SO(3)-invariant Finsler structures

Ján Brajerčík, Demeter Krupka

University of Prešov, Slovakia
Lepage Research Institute, Slovakia

28th International Summer School on Global Analysis and Its Applications, Brasov, Romania, 18.8.2026

Introduction

We deal with parameter invariant variational functionals on the manifold of 1-velocities $T^1 Q$ of a smooth manifold Q, which are also SO(3)-invariant.

Introduction

We deal with parameter invariant variational functionals on the manifold of 1-velocities T^1Q of a smooth manifold Q, which are also SO(3)-invariant.

We explicitly describe Lepage forms on T^1Q serving as an integrand in variational functionals.

Introduction

We deal with parameter invariant variational functionals on the manifold of 1-velocities T^1Q of a smooth manifold Q, which are also SO(3)-invariant.

We explicitly describe Lepage forms on T^1Q serving as an integrand in variational functionals.

Corresponding conservation law equations are discussed and illustrated by an example.

Preliminaries: fibred mechanics

Y a fibred manifold with 1-dimensional base X and projection π
$\dim Y = m + 1$

$J^r Y$ r-jet prolongation of Y, set of r-jets of sections γ of π

Preliminaries: fibred mechanics

Y a fibred manifold with 1-dimensional base X and projection π
$\dim Y = m + 1$

$J^r Y$ r-jet prolongation of Y, set of r-jets of sections γ of π

The canonical jet projections

$\pi^r : J^r Y \to X,\ \pi^r(J^r_x \gamma) = x,$

$\pi^{r,s} : J^r Y \to J^s Y,\ \pi^{r,s}(J^r_x \gamma) = J^s_x \gamma$

Preliminaries: fibred mechanics

Y a fibred manifold with 1-dimensional base X and projection π
$\dim Y = m + 1$

$J^r Y$ r-jet prolongation of Y, set of r-jets of sections γ of π

The canonical jet projections
$\pi^r : J^r Y \to X, \ \pi^r(J^r_x\gamma) = x,$
$\pi^{r,s} : J^r Y \to J^s Y, \ \pi^{r,s}(J^r_x\gamma) = J^s_x\gamma$

r-jet prolongation $J^r\gamma$ of γ is the mapping
$X \ni x \mapsto J^r\gamma(x) = J^r_x\gamma \in J^r Y$

Denoting $\pi(V) = U$ and $V^r = (\pi^{r,0})^{-1}(V)$ for an open subset V of Y, any fibred chart (V, ψ), $\psi = (t, y^\sigma)$, on Y, induces the associated charts

(U, φ), $\varphi = (t)$, on X,

(V^r, ψ^r), $\psi^r = (t, y^\sigma_{(0)}, y^\sigma_{(1)}, y^\sigma_{(2)}, \dots, y^\sigma_{(r)})$, on $J^r Y$, where

$y^\sigma_{(s)}(J^s_x \gamma) = D^s(y^\sigma \gamma \varphi^{-1})(\varphi(x))$, $0 \leq s \leq r$, $1 \leq \sigma \leq m$.

Denoting $\pi(V) = U$ and $V^r = (\pi^{r,0})^{-1}(V)$ for an open subset V of Y, any fibred chart (V, ψ), $\psi = (t, y^\sigma)$, on Y, induces the associated charts

(U, φ), $\varphi = (t)$, on X,

(V^r, ψ^r), $\psi^r = (t, y^\sigma_{(0)}, y^\sigma_{(1)}, y^\sigma_{(2)}, \dots, y^\sigma_{(r)})$, on $J^r Y$, where

$y^\sigma_{(s)}(J^s_x\gamma) = D^s(y^\sigma\gamma\varphi^{-1})(\varphi(x))$, $0 \leq s \leq r$, $1 \leq \sigma \leq m$.

For lower dimensions we use a modified notation; e.g. if $r = 2$: $\psi^2 = (t, y^\sigma, \dot{y}^\sigma, \ddot{y}^\sigma)$.

Denoting $\pi(V) = U$ and $V^r = (\pi^{r,0})^{-1}(V)$ for an open subset V of Y, any fibred chart (V, ψ), $\psi = (t, y^\sigma)$, on Y, induces the associated charts

(U, φ), $\varphi = (t)$, on X,
(V^r, ψ^r), $\psi^r = (t, y^\sigma_{(0)}, y^\sigma_{(1)}, y^\sigma_{(2)}, \dots, y^\sigma_{(r)})$, on $J^r Y$, where
$y^\sigma_{(s)}(J^s_x \gamma) = D^s(y^\sigma \gamma \varphi^{-1})(\varphi(x))$, $0 \leq s \leq r$, $1 \leq \sigma \leq m$.

For lower dimensions we use a modified notation; e.g. if $r = 2$: $\psi^2 = (t, y^\sigma, \dot{y}^\sigma, \ddot{y}^\sigma)$.

A tangent vector ξ at a point $y \in Y$ is said to be π-vertical, if $T\pi(\xi) = 0$.

A differential form ρ on Y is said to be π-horizontal, if for every point $y \in Y$ the contraction $i_\xi \rho(y)$ vanishes whenever $\xi \in T_y Y$ is a π-vertical vector.

Horizontalization of forms

Let $k \geq 1$. For any open set $W \subset Y$, we denote by

$\Omega^r_0 W$ the ring of smooth functions on W^r,

$\Omega^r_k W$ the $\Omega^r_0 W$-module of smooth k-forms on W^r

$\Omega^r_{k,X} W$ the submodule of $\Omega^r_k W$ of π^r-horizontal forms.

Horizontalization of forms

Let $k \geq 1$. For any open set $W \subset Y$, we denote by

$\Omega_0^r W$ the ring of smooth functions on W^r,

$\Omega_k^r W$ the $\Omega_0^r W$-module of smooth k-forms on W^r

$\Omega_{k,X}^r W$ the submodule of $\Omega_k^r W$ of π^r-horizontal forms.

A morphism of exterior algebras $h : \Omega_k^r W \to \Omega_{k,X}^{r+1} W$

$$hf = f \circ \pi^{r+1,r}, \quad hdt = dt, \quad hdy^\sigma_{(l)} = y^\sigma_{(l+1)} dt,$$

where $f : V^r \to \mathbf{R}$ is a function; h the π-horizontalization.

Horizontalization of forms

Let $k \geq 1$. For any open set $W \subset Y$, we denote by

$\Omega_0^r W$ the ring of smooth functions on W^r,

$\Omega_k^r W$ the $\Omega_0^r W$-module of smooth k-forms on W^r

$\Omega_{k,X}^r W$ the submodule of $\Omega_k^r W$ of π^r-horizontal forms.

A morphism of exterior algebras $h : \Omega_k^r W \to \Omega_{k,X}^{r+1} W$

$$hf = f \circ \pi^{r+1,r}, \quad hdt = dt, \quad hdy^{\sigma}_{(l)} = y^{\sigma}_{(l+1)} dt,$$

where $f : V^r \to \mathbf{R}$ is a function; h the π-horizontalization.

A vector field ξ on Y is said to be π-projectable if there is a vector field ξ_0 on X such that $T\pi \circ \xi = \xi_0 \circ \pi$.

Contact forms

A form $\rho \in \Omega_k^r W$ is contact, if $h\rho = 0$.

Contact forms

A form $\rho \in \Omega^r_k W$ is contact, if $h\rho = 0$.

1-forms

$$\omega^\sigma = dy^\sigma - \dot{y}^\sigma dt, \quad \dot{\omega}^\sigma = d\dot{y}^\sigma - \ddot{y}^\sigma dt,$$

are examples of contact forms on $V^2 \subset J^2 Y$.

Contact forms

A form $\rho \in \Omega^r_k W$ is contact, if $h\rho = 0$.

1-forms

$$\omega^\sigma = dy^\sigma - \dot{y}^\sigma dt, \quad \dot{\omega}^\sigma = d\dot{y}^\sigma - \ddot{y}^\sigma dt,$$

are examples of contact forms on $V^2 \subset J^2 Y$.

A form $\rho \in \Omega^1_k W$ has a unique decomposition

$$(\pi^{2,1})^* \rho = h\rho + p_1 \rho + p_2 \rho + \ldots + p_k \rho,$$

in which $p_k \rho$ contains, in any fibred chart, exactly k factors $\omega^\sigma, \dot{\omega}^\sigma$.

If $k = 1$, then $(\pi^{r+1,r})^* \rho = h\rho + p_1 \rho$,

if $k = 2$, then $(\pi^{r+1,r})^* \rho = p_1 \rho + p_2 \rho$.

We say that ρ is of order of contactness k, if $(\pi^{r+1,r})^* \rho = p_k \rho$.

Variational functional

By a Lagrangian of order r for Y we mean a 1-form $\lambda \in \Omega^r_{1,X} W$.
In fibred chart (V, ψ), $V \subset W$,

$$\lambda = \mathcal{L} dt.$$

$\mathcal{L} : V^r \to \mathbf{R}$ is the Lagrange function.

Variational functional

By a Lagrangian of order r for Y we mean a 1-form $\lambda \in \Omega^r_{1,X} W$. In fibred chart (V, ψ), $V \subset W$,

$$\lambda = \mathcal{L} dt.$$

$\mathcal{L} : V^r \to \mathbf{R}$ is the Lagrange function.

For any piece Ω of X with boundary $\partial\Omega$, λ gives rise to the variational functional

$$\Gamma_\Omega Y \ni \gamma \mapsto \lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\lambda \in \mathbf{R}, \qquad (1)$$

where $\Gamma_\Omega Y$ is the set of sections of Y, defined on Ω.

Variation of the variational functional

Let $V \subset Y$ be an open set, $\alpha : V \to Y$ an automorphism of Y, $U = \pi(V)$, and let $\alpha_0 : U \to X$ be the projection of α. Setting for every $J^r_x\gamma \in V^r$,

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}),$$

we obtain the r-jet prolongation $J^r\alpha : V^r \to J^rY$.

Variation of the variational functional

Let $V \subset Y$ be an open set, $\alpha : V \to Y$ an automorphism of Y, $U = \pi(V)$, and let $\alpha_0 : U \to X$ be the projection of α. Setting for every $J^r_x\gamma \in V^r$,

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}),$$

we obtain the r-jet prolongation $J^r\alpha : V^r \to J^rY$.

Choose $\gamma \in \Gamma_\Omega Y$ and a π-projectable vector field Ξ on Y with flow α_t, and consider the variation γ_t of γ, induced by Ξ. We get

$$(-\epsilon, \epsilon) \ni t \mapsto \lambda_{\alpha_{(0)t}(\Omega)}(\alpha_t\gamma\alpha^{-1}_{(0)t}) = \int_\Omega J^r\gamma^* J^r\alpha_t^*\lambda \in \mathbf{R}.$$

Variation of the variational functional

Let $V \subset Y$ be an open set, $\alpha : V \to Y$ an automorphism of Y, $U = \pi(V)$, and let $\alpha_0 : U \to X$ be the projection of α. Setting for every $J^r_x\gamma \in V^r$,

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}),$$

we obtain the r-jet prolongation $J^r\alpha : V^r \to J^rY$.

Choose $\gamma \in \Gamma_\Omega Y$ and a π-projectable vector field Ξ on Y with flow α_t, and consider the variation γ_t of γ, induced by Ξ. We get

$$(-\epsilon, \epsilon) \ni t \mapsto \lambda_{\alpha_{(0)t}(\Omega)}(\alpha_t\gamma\alpha_{(0)t}^{-1}) = \int_\Omega J^r\gamma^* J^r\alpha_t^*\lambda \in \mathbf{R}.$$

Differentiating this function at $t = 0$ we obtain

$$(\partial_{J^r\Xi}\lambda)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\lambda. \qquad (2)$$

The number (2) is the variation of λ_Ω at γ, induced by Ξ.

First variation formula

We say that a form $\Theta_\lambda \in \Omega_1^s W$ is a Lepage equivalent of λ on $W^r \subset J^r Y$, if the following conditions are satisfied:

(a) $h\Theta_\lambda = (\pi^{s+1,r})^* \lambda$, and

(b) $h i_\xi d\Theta_\lambda = 0$ for every $\pi^{s,0}$-vertical vector field ξ on W^s.

First variation formula

We say that a form $\Theta_\lambda \in \Omega_1^s W$ is a Lepage equivalent of λ on $W^r \subset J^r Y$, if the following conditions are satisfied:

(a) $h\Theta_\lambda = (\pi^{s+1,r})^* \lambda$, and

(b) $hi_\xi d\Theta_\lambda = 0$ for every $\pi^{s,0}$-vertical vector field ξ on W^s.

If $\Theta_\lambda \in \Omega_1^s W$ is a Lepage equivalent of a Lagrangian $\lambda \in \Omega_{1,X}^r W$, then

$$\int_\Omega J^r\gamma^* \partial_{J^r\Xi}\lambda = \int_\Omega J^s\gamma^* i_{J^s\Xi} d\Theta_\lambda + \int_{\partial\Omega} J^s\gamma^* i_{J^s\Xi}\Theta_\lambda.$$

First variation formula

We say that a form $\Theta_\lambda \in \Omega_1^s W$ is a Lepage equivalent of λ on $W^r \subset J^r Y$, if the following conditions are satisfied:

(a) $h\Theta_\lambda = (\pi^{s+1,r})^* \lambda$, and

(b) $h i_\xi d\Theta_\lambda = 0$ for every $\pi^{s,0}$-vertical vector field ξ on W^s.

If $\Theta_\lambda \in \Omega_1^s W$ is a Lepage equivalent of a Lagrangian $\lambda \in \Omega_{1,X}^r W$, then

$$\int_\Omega J^r\gamma^* \partial_{J^r\Xi} \lambda = \int_\Omega J^s\gamma^* i_{J^s\Xi} d\Theta_\lambda + \int_{\partial\Omega} J^s\gamma^* i_{J^s\Xi} \Theta_\lambda .$$

Theorem. For any Lepage equivalent $\Theta_\lambda \in \Omega_1^s W$ of a Lagrangian $\lambda \in \Omega_{1,X}^r W$,

$$J^r\gamma^* \partial_{J^r\Xi} \lambda = J^s\gamma^* i_{J^s\Xi} d\Theta_\lambda + d i_{J^s\Xi} J^s\gamma^* \Theta_\lambda .$$

Lepage equivalent of a Lagrangian

Theorem. Every Lagrangian $\lambda \in \Omega^1_{1,X} W$ has a unique Lepage equivalent Θ_λ. If in a fibred chart (V, ψ), $\psi = (t, y^\sigma)$, λ is expressed as $\lambda = \mathcal{L} dt$, then $\Theta_\lambda \in \Omega^1_1 W$ and has an expression

$$\Theta_\lambda = \mathcal{L} dt + \frac{\partial \mathcal{L}}{\partial \dot{y}^\sigma} \omega^\sigma .$$

Lepage equivalent of a Lagrangian

Theorem. Every Lagrangian $\lambda \in \Omega^1_{1,X} W$ has a unique Lepage equivalent Θ_λ. If in a fibred chart (V, ψ), $\psi = (t, y^\sigma)$, λ is expressed as $\lambda = \mathcal{L} dt$, then $\Theta_\lambda \in \Omega^1_1 W$ and has an expression

$$\Theta_\lambda = \mathcal{L} dt + \frac{\partial \mathcal{L}}{\partial \dot{y}^\sigma} \omega^\sigma.$$

Theorem. Let $W \subset Y$ be an open set, let $\lambda \in \Omega^1_{1,X} W$ be a Lagrangian. Then the form $d\Theta_\lambda$ has an expression

$$(\pi^{2,1})^* d\Theta_\lambda = E_\lambda + F_\lambda,$$

where $E_\lambda = E_\sigma(\mathcal{L})\, \omega^\sigma \wedge dt$ with

$$E_\sigma(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial y^\sigma} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}^\sigma}, \tag{3}$$

and F_λ is of order of contactness 2.

Extremal of a variational functional

Theorem. Let $\lambda \in \Omega^1_{1,X} W$ be a Lagrangian, Θ_λ the Lepage equivalent of λ, and γ a section of Y. The following conditions are equivalent:

(a) γ is an extremal of λ.

(b) For any fibred chart (V, ψ), $\psi = (t, y^\sigma)$, γ satisfies the system of partial differential equations

$$E_\sigma(\mathcal{L}) \circ J^2\gamma = 0, \quad 1 \le \sigma \le m. \tag{4}$$

Equations (4) are the Euler-Lagrange equations.

Invariant Lagrangians

Let $W \subset Y$ be an open set, let λ be a Lagrangian defined on $W^r \subset J^r Y$. Consider an automorphism $\alpha : W \to Y$ of Y, and its prolongation $J^r\alpha : W^r \to J^r Y$. We say that α is an invariance transformation of λ if $J^r\alpha^*\lambda = \lambda$.

The generator of invariance transformations of λ is a π-projectable vector field on Y, whose one-parameter group consists of invariance transformations of λ.

Invariant Lagrangians

Let $W \subset Y$ be an open set, let λ be a Lagrangian defined on $W^r \subset J^r Y$. Consider an automorphism $\alpha : W \to Y$ of Y, and its prolongation $J^r\alpha : W^r \to J^r Y$. We say that α is an invariance transformation of λ if $J^r\alpha^*\lambda = \lambda$.

The generator of invariance transformations of λ is a π-projectable vector field on Y, whose one-parameter group consists of invariance transformations of λ.

Lemma. A π-projectable vector field Ξ on Y generates invariance transformations of a Lagrangian λ for Y if and only if

$$\partial_{J^r\Xi}\lambda = 0. \tag{5}$$

Invariant Lagrangians

Let $W \subset Y$ be an open set, let λ be a Lagrangian defined on $W^r \subset J^r Y$. Consider an automorphism $\alpha : W \to Y$ of Y, and its prolongation $J^r\alpha : W^r \to J^r Y$. We say that α is an invariance transformation of λ if $J^r\alpha^*\lambda = \lambda$.

The generator of invariance transformations of λ is a π-projectable vector field on Y, whose one-parameter group consists of invariance transformations of λ.

Lemma. A π-projectable vector field Ξ on Y generates invariance transformations of a Lagrangian λ for Y if and only if

$$\partial_{J^r\Xi}\lambda = 0. \tag{5}$$

Theorem. Suppose that a Lagrangian λ is invariant with respect to a π-projectable vector field Ξ. Then for any Lepage equivalent ρ of order s of λ and any section γ of Y

$$J^s\gamma^* i_{J^s\Xi} d\rho + dJ^s\gamma^* i_{J^s\Xi}\rho = 0. \tag{6}$$

Noether's theory

Theorem. (First theorem of Emmy Noether) Let $\lambda \in \Omega^r_{1,X} W$ be a Lagrangian, ρ a Lepage equivalent of λ defined on $J^s Y$, and let γ be an extremal. Then for every generator of invariance transformations of λ

$$dJ^s\gamma^* i_{J^s\Xi}\rho = 0. \tag{7}$$

Noether's theory

Theorem. (First theorem of Emmy Noether) Let $\lambda \in \Omega^r_{1,X} W$ be a Lagrangian, ρ a Lepage equivalent of λ defined on $J^s Y$, and let γ be an extremal. Then for every generator of invariance transformations of λ

$$dJ^s\gamma^* i_{J^s\Xi}\rho = 0. \tag{7}$$

The function $i_{J^s\Xi}\rho$ is called a conserved current for a section $\gamma \in \Gamma_\Omega(\pi)$.

Formula (7) is called conservation law equation.

Manifolds of regular velocities

Q a smooth manifold of dimension m

$T^r Q$ the set of r-jets $J^r_0 \zeta$ with source $0 \in \mathbf{R}$ and target $\zeta(0) \in Q$, r-velocities

canonical projection $\tau^r : T^r Q \to Q$, $\tau^r(J^r_0 \zeta) = \zeta(0)$

$T^r Q$ with its canonical smooth structure is referred to as the manifold of r-velocities.

Manifolds of regular velocities

Q a smooth manifold of dimension m

$T^r Q$ the set of r-jets $J_0^r \zeta$ with source $0 \in \mathbf{R}$ and target $\zeta(0) \in Q$, r-velocities

canonical projection $\tau^r : T^r Q \to Q$, $\tau^r(J_0^r \zeta) = \zeta(0)$

$T^r Q$ with its canonical smooth structure is referred to as the manifold of r-velocities.

$\mathrm{Imm}\, T^r Q$ the open subset of $T^r Q$, consisting of r-jets whose representatives are immersions at $0 \in \mathbf{R}$

elements of the set $\mathrm{Imm}\, T^r Q$ are called regular r-velocities

$\mathrm{Imm}\, T^1 Q$ coincides with the slit tangent bundle of Q

Parameter invariance

$F : \mathrm{Imm}\, T^r Q \to \mathbf{R}$ a function

$S \subset \mathbf{R}$ a compact interval

$\zeta : S \to Q$ an immersion

These data define the integral

$$F_S(\zeta) = \int_S (F \circ T^r\zeta)(t)dt, \tag{8}$$

where $T^r\zeta(t) = J_0^r(\zeta \circ \mathrm{tr}_t)$ is the canonical lift of the curve ζ to $\mathrm{Imm}\, T^r Q$, and tr_t is the translation $s \mapsto s + t$ of $\mathbf{R}$.

Parameter invariance

$F : \mathrm{Imm}\, T^r Q \to \mathbf{R}$ a function

$S \subset \mathbf{R}$ a compact interval

$\zeta : S \to Q$ an immersion

These data define the integral

$$F_S(\zeta) = \int_S (F \circ T^r\zeta)(t)dt, \tag{8}$$

where $T^r\zeta(t) = J_0^r(\zeta \circ \mathrm{tr}_t)$ is the canonical lift of the curve ζ to $\mathrm{Imm}\, T^r Q$, and tr_t is the translation $s \mapsto s + t$ of **R.**

We say that the integral (8) is parameter-invariant, if for any curve $\zeta : S \to Q$, any open interval $I \subset S$ and any diffemorphism $\mu : J \to I$, such that $\mu(J) = I$ and $D\mu > 0$,

$$F_I(\zeta) = F_J(\zeta \circ \mu).$$

Positive homogeneous functions

Theorem. Suppose that the function $F : \mathrm{Imm}\, T^2 Q \to \mathbf{R}$ is differentiable. Then the following two conditions are equivalent.
(a) Integral (8) is parameter invariant.
(b) For any chart (W, χ), $\chi = (y^\sigma)$, on Q

$$\frac{\partial F}{\partial \dot{y}^\sigma} \dot{y}^\sigma + 2 \frac{\partial F}{\partial \ddot{y}^\sigma} \ddot{y}^\sigma = F, \quad \frac{\partial F}{\partial \ddot{y}^\sigma} \dot{y}^\sigma = 0. \tag{9}$$

Condition (9) is called the Zermelo condition.

Positive homogeneous functions

Theorem. Suppose that the function $F : \mathrm{Imm}\, T^2 Q \to \mathbf{R}$ is differentiable. Then the following two conditions are equivalent.
(a) Integral (8) is parameter invariant.
(b) For any chart (W, χ), $\chi = (y^\sigma)$, on Q

$$\frac{\partial F}{\partial \dot{y}^\sigma} \dot{y}^\sigma + 2 \frac{\partial F}{\partial \ddot{y}^\sigma} \ddot{y}^\sigma = F, \quad \frac{\partial F}{\partial \ddot{y}^\sigma} \dot{y}^\sigma = 0. \tag{9}$$

Condition (9) is called the Zermelo condition.

A function F satisfying the condition (9) is called positive homogeneous.

Setting

$$Y = \mathbf{R} \times Q,$$

one can consider variational problems on $\mathrm{Imm}\, T^r Q$ as a special case of variational problems on fibred manifolds.

Setting

$$Y = \mathbf{R} \times Q,$$

one can consider variational problems on $\mathrm{Imm}\, T^r Q$ as a special case of variational problems on fibred manifolds.

Sections $t \mapsto \gamma(t)$ of Y are canonically identified with curves $t \mapsto \zeta(t)$ in Q, where $\gamma(t) = (t, \zeta(t))$.

Then the r-jets $J^r_t \gamma$ are canonically identified with the pairs $(t, T^r \zeta(t)) \in \mathbf{R} \times T^r Q$; thus,

$$J^r Y = \mathbf{R} \times T^r Q. \tag{10}$$

Lepage forms on $\mathrm{Imm}\, T^1 Q$

We say that ρ on $\mathrm{Imm}\, T^1 Q$ is a Lepage form, if ρ is a Lepage form on $J^1(\mathbf{R} \times Q)$.

Lepage forms on $\mathrm{Imm}\, T^1 Q$

We say that ρ on $\mathrm{Imm}\, T^1 Q$ is a Lepage form, if ρ is a Lepage form on $J^1(\mathbf{R} \times Q)$.

In a chart (W^1, χ^1), $\chi^1 = (y^\sigma, \dot{y}^\sigma)$, on $T^1 Q$, arbitrary 1-form is expressed by

$$\rho = A_\sigma dy^\sigma + B_\sigma d\dot{y}^\sigma, \tag{11}$$

where A_σ, B_σ are functions on $\mathrm{Imm}\, T^1 Q$.

Lepage forms on $\mathrm{Imm}\, T^1 Q$

We say that ρ on $\mathrm{Imm}\, T^1 Q$ is a Lepage form, if ρ is a Lepage form on $J^1(\mathbf{R} \times Q)$.

In a chart (W^1, χ^1), $\chi^1 = (y^\sigma, \dot{y}^\sigma)$, on $T^1 Q$, arbitrary 1-form is expressed by

$$\rho = A_\sigma dy^\sigma + B_\sigma d\dot{y}^\sigma, \tag{11}$$

where A_σ, B_σ are functions on $\mathrm{Imm}\, T^1 Q$.

Theorem. Suppose ρ is 1-form on $\mathrm{Imm}\, T^1 Q$, in a chart expressed by (11). Then ρ is a Lepage form if and only if

$$\left(\frac{\partial A_\sigma}{\partial \dot{y}^\nu} - \frac{\partial B_\nu}{\partial y^\sigma} \right) \dot{y}^\sigma = 0, \quad \frac{\partial B_\sigma}{\partial \dot{y}^\nu} - \frac{\partial B_\nu}{\partial \dot{y}^\sigma} = 0. \tag{12}$$

Solution of the system (12)

$$A_\sigma = \frac{\partial P}{\partial y^\sigma} + Q_\sigma, \quad B_\sigma = \frac{\partial P}{\partial \dot{y}^\sigma}, \tag{13}$$

for some smooth functions P, Q_σ on $\operatorname{Imm} T^1 Q$ depending on $(y^\nu, \dot{y}^\nu)$, and (y^ν), respectively.

Solution of the system (12)

$$A_\sigma = \frac{\partial P}{\partial y^\sigma} + Q_\sigma, \quad B_\sigma = \frac{\partial P}{\partial \dot{y}^\sigma}, \tag{13}$$

for some smooth functions P, Q_σ on $\operatorname{Imm} T^1 Q$ depending on $(y^\nu, \dot{y}^\nu)$, and (y^ν), respectively.

Theorem. Let ρ be a Lepage form $\operatorname{Imm} T^1 Q$, in coordinates expressed by $\rho = A_\sigma dy^\sigma + B_\sigma d\dot{y}^\sigma$, where A_σ, B_σ are given (13). If P satisfies

$$\frac{\partial P}{\partial \dot{y}^\sigma} \dot{y}^\sigma = 0,$$

then $\mathcal{L}$ determined by $h\rho = \mathcal{L} dt$ is positive homogeneous function on $\operatorname{Imm} T^2 Q$.

Action of a Lie group on T^1Q

Let
$$\Phi : G \times Q \to Q, \quad (g, q) \mapsto \Phi_g(q)$$
be a left action of a Lie group G on a smooth manifold Q.

Action of a Lie group on $T^1 Q$

Let

$$\Phi : G \times Q \to Q, \quad (g, q) \mapsto \Phi_g(q)$$

be a left action of a Lie group G on a smooth manifold Q.

For $g \in G$ we define

$$T\Phi_g : TQ \to TQ$$

by

$$T\Phi_g(\xi) = (T_q\Phi_g)(\xi),$$

where $\xi \in T_qQ, \ \ q \in Q$.

Action of a Lie group on T^1Q

Let

$$\Phi : G \times Q \to Q, \quad (g, q) \mapsto \Phi_g(q)$$

be a left action of a Lie group G on a smooth manifold Q.

For $g \in G$ we define

$$T\Phi_g : TQ \to TQ$$

by

$$T\Phi_g(\xi) = (T_q\Phi_g)(\xi),$$

where $\xi \in T_qQ, \;\; q \in Q$.

Thus we have a left action of G on $T^1Q \equiv TQ$

$$\widetilde{\Phi} : G \times T^1Q \to T^1Q, \quad (g, \xi) \mapsto T\Phi_g(\xi).$$

Now consider $Q = \mathbf{R}^3$ and $G = \mathrm{SO}(3)$.

The special orthogonal group $\mathrm{SO}(3)$ of $\mathbf{R}^3$ consists of orthogonal matrices with determinant +1 representing rotations of $\mathbf{R}^3$ around a point $(0, 0, 0)$.

Such rotations are generated by the set of rotations around the axis x, y, z of the canonical frame in $\mathbf{R}^3$.

Now consider $Q = \mathbf{R}^3$ and $G = \mathrm{SO}(3)$.

The special orthogonal group $\mathrm{SO}(3)$ of $\mathbf{R}^3$ consists of orthogonal matrices with determinant +1 representing rotations of $\mathbf{R}^3$ around a point $(0, 0, 0)$.

Such rotations are generated by the set of rotations around the axis x, y, z of the canonical frame in $\mathbf{R}^3$.

Our aim is to determine all $\mathrm{SO}(3)$-invariant Lagrangians $\lambda = \mathcal{L} dt$ on $\mathbf{R} \times \mathrm{Imm}\, T^1\mathbf{R}^3$ with $\mathcal{L}$ defined on $\mathrm{Imm}\, T^1\mathbf{R}^3$.

Now consider $Q = \mathbf{R}^3$ and $G = \mathrm{SO}(3)$.

The special orthogonal group SO(3) of $\mathbf{R}^3$ consists of orthogonal matrices with determinant +1 representing rotations of $\mathbf{R}^3$ around a point $(0, 0, 0)$.

Such rotations are generated by the set of rotations around the axis x, y, z of the canonical frame in $\mathbf{R}^3$.

Our aim is to determine all SO(3)-invariant Lagrangians $\lambda = \mathcal{L}dt$ on $\mathbf{R} \times \mathrm{Imm}\, T^1\mathbf{R}^3$ with $\mathcal{L}$ defined on $\mathrm{Imm}\, T^1\mathbf{R}^3$.

According to the corresponding invariance theory, equivalently we find functions $\mathcal{L}$ on $\mathrm{Imm}\, T^1\mathbf{R}^3$ satisfying

$$\partial_{\Xi}\mathcal{L} = 0, \tag{14}$$

where Ξ is any generator of the action of SO(3) on $\mathrm{Imm}\, T^1\mathbf{R}^3$.

Generators of the action of SO(3) on $\mathbf{R}^3$

The generators of rotations around the coordinate axes z, x, and y in canonical coordinates on $\mathbf{R}^3$

$$\xi = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \eta = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \mu = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

Generators of the action of SO(3) on $\mathbf{R}^3$

The generators of rotations around the coordinate axes z, x, and y in canonical coordinates on $\mathbf{R}^3$

$$\xi = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \eta = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \mu = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

In the first spherical chart (V, Ψ) on $\mathbf{R}^3$, where $V = \mathbf{R}^3 \backslash \{(x, y, z) \in \mathbf{R}^3 \,|\, x \geq 0, y = 0\}$, $\Psi = (r, \varphi, \vartheta)$, defined by

$$x = r\cos\varphi\sin\vartheta, \quad y = r\sin\varphi\sin\vartheta, \quad z = r\cos\vartheta,$$

we have

$$\xi = \frac{\partial}{\partial\varphi}, \quad \eta = -\cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi} - \sin\varphi\frac{\partial}{\partial\vartheta},$$

$$\mu = -\sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\vartheta}.$$

SO(3)-invariant functions on $\mathrm{Imm}\, T^1\mathbf{R}^3$

- any solution $\mathcal{L}$ of the system

$$\partial_{\dot\xi}\mathcal{L} = 0, \quad \partial_{\dot\eta}\mathcal{L} = 0, \quad \partial_{\dot\mu}\mathcal{L} = 0, \tag{15}$$

where $\dot\xi, \dot\eta, \dot\mu$ are complete lifts of ξ, η, μ, respectively, to $T^1\mathbf{R}^3$

$$\begin{aligned}
\dot\xi &= \frac{\partial}{\partial\varphi}, \\
\dot\eta &= -\cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi} - \sin\varphi\frac{\partial}{\partial\vartheta} \\
&\quad + \left(\dot\vartheta\frac{\cos\varphi}{\sin^2\theta} + \dot\varphi\sin\varphi\cot\vartheta\right)\frac{\partial}{\partial\dot\varphi} - \dot\varphi\cos\varphi\frac{\partial}{\partial\dot\vartheta}, \\
\dot\mu &= -\sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\vartheta} \\
&\quad + \left(\dot\vartheta\frac{\sin\varphi}{\sin^2\theta} - \dot\varphi\cos\varphi\cot\vartheta\right)\frac{\partial}{\partial\dot\varphi} - \dot\varphi\sin\varphi\frac{\partial}{\partial\dot\vartheta}
\end{aligned}$$

Let us denote $V^1 = (\tau^1)^{-1}(V)$; $V^1 \subset \mathrm{Imm}\, T^1\mathbf{R}^3$.

Theorem. Every SO(3)-invariant function on V^1 is a smooth function of $r, \dot{r}, w$, where $w = \sqrt{\dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta}$.

Let us denote $V^1 = (\tau^1)^{-1}(V)$; $V^1 \subset \mathrm{Imm}\, T^1\mathbf{R}^3$.

Theorem. Every SO(3)-invariant function on V^1 is a smooth function of $r, \dot{r}, w$, where $w = \sqrt{\dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta}$.

Theorem. Any positively homogeneous SO(3)-invariant function $\mathcal{L}$ on V^1 is of the form

$$\mathcal{L}(r, \dot{r}, w) = w\, f\left(r, \frac{\dot{r}}{w}\right).$$

Example

$$\lambda = \mathcal{L}dt, \quad \mathcal{L} = w^2 = \dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta, \quad \text{on } \mathrm{Imm}\, T^1\mathbf{R}^3$$

Example

$$\lambda = \mathcal{L}dt, \quad \mathcal{L} = w^2 = \dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta, \quad \text{on } \operatorname{Imm} T^1\mathbf{R}^3$$

Hilbert form $\quad \Theta_\lambda = 2\dot{\vartheta}^2 d\vartheta + 2\dot{\varphi}\sin^2\vartheta d\varphi$

Example

$$\lambda = \mathcal{L} dt, \quad \mathcal{L} = w^2 = \dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta, \quad \text{on } \operatorname{Imm} T^1 \mathbf{R}^3$$

Hilbert form $\quad \Theta_\lambda = 2\dot{\vartheta}^2 d\vartheta + 2\dot{\varphi} \sin^2 \vartheta d\varphi$

$i_{\dot{\xi}} \Theta_\lambda \ \ldots \ \dot{\varphi} \sin^2 \vartheta = \text{const}$ corresponds with $E_\varphi(\mathcal{L})$

Example

$$\lambda = \mathcal{L} dt, \quad \mathcal{L} = w^2 = \dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta, \quad \text{on } \operatorname{Imm} T^1 \mathbf{R}^3$$

Hilbert form $\quad \Theta_\lambda = 2\dot{\vartheta}^2 d\vartheta + 2\dot{\varphi} \sin^2 \vartheta d\varphi$

$i_{\dot{\xi}} \Theta_\lambda \; \ldots \; \dot{\varphi} \sin^2 \vartheta = \text{const}$ corresponds with $E_\varphi(\mathcal{L})$

$i_{\dot{\eta}} \Theta_\eta, \; i_{\dot{\mu}} \Theta_\lambda \; \ldots \; \dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta \cos^2 \vartheta = \text{const}$

- D. Krupka, J. Brajerčík, Schwarzschild Spacetimes: Topology, Axioms 11 (2022), 693.
- D. Krupka, Introduction to Global Variational Geometry, Atlantis Studies in Variational Geometry, D. Krupka, H. Sun (Eds.), Atlantis Press, 2015.
- D. Krupka, Lepage forms in Kawaguchi spaces and the Hilbert form, Publ. Math. Debrecen, 84(1-2) (2014), 147-164.
- C. Pfeifer, M. N. R. Wohlfarth, Finsler geometric extension of Einstein gravity, Physical Review D - Particles, Fields, Gravitation, and Cosmology 85(6) (2012), 064009.
- Z. Urban, J. Brajerčík, The fundamental Lepage form in variational theory for submanifolds, Int. J. Geom. Meth. Mod. Phys. 15(6) (2018) 1850103, 1-30.
- Z. Urban, D. Krupka, Variational theory on Grassmann fibrations: Examples, Acta Math. Acad. Paed. Nyíregyhasiensis 31(1) (2015) 153-170.
- Z. Urban, D. Krupka, The Zermelo conditions and higher order homogeneous functions, Publ. Math. Debrecen 82(1) (2013) 59-76.