Teleparallel gravity through Noether's second theorem

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Abstract

We examine the gauge structure and the situation with matter coupling in teleparallel gravity. Using Noether's second theorem, we demonstrate that gauge symmetries are diffeomorphisms and local Lorentz transformations, and that the structure of Noether's identities implies the Levi-Civita coupling.

1. Introduction

It is well-known that teleparallel gravity, also known as the Teleparallel Equivalent of General Relativity (TEGR) is locally dynamically equivalent to General Relativity (GR) in the absence of matter. However, for decades, discussions have persisted about this equivalence when matter is present. These discussions primarily focus on the choice of coupling prescription, whether to use the teleparallel or the Levi-Civita one, and whether the description of spinor fields is consistent in TEGR.

TEGR is also often interpreted as a classical gauge theory of translations, i.e., it is claimed that its gauge structure is related to the group of translations $T(4)$.

The goal of this work [1] is to analyze the situation with matter coupling and gauge structure of TEGR.

2. Noether's second theorem

Let us consider an *action* over a fixed domain $U \subset M$

$$\mathcal{S}[\psi] = \int_U L(\psi, d\psi),$$

where $\psi = \{\psi^a\}$ is a collection of k differential forms of degrees p_a representing dynamical *fields* on a smooth 4-dimensional manifold M.

An infinitesimal variation of the action $\mathcal{S}[\psi]$ is

$$\delta\mathcal{S} = \int_U E_a \wedge \delta\psi^a + \int_{\partial U} \Theta,$$

where the $(4-p_a)$-forms E_a are the *Euler-Lagrange expressions* and the 3-form $\Theta := \Theta(\psi, \delta\psi)$ is linear in $\delta\psi$.

Let $\delta\psi^a$ be linearly parametrized by a set of ρ arbitrary smooth functions $\lambda^i(x)$ of spacetime and their first derivatives

$$\delta_\lambda\psi^a := R^a_i(\psi, d\psi)\lambda^i(x) + S^a_i(\psi, d\psi) \wedge d\lambda^i(x),$$

where $R^a_i(\psi, d\psi)$ are p_a-forms and $S^a_i(\psi, d\psi)$ are (p_a-1)-forms. Variations $\delta_\lambda\psi$ are called infinitesimal *gauge transformations* and $\{\lambda^i(x)\}$ generate ∞-dimensional Lie group $G_{\infty\rho}$ called *gauge group*.

The action $\mathcal{S}[\psi]$ is said to have a *gauge symmetry* $\delta_\lambda\psi$ iff it is *quasi-invariant* under $\delta_\lambda\psi$, i.e., it holds

$$\delta_\lambda\mathcal{S} = \int_{\partial U} K,$$

where K is a 3-form linear in $\delta_\lambda\psi$.

Noether's second theorem: If an action $\mathcal{S}[\psi]$ is quasi-invariant under the ∞-dim. Lie group $G_{\infty\rho}$, then there are ρ identities among the Euler-Lagrange expressions and their first derivatives.

These (off-shell) identities, called *Noether's identities* (or generalized Bianchi identities), are

$$E_a \wedge R^a_i + d\left(E_a \wedge S^a_i\right) = 0.$$

3. Teleparallel gravity

General Relativity is defined by (M, g, o_R, ∇_{LC}), where g is a metric tensor field, o_R is the right orientation and ∇_{LC} is the Levi-Civita connection, and the action is

$$\mathring{\mathcal{S}}[e] := -\frac{1}{2}\int_U \varepsilon_{abcd}\mathring{\Omega}^{ab} \wedge e^c \wedge e^d \implies \mathring{E}_a = \varepsilon_{abcd}\mathring{\Omega}^{bc} \wedge e^d.$$

The action $\mathring{\mathcal{S}}[e]$ is invariant under *local Lorentz transformations* and orientation-preserving *diffeomorphisms* of M. Noether's identities are

$$\mathring{E}_{[a} \wedge e^{b]} = 0, \qquad \mathring{\mathcal{D}}\mathring{E}_a = 0.$$

TEGR is defined by (M, g, o_R, ∇_T), where ∇_T is a flat metric-compatible linear connection, and the action is

$$\dot{\mathcal{S}}[e, \dot{\omega}] := \int_U T^a \wedge H_a \implies \dot{E}_a = -\dot{\mathcal{D}}H_a + \mathcal{J}_a,\ \dot{E}^a_b = 0,$$

where $H^a := \frac{1}{2} * (T^a - 2e^a \wedge i_b T^b + e^b \wedge i^a T_b)$, $\mathcal{J}_a := T^b \wedge \frac{\delta H_b}{\delta e^a}$, $\dot{\omega}$ is a flat metric-compatible linear connection 1-form, and T^a are *torsion* 2-forms of $\dot{\omega}$.

The action $\dot{\mathcal{S}}[e, \dot{\omega}]$ is invariant under *local Lorentz transformations* and orientation-preserving *diffeomorphisms* of M. Noether's identities are

$$\dot{E}_{[a} \wedge e^{b]} = 0, \qquad \dot{\mathcal{D}}\dot{E}_a + \dot{E}_c \wedge i_a T^c = 0.$$

Lagrangians of GR and TEGR are related as

$$-\frac{1}{2}\varepsilon_{abcd}\mathring{\Omega}^{ab} \wedge e^c \wedge e^d = T^a \wedge H_a + 2d(e^a \wedge *T_a),$$

therefore, GR and TEGR have the same field equations. Moreover, it turns out that Noether's identities in TEGR are equivalent to Noether's identities in GR.

4. Matter coupling in TEGR

Let $\dot{\mathcal{S}}_M[e, \dot{\omega}, \Psi]$ and $\mathring{\mathcal{S}}_M[e, \Psi]$ be matter actions of matter fields Ψ with teleparallel and Levi-Civita couplings. Their compactly supported variations are

$$\delta\dot{\mathcal{S}}_M = \int_U \left(\dot{M}_a \wedge \delta e^a + M^b_a \wedge \delta\dot{\omega}^a_b + \dot{\mathcal{E}}_i \wedge \delta\Psi^i\right),$$

where variations of $\dot{\omega}^a_b$ are in the teleparallel constrained form $\delta\dot{\omega}^a_b = \dot{\mathcal{D}}\lambda^a_b(x)$, $\lambda_{ab}(x) = \lambda_{[ab]}(x)$, and yield $\dot{M}^a_b = \dot{\mathcal{D}}M^a_b$ as the actual Euler-Lagrange expression

$$\delta\mathring{\mathcal{S}}_M = \int_U \left(\mathring{M}_a \wedge \delta e^a + \mathring{\mathcal{E}}_i \wedge \delta\Psi^i\right).$$

The matter on-shell Noether's identities of $\dot{\mathcal{S}}_M$ and $\mathring{\mathcal{S}}_M$ for local Lorentz and diffeomorphism invariance are

$$\dot{\mathcal{D}}M^a_b - \dot{M}_{[b} \wedge e^{a]} \overset{m}{\approx} 0, \quad \dot{\mathcal{D}}\dot{M}_a + \dot{M}_c \wedge i_a T^c \overset{m}{\approx} 0,$$
$$\mathring{M}_{[b} \wedge e^{a]} \overset{m}{\approx} 0, \quad \mathring{\mathcal{D}}\mathring{M}_a \overset{m}{\approx} 0.$$

Let $\dot{\mathcal{S}}_G[e, \dot{\omega}]$ be the gravitational action of a generic teleparallel theory. Its compactly supported variation is

$$\delta\dot{\mathcal{S}}_G = \int_U \left(\dot{E}_a \wedge \delta e^a + E^b_a \wedge \delta\dot{\omega}^a_b\right),$$

where again the teleparallel constrained variations yield $\dot{M}^a_b = \dot{\mathcal{D}}M^a_b$ as the actual Euler-Lagrange expression.

Noether's identities of $\dot{\mathcal{S}}_G$ for local Lorentz and diffeomorphism invariance are

$$\dot{\mathcal{D}}E^a_b - \dot{E}_{[b} \wedge e^{a]} = 0, \quad \dot{\mathcal{D}}\dot{E}_a + \dot{E}_c \wedge i_a T^c = 0.$$

Let actions $\dot{\mathcal{S}}_M$ and $\mathring{\mathcal{S}}_M$ be matter sources for $\dot{\mathcal{S}}_G$, then the corresponding Euler-Lagrange equations are

$$\dot{\mathcal{S}}_G + \dot{\mathcal{S}}_M \implies \dot{E}_a \overset{g}{\approx} -\dot{M}_a, \quad \dot{\mathcal{D}}E^a_b \overset{g}{\approx} -\dot{\mathcal{D}}M^a_b.$$
$$\dot{\mathcal{S}}_G + \mathring{\mathcal{S}}_M \implies \dot{E}_a \overset{g}{\approx} -\mathring{M}_a, \quad \dot{\mathcal{D}}E^a_b \overset{g}{\approx} 0.$$

Noether's identities for the gravitational field imply that when gravitational field equations are satisfied, we have

$$\dot{\mathcal{D}}M^a_b - \dot{M}_{[b} \wedge e^{a]} \overset{g}{\approx} 0, \quad \dot{\mathcal{D}}\dot{M}_a + \dot{M}_c \wedge i_a T^c \overset{g}{\approx} 0.$$
$$\mathring{M}_{[b} \wedge e^{a]} \overset{g}{\approx} 0, \quad \mathring{\mathcal{D}}\mathring{M}_a \overset{g}{\approx} 0.$$

Thus, for a generic teleparallel theory, both couplings are possible. Therefore, one coupling has to be postulated.

In the case of TEGR $\dot{\mathcal{S}}_G = \dot{\mathcal{S}}$, only the Levi-Civita coupling is possible since Noether's identities in TEGR are equivalent to the ones in GR, which allow only the Levi-Civita coupling.

5. Gauge theory of translations

A classical gauge theory of translations operates with the concept of *local translations* and *translational gauge field* [2]. Local translations are defined as elements of the ∞-dim. abelian Lie group $\mathbb{T}(4)$ consisting of smooth maps $S : U \to T(4)$. Translational gauge field h^a and its field strength $\mathcal{H}^a$ are defined as

$$h^a := \sigma^*\left(\omega^t\right)^a, \quad \mathcal{H}^a := \sigma^*\left(\Omega^t\right)^a = dh^a,$$

where ω^t is a connection 1-form on a principal $T(4)$-bundle, Ω^t is its curvature, and σ is a local section.

Let $\mathcal{S}[h]$ be invariant under $\mathbb{T}(4)$, then the corresponding Noether's identity for local translations is

$$dE_a = 0.$$

This identity is absent in TEGR; thus, TEGR cannot be interpreted as a classical gauge theory of translations.

Conclusion

We have applied Noether's second theorem to TEGR (in tetrad formalism) and demonstrated that

- gauge symmetries of TEGR are (orientation-preserving) diffeomorphisms of M and local Lorentz transformations, and no translational gauge symmetry is present due to the absence of corresponding Noether's identity for translational gauge symmetry.
- only the Levi-Civita coupling prescription for matter fields is possible since Noether's identities in TEGR are equivalent to the ones in GR. As a consequence, spinor fields are consistently described in TEGR, and TEGR and GR are locally dynamically equivalent theories in the presence of all matter.

References

[1] S. Brezina and M. Krššák, *Gauge theory and gravity through Noether's second theorem (work in progress)*,

[2] S. Brezina, E. Boffo, and M. Krššák, Teleparallel gravity from the principal bundle viewpoint, JHEP 05 (2026) 182.

This work was supported by VEGA grant 1/0565/25.