virtue of (11))} \\ &\qquad - \frac{d}{d\xi}\pi^{(1)}{}_n && \text{(by virtue of (7))} \\ &= \pi_n + \Gamma^q{}_{mn} u^m \pi^{(1)}_q && \text{(by virtue of (12)).} \end{aligned} \tag{13}$$

Differentiating (13) and applying the pattern (4) in order to express the ordinary derivatives of the variables $\pi$ and $u$ in terms, respectively, of the covariant derivatives $\pi'$ and $u'$, and implementing the guise (12), produces:

$$\begin{aligned} \frac{d}{d\xi}p_n &= \big(\pi'{}_n + \Gamma^l{}_{mn}\pi_l u^m\big) + \frac{\partial \Gamma^l{}_{mn}}{\partial x^k} u^k u^m \pi^{(1)}{}_l \\ &\quad + \big(\Gamma^l{}_{mn} u'^m - \Gamma^l{}_{mn}\Gamma^m{}_{qk} u^q u^k\big)\pi^{(1)}{}_l + \Gamma^l{}_{mn} u^m \big(\pi^{(1)\prime}{}_l + \Gamma^q{}_{kl}\pi^{(1)}{}_q u^k\big) \\ &= \pi'{}_n + \big(\pi^{(1)\prime}{}_l + \pi_l\big)\Gamma^l{}_{mn} u^m + \pi^{(1)}{}_l \Gamma^l{}_{mn} u'^m \\ &\quad + \pi^{(1)}{}_q u^m u^k \left(\Gamma^l{}_{mn}\Gamma^q{}_{lk} + \frac{\partial \Gamma^q{}_{mn}}{\partial x^k} - \Gamma^q{}_{ln}\Gamma^l{}_{mk}\right). \end{aligned}$$

*Step 4.* Now the Euler–Poisson expression (10) takes on the shape

$$\mathcal{E}_n = \frac{\partial \tilde{L}}{\partial x^n} - \big(\pi^{(1)\prime}{}_l + \pi_l\big)\Gamma^l{}_{mn} u^m - \pi^{(1)}{}_l \Gamma^l{}_{mn} u'^m - \pi'{}_n - \pi^{(1)}{}_l u^m u^k R_{nkm}{}^l \,.$$

Let us show, that the first four addends in this expression produce zero,— under the assumptions of the proposition we are now proving. For the sake of constructing the expression

$$\frac{\partial \tilde{L}}{\partial x^n} = \frac{\partial \tilde{L}}{\partial \gamma}\frac{\partial \gamma}{\partial x^n} + \frac{\partial \tilde{L}}{\partial \beta}\frac{\partial \beta}{\partial x^n} + \frac{\partial \tilde{L}}{\partial \alpha}\frac{\partial \alpha}{\partial x^n} \,, \tag{14}$$

using formula (5), we calculate:

$$\frac{\partial \gamma}{\partial x^n} = 2\Gamma^l{}_{mn} u^m u_l \,, \quad \frac{\partial \beta}{\partial x^n} = \Gamma^l{}_{mn} u^m u'{}_l + \Gamma^l{}_{mn} u'^m u_l \,, \quad \frac{\partial \alpha}{\partial x^n} = 2\Gamma^l{}_{mn} u'^m u'{}_l \,.$$

On the other hand, applying the definitions (7), we get:

$$(15)\qquad \pi^{(1)}_n = \frac{\partial\tilde L}{\partial\beta}u_n + 2\frac{\partial\tilde L}{\partial\alpha}{u'}_n\,,\quad {\pi^{(1)}}'_n + \pi_n = 2\frac{\partial\tilde L}{\partial\gamma}u_n + \frac{\partial\tilde L}{\partial\beta}{u'}_n\,.$$

Extracting these two expressions from (14) produces zero ♡.

**§ 2. The generalized variational equation of a structured particle in Riemannian space.** Now it is straightforward to obtain the equation of the extremal world line for the model (1). Recalling the expression of the first Frenet curvature,

$$k = \frac{\|u\wedge u'\|}{\|u\|^3}\,,$$

one sees that in terms of the invariants $\gamma$, $\beta$, and $\alpha$, the Lagrange function (1) takes the shape

$$L^k = \frac{\alpha\gamma-\beta^2}{\gamma^{5/2}} + A\gamma^{1/2}\,,$$

from where by means of the formulæ (15) together with the differential prolongation of the first of them,

$${\pi^{(1)}}'_n = \left(\frac{d}{d\xi}\frac{\partial\tilde L}{\partial\beta}\right)u_n + \frac{\partial\tilde L}{\partial\beta}{u'}_n + 2\left(\frac{d}{d\xi}\frac{\partial\tilde L}{\partial\alpha}\right){u'}_n + 2\frac{\partial\tilde L}{\partial\alpha}{u''}_n\,,$$

one immediately obtains:

$$(16)\quad \pi^{(1)} = \frac{2}{\|u\|^3}\,u' - \frac{2\,u\cdot u'}{\|u\|^5}\,u\,,$$

$$(17)\qquad \pi = \left(\frac{2\,u\cdot u''}{\|u\|^5} - \frac{u'\cdot u'}{\|u\|^5} - \frac{5\,(u\cdot u')^2}{\|u\|^7} + \frac{A}{\|u\|}\right)u + \frac{6\,u\cdot u'}{\|u\|^5}\,u' - \frac{2}{\|u\|^3}\,u''.$$

**§ 3. Relation to physics.**

**The Riewe equation.** The Euler–Poisson equation (8) for $L^k$ inherits the property of parametric ambivalence from the same property of the corresponding variational problem with the fundamental function (1) due to the fulfillment of the Zermelo conditions. Thus it is possible to pass to the natural parametrization in the expression (17) while substituting it in (8). Then one gets

$$\frac{D}{ds}\left[(-3\,u'_s\cdot u'_s + A)\,u_n - 2\,(u''_s)_n\right] = -{\pi^{(1)}}_l {R_{nkm}}^l u^m u^k\,.$$

The Riewe equation (2) follows from this expression in flat space-time on the surface $k = \text{const}$.

**The Dixon equations.** General relativistic top with inner angular momentum $S^{nm}$ in pseudo-Riemannian space-time is in common knowledge described by means of the system of first-order equations [6][3)]

$$\left\{ \begin{aligned} P'_n &= -\frac{1}{2}\, R_{nm}{}^{kl} u^m S_{kl}\,, \\ S'_{nm} &= \quad P_n u_m - P_m u_n\,. \end{aligned} \right. \tag{18}$$

By the skew-symmetric property of the Riemannian curvature tensor it easily follows that the first of the above equations is regained by putting $P = \pi$ and $S = u \wedge \pi^{(1)}$ in (8).

**Proposition 2** *Under the assumptions of the Proposition 1 the governing system of equations (18) does not depend on any particular appearance of the fundamental function L.*

This follows from formulæ (15) along with the similar formula for $\pi$ ♡.

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# References

[1] F. Bopp: Quantentheorie der Feldmechanik, *Zf. für Naturf.,* **1** (1946), 196–203.

[2] D. Lovelock: Classical relativistic dynamics of "spin" particles, *il Nuovo cim.,* **29** (1963), 1126–1142.

[3] F. Riewe: Relativistic classical spinning-particle mechanics, *il Nuovo cim.,* **8 B** (1972), 271–277.

[4] R. Matsyuk: A covering second–order Lagrangian for the relativistic top withouo forces, *Proc. Inst. Math. NAS of Ukraine,* **43** (2002), Part 2, 741–745.

[5] R. Matsyuk: Canonical formalism for quasi–classical particle "Zitterbewegung" in Ostrohrads̀kyj mechanics, *in Differential Geometry and Its Applications, Proc. Conf. Prague, August 30–September 3, 2004, Charles University, Prague,* 2005, 629–637.

[6] W. G. Dixon: Dynamics of extended bodies in general relativity I. Momentum and angular momentum, *Proc. Roy. Soc. London. Ser. A,* **314** (1970), 499–527.

[3)] The definition of the curvature tensor, adopted in the present communication, differs in sign from the one used in the Dixon's paper