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The interior Euler-Lagrange operator in field theory†

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Abstract The paper is devoted to the interior Euler-Lagrange operator in field theory, representing an important tool for constructing the variational sequence. We give a new invariant definition of this operator by means of a natural decomposition of spaces of differential forms, appearing in the sequence, which defines its basic properties. Our definition extends the well-known cases of the Euler-Lagrange class (Euler-Lagrange form) and the Helmholtz class (Helmholtz form). This linear operator has the property of a projector, and its kernel consists of contact forms. The result generalizes an analogous theorem valid for variational sequences over 1-dimensional manifolds and completes the known heuristic expressions by explicit characterizations and proofs.

Keywords Interior Euler-Lagrange operator, jet, Lagrangian, Euler-Lagrange expressions, Helmholtz conditions, variational sequence

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1 Introduction

In 1989, the variational sequence theory on finite order jet spaces was introduced by Krupka [6], primarily for the purpose of study of basic variational objects and their local and global properties. Later, the variational sequence was analysed in the particular case of fibred manifolds with 1-dimensional base (fibred mechanics, Krupka [7]). Related concepts of global variational theory (especially Lepage forms and variational bicomplexes) were also studied, see Krupka [9], Vitolo [14], and references therein. In particular, it was discovered that the Euler-Lagrange mapping, assigning to a Lagrangian (n-form) its Euler-Lagrange form ($(n+1)$-form, its coefficients are the Euler-Lagrange expressions) is a globally defined morphism in the variational sequence. Another important result of the variational sequence theory is the concept of Helmholtz class, a globalization of the well-known Helmholtz variationality conditions. The elements of the variational sequence are classes of differential forms on the underlying jet space, representing all known variational objects, such as total derivatives, Lagrangians, Euler-Lagrange equations, variationality conditions. Since the variational sequence is a quotient sequence, there arises a natural problem of representing classes by different geometric objects, e.g. differential forms. A mapping, assigning to a class its chart representative is known as the (local) interior Euler-Lagrange operator. This operator was considered by several authors in different ways: Anderson [1] introduced the interior Euler-Lagrange operator by means of (local) differential operators within the variational bicomplex theory; Krbek and Musilová [4] applied the integration by parts procedure;

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for further approaches see also Bauderon [2], Dedecker and Tulczyjew [3]. Uniqueness of the interior Euler-Lagrange operator in the context of the variational bicomplex on infinite jet spaces was studied by Mikulski [10].

On the other hand, Krupka and Šeděnková-Volná [8], [12] introduced the interior Euler-Lagrange operator by means of decomposition theory of spaces of contact forms, naturally appearing in the sequence. This approach resulted in an invariant construction of the operator. Our objective is to generalize the results of [8], obtained for variational sequences over 1-dimensional manifolds, to the case of n-dimensional base manifolds (field theory). This paper completes the results of the preprint Volná [13]. We give a new proof of the main theorem, characterizing basic properties of the interior Euler-Lagrange operator. Namely, we show that this $\mathbb{R}$-linear operator (a) preserves the classes of differential forms in the sequence, (b) has the kernel coinciding with the space of contact forms, and (c) has the property of a projector.

Our basic references on the variational sequence theory on finite order jet prolongations of fibred manifolds are Krupka [5, 7, 9] and Šeděnková-Volná [11, 12].

Throughout this paper, the standard multiindex notation as well as the Einstein summation convention are freely applied. The symbol $i_\xi \rho$ denotes the contraction of a differential form ρ by a vector field ξ.

2 Background

Throughout this paper we denote by Y a fibred manifold over n-dimensional base X with projection π, where $m = \dim Y - n$. Let $r \geq 0$. Let J^rY denote the r-jet prolongation of Y, and $\pi^{r,s} : J^rY \to J^sY$, $0 \leq s < r$, and $\pi^r : J^rY \to X$, denote the *canonical jet projections*. An element of J^rY, denoted by $J^r_x\gamma$, is the r-jet of a section γ of $\pi : Y \to X$ with source at a point $x \in X$. Recall that any fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, with $1 \leq i \leq n$, $1 \leq \sigma \leq m$, induces the associated charts (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r})$, on J^rY, and (U, φ), $\varphi = (x^i)$, on X, where $V^r = (\pi^{r,0})^{-1}(V)$, $U = \pi(V)$. The *r-jet prolongation* of a section γ of π is a section $J^r\gamma$ of π^r, defined by $J^r\gamma(x) = J^r_x\gamma$.

For an open set $W \subset Y$, we put $W^r = (\pi^{r,0})^{-1}(W)$. If $f : W^r \to \mathbb{R}$ is a function, then for any fibred chart (V, ψ) such that $V \subset W$, we denote by $d_i f : V^{r+1} \to \mathbb{R}$ the *i-th formal derivative* of f with respect to (V, ψ); in fibred coordinates $\psi = (x^i, y^\sigma)$,

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{l=0}^{r} \sum_{j_1 \leq j_2 \leq \dots \leq j_l} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \dots j_l}} y^\sigma_{j_1 j_2 \dots j_l i}.$$

Recall that a vector field Ξ on Y is called π*-vertical*, if $T\pi \cdot \Xi = 0$; this means in local coordinates $\Xi = \Xi^\sigma (\partial/\partial y^\sigma)$. The *s-jet prolongation* of a π-vertical vector field Ξ on Y is a vector field $J^s\Xi$ on J^sY, given by

$$J^s\Xi = \sum_{|J|=0}^{s} \Xi^\sigma_J \frac{\partial}{\partial y^\sigma_J}, \quad \text{where} \quad \Xi^\sigma_{Ki} = d_i \Xi^\sigma_K.$$

In fibred coordinates $\psi = (x^i, y^\sigma)$, we denote

$$\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n,$$
$$\omega_i = i_{\partial/\partial x^i} \omega_0 = (-1)^{i-1} dx^1 \wedge dx^2 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n.$$

We say that a differential k-form ρ on J^rY is *contact*, if it vanishes along the r-jet prolongation $J^r\gamma$ of every section γ of π. An important case are contact 1-forms; if (V, ψ), $\psi = (x^i, y^\sigma)$, is a fibred chart on Y, then a 1-form is contact if and only if it is a linear combination of the forms

(1) $\omega^\sigma_J = dy^\sigma_J - y^\sigma_{Ji} dx^i,$

where $J = (j_1, \dots, j_s)$, $0 \le s \le r-1$, $0 \le j_1, \dots, j_s \le n$. We get the *contact basis* of linear forms on V^r, constituted by dx^i, ω^σ_J, dy^σ_I, where $0 \le |J| \le r-1$ and $|I| = r$. A general differential k-form ρ on V^r is contact if and only if ρ is generated by 1-forms ω^σ_J, and by 2-forms $d\omega^\sigma_I$, with $0 \le |J| \le r-1$, $|I| = r-1$ (Krupka [6]). After the canonical lifting to $J^{r+1}Y$, every differential k-form ρ on J^rY has a unique decomposition, expressed by the sum of *l-contact components* $p_l\rho$ of ρ,

$$(\pi^{r+1,r})^*\rho = \sum_{l=0}^{k} p_l\rho = h\rho + \sum_{l=1}^{k} p_l\rho,$$

where $h\rho = p_0\rho$ is a *horizontal component* of ρ, and a k-form $p_l\rho$ on $J^{r+1}Y$ contains exactly l-factors ω^σ_J of the form (1); see e.g. [9]. Note that if ρ is a contact k-form, then also the exterior derivative $d\rho$ is contact, and the exterior product of two contact forms is again a contact form. This implies, in particular, that contact forms constitute a differential ideal in exterior algebra of differential forms, called the *contact ideal*.

By means of the previous definition of a contact form, it is evident that every k-form with $k \ge n+1$ would be contact. For the forms of degree $\ge n+1$, we apply a new definition of contactness.

Let $k \ge n+1$. A differential k-form ρ on J^rY is said to be *contact*, if for every point of J^rY there exist a fibred chart (V, ψ) on Y, an integer s, $s \ge r$, and a contact $(k-1)$-form η on V^s such that

(2) $p_{k-n}((\pi^{s,r})^*\rho - d\eta) = 0.$

Condition (2) is equivalent to saying that $(\pi^{s,r})^*\rho$ can be expressed in the form

(3) $(\pi^{s,r})^*\rho = \mu + d\eta,$

where μ is a k-form on V^s such that $p_{k-n}\mu = 0$, and η is a contact $(k-1)$-form on V^s. A k-form ρ, with $k \ge n+1$, satisfying $p_{k-n}\rho = 0$, is called *strongly contact*.

Let $k \ge 0$. Denote by Ω^r_k the direct image of the sheaf of smooth k-forms over J^rY by the jet projection $\pi^{r,0}$. We set $\Omega^r_{0,c} = \{0\}$, and

(4) $$\begin{aligned} &\Omega^r_{k,c} = \ker h, \quad \text{for } 1 \le k \le n, \\ &\Omega^r_{k,c} = \ker p_{k-n}, \quad \text{for } n+1 \le k \le N,\ N = \dim J^rY, \end{aligned}$$

and

(5) $\Theta^r_k = \Omega^s_{k,c} + d\Omega^s_{k-1,c},$

where $k \ge 1$, and $d\Omega^s_{k-1,c}$ is the image sheaf of $\Omega^s_{k-1,c}$ by d. Formula (5) means that a form ρ belongs to Θ^r_k if its canonical lift $(\pi^{s,r})^*\rho$ possesses a decomposition (3). Note that for $2 \le k \le n$, $\Theta^r_k = \Omega^r_{k,c}$. For every open set $W \subset Y$, Ω^r_kW (resp. $\Omega^r_{k,c}W$) is the Abelian group of k-forms (resp. contact k-forms ($1 \le k \le n$), strongly contact k-forms ($n+1 \le k \le N$)) on W^r, $d\Omega^r_{k-1,c}W$ is the Abelian group of forms which can be locally expressed as differentials of contact, resp. strongly contact $(k-1)$-forms on W^r, and Θ^r_kW is a subgroup of Ω^r_kW of contact forms. We get an *exact* sequence of sheaves of Abelian groups

(6) $0 \to \Theta^r_1 \to \Theta^r_2 \to \Theta^r_3 \to \cdots \to \Theta^r_M \to 0,$

in which all arrows denote the exterior differentiation d, and $M = m\binom{n+r-1}{n} + 2n - 1$ (see Krupka [6]). Sequence (6) is a subsequence of the de Rham sequence of differential forms

(7) $0 \to \mathbb{R} \to \Omega^r_0 \to \Omega^r_1 \to \Omega^r_2 \to \cdots \to \Omega^r_{N-1} \to \Omega^r_N \to 0,$

where $N = \dim J^rY$. The quotient sequence

(8) $0 \to \mathbb{R} \to \Omega_0^r \to \Omega_1^r/\Theta_1^r \to \cdots \to \Omega_M^r/\Theta_M^r \to \Omega_{M+1}^r \to \cdots \to \Omega_N^r \to 0$

is also exact. We call (8) the *r-th order variational sequence* on J^rY. The class of a differential form $\rho \in \Omega_k^r W$ in the variational sequence (8) is denoted by $[\rho]$.

The quotient mappings $E : \Omega_k^r/\Theta_k^r \to \Omega_{k+1}^r/\Theta_{k+1}^r$ are defined by

(9) $E([\rho]) = [d\rho]$,

and satisfy the condition $E^2 = 0$. The quotient mapping $E : \Omega_n^r/\Theta_n^r \to \Omega_{n+1}^r/\Theta_{n+1}^r$ coincides with the well-known *Euler-Lagrange mapping*, and $E : \Omega_{n+1}^r/\Theta_{n+1}^r \to \Omega_{n+2}^r/\Theta_{n+2}^r$ is the *Helmholtz-Sonin mapping*.

A *Lagrangian* of order r is a π^r-horizontal n-form λ. In a fibred chart, we write

(10) $\lambda = \mathscr{L}\omega_0$,

where $\mathscr{L}$ is a function on J^rY, called the *Lagrange function*.

Let ρ be an n-form on J^rY. A form ρ is called a *Lepage n-form* if $p_1 d\rho$ is a $\pi^{r+1,0}$-horizontal $(n+1)$-form. A Lepage form ρ is called a *Lepage equivalent* of a Lagrangian λ if $h\rho = \lambda$. It is known that in higher order mechanics, Lepage equivalents are uniquely determined by Lagrangians. We denote by θ_λ a Lepage equivalent of a Lagrangian λ. If $r = 1$, θ_λ is, e.g., the well known *Poincaré-Cartan form*, if $r > 1$, we have the *generalized Poincaré-Cartan form*. If λ has a local expression (10), then

(11) $p_1 d\theta_\lambda = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$,

where

(12) $E_\sigma(\mathscr{L}) = \sum_{l=0}^{r} (-1)^l d_{j_1} d_{j_2} \dots d_{j_l} \dfrac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 \dots j_l}}$.

The form (11) is called the *Euler-Lagrange form* with components (12), the *Euler-Lagrange expressions*.

3 The Interior Euler-Lagrange Operator

First, we formulate the following lemma on the structure of 1-contact $(n+1)$-forms.

Lemma 1 *Let ρ be a 1-contact $(n+1)$-form on $J^{r+1}Y$, expressed by*

(13) $\rho = \sum_{|J|=0}^{r} A_\sigma^J \omega_J^\sigma \wedge \omega_0$.

Then there exist a 1-contact ω^σ-generated $(n+1)$-form $I_1\rho$ on $J^{2r+1}Y$, a 1-contact n-form $J_1\rho$ on $J^{2r}Y$, and a 2-contact $(n+1)$-form $K_1\rho$ on $J^{2r+1}Y$ such that

(14) $(\pi^{2r+1,r+1})^*\rho = I_1\rho - dJ_1\rho + K_1\rho$,

where

(15)
$$I_1\rho = B_\sigma \omega^\sigma \wedge \omega_0, \quad J_1\rho = \sum_{|J|=0}^{r-1} B_\sigma^{Ji} \omega_J^\sigma \wedge \omega_i,$$
$$K_1\rho = \sum_{|J|=0}^{r-1} (pdB_\sigma^{Ji}) \wedge \omega_J^\sigma \wedge \omega_i,$$

and

$$(16)\quad \begin{aligned} B_\sigma &= \sum_{s=0}^{r} (-1)^s d_{i_1} d_{i_2} \dots d_{i_s} A_\sigma^{i_1 i_2 \dots i_s}, \\ B_\sigma^{Ji} &= \sum_{q=1}^{r-k} (-1)^{q+1} d_{i_{k+2}} d_{i_{k+3}} \dots d_{i_{k+q}} A_\sigma^{Jii_{k+2}i_{k+3}\dots i_{k+q}}, \end{aligned}$$

with $|J| = k$*,* $k = 0, 1, \dots, r-1$.

Proof. Suppose that a 1-contact $(n+1)$-form ρ has an expression (13). By a direct calculation we obtain for every multiindex J,

$$(17)\quad P_\sigma^{Ji} \omega_{Ji}^\sigma \wedge \omega_0 = -d_i P_\sigma^{Ji} \omega_J^\sigma \wedge \omega_0 + (pdP_\sigma^{Ji}) \wedge \omega_J^\sigma \wedge \omega_i - d(P_\sigma^{Ji} \omega_J^\sigma \wedge \omega_i).$$

Applying condition (17) to the form ρ, we separate all terms containing $\omega^\sigma \wedge \omega_0$. Then

$$\begin{aligned} &(\pi^{2r+1,r+1})^* \rho = \sum_{s=0}^{r} A_\sigma^{i_1 i_2 \dots i_s} \omega_{i_1 i_2 \dots i_s}^\sigma \wedge \omega_0 \\ &= A_\sigma \omega^\sigma \wedge \omega_0 + \sum_{s=1}^{r} \Big(-d_{i_s} A_\sigma^{i_1 \dots i_{s-1} i_s} \omega_{i_1 \dots i_{s-1}}^\sigma \wedge \omega_0 \\ &+ (pdA_\sigma^{i_1 \dots i_{s-1} i_s}) \wedge \omega_{i_1 \dots i_{s-1}}^\sigma \wedge \omega_{i_s} - d(A_\sigma^{i_1 \dots i_{s-1} i_s} \omega_{i_1 \dots i_{s-1}}^\sigma \wedge \omega_{i_s}) \Big) \\ &= A_\sigma \omega^\sigma \wedge \omega_0 + \sum_{s=1}^{r} \Big(d_{i_{s-1}} d_{i_s} A_\sigma^{i_1 \dots i_{s-2} i_{s-1} i_s} \omega_{i_1 \dots i_{s-2}}^\sigma \wedge \omega_0 \\ &- (pd(d_{i_s} A_\sigma^{i_1 \dots i_{s-2} i_{s-1} i_s})) \wedge \omega_{i_1 \dots i_{s-2}}^\sigma \wedge \omega_{i_{s-1}} \\ &+ d(d_{i_s} A_\sigma^{i_1 \dots i_{s-2} i_{s-1} i_s} \omega_{i_1 \dots i_{s-2}}^\sigma \wedge \omega_{i_{s-1}}) \\ &+ (pdA_\sigma^{i_1 \dots i_{s-1} i_s}) \wedge \omega_{i_1 \dots i_{s-1}}^\sigma \wedge \omega_{i_s} - d(A_\sigma^{i_1 \dots i_{s-1} i_s} \omega_{i_1 \dots i_{s-1}}^\sigma \wedge \omega_{i_s}) \Big) \\ &= \dots \\ &= \sum_{s=0}^{r} (-1)^s d_{i_1} \dots d_{i_{s-1}} d_{i_s} A_\sigma^{i_1 \dots i_{s-1} i_s} \omega^\sigma \wedge \omega_0 \\ &+ \sum_{s=1}^{r} pd \Big(\sum_{q=1}^{s} (-1)^{q+1} d_{i_{s-q+2}} \dots d_{i_s} A_\sigma^{i_1 \dots i_{s-q} i_{s-q+1} \dots i_s} \Big) \wedge \omega_{i_1 \dots i_{s-q}}^\sigma \wedge \omega_{i_{s-q+1}} \\ &- d\Big(\sum_{q=1}^{s} (-1)^{q+1} d_{i_{s-q+2}} \dots d_{i_s} A_\sigma^{i_1 \dots i_{s-q} i_{s-q+1} \dots i_s} \omega_{i_1 \dots i_{s-q}}^\sigma \wedge \omega_{i_{s-q+1}} \Big), \end{aligned}$$

and resummating the last expression over k with $k = s - q$, $k = 0, 1, \dots, r-1$, we obtain

$$\begin{aligned} &(\pi^{2r+1,r+1})^* \rho \\ &= \sum_{s=0}^{r} (-1)^s d_{i_1} \dots d_{i_{s-1}} d_{i_s} A_\sigma^{i_1 \dots i_{s-1} i_s} \omega^\sigma \wedge \omega_0 \\ &- d\Big(\sum_{k=0}^{r-1} \sum_{q=1}^{r-k} (-1)^{q+1} d_{i_{k+2}} \dots d_{i_{k+q}} A_\sigma^{i_1 \dots i_k i_{k+1} i_{k+2} \dots i_{k+q}} \omega_{i_1 \dots i_k}^\sigma \wedge \omega_{i_{k+1}} \Big) \end{aligned}$$

$$+\sum_{k=0}^{r-1} pd\Big(\sum_{q=1}^{r-k}(-1)^{q+1}d_{i_{k+2}}\dots d_{i_{k+q}}A_\sigma^{i_1\dots i_k i_{k+1} i_{k+2}\dots i_{k+q}}\Big)\wedge\omega^\sigma_{i_1\dots i_k}\wedge\omega_{i_{k+1}}$$

$$=B_\sigma\omega^\sigma\wedge\omega_0 - d\Big(\sum_{|J|=0}^{r-1}B_\sigma^{Ji}\omega^\sigma_J\wedge\omega_i\Big)+\sum_{|J|=0}^{r-1}(pdB_\sigma^{Ji})\wedge\omega^\sigma_J\wedge\omega_i$$

$$=I_1\rho - d(J_1\rho)+K_1\rho,$$

as required.

In the following lemma we show that the form $I_1\rho$ in decomposition (14) of Lemma 1 is uniquely determined.

Lemma 2 *Suppose ρ is a 1-contact $(n+1)$-form on $J^{r+1}Y$, expressed by (13), such that*

(18) $(\pi^{2r+1,r+1})^*\rho = \rho_0 - d\rho' + \rho''$,

where ρ_0 is a 1-contact ω^σ-generated $(n+1)$-form on $J^{2r+1}Y$, ρ' is a 1-contact n-form on $J^{2r}Y$, and ρ'' is a 2-contact $(n+1)$-form on $J^{2r+1}Y$. Then

(19) $\rho_0 = I_1\rho$

and

(20) $-d\rho'+\rho'' = -dJ_1\rho + K_1\rho$,

where the forms $I_1\rho$, $J_1\rho$ and $K_1\rho$ are given by (15).

Proof. By assumption (18) and by Lemma 1, (14), we have two decompositions of the form ρ, hence

(21) $0 = (\rho_0 - I_1\rho) - d(\rho' - J_1\rho) + (\rho'' - K_1\rho)$,

where $\rho_0 - I_1\rho$ is a 1-contact ω^σ-generated $(n+1)$-form on $J^{2r+1}Y$, $\rho' - J_1\rho$ is a 1-contact n-form on $J^{2r}Y$, and $\rho'' - K_1\rho$ is a 2-contact $(n+1)$-form on $J^{2r+1}Y$. Let us denote

(22) $\rho_0 - I_1\rho = M_\sigma\omega^\sigma\wedge\omega_0, \quad \rho' - J_1\rho = \sum_{|J|=0}^{2r-1} N_\sigma^{J,i}\omega^\sigma_J\wedge\omega_i.$

Now we apply the formula

$$d(\omega^\sigma_J\wedge\omega_{ij}) = \omega^\sigma_{Ji}\wedge\omega_j - \omega^\sigma_{Jj}\wedge\omega_i,$$

where $\omega_{ij} = i_{\partial/\partial x^j} i_{\partial/\partial x^i}\omega_0$. We get

$$N_\sigma^{j_1\cdots j_s,i}\omega^\sigma_{j_1\dots j_s}\wedge\omega_i$$

$$=\Big(\frac{1}{s+1}(N_\sigma^{j_1\cdots j_s,i}+N_\sigma^{ij_2\cdots j_s,j_1}+N_\sigma^{j_1 ij_3\cdots j_s,j_2}+\dots+N_\sigma^{j_1\cdots j_{s-1}i,j_s})$$

$$+\frac{1}{s+1}(N_\sigma^{j_1\cdots j_s,i}-N_\sigma^{ij_2\cdots j_s,j_1})+\frac{1}{s+1}(N_\sigma^{j_1\cdots j_s,i}-N_\sigma^{j_1 ij_3\cdots j_s,j_2})$$

$$+\dots+\frac{1}{s+1}(N_\sigma^{j_1\cdots j_s,i}-N_\sigma^{j_1\cdots j_{s-1}i,j_s})\Big)\omega^\sigma_{j_1\dots j_s}\wedge\omega_i$$

$$
\begin{aligned}
&= \frac{1}{s+1}(N_\sigma^{j_1\dots j_s,i} + N_\sigma^{ij_2\dots j_s,j_1} + N_\sigma^{j_1ij_3\dots j_s,j_2} + \dots + N_\sigma^{j_1\dots j_{s-1}i,j_s})\omega^\sigma_{j_1\dots j_s}\wedge\omega_i \\
&+ \frac{1}{s+1}N_\sigma^{j_1\dots j_s,i}d(\omega^\sigma_{j_2\dots j_s}\wedge\omega_{j_1i}) + \frac{1}{s+1}N_\sigma^{j_1\dots j_s,i}d(\omega^\sigma_{j_1j_3\dots j_s}\wedge\omega_{j_2i}) \\
&+ \dots + \frac{1}{s+1}N_\sigma^{j_1\dots j_s,i}d(\omega^\sigma_{j_1\dots j_{s-1}}\wedge\omega_{j_si}) \\
&= \tilde{N}_\sigma^{j_1\dots j_si}\omega^\sigma_{j_1\dots j_s}\wedge\omega_i \\
&+ \frac{1}{s+1}d(N_\sigma^{j_1\dots j_s,i}\omega^\sigma_{j_2\dots j_s}\wedge\omega_{j_1i}) + \frac{1}{s+1}d(N_\sigma^{j_1\dots j_s,i}\omega^\sigma_{j_1j_3\dots j_s}\wedge\omega_{j_2i}) \\
&+ \dots + \frac{1}{s+1}d(N_\sigma^{j_1\dots j_s,i}\omega^\sigma_{j_1\dots j_{s-1}}\wedge\omega_{j_si}) \\
&- \frac{1}{s+1}d(N_\sigma^{j_1\dots j_s,i})\wedge\omega^\sigma_{j_2\dots j_s}\wedge\omega_{j_1i} - \frac{1}{s+1}d(N_\sigma^{j_1\dots j_s,i})\wedge\omega^\sigma_{j_1j_3\dots j_s}\wedge\omega_{j_2i} \\
&- \dots - \frac{1}{s+1}d(N_\sigma^{j_1\dots j_s,i})\wedge\omega^\sigma_{j_1\dots j_{s-1}}\wedge\omega_{j_si},
\end{aligned}
$$

where

$$\tilde{N}_\sigma^{j_1\dots j_si} = \frac{1}{s+1}(N_\sigma^{j_1\dots j_s,i} + N_\sigma^{ij_2\dots j_s,j_1} + N_\sigma^{j_1ij_3\dots j_s,j_2} + \dots + N_\sigma^{j_1\dots j_{s-1}i,j_s}).$$

Computing 1-contact part of the right-hand side of equation (21), we obtain

$$
\begin{aligned}
0 &= \rho_0 - I_1\rho - p_1d(\rho' - J_1\rho) \\
&= M_\sigma\omega^\sigma\wedge\omega_0 - p_1d(N_\sigma^{J,i}\omega^\sigma_J\wedge\omega_i) \\
&= M_\sigma\omega^\sigma\wedge\omega_0 + d_i\tilde{N}_\sigma^{Ji}\omega^\sigma_J\wedge\omega_0 + \tilde{N}_\sigma^{Ji}\omega^\sigma_{Ji}\wedge\omega_0 \\
&- \frac{1}{s+1}d_{j_1}N_\sigma^{j_1\dots j_s,i}\wedge\omega^\sigma_{j_2\dots j_si}\wedge\omega_0 + \frac{1}{s+1}d_iN_\sigma^{j_1\dots j_s,i}\wedge\omega^\sigma_{j_2\dots j_sj_1}\wedge\omega_0 \\
&- \frac{1}{s+1}d_{j_2}N_\sigma^{j_1\dots j_s,i}\wedge\omega^\sigma_{j_1j_3\dots j_si}\wedge\omega_0 + \frac{1}{s+1}d_iN_\sigma^{j_1\dots j_s,i}\wedge\omega^\sigma_{j_1j_3\dots j_sj_2}\wedge\omega_0 \\
&- \dots \\
&- \frac{1}{s+1}d_{j_s}N_\sigma^{j_1\dots j_s,i}\wedge\omega^\sigma_{j_1\dots j_{s-1}i}\wedge\omega_0 + \frac{1}{s+1}d_iN_\sigma^{j_1\dots j_s,i}\wedge\omega^\sigma_{j_1\dots j_{s-1}j_s}\wedge\omega_0,
\end{aligned}
$$

hence the coefficients M_σ, $N_\sigma^{J,i}$, and $\tilde{N}_\sigma^{Ji}$ must satisfy the conditions

$$
\begin{aligned}
&M_\sigma + d_i\tilde{N}_\sigma^i = 0, \\
(23)\quad &d_i\tilde{N}_\sigma^{Jji} + \tilde{N}_\sigma^{Jj} + \frac{1}{|J|+2}d_i(N_\sigma^{Ji,j} - N_\sigma^{Jj,i}) = 0, \qquad 0 \le |J| \le 2r-2, \\
&\tilde{N}_\sigma^{j_1\dots j_{2r}} = 0.
\end{aligned}
$$

From the second equation we express

$$\tilde{N}_\sigma^i = -d_j\tilde{N}_\sigma^{ij} - \frac{1}{2}d_j(N_\sigma^{j,i} - N_\sigma^{i,j}),$$

and put these functions to the first equation of (23) to obtain

$$M_\sigma - d_id_j\tilde{N}_\sigma^{ij} = 0.$$

Note that the antisymmetric part of $\tilde{N}_\sigma^i$ vanishes because of the symmetry of d_id_j. We can repeat this procedure until we use the last condition $\tilde{N}_{\sigma}^{j_1\cdots j_{2r}} = 0$. Finally we get

$$M_\sigma = 0$$

hence $\rho_0 = I_1\rho$. Condition (20) now directly follows from (21). This completes the proof.

The preceding lemma gives us uniqueness of $I_1\rho$. We extend the operator I_1, acting on 1-contact $(n+1)$-forms, to operator I_k that will be defined on k-contact $(n+k)$-forms. To this purpose, we use inductive definition.

Let $k > 1$. Suppose ρ is a k-contact $(n+k)$-form on $J^{r+1}Y$, and $\Xi_1, \Xi_2, \dots, \Xi_k$ are arbitrary π-vertical vector fields on Y. We define a k-contact $(n+k)$-form $I_k\rho$ on $J^{2r+1}Y$ by

$$\begin{aligned} (24)\qquad & i_{J^{2r+1}\Xi_k}\dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I_k\rho \\ &= \frac{1}{k}\Big(i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}}\dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I_{k-1}(i_{J^{r+1}\Xi_1}\rho) \\ &- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}}\dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_1} I_{k-1}(i_{J^{r+1}\Xi_2}\rho) \\ &- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}}\dots i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_2} I_{k-1}(i_{J^{r+1}\Xi_3}\rho) \\ &- \dots \\ &- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_1}\dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I_{k-1}(i_{J^{r+1}\Xi_{k-1}}\rho) \\ &- i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_{k-1}}\dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I_{k-1}(i_{J^{r+1}\Xi_k}\rho)\Big) \end{aligned}$$

and a k-contact $(n+k-1)$-form $J_k\rho$ on $J^{2r}Y$ by

$$\begin{aligned} (25)\qquad & i_{J^{2r}\Xi_k}\dots i_{J^{2r}\Xi_2} i_{J^{2r}\Xi_1} J_k\rho \\ &= -\frac{1}{k}\Big(i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}}\dots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_2} J_{k-1}(i_{J^{r+1}\Xi_1}\rho) \\ &- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}}\dots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_1} J_{k-1}(i_{J^{r+1}\Xi_2}\rho) \\ &- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}}\dots i_{J^{2r}\Xi_1} i_{J^{2r}\Xi_2} J_{k-1}(i_{J^{r+1}\Xi_3}\rho) \\ &- \dots \\ &- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_1}\dots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_2} J_{k-1}(i_{J^{r+1}\Xi_{k-1}}\rho) \\ &- i_{J^{2r}\Xi_1} i_{J^{2r}\Xi_{k-1}}\dots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_2} J_{k-1}(i_{J^{r+1}\Xi_k}\rho)\Big). \end{aligned}$$

Finally, we define $K_k\rho$ by

$$(26)\quad K_k\rho = p_{k+1} dJ_k\rho.$$

Now let η be an arbitrary $(n+k)$-form on J^rY, $k \geq 1$. Using the previous definition of I_k we set

$$(27)\quad \mathscr{I}\eta = I_k p_k \eta.$$

An $\mathbb{R}$-linear mapping

$$(28)\quad \mathscr{I} : \Omega^r_{n+k}W \to \Omega^{2r+1}_{n+k}W,$$

defined by (27), is called the *interior Euler-Lagrange operator* (cf. Anderson [1], Krupka and Šedĕnková-Volná [8], Krbek and Musilová [4]). The form $\mathscr{I}\eta$ is called the *canonical representative* of η.

For our proof of the main theorem, we need the following lemma.

Lemma 3 *Let $k \geq 1$. Let $W \subset Y$ be an open set in a fibred manifold Y. Suppose that an*

$(n+k)$-form $\rho \in \Omega^r_{n+k} W$ is expressed in a fibred chart on W as

(29) $\rho = d(A^{I_1 \dots I_k i}_{\sigma_1 \dots \sigma_k} \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_k}_{I_k} \wedge \omega_i).$

Then $\mathscr{I}\rho = 0$.

Proof. We proceed by induction. First, let $k = 1$. By a direct computation, we have

$$p_1 \rho = -d_i A^{Ii}_{\sigma} \omega^{\sigma}_I \wedge \omega_0 - A^{Ii}_{\sigma} \omega^{\sigma}_{Ii} \wedge \omega_0.$$

Using definition (27) of $\mathscr{I}$, and Lemma 1, (15), we obtain

$$\begin{aligned}\mathscr{I}\rho = I_1(p_1\rho) &= -\Big(I_1(d_i A^{Ii}_{\sigma} \omega^{\sigma}_I \wedge \omega_0) + I_1(A^{Ii}_{\sigma} \omega^{\sigma}_{Ii} \wedge \omega_0)\Big) \\ &= -\sum_{s=0}^{r-1} (-1)^s d_{i_1} d_{i_2} \dots d_{i_s} d_i A^{i_1 i_2 \dots i_s i}_{\sigma} \omega^{\sigma} \wedge \omega_0 \\ &\quad - \sum_{s=1}^{r} (-1)^s d_{i_1} d_{i_2} \dots d_{i_s} A^{i_1 i_2 \dots i_s}_{\sigma} \omega^{\sigma} \wedge \omega_0 = 0.\end{aligned}$$

Suppose now $\mathscr{I}\rho = 0$ for every $(n+k)$-form (29), where $k \geq 1$. We show that the assertion is valid also for $(n+k+1)$-forms. Let $\rho \in \Omega^r_{n+k+1} W$ be of the form (29). Thus

$$\begin{aligned}\rho &= d(A^{I_1 \dots I_{k+1} i}_{\sigma_1 \dots \sigma_{k+1}} \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_i) \\ &= dA^{I_1 \dots I_{k+1} i}_{\sigma_1 \dots \sigma_{k+1}} \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_i + A^{I_1 \dots I_{k+1} i}_{\sigma_1 \dots \sigma_{k+1}} d(\omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}}) \wedge \omega_i,\end{aligned}$$

and for the $(k+1)$-contact component of ρ we get

$$\begin{aligned}p_{k+1}\rho &= (-1)^{k+1} d_i A^{I_1 \dots I_{k+1} i}_{\sigma_1 \dots \sigma_{k+1}} \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_0 \\ &\quad + (-1)^{k+1}(k+1) A^{I_1 I_2 \dots I_{k+1} i}_{\sigma_1 \sigma_2 \dots \sigma_{k+1}} \omega^{\sigma_1}_{I_1 i} \wedge \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_0.\end{aligned} \tag{30}$$

By definitions of $\mathscr{I}$ and I_{k+1} (27), (24), it is now sufficient to show that for an arbitrary vertical vector field Ξ on Y, $I_k(i_{J^{r+1}\Xi} p_{k+1}\rho)$ vanishes. Computing the contraction of (30), we have

$$\begin{aligned}i_{J^{r+1}\Xi} p_{k+1}\rho &= (-1)^{k+1}(k+1)\, d_i A^{I_1 I_2 \dots I_{k+1} i}_{\sigma_1 \sigma_2 \dots \sigma_{k+1}} \Xi^{\sigma_1}_{I_1} \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_0 \\ &\quad + (-1)^{k+1}(k+1) A^{I_1 I_2 \dots I_{k+1} i}_{\sigma_1 \sigma_2 \dots \sigma_{k+1}} \Xi^{\sigma_1}_{I_1 i} \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_0 \\ &\quad - (-1)^{k+1}(k+1) A^{I_1 I_2 \dots I_{k+1} i}_{\sigma_1 \sigma_2 \dots \sigma_{k+1}} \omega^{\sigma_1}_{I_1 i} \wedge (k \Xi^{\sigma_2}_{I_2} \omega^{\sigma_3}_{I_3} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}}) \wedge \omega_0 \\ &= (-1)^{k+1}(k+1)\, d_i(A^{I_1 I_2 \dots I_{k+1} i}_{\sigma_1 \sigma_2 \dots \sigma_{k+1}} \Xi^{\sigma_1}_{I_1}) \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_0 \\ &\quad + (-1)^{k+1}(k+1) A^{I_1 I_2 I_3 \dots I_{k+1} i}_{\sigma_1 \sigma_2 \sigma_3 \dots \sigma_{k+1}} k \Xi^{\sigma_1}_{I_1} \omega^{\sigma_2}_{I_2 i} \wedge \omega^{\sigma_3}_{I_3} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_0 \\ &= p_k \eta,\end{aligned}$$

where

$$\eta = d(\tilde{A}^{I_2 I_3 \dots I_{k+1} i}_{\sigma_2 \sigma_3 \dots \sigma_{k+1}} \omega^{\sigma_2}_{I_2} \wedge \omega^{\sigma_3}_{I_3} \wedge \dots \wedge \omega^{\sigma_{k+1}}_{I_{k+1}} \wedge \omega_i),$$

and

$$\tilde{A}^{I_2 I_3 \dots I_{k+1} i}_{\sigma_2 \sigma_3 \dots \sigma_{k+1}} = -(k+1) A^{I_1 I_2 I_3 \dots I_{k+1} i}_{\sigma_1 \sigma_2 \sigma_3 \dots \sigma_{k+1}} \Xi^{\sigma_1}_{I_1}.$$

But by our assumption, $\mathscr{I}\eta = 0$ hence $I_k(i_{J^{r+1}\Xi} p_{k+1}\rho) = I_k(p_k\eta) = 0$, as required.

The following theorem characterizes the main properties of the interior Euler-Lagrange operator $\mathscr{I}$.

Theorem 1 *Let $k \geq 1$ be an integer. Let $\pi : Y \to X$ be a fibred manifold over n-dimensional base X, and $W \subset Y$ be an open set.*
(a) *For every $\rho \in \Omega^r_{n+k}W$, $\mathscr{I}\rho$ lies in the same class as $(\pi^{2r+1,r})^*\rho$.*
(b) *The kernel of $\mathscr{I} : \Omega^r_{n+k}W \to \Omega^{2r+1}_{n+k}W$ coincides with $\Theta^r_{n+k}W$.*
(c) *$\mathscr{I} \circ \mathscr{I} = \mathscr{I}$ (up to the canonical jet projection).*

Proof. (a) When it is obvious from the context, we omit pull-back of forms by the canonical jet projection $\pi^{2r+1,r+1}$. We proceed by induction. If $k = 1$, then $\mathscr{I}\rho - \rho \in \Theta^{2r+1}_{n+1}$ is straightforward from decomposition (14) and formulas (15) of Lemma 1.

Suppose that $\mathscr{I}\rho - \rho \in \Theta^{2r+1}_{n+k}$ for some $k > 1$. We shall prove that the same condition holds for $k+1$. Let ρ be an arbitrary $(n+k+1)$-form on $W^r \subset J^rY$. From definition (27) we have $\mathscr{I}\rho = I_{k+1}(p_{k+1}\rho)$. For an arbitrary π-vertical vector field Ξ on Y, $i_{J^{r+1}\Xi}p_{k+1}\rho$ is a k-contact $(n+k)$-form hence, by our assumption, $\mathscr{I}(i_{J^{r+1}\Xi}p_{k+1}\rho) - i_{J^{r+1}\Xi}p_{k+1}\rho$ belongs to $\Theta^{2r+1}_{n+k}W$; we write

(31) $I_k(i_{J^{r+1}\Xi}p_{k+1}\rho) = i_{J^{r+1}\Xi}p_{k+1}\rho + \mu_\Xi,$

where $\mu_\Xi \in \Theta^{2r+1}_{n+k}W$. Substituting (31) in the definition of I_{k+1} (24), we get for arbitrary π-vertical vector fields $\Xi_1, \Xi_2, \dots, \Xi_{k+1}$,

$$
\begin{aligned}
& i_{J^{2r+1}\Xi_{k+1}} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I_{k+1}(p_{k+1}\rho) \\
&= \frac{1}{k+1}\Big(i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I_k(i_{J^{r+1}\Xi_1} p_{k+1}\rho) \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_1} I_k(i_{J^{r+1}\Xi_2} p_{k+1}\rho) \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_2} I_k(i_{J^{r+1}\Xi_3} p_{k+1}\rho) \\
&- \dots \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_1} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I_k(i_{J^{r+1}\Xi_k} p_{k+1}\rho) \\
&- i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I_k(i_{J^{r+1}\Xi_{k+1}} p_{k+1}\rho)\Big) \\
&= \frac{1}{k+1}\Big(i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} (i_{J^{r+1}\Xi_1} p_{k+1}\rho + \mu_{\Xi_1}) \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_1} (i_{J^{r+1}\Xi_2} p_{k+1}\rho + \mu_{\Xi_2}) \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_2} (i_{J^{r+1}\Xi_3} p_{k+1}\rho + \mu_{\Xi_3}) \\
&- \dots \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_1} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} (i_{J^{r+1}\Xi_k} p_{k+1}\rho + \mu_{\Xi_k}) \\
&- i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} (i_{J^{r+1}\Xi_{k+1}} p_{k+1}\rho + \mu_{\Xi_{k+1}})\Big) \\
&= i_{J^{2r+1}\Xi_{k+1}} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} p_{k+1}\rho + \frac{1}{k+1}\Big(i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} (\mu_{\Xi_1}) \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_1} (\mu_{\Xi_2}) \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_2} (\mu_{\Xi_3}) \\
&- \dots \\
&- i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_1} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} (\mu_{\Xi_k}) \\
&- i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} (\mu_{\Xi_{k+1}})\Big),
\end{aligned}
$$

and consequently

$$
\begin{aligned}
& i_{J^{2r+1}\Xi_{k+1}} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} (I_{k+1}(p_{k+1}\rho) - p_{k+1}\rho) \\
& = \frac{1}{k+1} \Big(i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} (\mu_{\Xi_1}) \\
& - i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_1} (\mu_{\Xi_2}) \\
& - \dots \\
& - i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} (\mu_{\Xi_{k+1}}) \Big) .
\end{aligned} \tag{32}
$$

Let $\xi_1, \dots, \xi_n \in T_x J^{2r+1}Y$ be tangent vectors to $J^{2r+1}Y$ at a point x. Expressing the value of the forms on both sides of (32) at x and $\xi_1, \dots, \xi_n$, we obtain

$$
\begin{aligned}
& (I_{k+1}(p_{k+1}\rho) - p_{k+1}\rho)(J^{2r+1}\Xi_{k+1}(x), \dots, J^{2r+1}\Xi_1(x), \xi_1, \dots, \xi_n) \\
& = \frac{1}{k+1} \big(\mu_{\Xi_1}(J^{2r+1}\Xi_{k+1}, J^{2r+1}\Xi_k, \dots, J^{2r+1}\Xi_3, J^{2r+1}\Xi_2, \xi_1, \dots, \xi_n) \\
& - \mu_{\Xi_2}(J^{2r+1}\Xi_{k+1}, J^{2r+1}\Xi_k, \dots, J^{2r+1}\Xi_3, J^{2r+1}\Xi_1, \xi_1, \dots, \xi_n) \\
& - \dots \\
& - \mu_{\Xi_{k+1}}(J^{2r+1}\Xi_1, J^{2r+1}\Xi_k, \dots, J^{2r+1}\Xi_3, J^{2r+1}\Xi_2, \xi_1, \dots, \xi_n) \big) .
\end{aligned} \tag{33}
$$

It now follows that the $(k+1)$-contact part of the left-hand side of (33) vanishes, thus the $(n+k+1)$-form $I_{k+1}(p_{k+1}\rho) - p_{k+1}\rho$ belongs to $\Theta^{2r+1}_{n+k+1}W$ (up to the canonical jet projection). In addition to this, it is easy to see that $p_{k+1}\rho - \rho \in \Theta^{2r+1}_{n+k+1}V$ hence we conclude with $\mathscr{I}\rho - \rho \in \Theta^{2r+1}_{n+k+1}W$.

(b) Let $\rho \in \Omega^r_{n+k}W$ be an arbitrary $(n+k)$-form. From assertion (a) of this theorem we have $\mathscr{I}\rho - \rho \in \Theta^{2r+1}_{n+k}V$. Suppose first that $\mathscr{I}\rho = 0$. Then it follows that $-\rho \in \Theta^{2r+1}_{n+k}W$. However, ρ is defined on $W^r \subset J^rY$ hence by definition (2) of contact forms we get $\rho \in \Theta^r_{n+k}W$.

Conversely, let $\rho \in \Theta^r_{n+k}W$. Then ρ is expressible as a sum of the following terms

$$
\begin{aligned}
& A^{I_1 \dots I_{k+s} i_1 \dots i_s}_{\sigma_1 \dots \sigma_{k+s}} \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_{k+s}}_{I_{k+s}} \wedge \omega_{i_1 \dots i_s}, \\
& d(A^{I_1 \dots I_{k+s-1} i_1 \dots i_s}_{\sigma_1 \dots \sigma_{k+s-1}} \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_{k+s-1}}_{I_{k+s-1}} \wedge \omega_{i_1 \dots i_s}),
\end{aligned} \tag{34}
$$

where $s = 1, 2, \dots, n$. The summands μ of (34) for which $p_k\mu = 0$, obey $\mathscr{I}\mu = 0$ directly from the definition of $\mathscr{I}$. It remains to show that the condition $\mathscr{I}\mu = 0$ is satisfied also for terms of the form

$$\mu = d(A^{I_1 \dots I_k i}_{\sigma_1 \dots \sigma_k} \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_k}_{I_k} \wedge \omega_i).$$

This was, however, already proved in Lemma 3. Thus, the kernel of $\mathscr{I}$ coincides with the space of contact forms $\Theta^r_{n+k}W$.

(c) Finally, we show that $\mathscr{I}$ has the property of a projector, up to a lift of the canonical jet projection. From (a) we have $\mathscr{I}\rho - \rho \in \Theta^{2r+1}_{n+k}W$, and property (b) of this theorem implies $\mathscr{I}(\mathscr{I}\rho - \rho) = 0$. From linearity of $\mathscr{I}$ we conclude that $\mathscr{I}(\mathscr{I}\rho) - \mathscr{I}\rho = 0$ for every $\rho \in \Omega^r_{n+k}W$. This means that $\mathscr{I} \circ \mathscr{I} = \mathscr{I}$.

Remark 1 Theorem 1, (b), implies that the image of $\Omega^r_{n+k}W$ by $\mathscr{I} : \Omega^r_{n+k}W \to \Omega^{2r+1}_{n+k}W$ is canonically isomorphic with the quotient group $\Omega^{2r+1}_{n+k}W / \Theta^{2r+1}_{n+k}W$. If $\rho \in \Omega^r_{n+k}W$, then the class $[\rho] \in \Omega^{2r+1}_{n+k}W / \Theta^{2r+1}_{n+k}W$ in the variational sequence can be identified with the globally defined differential form $\mathscr{I}\rho$. However, it may happen that a class of $\rho \in \Omega^r_{n+k}W$, an element of $\Omega^s_{n+k}W / \Theta^s_{n+k}W$ in the variational sequence of order s, $s < 2r+1$, is *not* a globally defined differential form (for an example, see [7]).

Remark 2 Instead of inductive definition (24) of operator I_k, it is also possible to use an equivalent definition by means of the operator I_1.

Let $k \geq 1$. Let ρ be a k-contact $(n+k)$-form on $J^{r+1}Y$, and $\Xi_1, \Xi_2, \dots, \Xi_k$ be arbitrary π-vertical vector fields on Y. We define a k-contact $(n+k)$-form $I_k\rho$ on $J^{2r+1}Y$ by

$$\begin{aligned}
i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I_k\rho = \frac{1}{k}\Big(& i_{J^{2r+1}\Xi_k} I_1 (i_{J^{r+1}\Xi_{k-1}} i_{J^{r+1}\Xi_{k-2}} \dots i_{J^{r+1}\Xi_3} i_{J^{r+1}\Xi_2} i_{J^{r+1}\Xi_1}\rho) \\
& - i_{J^{2r+1}\Xi_{k-1}} I_1 (i_{J^{r+1}\Xi_k} i_{J^{r+1}\Xi_{k-2}} \dots i_{J^{r+1}\Xi_3} i_{J^{r+1}\Xi_2} i_{J^{r+1}\Xi_1}\rho) \\
& - i_{J^{2r+1}\Xi_{k-2}} I_1 (i_{J^{r+1}\Xi_{k-1}} i_{J^{r+1}\Xi_k} \dots i_{J^{r+1}\Xi_3} i_{J^{r+1}\Xi_2} i_{J^{r+1}\Xi_1}\rho) \\
& - \dots \\
& - i_{J^{2r+1}\Xi_2} I_1 (i_{J^{r+1}\Xi_{k-1}} i_{J^{r+1}\Xi_{k-2}} \dots i_{J^{r+1}\Xi_3} i_{J^{r+1}\Xi_k} i_{J^{r+1}\Xi_1}\rho) \\
& - i_{J^{2r+1}\Xi_1} I_1 (i_{J^{r+1}\Xi_{k-1}} i_{J^{r+1}\Xi_{k-2}} \dots i_{J^{r+1}\Xi_3} i_{J^{r+1}\Xi_2} i_{J^{r+1}\Xi_k}\rho)\Big).
\end{aligned}$$

4 Examples

We conclude this paper with two examples of canonical representatives of differential forms, canonically isomorphic with classes in the variational sequence, well-known in the calculus of variations. From the variational sequence theory we observe that classes of forms represent Lagrangians (n-forms), Euler-Lagrange expressions ($(n+1)$-forms), Helmholtz variationality conditions ($(n+2)$-forms), etc.

At first, consider a Lagrangian λ of order r on J^rY, given in a fibred chart by $\lambda = \mathscr{L}\omega_0$ on J^rY. λ represents the class of a 1-form (the Lagrange class), defined by its horizontal component. We find the canonical representative of the $(n+1)$-form $d\lambda$. By our definition of $\mathscr{I}$, we compute

$$p_1(d\lambda) = pd\mathscr{L} \wedge \omega_0 = \Big(\sum_{l=0}^{r} \frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2 \cdots j_l}} \omega^\sigma_{j_1 j_2 \cdots j_l}\Big) \wedge \omega_0,$$

and obtain by Lemma 1, (15), the *Euler-Lagrange form* of $d\lambda$,

(35) $\mathscr{I}(d\lambda) = I_1(p_1 d\lambda) = E_\sigma(\mathscr{L})\,\omega^\sigma \wedge \omega_0,$

where

(36) $E_\sigma(\mathscr{L}) = \sum_{l=0}^{r} (-1)^l d_{j_1} d_{j_2} \dots d_{j_l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2 \cdots j_l}}.$

The mapping assigning to the Lagrange class its Euler-Lagrange expressions (36) represents the well-known Euler-Lagrange mapping.

Let ε be a 1-contact, $\pi^{r,0}$-horizontal form on J^rY (called the *source form*), an element of image of the Euler-Lagrange mapping. In a fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, ε is of the form $\varepsilon = \varepsilon_\nu \omega^\nu \wedge \omega_0$. We have

$$p_2 d\varepsilon = \sum_{l=0}^{r} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \cdots j_l}} \omega^\sigma_{j_1 j_2 \cdots j_l} \wedge \omega^\nu \wedge \omega_0.$$

By the definition of the operator $I_2\rho$ (24), for arbitrary π-vertical vector fields Ξ_1, Ξ_2 on Y we get

$$i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I_2 \rho = \frac{1}{2}\left(i_{J^{2r+1}\Xi_2} I_1 (i_{J^{r+1}\Xi_1}\rho) - i_{J^{2r+1}\Xi_1} I_1 (i_{J^{r+1}\Xi_2}\rho)\right).$$

Hence, we obtain a chart expression of $I_2(p_2 d\varepsilon)$ of the form

$$\begin{aligned}\mathscr{I}(d\varepsilon) &= I_2(p_2 d\varepsilon)\\ &= \frac{1}{2}\sum_{k=0}^{r} H^{j_1 j_2 \cdots j_k}_{\nu\sigma}(\varepsilon)\,\omega^{\sigma}_{j_1 j_2 \cdots j_k} \wedge \omega^{\nu} \wedge \omega_0,\end{aligned}$$

where

(37)

$$H^{j_1 j_2 \cdots j_k}_{\nu\sigma}(\varepsilon) = \frac{\partial \varepsilon_\nu}{\partial y^{\sigma}_{j_1 j_2 \cdots j_k}} - (-1)^k \frac{\partial \varepsilon_\sigma}{\partial y^{\nu}_{j_1 j_2 \cdots j_k}} - \sum_{p=k+1}^{r} (-1)^p \binom{p}{k} d_{j_{k+1}} d_{j_{k+2}} \ldots d_{j_p} \frac{\partial \varepsilon_\sigma}{\partial y^{\nu}_{j_1 j_2 \cdots j_p}}.$$

$\mathscr{I}(d\varepsilon)$ is a globally defined object in the variational sequence, called the *Helmholtz form*, with coefficients (37), the *Helmholtz expressions* (cf. Krupka [9], Krbek and Musilová [4], Šeděnková-Volná [12]).

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