THE NEXT VARIATIONAL PROLONGATION OF THE EUCLIDEAN SPACE.

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Abstract

The unique third-order invariant variational equation in three-dimensional (pseudo)Euclidean space is derived.

Introduction. The puzzle of what construction should be taken as the local model of a higher-order Kawaguchi space up to now does not have a common solution. From the point of view of extremal paths approach one may assert that for the role of the candidate for the very next order generalization of the celebrated (pseudo)Riemannian geometry might be taken a space, the exremals in which do satisfy a third-order differential equation. In plain geometry the first one such equation which drops in on one's mind is that of the (geodesic) circle, $\frac{dk}{ds} = 0$. In 1969 Ukrainian mathematician Skorobohat'ko suggested building up a geometry in the Euclidean plain, where geodesics should pass through n arbitrarily given points and therefore be solutions of a higher-order differential equation of the type $\frac{d^r k}{ds^r} = 0$ for some r [1, 2]. Inspired by these ideas I tried to solve the inverse variational problem for a third order differential equation in (pseudo)Euclidean space. The fact that one starts from the (pseudo)Euclidean geometry suggests that the higher-order equation of geodesic paths one looks for should inherit (pseudo)Euclidian symmetry. It turns out that in case of three-dimensional space this problem admits definite solution. Due to the very kind support of the Organizes of the Conference I take this opportunity to present the corresponding statements at this talk. The construction to be proposed here deviates from the notion of Kawaguchi space in that there does not exist an intrinsically defined integrand for the variational problem, although the variational Euler-Poisson equation itself is well defined. On other hand, to produce a strictly third-order equation, the integrand should be affine in second order derivatives. This latter feature relates it to the case of special Kawaguchi space with $p = 1$. Also in three-dimensional space the (vector) variational equation is necessary degenerate, so one may chose to prefer that of parameter-indifference generacy, again thus meeting the terms of Kawaguchi space. I dedicate this special case of third-order variational space to the name of professor Vitaliy Skorobohat'ko.

§ 1. Preliminary agreements. The shortest constructive way to treat the inverse problem of variational calculus is to introduce the operator of Lagrange differential δ. In calculable form it was done by Tulczyjew in [3] for the autonomous variational problem and modified by Kolář in [4] for the case of non-autonomous one. As our substantial considerations here will concentrate on (pseudo)Euclidean case, we shall not emphasize general significance of the notions introduced, but the Reader will easily understand, what constructions work perfectly well on general differential manifolds. Let, therefore an $n+1$-dimensional manifold M be parameterized by local coordinates $t \equiv x^0$, x^i, $i \in \overline{1,n}$ and consider the space of r-th order velocities $T^rM = J^r(\mathbb{R}, M)(0)$, those being r-th order jets with source zero from $\mathbb{R}$ to M. Let $x^\alpha, u^\alpha, \dot{u}^\alpha, \dots u^\alpha_{r-1}$, $\alpha \in \overline{0,n}$, be local coordinates in T^rM. Germs of one-dimensional submanifolds in M give rise to another space—that of one-dimensional contact elements in M, denoted by $C^r(M,1)$. This latter space locally is parameterized by coordinates $t, x^i, v^i, v'^i, \dots v^i_{r-1}$. From time to time notations u^α_{-1} and v^i_{-1} will be used instead of x^α and x^i respectively. The projection

$$\wp : \tilde{T}^rM \to C^r(M,1) \tag{1}$$

from non-zero velocities space $\tilde{T}^rM$ to the space of contact elements is that of quotient projection under the right action of the reparametrization group $GL^r(\mathbb{R})$ on the space T^rM. In the third order this projection is given by the expressions

$$\begin{aligned} v^i &= \frac{u^i}{u^0} \\ v'^i &= \frac{\dot{u}^i}{(u^0)^2} - \frac{\dot{u}^0}{(u^0)^3} u^i \\ v''^i &= \frac{\ddot{u}^i}{(u^0)^3} - 3\,\frac{\dot{u}^0}{(u^0)^4}\dot{u}^i + 3\,\frac{(\dot{u}^0)^2}{(u^0)^5}u^i - \frac{\ddot{u}^0}{(u^0)^4}u^i\,. \end{aligned} \tag{2}$$

The generalization of these formulae to arbitrary order of jets may be found in [6]. It can be deduced from general transformation rules for higher order derivatives as presented, for instance, in [5]. The contact elements manifold $C^r(M,1)$ locally is built as the jet bundle $J^r(\mathbb{R}, \mathbb{R}^n)$. Any local Lagrange density on $C^r(M,1)$ is therefore best represented by a semi-basic differential one-form

$$\Lambda = L(t, x^i, v^i, \dots v^i_{k-1})dt\,. \tag{3}$$

The corresponding local Euler-Poisson equations,

$$\mathsf{E}_i(t, x^i, v^i, \dots v^i_{r-1}) = 0 \tag{4}$$

naturally fit in with the conception of a vector differential one-form

$$e = \mathsf{E}_i dx^i \otimes dt\,. \tag{5}$$

On the space T^kM one may pose an autonomous variational problem by introducing a Lagrange function $\mathcal{L}(x^\alpha, u^\alpha, \dots u^\alpha_{k-1})$, the corresponding Euler-Poisson equations of which,

$$\mathcal{E}_\alpha(x^\alpha, u^\alpha, \dots u^\alpha_{r-1}) \tag{6}$$

fall into the shape of globally well defined differential form

$$\varepsilon = \mathcal{E}_\alpha dx^\alpha\,. \tag{7}$$

The following assertion is true:

Proposition 1 *The differential forms e from (5) and*

$$\varepsilon = -u^i \mathsf{E}_i dx^0 + u^0 \mathsf{E}_i dx^i \tag{8}$$

both satisfy variational criterion simultaneously, if either does. The corresponding local Lagrangians are related by the formula

$$\mathcal{L} = u^0 L\,. \tag{9}$$

In addition, one observes that the function (9) satisfies the Zermelo conditions, and each such $\mathcal{L}$ passes to quotient along the projection (1). A few words on variational criteria deserve saying then.

§ 2. Variational criterion. One reason for casting the system of Euler-Poisson equations in the shape of exterior differential forms is that in the algebra of differential forms the operator δ called Lagrange differential may be introduced. It satisfies $\delta^2 = 0$, due to what the criterion of the existence of a local Lagrange function for, say, the system of equations (4) is expressed as $\delta e = 0$ for e in (5).

Consider the graded algebra of differential forms on the space $J^r(\mathbb{R}, Q) \approx \mathbb{R}\times T^r Q$ of jets from $\mathbb{R}$ to arbitrary manifold Q. Let us recall that an operator D is called a derivation of degree q if for any differential form ϖ of degree p and any other differential form ω the differential form $D\varpi$ is of degree $p+q$ and the Leibniz rule $D(\varpi\wedge\omega) = D\varpi\wedge\omega + (-1)^{pq}\varpi\wedge D\omega$ holds. Let us recall some familiar operators acting on forms. The operator of vertical differential d_v is first defined on the ring of functions as $d_v f = \sum_i \frac{\partial f}{\partial x^i} dx^i + \sum_{s=0}^{r-1}\sum_i \frac{\partial f}{\partial v^i_s} dv^i_s$, $\{x^i\} \in Q$, and then extended as a derivation of degree 1 by means of the coboundary property $d_v^2 = 0$. The total derivative D_t is also first defined on the ring of functions as $D_t f = \frac{\partial f}{\partial t} + \sum_i v^i \frac{\partial f}{\partial x^i} + \sum_{s=0}^{r-1}\sum_i v^i_{s+1}\frac{\partial f}{\partial v^i_s}$, and then extended as a derivative of degree zero by means of the commutation relation $D_t d_v = d_v D_t$. Following Tulczyjew, we need one more derivation of degree zero, denoted here by ι, and defined by its action on functions and forms as $\iota f = 0$, $\iota dx^i = 0$, $\iota dv^i_s = (s+1) dv^i_{s-1}$, $s \in \overline{0, r-1}$. Let us denote by ι^0 the operator of evaluating the degree of a differential form and by D^s the iterated D. The Lagrange δ is first introduced by its action in the algebra of differential forms on $T^r Q$, eventually with coefficients depending on the time $t \in \mathbb{R}$,

$$\delta = \sum_{s=0}^{r} \frac{(-1)^s}{s!} D_t^s \iota^s d_v\,,$$

and afterwards trivially extended to the graded module of semi-basic with respect to $\mathbb{R}$ differential forms on $J^r(\mathbb{R}, Q)$ (actually one-forms) with coefficients in the bundle of graded algebras $\wedge T^*(T^r Q) \to T^r Q$ by means of the prescriptions:

$$\delta(\omega \otimes dt) = \delta(\omega) \otimes dt\,.$$

The property $\delta^2 = 0$ holds. One may apply either the notion of the (above defined time-extended) Lagrange differential to forms on the jet space $J^r(\mathbb{R}, \mathbb{R}^n)$, setting $Q = \mathbb{R}^n$, or the notion of the "truncated" time-independent Lagrange differential to the forms both on the manifold $T^r M$ as well as on the manifold $T^r(\mathbb{R}\times\mathbb{R}^n)$ setting $Q = M$ and $Q = \mathbb{R}\times\mathbb{R}^n$ respectively. Thus locally the notion of the Lagrange differential is applicable to both sides of the projection (1), whereas globally it is well defined on the left hand side solely. In each case the differential forms (5) and (7) that represent the Euler-Poisson equations are in fact semi-basic also with respect to Q. In terms of the operators introduced above this means that ιe and $\iota\varepsilon$ both are zero.

Proposition 2 *Considering formulae (3), (5), (8) and (9), if $e = \delta\Lambda$, then $\varepsilon = \delta\mathcal{L}$. The variational criterion for (5) consists in $\delta e = 0$ and is equal to $\delta\varepsilon = 0$.*

The criterion $\delta e = 0$ now can be expressed in coordinates. After some permutations of indices and some interchanges in the order of sequential sums one gets in a way similar to that of [7]

$$\delta e = \sum_{s=0}^{r} \left(\frac{\partial \mathsf{E}_i}{\partial v^j_{s-1}} - \sum_{k=s}^{r} (-1)^k \frac{k!}{(k-s)!s!} D_t^{k-s} \frac{\partial \mathsf{E}_j}{\partial v^i_{k-1}} \right) dv^j_{s-1} \wedge dx^i \,,$$

from where the following system of partial differential equations follows:

$$\frac{\partial \mathsf{E}_i}{\partial x^j} - \frac{\partial \mathsf{E}_j}{\partial x^i} + \sum_{k=0}^{r} (-1)^k D_t^k \left(\frac{\partial \mathsf{E}_i}{\partial v^j_{k-1}} - \frac{\partial \mathsf{E}_j}{\partial v^i_{k-1}} \right) = 0 \,; \tag{10a}$$

$$\frac{\partial \mathsf{E}_i}{\partial v^j_{s-1}} - \sum_{k=s}^{r} (-1)^k \frac{k!}{(k-s)!s!} D_t^{k-s} \frac{\partial \mathsf{E}_j}{\partial v^i_{k-1}} = 0 \qquad 1 \leqslant s \leqslant r \,. \tag{10b}$$

The above system of equation is equivalent to the following one (obtained from (10b) by extending the range of s to include $s = 0$):

$$\frac{\partial \mathsf{E}_i}{\partial v^j_{s-1}} - \sum_{k=s}^{r} (-1)^k \frac{k!}{(k-s)!s!} D_t^{k-s} \frac{\partial \mathsf{E}_j}{\partial v^i_{k-1}} = 0 \qquad 0 \leqslant s \leqslant r \,. \tag{11}$$

Proof. The antisymmetrization of (11) at $s = 0$ produces the equation (10a). On the contrary, in equation (10a) separate the summand with $k = 0$:

$$2\,\frac{\partial \mathsf{E}_i}{\partial x^j} - 2\,\frac{\partial \mathsf{E}_j}{\partial x^i} + \sum_{k=1}^{r} (-1)^k D_t^k \frac{\partial \mathsf{E}_i}{\partial v^j_{k-1}} - \sum_{k=1}^{r} (-1)^k D_t^k \frac{\partial \mathsf{E}_j}{\partial v^i_{k-1}} = 0 \,. \tag{12}$$

Under the first sum sign substitute $\dfrac{\partial \mathsf{E}_i}{\partial v^j_{k-1}}$ from equation (10b):

$$\sum_{k=1}^{r} (-1)^k D_t^k \frac{\partial \mathsf{E}_i}{\partial v^j_{k-1}} = \sum_{k=1}^{r} (-1)^k D_t^k \sum_{s=k}^{r} (-1)^s \frac{s!}{(s-k)!k!} D_t^{s-k} \frac{\partial \mathsf{E}_j}{\partial v^i_{s-1}} \,.$$

Interchange the summation order: $\sum_{k=1}^{r} \sum_{s=k}^{r} = \sum_{\substack{s,k=1 \\ s \geqslant k}}^{r} = \sum_{s=1}^{r} \sum_{k=1}^{s}$. Calculate the sum over k:

$$\sum_{k=1}^{s} (-1)^k \frac{s!}{(s-k)!k!} = \sum_{k=0}^{s} (-1)^k \binom{s}{k} - \binom{s}{0} = 0 - 1 = -1 \,.$$

Ultimately equation (10a) becomes

$$2\,\frac{\partial \mathsf{E}_i}{\partial x^j} - 2\,\frac{\partial \mathsf{E}_j}{\partial x^i} - \sum_{k=1}^{r}(-1)^k D_t^k\,\frac{\partial \mathsf{E}_j}{\partial v^i_{k-1}} - \sum_{k=1}^{r}(-1)^k D_t^k\,\frac{\partial \mathsf{E}_j}{\partial v^i_{k-1}} = 0\,,$$

which coincides with doubled equation (11) at $s = 0$. The criterion (11) has been obtained by different authors. The Reader may consult the book [8] by Olga Krupková for a recent review.

Let us focus on third order variational equations. It is obvious that the Euler-Poisson expressions are of affine type in the highest derivatives. We utilize some familiar vector notations: the lower dot symbol will denote the contraction between a row-array and the subsequent column-array and sometimes also will stand for the matrix multiplication between a matrix and the subsequent column-array. From the system of partial differential equations (11) it is possible to deduce that the most general form of the Euler-Poisson equation of the third order reads:

$$\mathbf{A}\,.\,\mathbf{v}'' + (\mathbf{v}'.\,\boldsymbol{\partial}_{\mathbf{v}})\,\mathbf{A}\,.\,\mathbf{v}' + \mathbf{B}\,.\,\mathbf{v}' + \mathbf{c} \;=\; \mathbf{0}\,, \tag{13}$$

where the skew-symmetric matrix $\mathbf{A}$, the symmetric matrix $\mathbf{B}$, and a column $\mathbf{c}$ all depend on t, x^i, and v^i and satisfy the following system of partial differential equations:

$$\partial_{v}{}_{[i}\mathsf{A}_{jl]} = 0 \tag{14a}$$
$$2\,\mathsf{B}_{[ij]} - 3\,\mathbf{D_1}\,\mathsf{A}_{ij} = 0 \tag{14b}$$
$$2\,\partial_{v}{}_{[i}\mathsf{B}_{j]\,l} - 4\,\partial_{x}{}_{[i}\mathsf{A}_{j]\,l} + \partial_{x}{}_{l}\,\mathsf{A}_{ij} + 2\,\mathbf{D_1}\,\partial_{v}{}_{l}\,\mathsf{A}_{ij} = 0 \tag{14c}$$
$$\partial_{v}{}_{(i}\mathsf{c}_{j)} - \mathbf{D_1}\,\mathsf{B}_{(ij)} = 0 \tag{14d}$$
$$2\,\partial_{v}{}_{l}\,\partial_{v}{}_{[i}\mathsf{c}_{j]} - 4\,\partial_{x}{}_{[i}\mathsf{B}_{j]\,l} + \mathbf{D_1}{}^2\,\partial_{v}{}_{l}\,\mathsf{A}_{ij} + 6\,\mathbf{D_1}\,\partial_{x}{}_{[i}\mathsf{A}_{jl]} = 0 \tag{14e}$$
$$4\,\partial_{x}{}_{[i}\mathsf{c}_{j]} - 2\,\mathbf{D_1}\,\partial_{v}{}_{[i}\mathsf{c}_{j]} - \mathbf{D_1}{}^3\,\mathsf{A}_{ij} = 0\,. \tag{14f}$$

Here the differential operator $\mathbf{D_1}$ is the lowest order truncated operator of total derivative D_t,

$$\mathbf{D_1} = \partial_t + \mathbf{v}\,.\,\boldsymbol{\partial}_{\mathbf{x}}\,.$$

Alongside with the differential form (5) it is convenient to introduce the so-called Lepagian equivalent to it, whose coefficients do not depend on third-order derivatives:

$$\epsilon = \mathsf{A}_{ij}dx^i \otimes dv'^j + \mathsf{k}_i dx^i \otimes dt, \quad \text{where}$$
$$\mathbf{k} = (\mathbf{v}'.\,\boldsymbol{\partial}_{\mathbf{v}})\,\mathbf{A}\,.\,\mathbf{v}' + \mathbf{B}\,.\,\mathbf{v}' + \mathbf{c}\,. \tag{15}$$

This vector-valued differential one-form (taking values in $T^*\mathbb{R}^n$) may be thought of as an interpretation of the Lepagian form, alternative to that considered in [8].

Since we are interested in holonomic local curves in $C^3(M,1)$, it is a common point that the vector-valued differential one-forms (15) and (5) are treated as equal with respect to the contact module on $J^3(\mathbb{R},\mathbb{R}^n)$:

$$\epsilon - e = \mathsf{A}_{ij}dx^i \otimes \theta^j_{\mathbf{3}}\,,$$

where the vector-valued contact one-forms

$$\boldsymbol{\theta_1} = d\mathbf{x} - \mathbf{v}dt\,, \quad \boldsymbol{\theta_2} = d\mathbf{v} - \mathbf{v}'dt\,, \quad \boldsymbol{\theta_3} = d\mathbf{v}' - \mathbf{v}''dt \tag{16}$$

generate the contact module on $J^3(\mathbb{R}, \mathbb{R}^n)$.

§ 3. **Euclidean symmetry.** Since we simultaneously consider both the true Euclidean and the pseudo-Euclidean cases, let us fix some notations. By η the sign $+$ or $-$ of the component g_{00} of the canonical diagonal metric tensor will be denoted. Centered dot will mean scalar product between matrices which represent tensors or between arrays which represent vectors—with respect to the (pseudo)Euclidean canonical metric tensor. Thus the scalar product is merely the contraction that involves the metric tensor. The infinitesimal generator X of the (pseudo)Euclidean transformation in three-dimensional space may be parametrized by means of a skew-symmetric matrix $\boldsymbol{\Omega}$ and some vector $\boldsymbol{\pi}$:

$$\begin{aligned} X \;\; = \;\; & -(\boldsymbol{\pi} \cdot \mathbf{x})\, \partial_t + \eta\, t\, \boldsymbol{\pi}\,.\,\boldsymbol{\partial}_{\mathbf{x}} + \boldsymbol{\Omega} \cdot (\mathbf{x} \wedge \boldsymbol{\partial}_{\mathbf{x}}) \\ & + \eta\, \boldsymbol{\pi}\,.\,\boldsymbol{\partial}_{\mathbf{v}} + (\boldsymbol{\pi} \cdot \mathbf{v})\, \mathbf{v}\,.\,\boldsymbol{\partial}_{\mathbf{v}} + \boldsymbol{\Omega} \cdot (\mathbf{v} \wedge \boldsymbol{\partial}_{\mathbf{v}}) \\ & + 2\, (\boldsymbol{\pi} \cdot \mathbf{v})\, \mathbf{v}'.\,\boldsymbol{\partial}_{\mathbf{v}'} + (\boldsymbol{\pi} \cdot \mathbf{v}')\, \mathbf{v}\,.\,\boldsymbol{\partial}_{\mathbf{v}'} + \boldsymbol{\Omega} \cdot (\mathbf{v}' \wedge \boldsymbol{\partial}_{\mathbf{v}'})\,. \end{aligned}$$

It is possible to cast the idea of the symmetry of the equation (13) into the framework of exterior differential system invariance concept. The system to handle is generated by the vector-valued Phaff form $\boldsymbol{\epsilon}$ from (15) and the contact vector-valued differential forms $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$ from (16). Let $X(\boldsymbol{\epsilon})$ denote the Lie derivative of the vector-valued differential form $\boldsymbol{\epsilon}$ along the vector field X. The invariance condition consists in that there may be found some matrices $\boldsymbol{\Phi}$, $\boldsymbol{\Xi}$, and $\boldsymbol{\Pi}$ depending on $\mathbf{v}$ and $\mathbf{v}'$ such that

$$X(\boldsymbol{\epsilon}) = \boldsymbol{\Phi}\,.\,\boldsymbol{\epsilon} + \boldsymbol{\Xi}\,.\,(d\mathbf{x} - \mathbf{v}dt) + \boldsymbol{\Pi}\,.\,(d\mathbf{v} - \mathbf{v}'dt). \tag{17}$$

We also assert that $\mathbf{A}$ and $\mathbf{k}$ in (15) do not depend neither on t nor on $\mathbf{x}$.

The identity (17) splits into more identities, obtained by evaluating the coefficients of the differentials dt, $d\mathbf{x}$, $d\mathbf{v}$, and $d\mathbf{v}'$ independently:

$$\begin{aligned} \big(\boldsymbol{\pi}\,.\,\boldsymbol{\partial}_{\mathbf{v}} + (\boldsymbol{\pi} \cdot \mathbf{v})\, \mathbf{v}\,.\,\boldsymbol{\partial}_{\mathbf{v}} + \boldsymbol{\Omega} \cdot (\mathbf{v} \wedge \boldsymbol{\partial}_{\mathbf{v}})\big)\, \mathbf{A} & \\ + 2\, (\boldsymbol{\pi} \cdot \mathbf{v})\mathbf{A} + (\mathbf{A}\mathbf{v}) \otimes \boldsymbol{\pi} - \mathbf{A}\boldsymbol{\Omega} = \boldsymbol{\Phi}\mathbf{A}\,; & \quad (18) \\ 2\, (\mathbf{A}\mathbf{v}') \otimes \boldsymbol{\pi} + (\boldsymbol{\pi} \cdot \mathbf{v}')\, \mathbf{A} \; = \boldsymbol{\Pi}\,; & \quad (19) \\ -\mathbf{k} \otimes \boldsymbol{\pi} = \boldsymbol{\Xi}\,; & \quad (20) \\ X(\mathbf{k}) = \boldsymbol{\Phi}\mathbf{k} - \boldsymbol{\Xi}\mathbf{v} - \boldsymbol{\Pi}\mathbf{v}'\,. & \quad (21) \end{aligned}$$

In the above the '$\otimes$' symbol means the tensor (sometimes named as 'direct') product of matrices; the associative matrix multiplication is represented by joint writing.

A skew-symmetric two-by-two matrix always has the inverse, so the 'Lagrange multipliers' $\boldsymbol{\Phi}$, $\boldsymbol{\Xi}$, and $\boldsymbol{\Pi}$ may explicitly be defined from the equations (18–20) and then substituted into (21). Subsequently, the equation (21) splits into the following identities by the powers of the variable $\mathbf{v}'$ and by the parameters $\boldsymbol{\Omega}$ and $\boldsymbol{\pi}$ (take notice of the derivative matrix $\mathbf{A}' = (\mathbf{v}'.\,\boldsymbol{\partial}_{\mathbf{v}})\, \mathbf{A}$; also the vertical arrow sign points to the very last factor to which the aforegoing differential operator still applies):

$$\big(\boldsymbol{\Omega}\cdot(\mathbf{v}\wedge\partial_{\mathbf{v}})\big)\,\mathbf{A}'\mathbf{v}' + \big(\boldsymbol{\Omega}\cdot(\mathbf{v}'\wedge\partial_{\mathbf{v}})\big)\,\mathbf{A}\mathbf{v}' - (\mathbf{v}'\,.\,\partial_{\mathbf{v}})\,\mathbf{A}\boldsymbol{\Omega}\mathbf{v}'$$
$$= \big(\boldsymbol{\Omega}\cdot(\mathbf{v}\wedge\partial_{\mathbf{v}})\big)\,\overset{\downarrow}{\mathbf{A}}\mathbf{A}^{-1}\mathbf{A}'\mathbf{v}' - \mathbf{A}\boldsymbol{\Omega}\mathbf{A}^{-1}\mathbf{A}'\mathbf{v}'\,; \tag{22}$$

$$\big(\boldsymbol{\Omega}\cdot(\mathbf{v}\wedge\partial_{\mathbf{v}})\big)\,\mathbf{B} - \mathbf{B}\boldsymbol{\Omega} = \big(\boldsymbol{\Omega}\cdot(\mathbf{v}\wedge\partial_{\mathbf{v}})\big)\,\overset{\downarrow}{\mathbf{A}}\mathbf{A}^{-1}\mathbf{B} - \mathbf{A}\boldsymbol{\Omega}\mathbf{A}^{-1}\mathbf{B}\,; \tag{23}$$

$$\big(\boldsymbol{\Omega}\cdot(\mathbf{v}\wedge\partial_{\mathbf{v}})\big)\,\mathbf{c} = \big(\boldsymbol{\Omega}\cdot(\mathbf{v}\wedge\partial_{\mathbf{v}})\big)\,\overset{\downarrow}{\mathbf{A}}\mathbf{A}^{-1}\mathbf{c} - \mathbf{A}\boldsymbol{\Omega}\mathbf{A}^{-1}\mathbf{c}\,; \tag{24}$$

$$\big(\boldsymbol{\pi}\,.\,\partial_{\mathbf{v}} + (\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{v}\,.\,\partial_{\mathbf{v}}\big)\,\mathbf{A}'\mathbf{v}' + (\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{A}'\mathbf{v}' + (\boldsymbol{\pi}\cdot\mathbf{v}')\,(\mathbf{v}\,.\,\partial_{\mathbf{v}})\,\mathbf{A}\mathbf{v}' + (\boldsymbol{\pi}\cdot\mathbf{v}')\,\mathbf{A}'\mathbf{v}$$
$$= \big(\boldsymbol{\pi}\,.\,\partial_{\mathbf{v}} + (\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{v}\,.\,\partial_{\mathbf{v}}\big)\,\overset{\downarrow}{\mathbf{A}}\mathbf{A}^{-1}\mathbf{A}'\mathbf{v}' + (\boldsymbol{\pi}\mathbf{A}^{-1}\mathbf{A}'\mathbf{v}')\,\mathbf{A}\mathbf{v} - 3\,(\boldsymbol{\pi}\cdot\mathbf{v}')\,\mathbf{A}\mathbf{v}'\,; \tag{25}$$

$$\big(\boldsymbol{\pi}\,.\,\partial_{\mathbf{v}} + (\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{v}\,.\,\partial_{\mathbf{v}}\big)\,\mathbf{B} + (\mathbf{B}\mathbf{v})\otimes\boldsymbol{\pi}$$
$$= \big(\boldsymbol{\pi}\cdot\partial_{\mathbf{v}} + (\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{v}\,.\,\partial_{\mathbf{v}}\big)\,\overset{\downarrow}{\mathbf{A}}\mathbf{A}^{-1}\mathbf{B} + (\mathbf{A}\mathbf{v})\otimes\boldsymbol{\pi}\mathbf{A}^{-1}\mathbf{B} + (\boldsymbol{\pi}\cdot\mathbf{v})\mathbf{B}\,; \tag{26}$$

$$\big(\boldsymbol{\pi}\,.\,\partial_{\mathbf{v}} + (\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{v}\,.\,\partial_{\mathbf{v}}\big)\,\mathbf{c}$$
$$= \big(\boldsymbol{\pi}\,.\,\partial_{\mathbf{v}} + (\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{v}\,.\,\partial_{\mathbf{v}}\big)\,\overset{\downarrow}{\mathbf{A}}\mathbf{A}^{-1}\mathbf{c} + 3\,(\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{c} + (\boldsymbol{\pi}\mathbf{A}^{-1}\mathbf{c})\,\mathbf{A}\mathbf{v}\,. \tag{27}$$

Straightforward but cumbersome routine calculations accompanying the simultaneous solving of the partial differential equations (22) and (25) with respect to the unknown function A_{12} produce the unique output of

$$\mathsf{A}_{12} = \frac{\mathrm{const}}{(1 + v_1v^1 + v_2v^2)^{3/2}}\,.$$

We remind that the system of the equations $\{(22)–(27)\}$ and the system (14) must be solved simultaneously. Thus, the equation (14a) becomes trivial now.

Under the assumption of $\mathbf{B}$ being a symmetric matrix (see (14b)), the solution of the equations $\{(23),\,(26)\}$ is:

$$\mathsf{B}_{ij} = \mathrm{const}\cdot(1 + \mathbf{v}\cdot\mathbf{v}))^{-3/2}\big(v_iv_j - (1 + \mathbf{v}\cdot\mathbf{v})\,g_{ij}\big)\,.$$

This automatically satisfies the equation (14c) too. In what concerns the subsystem $\{(24),\,(27)\}$, only the trivial solution $\mathbf{c} = \mathbf{0}$ exists.

We are ready now to formulate the summary of the above development in terms of a proposition:

Proposition 3 *The invariant parameter-indifferent Euler-Poisson equation in three-dimensional (pseudo)Euclidean space is:*

$$-\frac{*\mathbf{v}''}{(1 + \mathbf{v}\cdot\mathbf{v})^{3/2}} + 3\,\frac{*\mathbf{v}'}{(1 + \mathbf{v}\cdot\mathbf{v})^{5/2}}\,(\mathbf{v}\cdot\mathbf{v}') + \frac{\mu}{(1 + \mathbf{v}\cdot\mathbf{v})^{3/2}}\,\big((1 + \mathbf{v}\cdot\mathbf{v})\,\mathbf{v}' - (\mathbf{v}'\cdot\mathbf{v})\,\mathbf{v}\big) = \mathbf{0}\,. \tag{28}$$

The arbitrary constant μ serves to parameterize the set of all the variational equations (28). The definition of the 'star operator' is common. Thus, $*1 = \mathbf{e}_{(1)}\wedge\mathbf{e}_{(2)}$, whereas $*\,(\mathbf{e}_{(1)}\wedge\mathbf{e}_{(2)}) = 1$ if the (pseudo)orthonormal frame $\{\mathbf{e}_{(1)}\,,\ \mathbf{e}_{(2)}\}$ carries the positive orientation; also $(*\mathbf{w})_i = \varepsilon_{ji}w^j$ for a two-dimensional vector $\mathbf{w}$.

I know two different ($j = 1, 2$) Lagrange functions which produce the equation (28),

$$L_{(j)} = \frac{*(\mathbf{v}' \wedge \mathbf{e}_{(j)})}{(1 + \mathbf{v} \cdot \mathbf{v})^{1/2}(1 + g_{jj}\|\mathbf{v} \wedge \mathbf{e}_{(j)}\|^2)} v^j - \mu\,(1 + \mathbf{v} \cdot \mathbf{v})^{1/2}\,. \tag{29}$$

These differ by the total time derivative:

$$L_{(2)} - L_{(1)} = \frac{d}{dt} \arctan \frac{v^1 v^2}{\sqrt{1 + v_j v^j}}\,.$$

Remark 1. Equation (28) describes helices with second curvature equal to $\|\mu\|$.

Remark 2. The point symmetries of the equation (28) are exhausted by (pseudo)Euclidean transformations if $\mu \neq 0$. Otherwise they precisely consist of conformal ones [9].

Remark 3. There does not exist an invariant affine second-order Lagrange function in (pseudo)Euclidean space of dimension greater than 2 (strictly speaking, this was proved for the signature not equal 2) [10].

With the Proposition(1) in hand and applying formula (2) it is not difficult to put down the "homogeneous" counterpart (6) of equation (28). It reads:

$$-\frac{\ddot{\boldsymbol{u}} \times \boldsymbol{u}}{\|\boldsymbol{u}\|^3} + 3\,\frac{\dot{\boldsymbol{u}} \times \boldsymbol{u}}{\|\boldsymbol{u}\|^5}\,(\dot{\boldsymbol{u}} \cdot \boldsymbol{u}) - \frac{\mu}{\|\boldsymbol{u}\|^3}\,[(\boldsymbol{u} \cdot \boldsymbol{u})\,\dot{\boldsymbol{u}} - (\dot{\boldsymbol{u}} \cdot \boldsymbol{u})\,\boldsymbol{u}] = \mathbf{0}\,. \tag{30}$$

Furthermore, by same means of (9) one may deduce a general formula for the family ($\beta = 0, 1, 2$) of the Lagrange functions which produce the right hand side of (30):

$$\mathcal{L}_{(\beta)} \;=\; \frac{u^\beta[\dot{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{e}_{(\beta)}]}{\|\boldsymbol{u}\|\,\|\boldsymbol{u} \times \boldsymbol{e}_{(\beta)}\|^2} \;-\; m\,\|\boldsymbol{u}\| \;+\; \dot{\boldsymbol{u}}\,\boldsymbol{.}\,\boldsymbol{\partial}_{\boldsymbol{u}}\,\phi \;+\; \boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{u}\,, \tag{31}$$

where an arbitrary row vector $\boldsymbol{a}$ is constant and a function ϕ depending on the variable $\boldsymbol{u}$ is subject to the constraint $\boldsymbol{u}\,\boldsymbol{.}\,\boldsymbol{\partial}_{\boldsymbol{u}}\,\phi \;=\; 0$. Recall also the notation [, ,] for the parallelepipedal product of three vectors. The vector $\boldsymbol{e}_{(\beta)}$ denotes the β-th component of the (pseudo)Euclidean frame. *Each* $\mathcal{L}_{(\beta)}$ *fits in.*

The problem of finding invariant variational equations in some special cases, discussed in this talk, might have been formulated in still more recent framework of invariant variational bicomplexes (cf. for example [11]). Unfortunately, the threshold of the non-existence of invariant Lagrangian functions diminishes the effectiveness of the corresponding machinery, which from the very beginning suggests the invariance of the full bicomplex. Similar difficulties arise when one starts to apply notions developed for Kawaguchi spaces. For example, the metric 'tensor' calculated from the Lagrange function (31) does not designate any geometric object. Only quantities, built of the invariant momentum

$$\mathcal{P} \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \boldsymbol{u}} - \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}}\right)^{\boldsymbol{\cdot}} = \frac{\dot{\boldsymbol{u}} \times \boldsymbol{u}}{\|\boldsymbol{u}\|^3} + \mu\,\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$$

would play any significant role in a generally covariant theory. Several such quantities were introduced in chapter 2 of paper [12].

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