VARIATIONALITY OF GEODESIC CIRCLES IN TWO DIMENSIONS

R. Ya. MATSYUK*

Institute for Applied Problems in Mechanics and Mathematics, 15 Dudayev St., L'viv, 79005, Ukraine
**E-mail: matsyuk@lms.lviv.ua*

This note treats the notion of Lagrange derivative for the third order mechanics in the context of covariant Riemannian geometry. The variational differential equation for geodesic circles in two dimensions is obtained. The influence of the curvature tensor on the Lagrange derivative leads to the emergence of the notion of quasiclassical spin in the pseudo-Riemannian case.

Keywords: Ostrohrads'kyj mechanics; Inverse variational problem; Concircular geometry; Classical spin.

1. Introduction

This is a note on the variational formulation for the differential equations of geodesic circles in two-dimensional Riemannian space, although the results apply straightforward to the pseudo-Riemannian case. The geodesic curves $x^i(t)$ obey with respect to the natural parameter s the third order differential equation[1]

$$\frac{D^3x^i}{ds^3} + g_{lj}\frac{D^2x^l}{ds^2}\frac{D^2x^j}{ds^2}\frac{Dx^i}{ds} = 0\,,$$

and they are exactly characterized by the property that the (signed) Frenet curvature k keeps constant along them. In view of the Proposition 2.1 below we could have immediately stated that the variational functional $\int k dt$ provides an answer to the problem, all the more that in two dimensions $\sqrt{k^2}$ depends linearly on the second derivatives of the coordinates along the curve thus producing exactly *the third order* variational (called Euler-Poisson) equation.

However, we wish to investigate, to what extent this answer in predefined by the limiting case of the Euclidean space — the local model of the Riemannian one. With this idea in mind we start by recalling one solution of the *invariant inverse variational problem* in two-dimensional Euclidean space for a third order variational equation possessing the first integral k. Before proceeding further, it is necessary to agree about some notations and to recall some basic calculus on the second order Ehresmann velocity space $T^2M \stackrel{\text{def}}{=} J^2_0(\mathbb{R}, M)$ of jets from $\mathbb{R}$ to our manifold M starting at $0 \in \mathbb{R}$ (as possible source of references we can recommend, for example, Refs. 2–4)

2. Calculus on the higher order velocities space

Let $u^i, \dot{u}^i$ denote the standard fiber coordinates in T^2M. In case of an affine space M, we use the vector notations $\boldsymbol{u}, \dot{\boldsymbol{u}}$ for that tuple. In future we shall profoundly also use another tuple of coordinates, namely, $\boldsymbol{u}, \boldsymbol{u'}$, where

$$u'^i = \dot{u}^i + \Gamma^i_{lj} u^l u^j \tag{1}$$

stands for the covariant derivative of $\boldsymbol{u}$. Let us recall some operators acting in the algebra of differential forms, defined on the velocity spaces of the sequential orders:

- The total derivative:
  $$d_T f = u^i \frac{\partial f}{\partial x^i} + \dot{u}^i \frac{\partial f}{\partial u^i} + \ddot{u}^i \frac{\partial f}{\partial \dot{u}^i} \dots$$
  This is a derivation of degree zero and of the type d, *i.e.* who commutes with the exterior differential: $dd_T = d_T d$.
- For each $k = 1, 2, 3$, let $u^i_{(k)} = \overset{\overbrace{\dots}^{k}}{u}{}^i$, $u^i_{(0)} = u^i$, $x^i_{(k)} = u^i_{(k-1)}$, and $x^i_{(0)} = x^i$. For each $r = 0, 1, 2, 3, \dots$, we recall the following derivations of degree zero and of the type i, *i.e.* who produce zeros while acting on the ring of functions:
  $$\iota_r(f) = 0\,,$$
  $$\iota_r(dx^i_{(k)}) = \frac{k!}{(k-r)!} dx^i_{(k-r)}, \quad \text{and} \quad \iota_r(dx^i_{(k)}) = 0, \quad \text{if} \quad r > k. \tag{2}$$
- The Lagrange derivative δ:
  $$\delta = (\iota_0 - d_T \iota_1 + \frac{1}{2!} d_T^2 \iota_2 - \frac{1}{3!} d_T^3 \iota_3 + \dots)\, d\,, \tag{3}$$
  who satisfies $\delta^2 = 0$.

Let some system of the third order ordinary differential equations $\mathcal{E}_i(x^j, u^j, \dot{u}^j, \ddot{u}^j)$ be put in the shape of a covariant object ϵ:

$$\epsilon = \mathcal{E}_i(x^j, u^j, \dot{u}^j, \ddot{u}^j) dx^i\,.$$

The variationality criterion reads: If $\delta\epsilon = 0$, then the system $\mathcal{E}_i$ is variational, *i.e.* locally there exists some function L, such that $\epsilon = \delta L$.

The left action of the prolonged group $GL_{(2)}(\mathbb{R}) \stackrel{\text{def}}{=} \overset{\circ}{J}{}^2_0(\mathbb{R},\mathbb{R})_0$ of parameter transformations (invertible transformations of the independent variable t) on T^2M gives rise to the so-called fundamental fields on T^2M:

$$\zeta_1 = u^i \frac{\partial}{\partial u^i} + 2\dot{u}^i \frac{\partial}{\partial \dot{u}^i}, \quad \zeta_2 = u^i \frac{\partial}{\partial \dot{u}^i}\,.$$

A function f defined on T^2M does not depend on the change of independent variable t (so–called parameter–independence) if and only if

$$\zeta_1 f = 0, \quad \zeta_2 f = 0\,. \tag{4}$$

On the other hand, a function L defined on T^2M constitutes a parameter–independent variational problem with the functional $\int L(x^j, u^j, \dot{u}^j)dt$ if and only if the following Zermelo conditions are satisfied:

$$\zeta_1 L = L, \quad \zeta_2 L = 0\,. \tag{5}$$

Let us introduce the generalized momenta:

$$p_i^{(1)} = \frac{\partial L}{\partial \dot{u}^i}, \quad p_i = \frac{\partial L}{\partial u^i} - d_T p_i^{(1)}\,.$$

These satisfy the relation:

$$p^{(1)}{}_i du^i + p_i dx^i = \iota_1 dL - \frac{1}{2} d_T \iota_2 dL\,. \tag{6}$$

The Euler–Poisson equation is given by $\delta L = 0$, or, equivalently, by

$$\dot{p}_i dx^i = \frac{\partial L}{\partial x^i} dx^i\,.$$

The Hamilton function is given by:

$$H = p_i^{(1)} \dot{u}^i + p_i u^i - L\,.$$

Lemma 2.1.

$$H = \zeta_1 L - d_T \zeta_2 L - L\,.$$

Proposition 2.1. *If a function L_{II} is parameter–independent and a function L_{I} constitutes a parameter-independent variational problem, then L_{II} is constant along the extremals of $L = L_{\mathrm{II}} + L_{\mathrm{I}}$.*

Proof. By Lemma 2.1 and in course of the properties (4) and (5) we calculate $H_{L_{\mathrm{II}}+L_{\mathrm{I}}} = \zeta_1(L_{\mathrm{II}} + L_{\mathrm{I}}) - d_T\zeta_2(L_{\mathrm{II}} + L_{\mathrm{I}}) - L = -L_{\mathrm{II}}$. But as far as the Hamilton function is constant of motion, so is the L_{II}. $\square$

3. The Lagrange derivative in Riemannian space

In Riemannian space with symmetric connection the covariant differential of a vector field $\boldsymbol{\xi}$ is a vector field valued semibasic differential form $D\boldsymbol{\xi}$ calculated according to the formula

$$(D\boldsymbol{\xi})^i = d\xi^i + \Gamma^i_{lj}\xi^j dx^l. \tag{7}$$

The fundamental application of the curvature tensor, from which this note profits, provides the commutator of the subsequent derivations, the one that substitutes the known Schwarz lemma:

$$(D\boldsymbol{u})'^i = (D(\boldsymbol{u}'))^i - R_{ljq}{}^i u^j u^q dx^l\,. \tag{8}$$

We also recall that, on the other hand, the first order derivations commute:

$$(dx)' = D\boldsymbol{u}\,. \tag{9}$$

Given some local coordinate expression of a function,

$$L(x^i, u^i, \dot{u}^i)\,,$$

we wish to introduce generalized momenta π_i and $\pi^{(1)}{}_i$, calculated with respect to the alternative set of coordinates in T^2M, namely, x^i,u^i, u'^i, where the transition functions are presented by (1).

Definition 3.1. Let

$$\pi^{(1)}{}_i = \frac{\partial L}{\partial u'^i}\,, \quad \pi_i = \frac{\partial L}{\partial u^i} - \pi^{(1)\prime}{}_i\,.$$

Proposition 3.1. *In Riemannian space the generalized momenta satisfy the relation, analogous to (6):*

$$\boldsymbol{\pi}^{(1)} D\boldsymbol{u} + \boldsymbol{\pi} dx = \iota_1 dL - \frac{1}{2}(\iota_2 dL)'$$

Proof. First let us calculate by the reason of formulas (2) and (1):

$$\iota_1 d\boldsymbol{u} = dx, \quad \iota_1 d\boldsymbol{u}' = 2D\boldsymbol{u}, \quad \iota_2 d\boldsymbol{u}' = 2dx\,.$$

For the differential of Lagrange function,

$$dL = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial \boldsymbol{u}} d\boldsymbol{u} + \frac{\partial L}{\partial \boldsymbol{u}'} d\boldsymbol{u}'\,, \tag{10}$$

we then check:

$$\iota_2 dL = 2\,\frac{\partial L}{\partial \boldsymbol{u}'} dx\,.$$

Now calculate:

$$\begin{aligned}\iota_1 dL &= \frac{\partial L}{\partial \boldsymbol{u}} dx + 2\, \frac{\partial L}{\partial \boldsymbol{u'}} D\boldsymbol{u} \\ &= \frac{\partial L}{\partial \boldsymbol{u}} dx + 2\, \left(\frac{\partial L}{\partial \boldsymbol{u'}} dx\right)' - 2\, \left(\frac{\partial L}{\partial \boldsymbol{u'}}\right)' dx \\ &= \frac{\partial L}{\partial \boldsymbol{u}} dx + (\iota_2 dL)' - 2\, \left(\frac{\partial L}{\partial \boldsymbol{u'}}\right)' dx\,,\end{aligned}$$

from where and from the Definition 3.1 it follows immediately that

$$\begin{aligned}\boldsymbol{\pi} dx &= \iota_1 dL - (\iota_2 dL)' + \left(\frac{\partial L}{\partial \boldsymbol{u'}}\right)' dx \\ &= \iota_1 dL - \frac{1}{2}(\iota_2 dL)' - \frac{1}{2}(\iota_2 dL)' + (\boldsymbol{\pi}^{(1)})' dx \\ &= \iota_1 dL - \frac{1}{2}(\iota_2 dL)' - (\boldsymbol{\pi}^{(1)} dx)' + (\boldsymbol{\pi}^{(1)})' dx \\ &= \iota_1 dL - \frac{1}{2}(\iota_2 dL)' - \boldsymbol{\pi}^{(1)} D\boldsymbol{u}\end{aligned}$$

by virtue of (9). □

Proposition 3.2. *In Riemannian space the Euler–Poisson equation for a second order Lagrange function reads:*

$$\boldsymbol{\pi}' dx + \pi^{(1)}{}_i R_{ljq}{}^i u^j u^q dx^l = \frac{\partial L}{\partial x^l} dx^l - \frac{\partial L}{\partial u^i} \Gamma^i_{lj} u^j dx^l - \frac{\partial L}{\partial u'^i} \Gamma^i_{lj} u'^j dx^l \quad (11)$$

Proof. From (3) and from Proposition 3.1 we obtain

$$\delta L = \iota_0 dL - d_T(\boldsymbol{\pi} dx + \boldsymbol{\pi}^{(1)} D\boldsymbol{u}).$$

While the expression in the parenthesis constitutes a geometrical invariant, it is possible to replace d_T by the covariant derivative, after what by direct calculation we obtain in virtue of (9) and of (8):

$$\begin{aligned}\delta L = \iota_0 dL - (\boldsymbol{\pi} dx + \boldsymbol{\pi}^{(1)} D\boldsymbol{u})' &= \iota_0 dL - \boldsymbol{\pi}' dx - \left(\boldsymbol{\pi} + \boldsymbol{\pi}^{(1)\prime}\right) D\boldsymbol{u} - \boldsymbol{\pi}^{(1)} (D\boldsymbol{u})' \\ &= \iota_0 dL - \boldsymbol{\pi}' dx - \frac{\partial L}{\partial \boldsymbol{u}} D\boldsymbol{u} - \pi^{(1)}{}_i \left(D(\boldsymbol{u'})^i - R_{ljq}{}^i u^j u^q dx^l\right),\end{aligned}$$

and the proof ends by substituting (10) into $i_0 dL \equiv dL$ here and by applying (7). □

4. The two-dimensional variational concircular geometry

As promised, we first cite one result, concerning the invariant inverse variational problem in two dimensional Euclidean space.[5]

Proposition 4.1. *Let some system of third order differential equations*

$$\mathcal{E}_i(x^j, u^j, \dot{u}^j, \ddot{u}^j) = 0 \tag{12}$$

satisfy the conditions:

(i) $\delta\,\mathcal{E}_i dx^i \,=\, 0$
(ii) The system (12) possesses Euclidean symmetry
(iii) The Euclidean geodesics $\dot{\mathbf{u}} = \mathbf{0}$ *enter in the set of solutions of (12)*
(iv) $d_T k = 0$ *along the solutions of (12)*

Then

$$\mathcal{E}_i = \frac{e_{ij}\ddot{u}^j}{\|\boldsymbol{u}\|^3} - 3\,\frac{(\dot{\boldsymbol{u}}\cdot\boldsymbol{u})}{\|\boldsymbol{u}\|^5} e_{ij}\dot{u}^j + m\,\frac{\|\boldsymbol{u}\|^2\dot{u}_i - (\dot{\boldsymbol{u}}\cdot\boldsymbol{u})u_i}{\|\boldsymbol{u}\|^3}\ .$$

This system may be obtained from the Lagrange function

$$L = \frac{e_{ij}u^i\dot{u}^j}{\|\boldsymbol{u}\|^3} - m\,\|\boldsymbol{u}\|\,. \tag{13}$$

The first addend in (13) is sometimes called *the signed Frene curvature* in $\mathbb{E}^2$. This, along with the observation that in two dimensional Riemannian space the Frenet curvature

$$k = \frac{\|\boldsymbol{u}\wedge\boldsymbol{u'}\|}{\|\boldsymbol{u}\|^3} = \pm\frac{*\,(\boldsymbol{u}\wedge\boldsymbol{u'})}{\|\boldsymbol{u}\|^3} \tag{14}$$

depends linearly on $\boldsymbol{u'}$ and thus produces at most third order Euler–Poisson equation, suggests the next assertion, based on Proposition 2.1:

Proposition 4.2. *The variational functional* $\int (k - m\,\|\boldsymbol{u}\|)\,dt$ *produces geodesic circles in two dimensional Riemannian space.*

It remains to calculate the Euler–Poisson expression for the Lagrange function (14). In the process of calculations it is convenient to profit from the exeptional properties of vector operations in two dimensions. Namely, the following two relations for arbitrary vectors hold:

$$(\boldsymbol{a}\wedge\boldsymbol{b})\boldsymbol{\cdot}(\boldsymbol{v}\wedge\boldsymbol{w}) = \pm\,\|\boldsymbol{a}\wedge\boldsymbol{b}\|\|\boldsymbol{v}\wedge\boldsymbol{w}\|\,,$$

and

$$\|\boldsymbol{a}\wedge\boldsymbol{b}\|(\boldsymbol{b}\cdot\boldsymbol{c}) + \|\boldsymbol{b}\wedge\boldsymbol{c}\|(\boldsymbol{a}\cdot\boldsymbol{b}) = \|\boldsymbol{a}\wedge\boldsymbol{c}\|(\boldsymbol{b}\cdot\boldsymbol{b})\,.$$

The above simplifications bring much release to otherwise very laborious calculations.

We start with the momentum $\boldsymbol{\pi}^{(1)}$:

$$\pm\,\boldsymbol{\pi}^{(1)}dx = -\frac{(dx\wedge\boldsymbol{u})\boldsymbol{\cdot}(\boldsymbol{u}\wedge\boldsymbol{u'})}{\|\boldsymbol{u}\|^3\|\boldsymbol{u}\wedge\boldsymbol{u'}\|} = -\frac{\|dx\wedge\boldsymbol{u}\|}{\|\boldsymbol{u}\|^3};$$

$$\pm\,\boldsymbol{\pi}^{(1)\prime}dx = -\frac{\|dx\wedge\boldsymbol{u'}\|}{\|\boldsymbol{u}\|^3} + 3\,\frac{\|dx\wedge\boldsymbol{u}\|}{\|\boldsymbol{u}\|^5}(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{u'}).$$

Based on Definition 3.1 we now calculate $\boldsymbol{\pi}$:

$$\pm\boldsymbol{\pi}dx = 2\,\frac{\|dx\wedge\boldsymbol{u'}\|}{\|\boldsymbol{u}\|^3} - 3\,\frac{\|dx\wedge\boldsymbol{u}\|(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{u'})}{\|\boldsymbol{u}\|^5} - 3\,\frac{(dx\boldsymbol{\cdot}\boldsymbol{u})\|\boldsymbol{u}\wedge\boldsymbol{u'}\|}{\|\boldsymbol{u}\|^5} = -\frac{\|dx\wedge\boldsymbol{u'}\|}{\|\boldsymbol{u}\|^3}$$

In terms of the Hodge star operator the derivative of the momentum $\boldsymbol{\pi}$ may be put in the form

$$\boldsymbol{\pi'} = \frac{*\boldsymbol{u''}}{\|\boldsymbol{u}\|^3} - 3\,\frac{*\boldsymbol{u'}}{\|\boldsymbol{u}\|^5}(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{u'})\,,$$

which agrees with the flat Euclidean case.

For the Lagrange function (14) it is easy to verify that

$$\frac{\partial k}{\partial x^l}dx^l - \frac{\partial k}{\partial u^i}\Gamma^i_{lj}u^j dx^l - \frac{\partial k}{\partial u'^i}\Gamma^i_{lj}u' j dx^l = 0.$$

The proof consists in direct calculations and founds on the skew-symmetric property of the Christoffel symbols in Riemannian geometry:

$$g_{jl}\Gamma^l_{qi} + g_{il}\Gamma^l_{qj} = \frac{\partial g_{ij}}{\partial x^q}.$$

Going back to the Euler–Poisson equation (11) it is now facile to obtain the variational equation for the full Lagrange function $L = k - m\|\boldsymbol{u}\|$:

$$-\frac{*\boldsymbol{u''}}{\|\boldsymbol{u}\|^3} + 3\,\frac{*\boldsymbol{u'}}{\|\boldsymbol{u}\|^5}(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{u'}) \;+ m\,\frac{\|\boldsymbol{u}\|^2\boldsymbol{u'} - (\boldsymbol{u'}\cdot\boldsymbol{u})\boldsymbol{u}}{\|\boldsymbol{u}\|^3} = \pi^{(1)}{}_i R_{ljq}{}^i u^j u^q. \quad (15)$$

The term on the right in pseudo-Riemannian case physically may be interpreted as a spin force[7] if, following Ref. 6, we formally introduce spin tensor as $S = (\boldsymbol{u}\wedge\boldsymbol{u'})$.

In fact, one checks that in terms of the tensor S the right hand side of equation (15) may be rewritten as $\dfrac{R_{ljqi}u^j S^{qi}}{\|\boldsymbol{u}\|\|\boldsymbol{u}\wedge\boldsymbol{u'}\|}$.

5. Acknowledgments

This work was supported by the Grant GAČR 201/06/0922 of Czech Science Foundation.

References

1. K. Yano, *Proc. Impt. Academy. Tokyo* **16**, 195 (1940).
2. L. A. Ibort and C. López-Lacasta, *J. Phys. A* **23**, 4779 (1990).
3. M. de Léon and Paulo R. Rodrigues, *Generalized Classical Mechanics and Field Theory* (Elsevier, Amsterdam, 1985).
4. W. Tulczyjew, *C. R. Acad. Sci. Paris. Série A* **280**, 1295 (1975).
5. R. Ya. Matsyuk, *Mat. Metody Fiz.-Mekh. Polya* **16**, 84 (1982) (in Russian).
6. S. G. Leiko, *Russian Acad. Sci. Dokl. Math.* **46**, 84 (1993).
7. M. Mathisson, *Acta phys. polon.* **6**, 163 (1937).