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# ON THE STRUCTURE OF FINSLER AND AREAL SPACES

ERICO TANAKA AND DEMETER KRUPKA

*Abstract.* We study underlying geometric structures for integral variational functionals, depending on submanifolds of a given manifold. Applications include (first order) variational functionals of Finsler and areal geometries with integrand the Hilbert 1-form, and admit immediate extensions to higher-order functionals.

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## 1. INTRODUCTION

This paper is a contribution to the theory of integral variational functionals, depending on submanifolds of a given manifold $X$. The theory is based on geometric notions such as the bundles of (skew-symmetric) multivectors, and Grassmann fibrations. Conceptually, it extends local parametric integrals of Finsler–Kawaguchi and areal geometries (see, e. g., Chern, Chen, Lam [1], Davies [3], Kawaguchi [4], and Tamassy [6]) to global functionals, depending on (global) submanifolds. In Section 2 we summarize integration theory of differential forms along submanifolds. Section 3 is devoted to vector bundles of $k$-vectors; we show how mappings of Euclidean spaces into manifolds (*parametrisations*) can be lifted to the bundles of $k$-vectors. In Section 4 we introduce, using the Plücker embedding, underlying spaces for parameter-invariant variational problems, the Grassmann fibrations. In Section 5 we show that any $k$-form on the Grassmann fibration defines an integral variational functional, depending on $k$-dimensional submanifolds. An example is the *Hilbert form*, a well-known first-order construction in Finsler geometry and its generalisations (Chern, Chen, Lam [1], Crampin, Saunders [2]).

It should be pointed out that the theory can be further generalised. To this end, one should consider higher-order Grassmann fibrations endowed with Lagrangians satisfying the relevant homogeneity conditions (Zermelo conditions, see, e. g., Saunders [5], and Urban and Krupka [8]).

## 2. Integration over submanifolds

Let $X$ be an $n$-dimensional manifold, $S$ a subset of $X$, $x_0 \in S$ a point. A chart $(U, \varphi)$,$\varphi = (x^i)$, at $x_0$ is a *submanifold chart* for $S$, if there exists a non-negative integer $k \leqslant n$ such that $\varphi(U \cap S) = \{x \in U | x^{k+1}(x) = c_1, x^{k+2}(x) = c_2, \dots, x^n(x) = c_{n-k}\}$. If such a chart exists, we say that $S$ is a *submanifold* of $X$ at the point $x_0$; $k$ is the *dimension* of $S$ at $x_0$. If such a submanifold chart exists at every point of $X$, we say $S$ is a *submanifold* of $X$ and call $k$ the *dimension* of $S$.

Denote by $(t^1, t^2, \dots, t^n)$ the canonical coordinates on the Euclidean space $\mathbb{R}^n$, and $\mathbb{R}^n_{(-)} = \{t_0 \in \mathbb{R}^n | t^n(t_0) \leqslant 0\}$, $\partial \mathbb{R}^n_{(-)} = \{t_0 \in \mathbb{R}^n_{(-)} | t^n(t_0) = 0\}$. $\mathbb{R}^n_{(-)}$ is the *halfspace* of $\mathbb{R}^n$, $\partial \mathbb{R}^n_{(-)}$ is the *boundary* of $\mathbb{R}^n_{(-)}$. Let $\Omega$ be a non-void subset of $X$, and $x_0 \in \Omega$ a point. A chart $(U, \varphi)$ at $x_0$ is said to be *adapted* to $\Omega$, if the set $\varphi(U \cap \Omega)$ is an open set in $\mathbb{R}^n_{(-)}$. $\Omega$ is a *piece* of $X$, if it is compact and each point $x \in \Omega$ admits a chart, adapted to $\Omega$.

Let $\eta$ be a $k$-form on $X$. Our aim now will be to introduce an integral of $\eta$ on a piece of a $k$-dimensional submanifold $S$ ($k$-*piece* of a $X$). Express $\eta$ in a submanifold chart $(U, \varphi)$, $\varphi = (x^i)$, as $\eta = \eta_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$. Then, restricting $\eta$ to $S$ we get from the equations $x^{k+1} = 0, x^{k+2} = 0, \cdots, x^n = 0$

$$\eta = f dx^1 \wedge dx^2 \wedge \dots \wedge dx^k,$$

where we write $f = f(x^{i_1}, x^{i_2}, \dots, x^{i_k})$ for the component of $\eta$ restricted to $S$. From now on we suppose that $S$ is *orientable*, and is endowed with an orientation $\mathrm{Or}_S X$; only submanifold charts on $X$ belonging to $\mathrm{Or}_S X$ are used. The integral of $\eta$ on a compact set $\Omega \subset S$ is defined in a standard way. There exist a finite family $\{(U_1, \varphi_1), (U_2, \varphi_2), \dots, (U_N, \varphi_N)\}$ of submanifold charts on $X$, such that the family $\{U_1 \cap S, U_2 \cap S, \dots, U_N \cap S\}$ covers $\Omega$. Let $\{\chi_1, \chi_2, \dots, \chi_N\}$ be a partition of unity, subordinate to this covering. Then,

$$\int_\Omega \eta = \sum_{j=1}^{N} \int_{\mathrm{supp}\, \chi_j \cap \Omega} \chi_j \eta.$$

The following basic properties of the integral are needed in the calculus of variations.

**Lemma 1** (transformation of integration domain)**.** *Let $X$ and $Y$ be two smooth $n$-dimensional oriented manifolds, $\alpha : X \to Y$ an orientation-preserving diffeomorphism. Then*

$$\int_\Omega \eta = \int_{\alpha^{-1}(\Omega)} \alpha * \eta$$

*for any compact set $\Omega \subset S$ and any continuous differential $n$-form on $Y$.*

**Lemma 2** (Leibniz rule)**.** *Let $X$ be an oriented $n$-dimensional manifold, $\eta_t$ a family of $n$-forms on $X$, differentiable on a real parameter $t$, $\Omega \subset S$ a compact set.*

*Then, the function* $I \ni t \mapsto \int_\Omega \eta_t \in \mathbb{R}$ *is differentiable, and*

$$\frac{d}{dt}\int_\Omega \eta_t = \int_\Omega \frac{d\eta_t}{dt}.$$

**Lemma 3** (Stokes formula)**.** *Let* $X$ *be an* $n$*-dimensional manifold,* $S$ *a* $k$*-dimensional oriented submanifold of* $X$*,* $\eta$ *a* $(k-1)$*-form on* $X$*. Let* $\Omega$ *be a piece of* $S$ *with boundary* $\partial\Omega$*, endowed with induced orientation. Then*

$$\int_{\partial\Omega} \eta = \int_\Omega d\eta.$$

## 3. Bundles of $k$-vectors

Let $X$ be an $n$-dimensional manifold, $\Lambda^k T_x X$ the $k$-exterior product of the tangent space $T_x X$, $x \in X$ a point. We put

$$\Lambda^k TX = \bigcup_{x\in X} \Lambda^k T_x X.$$

This set has a natural vector bundle structure over $X$, with type fibre $\Lambda^k \mathbb{R}^n$. We denote by $\tau^k$ the vector bundle projection of $\Lambda^k TX$.

Let $X$ (resp. $Y$) be a smooth manifold of dimension $n$ (resp. $m$), and let $f : X \to Y$ be a differentiable mapping. Choose a point $x \in X$ and a $k$-vector $\Xi \in \Lambda^k T_x X$. Then, choose a chart $(U, \varphi)$, $\varphi = (x^i)$, at $x$ and a chart $(V, \psi)$, $\psi = (y^\sigma)$, at $f(x) \in Y$ such that $f(U) \subset V$. Expressing $\Xi$ in components and setting

$$\Lambda^k T_x f \cdot \Xi = \frac{1}{(k!)^2}\left(\frac{\partial y^{\sigma_1} f\varphi^{-1}}{\partial x^{i_1}}\right)_{\varphi(x)}\left(\frac{\partial y^{\sigma_2} f\varphi^{-1}}{\partial x^{i_2}}\right)_{\varphi(x)}\cdots\left(\frac{\partial y^{\sigma_k} f\varphi^{-1}}{\partial x^{i_k}}\right)_{\varphi(x)}$$
$$\cdot\, \Xi^{i_1 i_2\ldots i_k}\left(\frac{\partial}{\partial y^{\sigma_1}}\right)_{f(x)}\wedge\left(\frac{\partial}{\partial y^{\sigma_2}}\right)_{f(x)}\wedge\ldots\wedge\left(\frac{\partial}{\partial y^{\sigma_k}}\right)_{f(x)},$$

we get a $k$-vector $\Lambda^k T_x f \cdot \Xi \in \Lambda^k T_{f(x)} Y$, and a vector bundle homomorphism $\Lambda^k Tf : \Lambda^k TX \to \Lambda^k TY$ over $f$ (the *lift* of $f$).

It is easily seen that differentiable mappings of a Euclidean space into a manifold can be canonically lifted to the bundles of $k$-vectors. For this purpose we use the *canonical* $k$*-vector field* on $\mathbb{R}^n$

$$\mathbb{R}^n \ni t \to \theta(t) = \frac{1}{k!}\varepsilon^{i_1 i_2\ldots i_k}\left(\frac{\partial}{\partial t^{i_1}}\right)_t\wedge\left(\frac{\partial}{\partial t^{i_2}}\right)_t\wedge\ldots\wedge\left(\frac{\partial}{\partial t^{i_k}}\right)_t \in \Lambda^k T\mathbb{R}^n.$$

Identifying $\Lambda^k T\mathbb{R}^n$ with $\mathbb{R}^n \times \Lambda^k \mathbb{R}^n$, the canonical section becomes the mapping $t \to (t, \varepsilon^{i_1 i_2\ldots i_k})$.

Consider a differentiable mapping $f : U \to Y$, where $U$ is an open subset of $\mathbb{R}^n$. For any point $t \in U$, $\Lambda^k T_t f \cdot \theta(t)$ is an element of the vector space $\Lambda^k T_{f(t)} Y$. We get the *canonical lift* $\Lambda^k f$ of $f$ to $\Lambda^k TY$, defined by

$$\Lambda^k f = \Lambda^k Tf \cdot \theta.$$

The canonical lift of the *parametrisation* $U \ni t \to (\psi^{-1} \circ \iota_{k,m})(t) \in V \cap S$ is expressed in a chart $(V, \psi)$, $\psi = (y^\sigma)$, as

$$\Lambda^k(\psi^{-1} \circ \iota_{k,m})(t) = \left(\frac{\partial}{\partial y^1}\right)_{\psi^{-1}\circ\iota_{k,m}(t)} \wedge \left(\frac{\partial}{\partial y^2}\right)_{\psi^{-1}\circ\iota_{k,m}(t)} \wedge \ldots \wedge \left(\frac{\partial}{\partial y^k}\right)_{\psi^{-1}\circ\iota_{k,m}(t)}. \quad (3.1)$$

Formula (3.1) also defines the mapping $V \ni y \to (\Lambda^k \psi)(y) \in (\tau^k)^{-1}(V)$ by

$$\Lambda^k \psi = \Lambda^k(\psi^{-1} \circ \iota_{k,m}) \circ \mathrm{pr}_{m,k} \psi, \quad (3.2)$$

the *canonical section along* $S$, associated with $(V, \psi)$. $\Lambda^k \psi$ is expressed by

$$(y^1, y^2, \ldots, y^k, y^{k+1}, y^{k+2}, \ldots, y^m) \to \Lambda^k(\psi^{-1} \circ \iota_{k,m})(y^1, y^2, \ldots, y^k) = ((y^1, y^2, \ldots, y^k, 0, 0, \ldots, 0), (1, 0, 0, \ldots, 0)).$$

Writing in the multi-index notation $((\tau^r)^{-1}(V), \Phi)$, $\Phi = (\dot{y}^I)$, and setting $I_0 = (1, 2, \ldots, k)$, we get the image of this mapping as a subset of $(\tau^r)^{-1}(V)$, defined by the equations $y^{k+1} = 0$, $y^{k+2} = 0, \ldots, y^m = 0$, $\dot{y}^{I_0} = 1$, $\dot{y}^I = 0$, $I \neq I_0$.

**Lemma 4.** *Let* $(V, \psi)$, $\psi = (y^\sigma)$, *and* $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, *be two charts on* $Y$, *adapted to* $S$, *such that* $V \cap \bar{V} \neq \varnothing$.

(1) *The canonical sections along* $S$ *satisfy*

$$\Lambda^k \bar{\psi} = \det\left(\frac{\partial y^i}{\partial \bar{y}^j}\right)_{\bar{\psi}(y)} \Lambda^k \psi.$$

(2) *The differential forms* $dy^\sigma$ *and* $dy^I$ *satisfy* $(\Lambda^k \psi)^* dy^i = dy^i$, $1 \leqslant i \leqslant k$, $(\Lambda^k \psi)^* dy^\nu = 0$, $k+1 \leqslant \nu \leqslant m$, $(\Lambda^k \psi)^* d\dot{y}^I = 0$. *In particular, on the set* $V \cap \bar{V}$,

$$(\Lambda^k \bar{\psi})^* d\bar{y}^1 \wedge d\bar{y}^2 \wedge \ldots \wedge d\bar{y}^k = \det\left(\frac{\partial \bar{y}^i}{\partial y^j}\right)_{\psi(y)} (\Lambda^k \psi) * dy^1 \wedge dy^2 \wedge \ldots \wedge dy^k. \quad (3.3)$$

## 4. Grassmann Fibrations

Consider the vector bundle $\Lambda^k TY$ and the subset $\Lambda^k_0 TY \subset \Lambda^k TY$, consisted of *non-zero* $k$-vectors. We have an equivalence relation on $\Lambda^k_0 TY$ "$\Xi_1$ *is equivalent with* $\Xi_2$, *if there exists a real number* $\lambda > 0$ *such that* $\Xi_1 = \lambda \Xi_2$". The quotient set has the structure of a fibration over $Y$, called the *Grassmann fibration* of degree $k$, and is denoted by $G^k Y$.

To describe the structure of the set $G^kY$, we proceed similarly as in the case of classical projective spaces. If in a chart $(V, \psi)$, $\psi = (y^\sigma)$,

$$\Xi_i = \frac{1}{k!}\Xi_i^{\sigma_1\sigma_2\ldots\sigma_k}\left(\frac{\partial}{\partial y^{\sigma_1}}\right)_y \wedge \left(\frac{\partial}{\partial y^{\sigma_2}}\right)_y \wedge \ldots \wedge \left(\frac{\partial}{\partial y^{\sigma_k}}\right)_y, \quad i = 1, 2,$$

are two nonzero $k$-vectors, then $\Xi_1$ is equivalent with $\Xi_2$ if and only if in this chart, $\Xi_1^{\sigma_1\sigma_2\ldots\sigma_k} = \lambda\Xi_2^{\sigma_1\sigma_2\ldots\sigma_k}$ for some $\lambda > 0$ and all $\sigma_1, \sigma_2, \ldots, \sigma_k$. We denote $V^{\nu_1\nu_2\ldots\nu_k} = \{\Xi \in (\tau^k)^{-1}(V) \mid \Xi^{\nu_1\nu_2\ldots\nu_k} > 0\}$. Then, a $k$-vector belonging to the set $V^{\nu_1\nu_2\ldots\nu_k} \subset \Lambda_0^kTY$ can be expressed by

$$\begin{aligned}\Xi = {} & \Xi^{\nu_1\nu_2\ldots\nu_k}\left(\frac{\partial}{\partial y^{\nu_1}}\right)_y \wedge \left(\frac{\partial}{\partial y^{\nu_2}}\right)_y \wedge \ldots \wedge \left(\frac{\partial}{\partial y^{\nu_k}}\right)_y \\ & + \frac{1}{k!}\sum_{(\tau_1\tau_2\ldots\tau_k)\neq(\nu_1\nu_2\ldots\nu_k)} \Xi^{\tau_1\tau_2\ldots\tau_k}\left(\frac{\partial}{\partial y^{\tau_1}}\right)_y \wedge \left(\frac{\partial}{\partial y^{\tau_2}}\right)_y \wedge \ldots \wedge \left(\frac{\partial}{\partial y^{\tau_k}}\right)_y\end{aligned}$$

(no summation through $\nu_1, \nu_2, \ldots, \nu_k$). Denoting by $\operatorname{sgn}\Xi^{\nu_1\nu_2\ldots\nu_k}$ the sign of the component $\Xi^{\nu_1\nu_2\ldots\nu_k}$, we can write $\Xi^{\nu_1\nu_2\ldots\nu_k} = \operatorname{sgn}\Xi^{\nu_1\nu_2\ldots\nu_k} \cdot |\Xi^{\nu_1\nu_2\ldots\nu_k}|$ and

$$\begin{aligned}\Xi = {} & \operatorname{sgn}\Xi^{\nu_1\nu_2\ldots\nu_k} \cdot |\Xi^{\nu_1\nu_2\ldots\nu_k}|\left(\frac{\partial}{\partial y^{\nu_1}}\right)_y \wedge \left(\frac{\partial}{\partial y^{\nu_2}}\right)_y \wedge \ldots \wedge \left(\frac{\partial}{\partial y^{\nu_k}}\right)_y \\ & + \frac{|\Xi^{\nu_1\nu_2\ldots\nu_k}|}{k!}\sum \frac{\Xi^{\tau_1\tau_2\ldots\tau_k}}{|\Xi^{\nu_1\nu_2\ldots\nu_k}|}\left(\frac{\partial}{\partial y^{\tau_1}}\right)_y \wedge \left(\frac{\partial}{\partial y^{\tau_2}}\right)_y \wedge \ldots \wedge \left(\frac{\partial}{\partial y^{\tau_k}}\right)_y,\end{aligned}$$

with the summation through $(\tau_1\tau_2\ldots\tau_k) \neq (\nu_1\nu_2\ldots\nu_k)$. But $\operatorname{sgn}\Xi^{\nu_1\nu_2\ldots\nu_k} = 1$, so we see the class of $\Xi$ can be represented as

$$\begin{aligned}[\Xi] = {} & \left(\frac{\partial}{\partial y^{\nu_1}}\right)_y \wedge \left(\frac{\partial}{\partial y^{\nu_2}}\right)_y \wedge \ldots \wedge \left(\frac{\partial}{\partial y^{\nu_k}}\right)_y \\ & + \frac{1}{k!}\sum \frac{\Xi^{\tau_1\tau_2\ldots\tau_k}}{\Xi^{\nu_1\nu_2\ldots\nu_k}}\left(\frac{\partial}{\partial y^{\tau_1}}\right)_y \wedge \left(\frac{\partial}{\partial y^{\tau_2}}\right)_y \wedge \ldots \wedge \left(\frac{\partial}{\partial y^{\tau_k}}\right)_y.\end{aligned}$$

We set for any $\Xi \in V^{\nu_1\nu_2\ldots\nu_k}$

$$\begin{aligned}& w^\sigma(\Xi) = y^\sigma(\Xi), \quad w^{\nu_1\nu_2\ldots\nu_k}(\Xi) = \dot{y}^{\nu_1\nu_2\ldots\nu_k}(\Xi), \\ & w^{\sigma_1\sigma_2\ldots\sigma_k}(\Xi) = \frac{\dot{y}^{\sigma_1\sigma_2\ldots\sigma_k}(\Xi)}{\dot{y}^{\nu_1\nu_2\ldots\nu_k}(\Xi)}, \quad (\sigma_1\sigma_2\ldots\sigma_k) \neq (\nu_1\nu_2\ldots\nu_k).\end{aligned} \tag{4.1}$$

The pair $(V^{\nu_1\nu_2\ldots\nu_k}, \Psi^{\nu_1\nu_2\ldots\nu_k})$, $\Psi^{\nu_1\nu_2\ldots\nu_k} = (w^\sigma, w^{\nu_1\nu_2\ldots\nu_k}, w^{\sigma_1\sigma_2\ldots\sigma_k})$, where the indices satisfy $(\sigma_1\sigma_2\ldots\sigma_k) \neq (\nu_1\nu_2\ldots\nu_k)$, is a chart on $\Lambda_0^kTY$; we call this chart $(\nu_1\nu_2\ldots\nu_k)$-*associated* with $(V, \psi)$. The pair $(V^{\nu_1\nu_2\ldots\nu_k}, W^{\nu_1\nu_2\ldots\nu_k})$, $W^{\nu_1\nu_2\ldots\nu_k} = (w^\sigma, w^{\sigma_1\sigma_2\ldots\sigma_k})$, $(\sigma_1\sigma_2\ldots\sigma_k) \neq (\nu_1\nu_2\ldots\nu_k)$, is a fibred chart on

$G^kY$. Writing formulas (4.1) in a different way, we have the transformation equations

$$w^\sigma = y^\sigma, \quad w^{\nu_1\nu_2\dots\nu_k} = \dot{y}^{\nu_1\nu_2\dots\nu_k}, \quad w^{\sigma_1\sigma_2\dots\sigma_k} = \frac{\dot{y}^{\sigma_1\sigma_2\dots\sigma_k}}{\dot{y}^{\nu_1\nu_2\dots\nu_k}}.$$

The projection $\kappa^k : \Lambda^k TY \to G^kY$ of $\Lambda^k TY$ onto $G^kY$ is the Cartesian projection $(w^\sigma, w^{\nu_1\nu_2\dots\nu_k}, w^{\sigma_1\sigma_2\dots\sigma_k}) \to (w^\sigma, w^{\sigma_1\sigma_2\dots\sigma_k})$. Combining $\Lambda^k(\psi^{-1}\iota_{k,m})$ and $\kappa^k$ we get the canonical lift of $\psi^{-1}\iota_{k,m}$ to the Grassmann fibration,

$$G^k(\psi^{-1}\iota_{k,m}) = \kappa^k \circ \Lambda^k(\psi^{-1}\iota_{k,m}). \tag{4.2}$$

**Lemma 5.** *Let $(V, \psi)$, $\psi = (y^\sigma)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, be two rectangle charts, adapted to $S$ at a point $y \in Y$. Suppose that $(V, \psi)$ and $(\bar{V}, \bar{\psi})$ are consistently oriented. Then*

$$G_k(\bar{\psi}^{-1}\iota_{k,m}) = G^k(\psi^{-1}\iota_{k,m}). \tag{4.3}$$

We set

$$G^kS = \{[\Xi] \in G^kY | [\Xi] = G^k(\psi^{-1}\iota_{k,m})(\mathrm{pr}_{m,k}\psi(y)), y \in S\}. \tag{4.4}$$

To a given chart $(V, \psi)$, $\psi = (y^\sigma)$, we associate the induced chart $((\tau^k)^{-1}(V), \Phi)$, $\Phi = (y^\sigma, \dot{y}^{\sigma_1\sigma_2\dots\sigma_k})$, on $\Lambda^k TY$; the associated charts on the Grassmann fibration $G^kY$ are $(V_0^{\nu_1\nu_2\dots\nu_k}, W^{\nu_1\nu_2\dots\nu_k})$,

$$W^{\nu_1\nu_2\dots\nu_k} = (w^\sigma, w^{\sigma_1\sigma_2\dots\sigma_k}),$$

with $(\sigma_1\sigma_2\dots\sigma_k) \neq (\nu_1\nu_2\dots\nu_k)$. Then, it is easily seen that each of the charts $(V_0^{\nu_1\nu_2\dots\nu_k}, W^{\nu_1\nu_2\dots\nu_k})$ is adapted to the submanifold $G^kS$.

**Theorem 1.** *Suppose $S$ is oriented. Then, the subset $G^kS$ of the Grassmann fibration $G^kY$ is a $k$-dimensional oriented submanifold, diffeomorphic with $S$.*

Theorem 1 allows us to integrate over $k$-dimensional submanifolds of $Y$ directly on the Grassmann fibration $G^kY$.

## 5. Variational functionals depending on submanifolds

As before, we write $G^kS$ (resp., $G^k\Omega$) for the canonical lift of a $k$-dimensional submanifold $S \subset Y$ (resp., $k$-piece $\Omega \subset Y$) to the Grassmann fibration $G^kY$. Denote by $\Gamma^kY$ the set of all $k$-pieces $\Omega$ of the manifold $Y$.

Let $\eta$ be a $k$-form on the Grassmann fibration $G^kY$. The form $\eta$ defines the *variational functional*

$$\Gamma^kY \ni \Omega \to \eta_\Omega(S) = \int_{G^k\Omega} \eta \in \mathbb{R}. \tag{5.1}$$

We roughly describe in this paper this construction for $k = 1$, representing variational functionals of *Finsler geometry* in terms of differential forms (cf. Urban and

Krupka [7]). Consider the tangent bundle $\Lambda^1 TY = TY$, a chart $(V, \psi)$, $\psi = (y^\sigma)$, on $Y$, and the associated chart $(\tau^1)^{-1}(V)$, $\Psi = (y^\sigma, \dot{y}^\sigma)$, on $TY$. A function $F : TY \to \mathbb{R}$ satisfies the *homogeneity condition*, if it satisfies

$$F(\lambda\xi) = \lambda F(\xi)$$

for all tangent vectors $\xi$ and every positive $\lambda \in R$. The same can be stated in coordinates, requiring that

$$F(y^\nu, \lambda\dot{y}^\nu) = \lambda F(y^\nu, \dot{y}^\nu).$$

**Theorem 2.**

(1) *For any function $F : TY \to \mathbb{R}$, the chart expressions*

$$\eta = \frac{\partial F}{\partial \dot{y}^\nu} dy^\nu \tag{5.2}$$

*define a global 1-form on $TY$.*

(2) *If $F$ satisfies the homogeneity condition, then $\eta$ is projectable on the Grassmann fibration $G^1 TY$.*

(3) *If $F$ satisfies the homogeneity condition, then, for any curve $\zeta : I \to Y$*

$$(\Lambda^1\zeta) * \eta = (F \circ \Lambda^1\zeta)dt. \tag{5.3}$$

The form $\eta$ (5.2) is known as the *Hilbert form* (Chern, Chen and Lam [1], Crampin and Saunders [2]). Theorem 2 (2) characterizes its basic property when $F$ is positive homogeneous: namely, in this case the Hilbert form is defined on the *Grassmann fibration* $G^1 TY$. One can also easily verify that $\eta$ is a special case of the *Lepage-Cartan form*. This fact completely determines the behaviour of the variational functional (5.1) under variations of submanifolds, extremal submanifolds, and their invariance properties.


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*Authors' addresses*

**Erico Tanaka**
Ochanomizu University, Department of Physics, 2-1-1 Ootsuka Bunkyo, 112-0086 Tokyo, Japan and
Palacky University, Department of Mathematics, 17. listopadu 12, 77146 Olomouc, Czech Republic
*E-mail address:* erico@cosmos.phys.ocha.ac.jp

**Demeter Krupka**
Lepage Research Institute, 783 42 Slatinice, Czech Republic
and
School of Mathematics, Beijing Institute of Technology, 5 South Zhongguancun Street, Haidian Zone, Beijing 100081, China
and
Department of Mathematics, University of Ostrava, 30. dubna 22, 701 03 Ostrava, Czech Republic
*E-mail address:* demeter.krupka@lepageri.eu