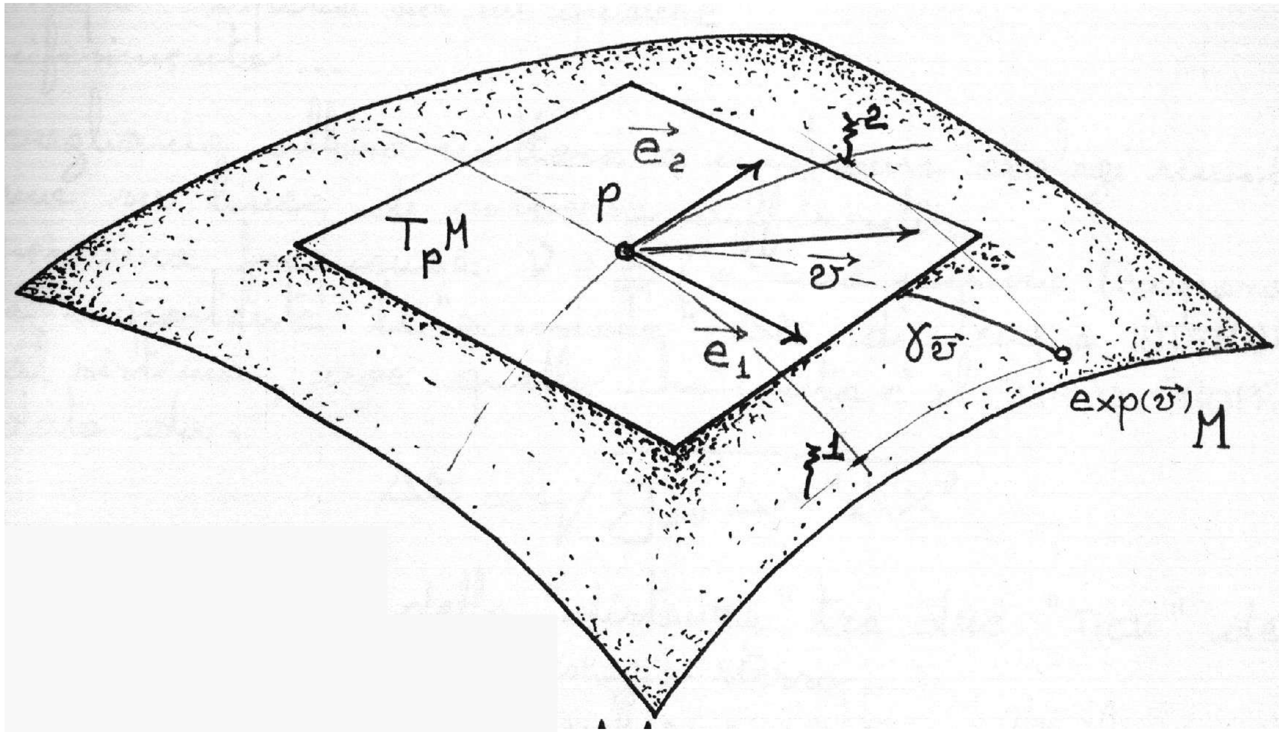
The Metric-Affine Gravity
Theory and Applications

Salvatore Capozziello



OUTLINES OF THE LECTURES

1.THE EQUIVALENCE PRINCIPLE

2. A GEOMETRIC EXCURSUS

3. THE DEBATE ON THEORIES OF GRAVITY

4.METRIC-AFFINE THEORIES

5.TETRADS AND SPIN CONNECTIONS

6.GEOMETRIC TRINITY: THE LAGRANGIAN APPROACH

7.GEOMETRIC TRINITY: THE FIELD EQUATIONS

8.GEOMETRIC TRINITY: THE SOLUTIONS

9.EXTENSIONS OF GEOMETRIC TRYNITY

10. A SUMMARY AND OPEN ISSUES

Recommended Textbooks

- **A Mathematical Journey to Relativity (Second Edition)**
  W.G. Boskoff and S. Capozziello, Springer, 2024

- **Beyond Einstein Gravity**
  S. Capozziello and V. Faraoni, Springer, 2011

- **General Relativity**
  N. Straumann, Springer 2004

- **Noether Symmetries in Theories of Gravity**
  F. Bajardi and S. Capozziello, Cambridge, 2022

- **Applications of Lie groups to Differential Equations**
  P. J. Oliver, Cambridge , Springer, 1986

1 LECTURE

The Equivalence Principle

General Relativity *is a theory of gravity, formulated by Albert Einstein between 1907 and 1915, where the spacetime geometry is dealt with the same standard of dynamical variables. The physical basis of the theory is the* ***Equivalence Principle***

Gravity and Acceleration

- Einstein has an extraordinary intuition
- He said: «**The happiest thought of my life!**»
- **The effects of the acceleration are indistinguishable from the effects of gravity!**
- A traveler far away from any gravitational field moves along a straight line.
- A traveler in acceleration is like one that locally "feels" the presence of a gravitational field.
- Locally, **acceleration is equivalent to a gravitational field!**
- This is an extension of the **Galileo Equivalence Principle**
- $$\boldsymbol{m_i = m_g}$$

Gravity and Acceleration

- Einstein does not interpret the gravitational field as due to an effective force field, but to the curvature of spacetime.
- Inertial observers move always along the shorter spacetime lines (geodesics), but the results of the measures differ from those envisaged by the Euclidean geometry, due to the intrinsic curvature of spacetime, caused by the presence of gravitational masses.
- The gravitational action, for example, by the Sun or by the Earth is performed not through the 'physical action' on material bodies, but in the form of a modification of spacetime geometry.
- The changes induced by the presence in space of gravitational masses allow us to interpret geometrically the gravitational interaction.
- This is an effect due to the **Equivalence Principle**
- The geodesic structure and the metric structure of spacetime coincide!!!

The Equivalence Principle

Let us consider the motion of a non relativistic particle moving in a constant gravitational field.

According to the Newtonian mechanics, the equations of motion are

$$m_I \frac{d^2\vec{x}}{dt^2} = m_G \vec{g} + \sum_k \vec{F}_k \longrightarrow$$ forces acting on the particle

Let us make the following coordinate transformation

$$\vec{x}' = \vec{x} - \frac{1}{2}\vec{g}t^2 \qquad\qquad t' = t.$$

In this new reference frame, it is

$$m_I \left[\frac{d^2\vec{x}'}{dt^2} + \vec{g}\right] = m_G \vec{g} + \sum_k \vec{F}_k.$$

Since the Equivalence Principle $m_I = m_G$, and since this is true for any particle, it is

$$\boxed{m_I \frac{d^2\vec{x}'}{dt^2} = \sum_k \vec{F}_k}$$

The Equivalence Principle

An observer **O**, in free fall in a gravitational field, feels the same laws of physics as an observer **O'** which is not in a gravitational field

This result follows from the equivalence, experimentally tested, of the inertial and gravitational mass.

- If $\mathbf{m_I}$ were different from $\mathbf{m_G}$, or if their ratio were not constant and the same for all bodies, this would not be true, because we could not simplify the term in **g** in the above equation
- If **g** were not constant, the equation would contain additional terms with derivatives of **g**.
- However, we can consider an interval of time so short that **g** is constant and equation holds.
- Consider a particle at rest in this frame and no force $\mathbf{F}_k$ acting on it. Under this assumption, according to the above equation, it remains at rest forever
- Therefore we can define this reference frame as a **local inertial frame (LIF)**.
- If the gravitational field is constant and unifom everywhere, the coordinate transformation defines a local inertial frame that covers the whole space-time.
- **If this is not the case, we can set up a LIF only in the neighborhood of a given point.**

The Equivalence Principle

- Gravity is different from the other interactions since all bodies, given the same initial velocity, follow the same trajectory in a gravitational field, regardless of their internal constitution.

- This is not the case, for example, for Electromagnetism which acts on charged but not on neutral bodies, and, in any spacetime event, the trajectories of charged particles depend on the ratio between the charge and the mass, which is not the same for all particles

- Similarly, other forces, like the strong and weak interactions, affect different particles differently!

The Equivalence Principle

The covariant derivative of $g_{\mu\nu}$ is always zero if EP holds

$$g_{\mu\nu;\alpha} = 0$$

it follows that

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^{\nu}_{\alpha\mu} g_{\nu\beta} - \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} = 0,$$

therefore

$$g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\alpha\mu} g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu}$$

Let us now consider the following equations

$$g_{\alpha\mu,\beta} = \Gamma^{\nu}_{\alpha\beta} g_{\nu\mu} + \Gamma^{\nu}_{\mu\beta} g_{\alpha\nu},$$

$$-g_{\beta\mu,\alpha} = -\Gamma^{\nu}_{\beta\alpha} g_{\nu\mu} - \Gamma^{\nu}_{\mu\alpha} g_{\beta\nu},$$

we have

$$\begin{aligned} g_{\alpha\beta,\mu} \quad &+ \quad g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = (\Gamma^{\nu}_{\alpha\mu} - \Gamma^{\nu}_{\mu\alpha}) g_{\nu\beta} + \\ &+ \quad (\Gamma^{\nu}_{\beta\mu} + \Gamma^{\nu}_{\mu\beta}) g_{\alpha\nu} + (\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\beta\alpha}) g_{\nu\mu}, \end{aligned}$$

where we used $g_{\alpha\beta} = g_{\beta\alpha}$

The Equivalence Principle

Since the connections are symmetric in β and γ, it follows that

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2\Gamma^{\nu}_{\beta\mu} g_{\alpha\nu}$$

If we multiply by $g^{\alpha\gamma}$ and remember that since $g^{\alpha\gamma}$ is the inverse of $g_{\alpha\gamma}$

$$g^{\alpha\gamma} g_{\alpha\nu} = \delta^{\gamma}_{\nu},$$

we finally find

$$\boxed{\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})}$$

This expression allows to compute the affine connections in terms of the components of the metric. It holds thanks to the Equivalence Principle.

Causal Structure coincides with Geodesic Structure

The Equivalence Principle

Two formulations:

- § **The Strong Equivalence Principle:** In an arbitrary gravitational field, at any given spacetime point, we can choose a local inertial reference frame such that, in a sufficiently small region surrounding that point, all physical laws take the same form they would take in absence of gravity, namely the form prescribed by Special Relativity.

- § **The Weak Equivalence Principle:** The same as before, but it refers to the laws of motion of freely falling bodies, instead of all physical laws.

The previous formulations of the Equivalence Principle resembles to the axiom that Gauss chose as the basis for non-Euclidean geometries, namely: for any given point in space, there exists an Euclidean local reference frame such that, in a sufficiently small region surrounding that point, the distance between two points is given by the Pythagoras theorem.

The Equivalence Principle

The **Equivalence Principle** states that in a local inertial frame all laws of physics must coincide with those of Special Relativity, and consequently, in this frame, the distance between two points must coincide with the Minkowski metr ic

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2$$

We therefore expect that the equations of gravity will look very similar to those of the **Riemann geometry**

In particular, as Gauss defined the inner properties of curved surfaces in terms of the derivatives $\partial\xi^{\alpha}/\partial x_{\mu}$ where ξ^{α} are the "locally Euclidean coordinates" and x_{μ} are arbitrary coordinates, in a similar way, we expect that the effects of a gravitational field will be described in terms of the derivatives $\partial\xi^{\alpha}/\partial x_{\mu}$ where now ξ^{α} are the "locally inertial coordinates", and x_{μ} are arbitrary coordinates.

All these facts follow from the Equivalence Principle

The Equivalence Principle

We need to investigate the EP for:

- discriminating among theories of gravity
- its validity at classical and quantum level
- investigating geodesic and causal structure
- the Schiff Conjecture: WEP, LLI, LPI hold at the same time

The Equivalence Principle

The SEP extends the Einstein EP by including all the laws of physics in its terms:

- WEP is valid for self-gravitating bodies as well as for test bodies (Gravitational Weak Equivalence Principle);
- the outcome of any local test experiment is independent of the velocity of the free-falling apparatus;
- the outcome of any local test experiment is independent of where and when in the Universe it is performed.

The SEP contains the Einstein Equivalence Principle, when gravitational forces are neglected.

The Equivalence Principle

- Many authors claim that the only theory coherent with the SEP is GR
- An extremely important issue is related to the consistency of Equivalence Principle with respect to the Quantum Mechanics .
- Some phenomena, like neutrino oscillations could violate it if induced by the gravitational field.
- GR is not the only theory of gravitation and, several alternative theories of gravity have been investigated from the 60's, considering the space-time to be "special relativistic" at a background level and treating gravitation as a Lorentz-invariant field on the background.

The Equivalence Principle

Two different classes of experiments have to be considered:

- the first ones testing the foundation of gravitation theory (among them the EP)

- the second ones testing the metric theories of gravity where space-time is endowed with a metric tensor (EP could be violated at quantum level).

For several fundamental reasons extra fields might be necessary to describe the gravitation (e.g QG at UV scales or DM and DE at IR scales, e.g. scalar fields or higher-order corrections in curvature invariants.

The Equivalence Principle

Two sets of equations can be taken into account:

- **The first ones couple the gravitational fields to the non–gravitational contents of the Universe, i.e. the matter distribution, the electromagnetic fields, etc…**

- **The second set of equations gives the evolution of non–gravitational fields.**

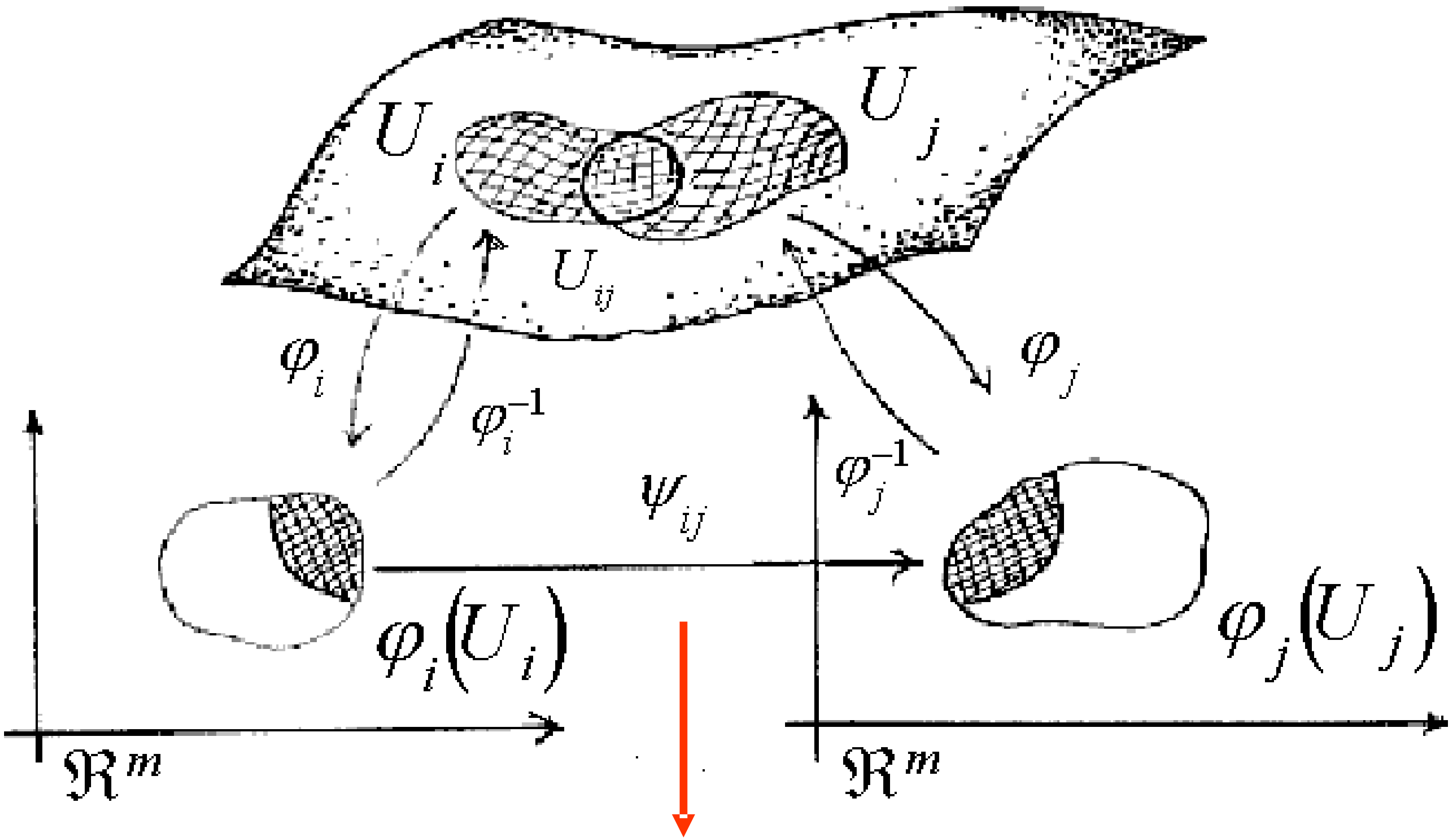
Within the framework of metric theories, these laws depend
only on the metric: this is a consequence of the EEP
and the so-called ”minimal coupling”.

- **We do not know if EP holds at quantum level**
- **We do not know if the Schiff conjecture always holds**

2 LECTURE

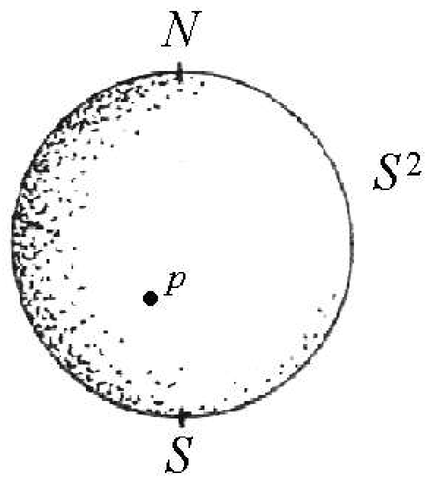
A geometric excursus

Open Charts:

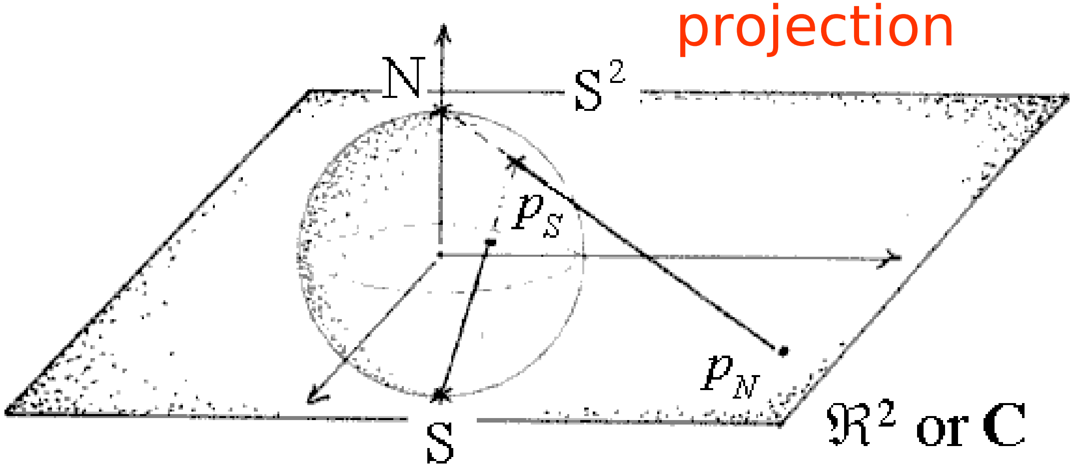


The same point **(= event**) is contained in more than one open chart. Its description in one chart is related to its description in another chart by a **transition function**

Gluing together a Manifold: the example of the sphere



Stereographic projection



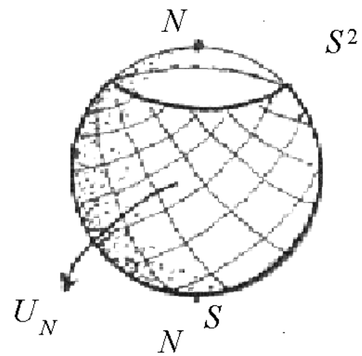
The transition function

$$\psi_{NS} : z_N = \frac{1}{z_S}$$

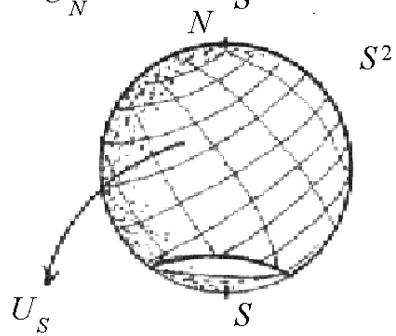
on

$$U_N \bigcap U_S$$

$$\varphi_N(P) = z_N \in \mathbb{C}$$



$$\varphi_S(P) = z_S \in \mathbb{C}$$



Differentiable structure

Definition 2.3 << *Let M be a topological Hausdorff space. A differentiable structure of dimension m on M is an atlas $\mathcal{A} = \bigcup_{i\in A}(U_i, \varphi_i)$ of open charts (U_i, φ_i) where $\forall i \in A$, $U_i \subset \mathcal{M}$ is an open subset and*

$$\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^m \tag{2.5}$$

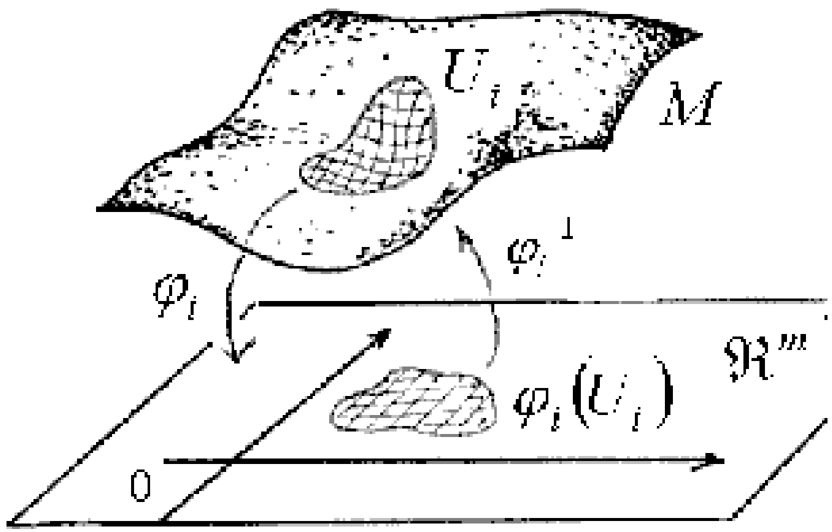
is a homeomorphism of U_i in $\mathbb{R}^m$ namely a continuous, invertible map onto an open subset of $\mathbb{R}^m$ such that the inverse map

$$\varphi_i^{-1} : \varphi_i(U_i) \to U_i \subset \mathcal{M} \tag{2.6}$$

is also continuous (see fig.1). The atlas must fulfill the following axioms:

$\mathbf{M_1}$ *It covers $\mathcal{M}$, namely*

$$\bigcup_i U_i = \mathcal{M} \tag{2.7}$$



Differentiable structure

so that each point of $\mathcal{M}$ is contained at least in one chart and generically in more than one: $\forall\, p \in \mathcal{M} \mapsto \exists\, (U_i, \varphi_i)\ /p \in U_i$.

$\mathbf{M_2}$ *Chosen any two charts* (U_i, φ_i), (U_j, φ_j) *such that* $U_i \bigcap U_j \neq \emptyset$ *on the intersection*

$$U_{ij} \stackrel{\text{def}}{=} U_i \bigcap U_j \tag{2.8}$$

there exist two homeomorphisms:

$$\begin{aligned} \varphi_i &: U_{ij} \bigcap U_j \rightarrow \varphi_i(U_{ij}) \subset \mathbb{R}^m \\ \varphi_j &: U_{ij} \bigcap U_j \rightarrow \varphi_j(U_{ij}) \subset \mathbb{R}^m \end{aligned} \tag{2.9}$$

and the composite map:

$$\begin{aligned} \psi_{ij} &\stackrel{\text{def}}{=} \varphi_j \circ \varphi_i^{-1} \\ \psi_{ij} &: \varphi_i(U_{ij}) \subset \mathbb{R}^m \rightarrow \varphi_j(U_{ij}) \subset \mathbb{R}^m \end{aligned} \tag{2.10}$$

named the transition function which is actually an m-tuplet of m real functions of m real variables is requested to be differentiable (see fig.2)

$\mathbf{M_3}$ *The collection* $(U_i, \varphi_i)_{i \in A}$ *is the maximal family of open charts for which both* M_1 *and* M_2 *hold true.*

>>

Manifolds

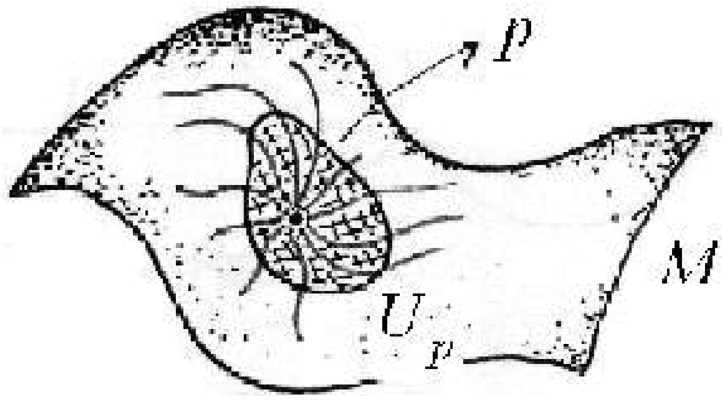
Definition 2.4 << *A differentiable manifold of dimension m is a topological space M that admits at least one differentiable structure $(U_i, \varphi_i)_{i\in A}$ of dimension m.* >>

The definition of a differentiable manifold is constructive in the sense that it provides a way to construct it explicitly. What one has to do is to give an atlas of open charts (U_i, φ_i) and the corresponding transition functions ψ_{ij} which should satisfy the necessary consistency conditions:

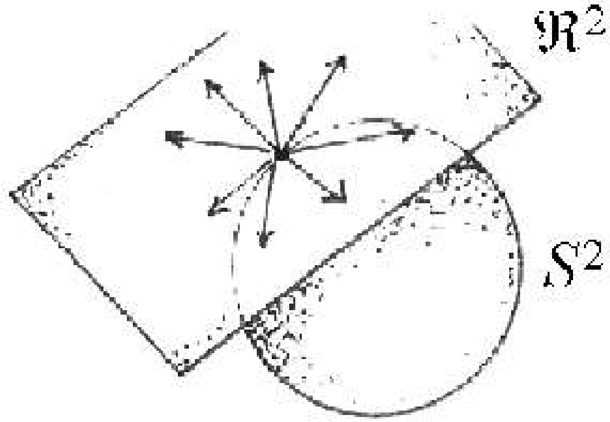
$$\begin{array}{ll} \forall\, i,j & \psi_{ij} = \psi_{ji}^{-1} \\ \forall\, i,j,k & \psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \mathbb{1} \end{array}$$

In other words a general recipe to construct a manifold is to specify the open charts and how they are *glued* together. The properties assigned to a manifold are the properties fulfilled by its transition functions. In

Tangent spaces and vector fields



In a neighborhood U_p of each point $p \in \mathcal{M}$ we consider the curves that go through p



The tangent space in a generic point of an $\mathbb{S}^2$ sphere

$$\forall p \in \mathcal{M} : p \mapsto T_p\mathcal{M} \quad \dim T_p\mathcal{M} = m$$

$T_p\mathcal{M}$ parametrizes the possible directions in which a curve starting at p can depart.

$$\vec{t}_p = c^\mu \frac{\vec{\partial}}{\partial x^\mu} ! \quad T_p\mathcal{M}$$

A tangent vector is a 1st order differential operator

Under change of local coordinates

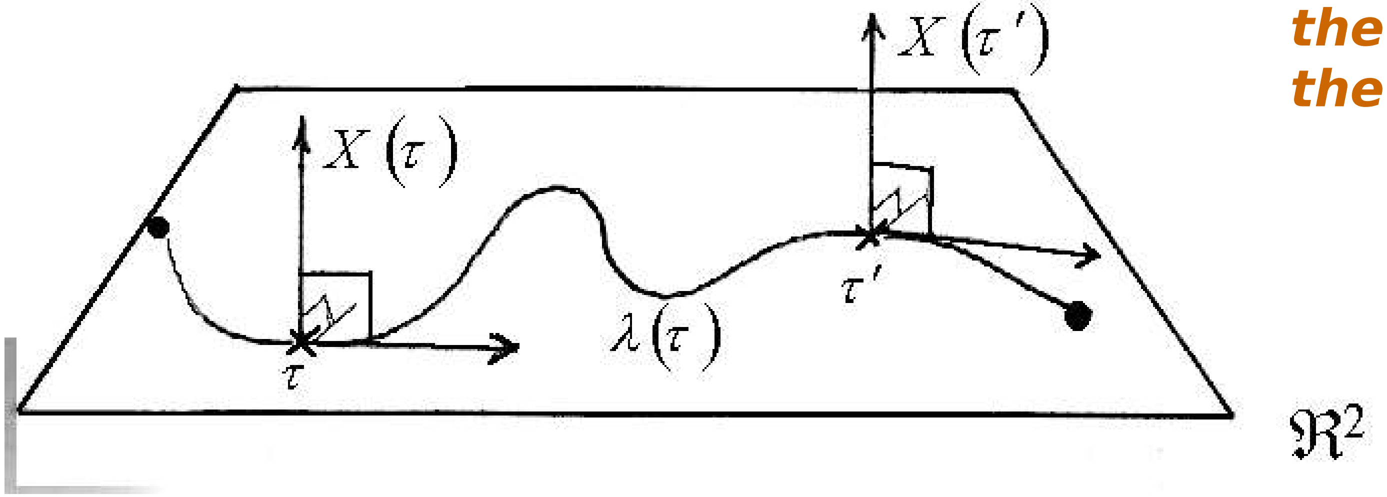
$$x^\mu = x^\mu(y) \quad ; \quad y^\alpha = y^\alpha(x)$$

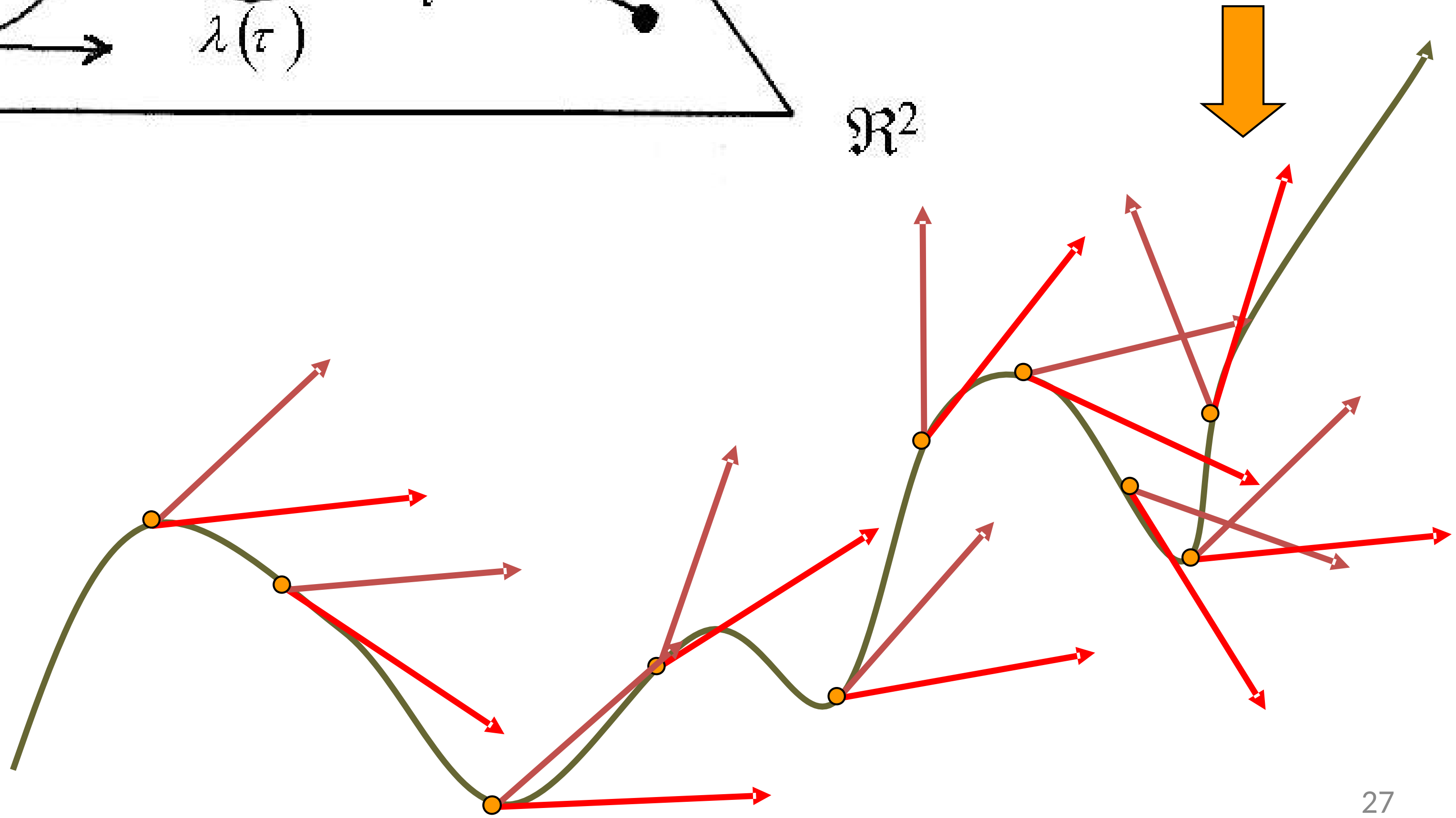
$$\vec{t}_p = c^\mu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) \frac{\vec{\partial}}{\partial y^\alpha} = c^\alpha \frac{\vec{\partial}}{\partial y^\alpha}$$

$$c^\alpha \equiv c^\mu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)$$

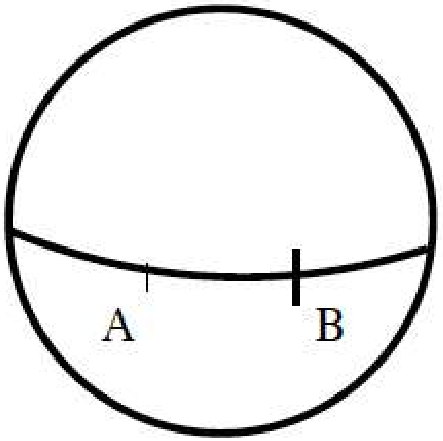
Parallel Transport

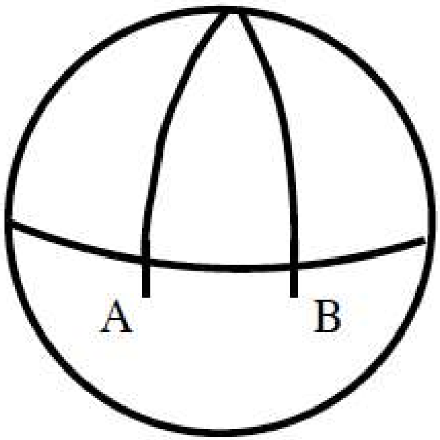
A vector field is parallel transported along a curve, when it mantains a constant angle with the tangent vector to the curve





Parallel Transport





- Two lines which start parallel do not remain parallel when prolonged on a curved manifold:

- *Let us consider two segments in A and B, perpendicular to the equator, i.e. parallel.*

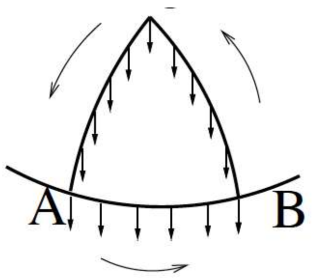
  - *The same lines when prolonged do not remain parallel.*

What happens when we parallel transport a vector along a path?

Parallel Transport

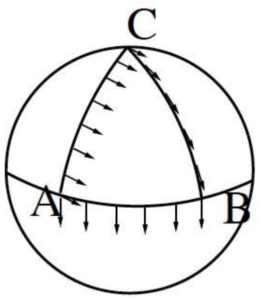
Parallel Transport means that for each infinitesimal displacement, the displaced vector must be parallel to the original one, and must have the same length.
Let us consider first the case when the path belongs to a flat space

a) Flat space



When we return to A, the displaced vector coincides with the original vector in A

b) On a sphere



When the vector goes back to A, it is rotated by 90 degrees. This is a consequence of the curvature of the sphere.

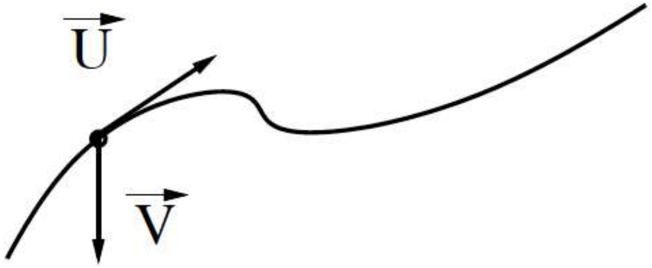
On a curved manifold, it is impossible to define a globally parallel vector field. The parallel transport of a vector depends on the path along which it is transported.

Parallel Transport

Let us now compute how a vector changes when it is parallel transported

Consider a curve of parameter λ and a vector field V defined at every point of the curve

Be $\vec{U} \rightarrow \left\{ \frac{dx^\alpha}{d\lambda} \right\}$ the tangent vector to the curve



At every point of the curve, we can choose a local inertial frame $\{\xi^\alpha\}$.

In this frame, if we move V along the curve of an infinitesimal dλ, parallel to itself and keeping its length unchanged, its components do not change

$$\frac{dV^\alpha}{d\lambda} = 0$$

But $$\frac{dV^\alpha}{d\lambda} = \frac{\partial V^\alpha}{\partial \xi^\beta} \frac{d\xi^\beta}{d\lambda} = U^\beta V^\alpha{}_{,\beta} = 0.$$

Since we are in a local inertial frame, ordinary and covariant derivative coincide and therefore we can write

$$U^\beta V^\alpha{}_{;\beta} = 0$$

If this equation is true in a local inertial frame, since it is a tensor equation, it must be true in any other frame

This is the frame-invariant definition of the parallel transport of V along the curve identified by the tangent vector U.

Parallel Transport

The above equation is written in terms of the components of V and U ; if we want to write it in a frame-independent form, we shall write

$$\nabla_{\vec{U}} \vec{V} = 0$$

which means that the covariant derivative along the direction of the vector U is zero.

Written explicitly for a generic reference frame with coordinates $\{x^{\alpha}\}$, it gives

$$\begin{aligned}
&\left(\nabla_{\vec{U}} \vec{V}\right)^{\alpha} \equiv U^{\beta} V^{\alpha}{}_{;\beta} \\
&= \frac{dx^{\beta}}{d\lambda}\left[\frac{\partial V^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}{}_{\beta\nu} V^{\nu}\right] = \frac{dV^{\alpha}}{d\lambda} + \Gamma^{\alpha}{}_{\beta\nu} V^{\nu} U^{\beta} = 0.
\end{aligned}$$

Thus, contrary to what happens in flat space, the components of a vector parallel transported along a curve in curved space do change, and the change is given by

$$\boxed{\frac{dV^{\alpha}}{d\lambda} = -\Gamma^{\alpha}{}_{\beta\nu} V^{\nu} U^{\beta}.}$$

Parallel Transport and Geodesic Equation

A different derivation of the geodesic equation makes use of the notion of covariant derivative

Let us consider a “free” particle, with worldline $x^\mu(\tau)$ and four-velocity (i.e. tangent vector to the worldline) $U^\mu = dx^\mu/d\tau$

By the **Equivalence Principle**, at any point of the worldline, we can define a local inertial frame $\{x^{\alpha'}\}$, where the laws of Special Relativity hold; then, in this frame, the particle four-acceleration is zero, i.e.

$$\frac{dU^{\mu'}}{d\tau} = \frac{dx^{\alpha'}}{d\tau}\frac{\partial U^{\mu'}}{\partial x^{\alpha'}} = U^{\alpha'} U^{\mu'}{}_{,\alpha'} = 0$$

In a local inertial frame, ordinary and covariant derivatives coincide, thus $U^{\alpha'} U^{\mu'}{}_{;\alpha'} = 0$

This is a tensor equation, and the **covariance principle** establishes that it holds in any coordinate frame; therefore, in a generic frame, we can write $U^\alpha U^\mu{}_{;\alpha} = 0$

indeed $$U^\alpha U^\mu{}_{;\alpha} = U^\alpha U^\mu{}_{,\alpha} + U^\alpha \Gamma^\mu{}_{\alpha\beta} U^\beta$$

by substituting $U^\mu = dx^\mu/d\tau$ this equation becomes

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0}$$

Parallel Transport and Geodesic Equation

The parameter along a geodesic needs not to be the proper time.
Be s the new parameter chosen to parameterize the geodesic. Since

$$\frac{d}{d\tau} = \frac{d}{ds}\frac{ds}{d\tau};$$

τ and s are affine parameters

then

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\left[\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right] = -\left[\frac{d^2s}{d\tau^2}\Big/\left(\frac{ds}{d\tau}\right)^2\right]\frac{dx^\alpha}{ds}$$

From this equation, we see that the new curve is a geodesic, only if the new parameter is related to the proper time τ by a linear transformations

$$s = a\lambda + b, \qquad a, b = \text{const}$$

In this case, the right hand side of the above equation vanishes

This equation has a more general validity since a geodesic can be either timelike, spacelike or null

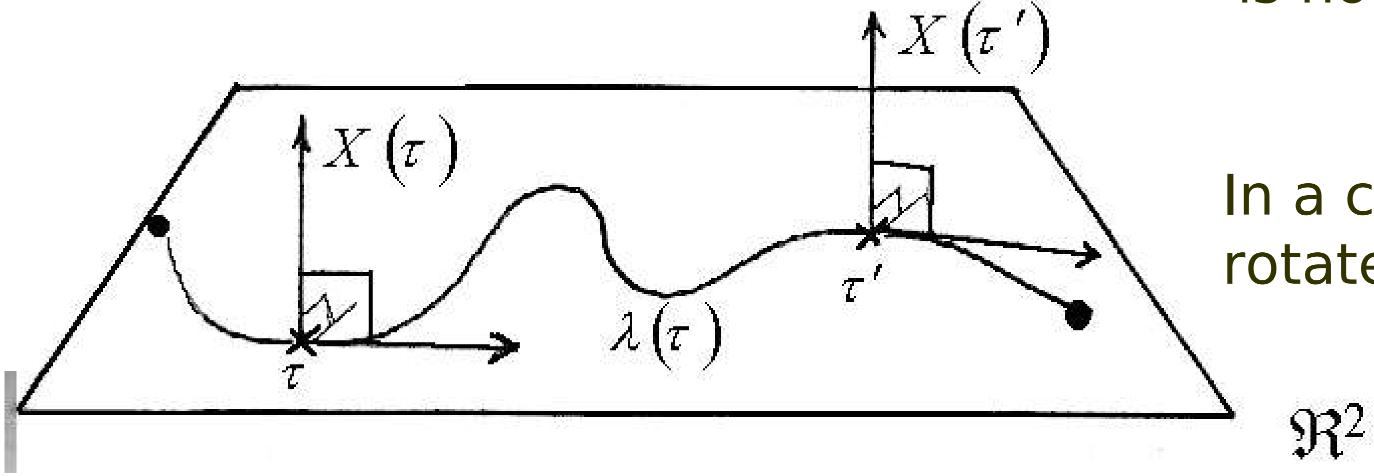
Thus, a curve C with tangent vector U is a geodesic if $\boxed{\nabla_{\vec{U}}\vec{U} = 0}$

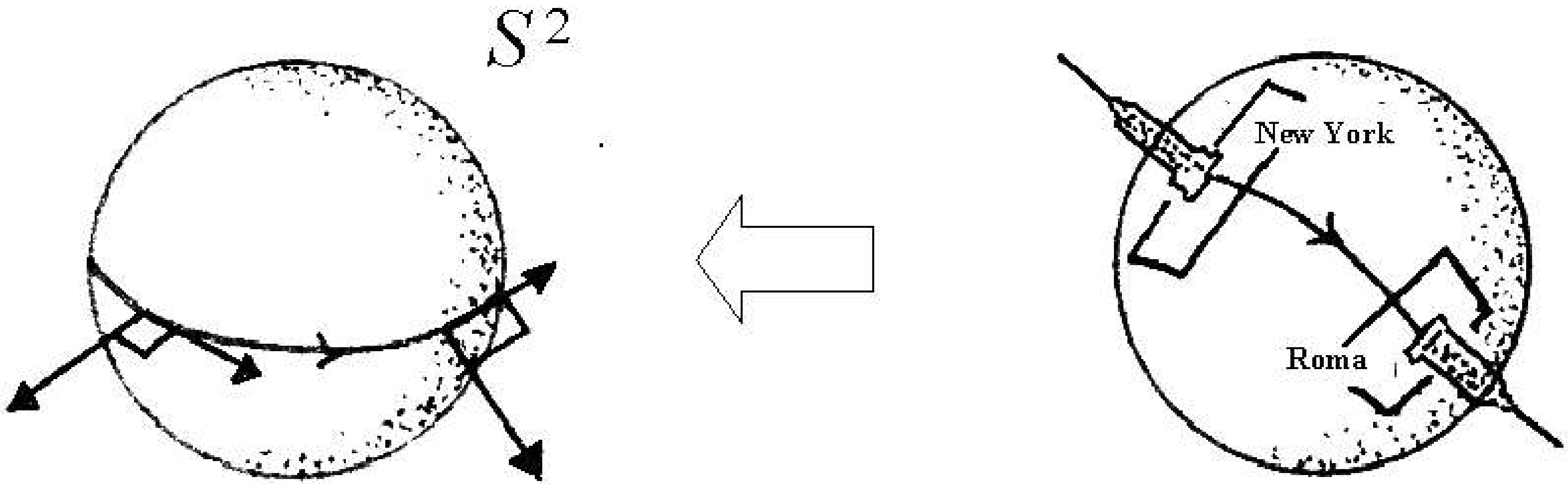
*For this reason we say that**:***
geodesics are those curves which parallel-transport their own tangent vectors

The difference between flat and curved manifo

In a flat manifold, while transported, the vector is not rotated.

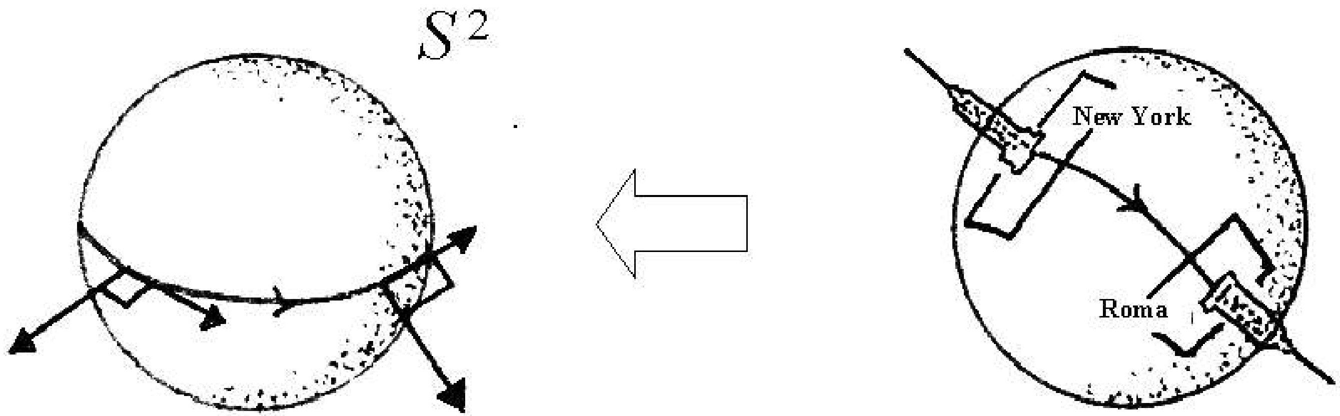
In a curved manifold it is rotated**:**



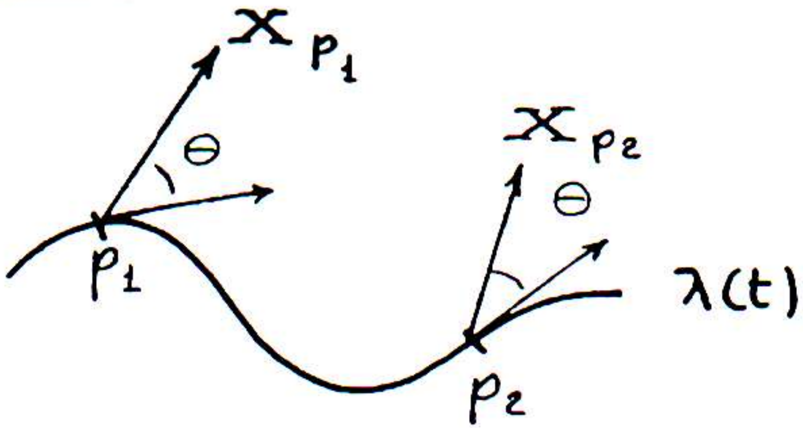


What are the straight lines?

They are the geodesics, curves that do not change length under small deformations. **These are the curves along which we have parallel transported our vectors**



On a sphere geodesics are maximal circles

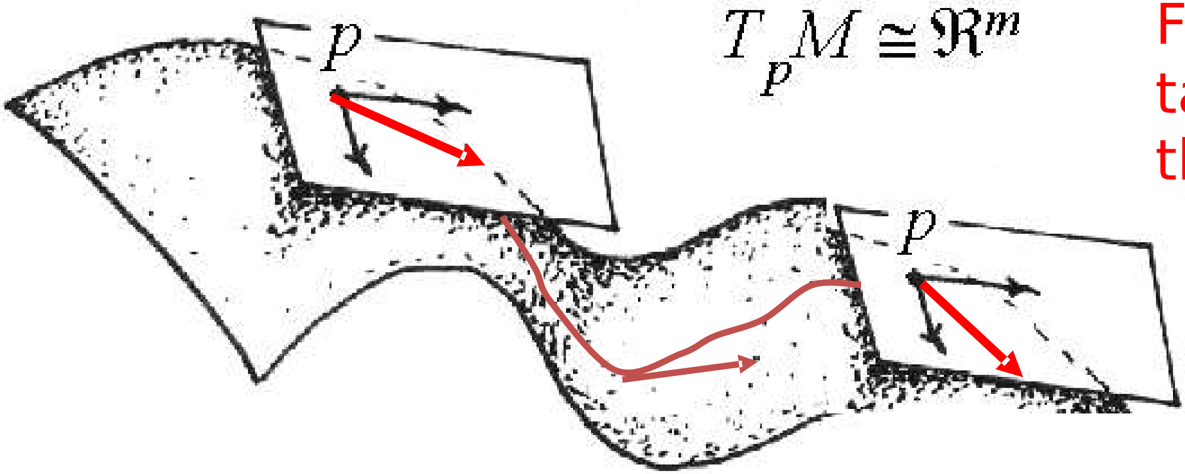


In the parallel transport, the angle with the tangent vector remains fixed. On geodesics the tangent vector is transported parallel to itself.

Connection and covariant derivative

A connection is a map

$\# \;\; "!!/'' \quad \forall !'' \quad ! \quad !''$



$T_pM \cong \Re^m$

From the product of the tangent bundle with itself to the tangent bundle

with defining properties:

1 $!_{!}\left('' + \#\right) = !_{!}\, '' + !_{!}\, \#$

2 $!_{(!\, '+\,)}\# = !_{!}\, \# + !_{''}\, \#$

3 $!_{(!'')}\# = !' \;\; {}_{''}\, \#$

4 $!_{!}\left(''\#\right) = ! \left[' \;\; \right]\# + !'_{!}\, \#$

In a basis...

If one has a basis of tangent vector fields $\{\vec{e}_\mu\}$ such that any vector field is written as:

$$\vec{X} = X^\mu(x)\,\vec{e}_\mu$$

then the connection is defined by giving **its coefficients**:

$$\nabla_{\vec{e}_\mu}\vec{e}_\nu = \Gamma^\rho_{\mu\nu}\vec{e}_\rho$$

Correspondingly we obtain:

$$\begin{aligned}\nabla_{\vec{e}_\nu}\vec{X} &= (\nabla_\mu X^\rho)\,\vec{e}_\rho \\ \nabla_\mu X^\rho &= \partial_\mu X^\rho - \Gamma^\rho_{\mu\sigma}\,X^\sigma\end{aligned}$$

This defines the covariant derivative of a (controvariant) vector

Torsion and Curvature

! (′ #!) ! ∀ #∀ #

$$! \left(' \#! \quad \right) ! \nabla \# \nabla \#_{\#} \quad \left['' !\# \right]$$

Torsion Tensor

$$! \left(' \#!\$! \quad \right) = ! \; !_{,,} \quad {}_{\#}\$ \nabla ! \; !_{\#} \quad {}_{,,} \$ \nabla ! \; {}_{\left['' !\# \right]}\$$$

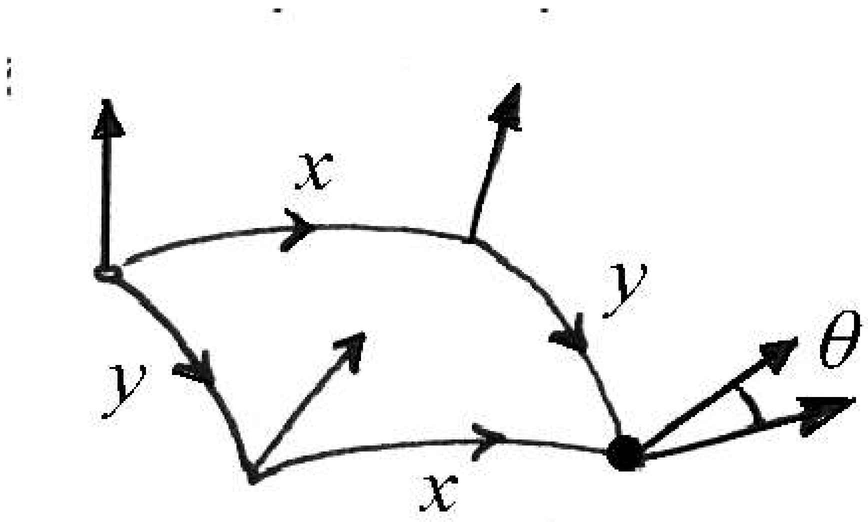
Curvature Tensor

In a basis we obtain:

$$
\begin{aligned}
T^{\mu}_{\rho\sigma} &= \Gamma^{\mu}_{\rho\sigma} - \Gamma^{\mu}_{\sigma\rho} \\
R^{\mu}_{\nu\rho\sigma} &= \partial_{\nu}\Gamma^{\mu}_{\rho\sigma} - \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \Gamma^{\mu}_{\nu\delta}\Gamma^{\delta}_{\rho\sigma} + \Gamma^{\mu}_{\rho\delta}\Gamma^{\delta}_{\nu\sigma}
\end{aligned}
$$

The Riemann curvature tensor

To see the real effect of curvature we must consider the parallel transport along loops



After transport along a loop, the vector does not come back to the original position but it is rotated of some angle.

If we have a metric........

An affine connection, namely a rule for the parallel transport can be arbitrarily given, but if we have a metric, then this induces a canonical special connection:

THE LEVI CIVITA CONNECTION

The Levi Civita connection is determined by requirement that:

$$\boxed{\begin{aligned} 0 &= \nabla_\rho g_{\mu\nu} \\ 0 &= T^{\mu}_{\rho\sigma} = \Gamma^{\mu}_{\rho\sigma} - \Gamma^{\mu}_{\sigma\rho} \end{aligned}}$$

This connection is the one which emerges from the variational principle of geodesics!

Now we can state the.......

Appropriate formulation of the Equivalence Principle:

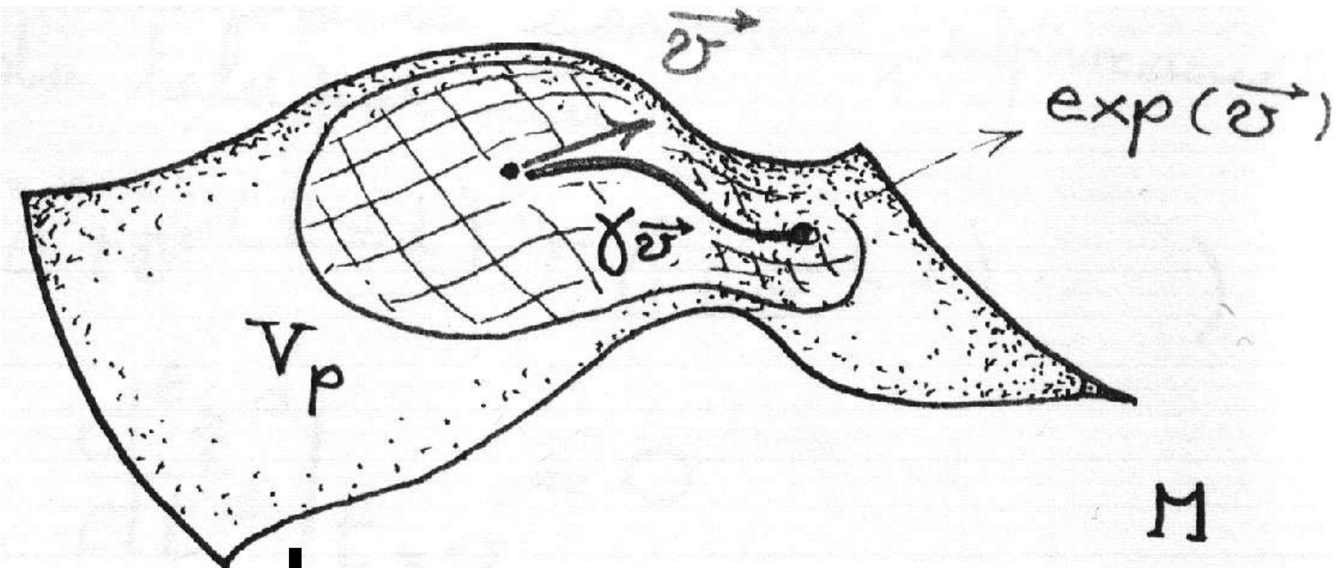
At any event " ! ! of space-time we can find a reference frame where the Levi Civita connection vanishes at that point. Such a frame is provided by the harmonic or locally inertial coordinates and it is such that the gravitational field is locally removed. Yet the gradient of the gravitational field cannot be removed if it exists.

In other words curvature can never be removed, since it is a tensor

Specifically: Harmonic Coordinates and the exponential map

∀!#! "$! ⟹ !!„"!!

#$%!!#$!   !   "$!   !

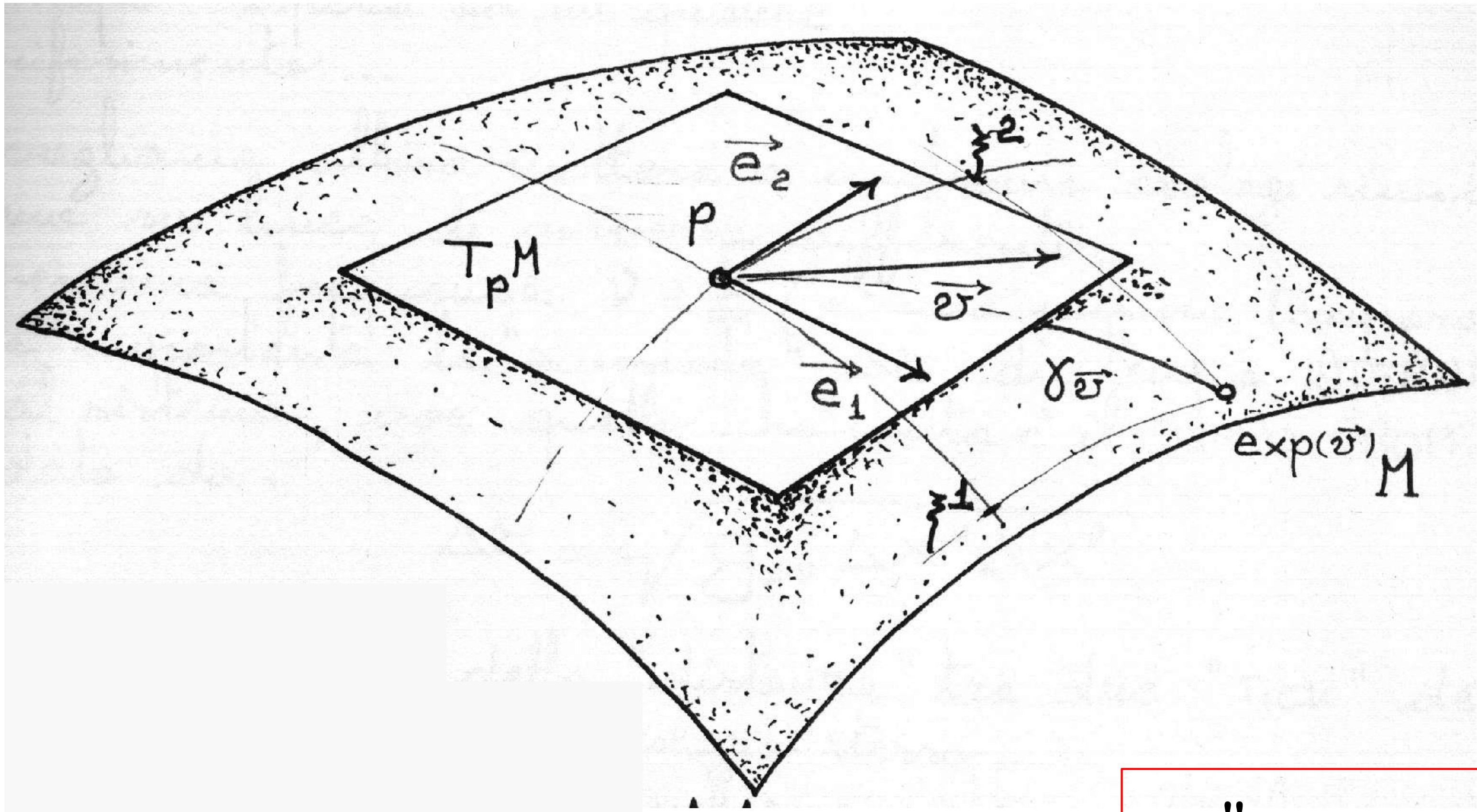


Follow the geodesics that admit the vector *v* as tangent and passes through p up to the value t=1 of the affine parameter. The point you reach is the image of *v* in the manifold

#$%["] = !!„"!!

!# = "#! *are the harmonic coordinates*

A view of the locally inertial frame



The geodesic equation, by definition, reduces in this frame to:

$$\frac{!\ "\ !}{!\,\prime\prime\,"} = !$$

3

LECTURE

The debate on Theories of Gravity

The physical world of General Relativity

Physics	Geometry
• The **when** and the **where** of any physical phenomenon constitute an **event.** • The set of all events is a continuous space, named **space-time** • **Gravitational phenomena** are manifestations of the **geometry** of space-time • **Point-like particles** move in space—time following special world-lines that are **"straight"** • The **laws of physics** are the same for **all observers**	• An **event** is a **point** in a topological space • **Space-time** is a **differentiable manifold M** • **The gravitational field** is a **metric g** on **M** • **Straight lines** are **geodesics** • **Field equations** are **generally covariant** under diffeomorphisms

According to Einstein, the mathematical model of spacetime is a pair:

(″ ! !)

Differentiable Manifold

Metric

General Relativity

It is a metric theory of gravity based on:

- Equivalence Principle
- Relativity Priciple
- Isometries
- Metric structure
- Geodesic structure
- General covariance
- Local Lorentz Invariance
- Causality Principle
- Spacetime curvature (dynamics)

Celestial mechanics, GWs, BH, cosmology constituted the success of GR.

The main consequence of this result is that the GEODESIC STRUCTURE and the CAUSAL STRUCTURE of spacetime coincide thanks to the Equivalence Principle. The first is given by the connections $\Gamma^{\alpha}_{\beta\gamma}$ and the second by the metric $g_{\alpha\beta}$. This is not true if the Equivalence Principle does not hold.

Or a Triple as pointed out by Palatini?

$$\left({''} \;\; ! \quad ! \right) + \Gamma$$

Differentiable Manifold

Metric

Connection

We need to review these fundamental concepts

We are dealing with metric-affine structure if

Christoffel symbols

≠

Levi Civita connection

In such a case the metric (causal) structure and
the geodesic structure could not coincide as pointed out by
Palatini!

Differentiable Manifold = Spacetime

Metric = Gravitational Potentials

Connections = Forces

Proposals for extending General Relativity

Although GR was not yet validated, some authors advanced proposals to extend it to general pictures

In 1918, Weyl studied the question of how to connect **gravity** and **electromagnetism** in a single and coherent geometric theory. To achieve this objective, he introduced an additional gauge field, which singles out a unique ***length connection***, whose four **additional degrees of freedom** (DoFs) are associated to the **electromagnetic potentials**. The consequence is that, during a parallel transport, both direction and length of vectors vary. However, Weyl's theory revealed to be in conflict not only with experiments (e.g. the null mass of photon), but even, in a more fundamental way, with **Quantum Mechanics.**

In 1922, **Cartan** considered a natural extension of GR, constituted not only by the **Levi-Civita connection**, but also by the ***torsion tensor*** (essentially the antisymmetric part of a metric compatible affine connection). Given these premises, he developed all the ensuing geometric formulation, where he suggested that the **torsion can be physically related to the intrinsic (quantum) angular momentum of matter** and it vanishes as soon as vacuum regions are considered.

In 1930, Einstein himself proposed some modifications to his theory. Fascinated by teleparallelism and tetrad formalism, he initiated a prolific and extensive correspondence mainly with **Cartan**, **Weitzenböck**, and **Lanczos**. Indeed, since the **tetrad fields posses sixteen independent components**, he associated **ten of them to the metric tensor**, whereas the other **six** were believed to be **linked to a separate connection**, entrusted to model electromagnetic potentials. Unfortunately, he failed in his attempt, but his studies shed new light on the importance of **additional DoFs**, which theoretically belong to the **Lorentz group** and physically are a consequence of the **local Lorentz invariance**.

Proposals for extending General Relativity

Around 1960, **Kibble** and **Sciama** revisited the Cartan theory formulating it within the gauge theory of the Poincaré group. This approach can be extended to the more **general affine group**, leading thus to the **metric-affine gauge gravity.**

It was growing the awareness that ***affinity*** and ***metricity*** could be considered as two different and independent concepts, where the affine connection could not respect *a priori* the metric postulate. This perspective is considered into the so-called ***Palatini approach***, where GR is described by a metric tensor and an affine connection, considered as two different geometric structures. Varying the Einstein-Hilbert action with respect to the metric, the Einstein field equations are recovered; whereas, varying it with respect to the affine connection, the **metric compatibility condition** is naturally obtained and the **Levi-Civita connection** is restored. This shows that GR structure entails metric compatibility, and the affine connection can be considered as a true dynamical field. In GR extensions the connection cannot concide with the Levi-Civita in the Palatini formalism.

There are theories of gravity, beyond the Einstein picture, where the field equations, besides the scalar curvature, can be formulated in terms of other geometric invariants. Furthermore, the affine connections were not considered anymore with an ancillary role with respect to the metric tensor, but, contrarily, they assumed a dynamical fundamental role. These concept give rise to the **Extended and Alternative Theories of Gravity.**

Why extending General Relativity?

GR revealed to be extraordinarily successful, because passed several astrophysical and cosmological observational probes, like: the Solar System tests, the direct detection of gravitational waves, the recent black hole imaging and etc.

Despite these achievements, the GR theory exhibits various **pathological issues**, like: from galaxies to cosmic evolution, the infrared behavior of gravitational field presents the **Dark Matter** and **Dark Energy** problems, and the tensions in cosmological parameters like H_0; at ultraviolet scales, the lack of renormalizability and unitarity of gravitational field points out that a final, self-consistent theory of **Quantum Gravity** is not at hand.

In general, the formulation of a new theory of gravity to solve the aforementioned issues is not a simple task. There are principles, constraints, mathematical consistencies, and the matching with observations that any novel approach must necessarily fulfill before being accepted as a self-consistent picture. This is one of the thorny theoretical challenges of modern physics.

In this perspective, it is possible to focus on *GR* and its **dynamically equivalent formulations**, in view to put in evidence similarities and differences towards a unified view of gravitational interaction.

We must suitably discriminate among theories of gravity!

$$G_{\mu\nu} = (8\pi G) T_{\mu\nu}$$

$G_{\mu\nu} \Rightarrow \tilde{G}_{\mu\nu}$ $T_{\mu\nu} \Rightarrow \tilde{T}_{\mu\nu}$

Extended Theories of Gravity **Dark Energy and Dark Matter**

WHY?

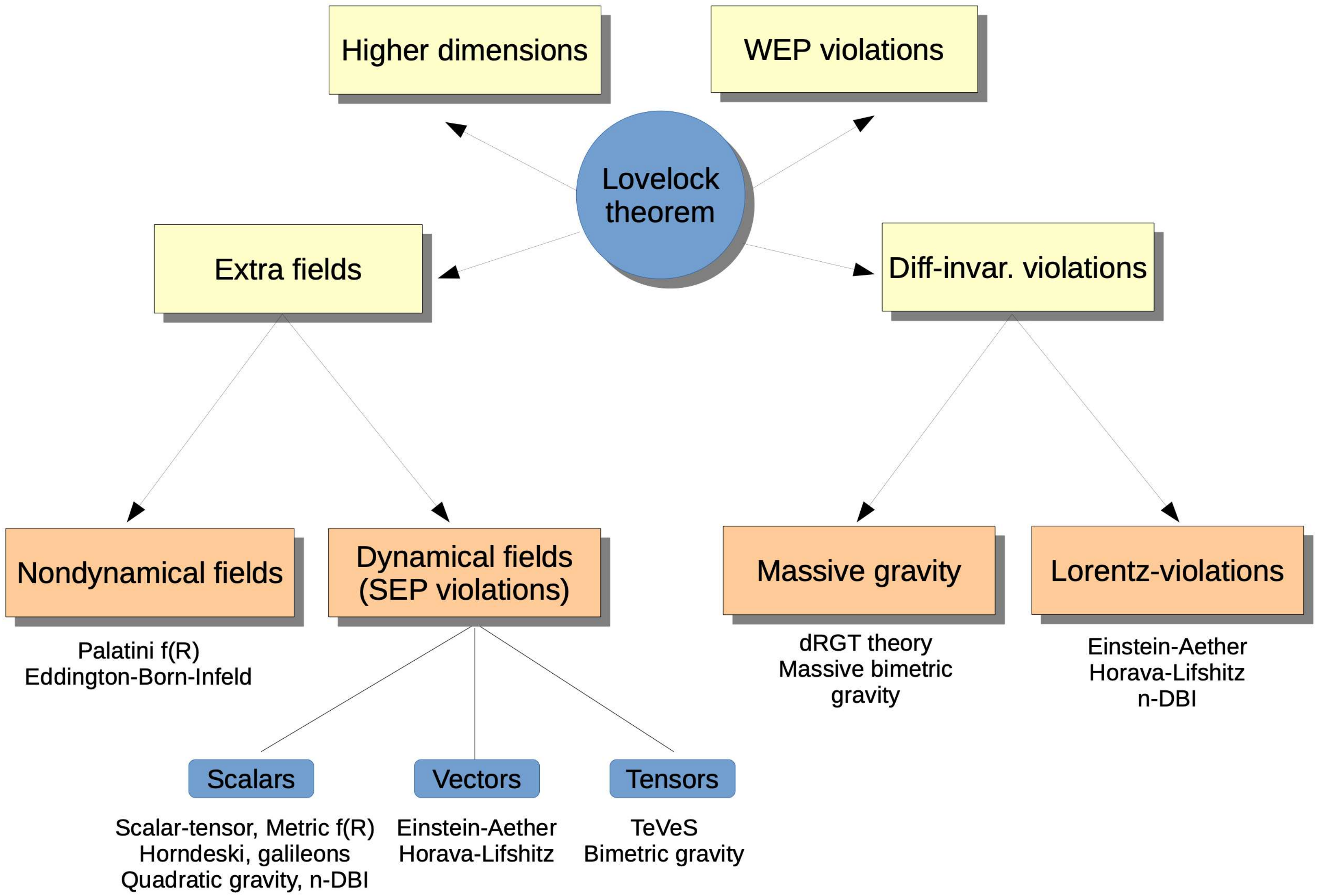
- § **QFT on curved spacetimes**
- § **String/M-theory corrections**
- § **Brane-world models**

$$R,\ R^{\mu\nu}R_{\mu\nu},\ R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta},\ R\Box^{l}R,$$

- § **Cosmological constant (Λ)**
- § **Time varing Λ**
- § **Scalar field theories**
- § **Phantom fields**
- § **Phenomenological Theories**
- § **Exotic matter**

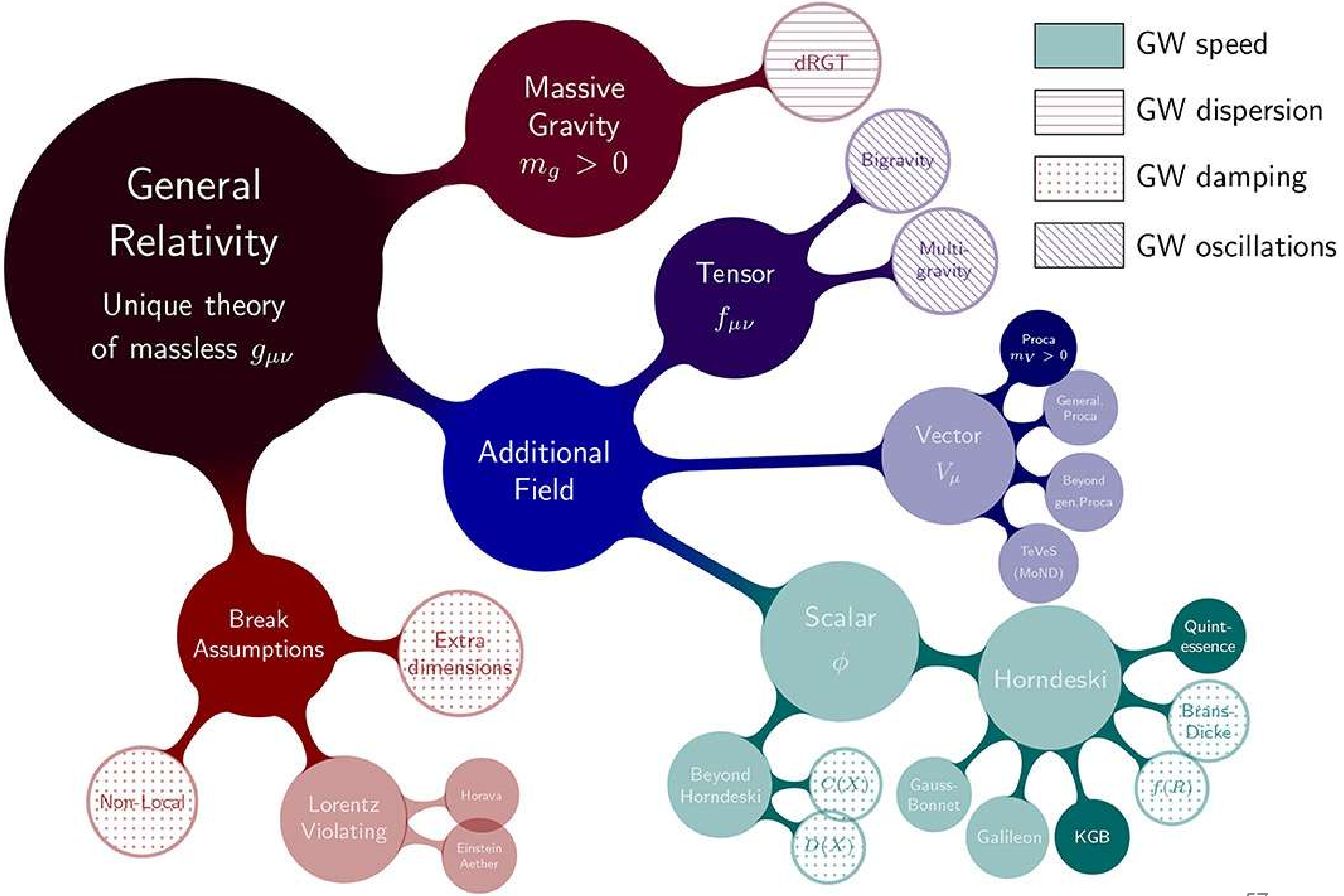
Curvature and geometric invariants have to be taken into account

Most theories can be reduced to GR +scalar fields by the Lovelock Theorem



E. Berti et al. CQG 32 (2015) 243001

Modified gravity roadmap



4

LECTURE

Metric-Affine Theories

GOALS

Let us discuss equivalent representations of gravity in the framework of **metric-affine geometries** pointing out basic concepts from where these theories stem out. In particular, we take into account tetrads and spin connection to describe the so called ***Geometric Trinity of Gravity***. Specifically, we consider **General Relativity**, constructed upon the metric tensor and based on the curvature scalar *R*; **Teleparallel Equivalent of General Relativity**, formulated in terms of torsion scalar *T* and relying on tetrads and spin connection; **Symmetric Teleparallel Equivalent of General Relativity**, built up on non-metricity scalar *Q*, constructedstarting from metric tensor and affine connection. General Relativity is formulated as a geometric theory of gravity based on metric, whereas teleparallel approaches configure as gauge theories, where gauge choices permit not only to simplify calculations, but also to give deep insight into the basic concepts of gravitational field. Specifically, we point out how the foundation principles of General Relativity (i.e., the **Equivalence Princip**le and the **General Covariance**) can be seen from the teleparallel point of view. These theories are dynamically equivalent and this feature can be demonstrated under three different standards: (1) the **variational method**; (2) the **field equations**; (3) the **solutions**. Regarding the second point, we provide a procedure starting from the (generalised) **second Bianchi identity** and then deriving the field equations. Referring to the third point, we compare spherically symmetric solutions in vacuum recovering the Schwarzschild metric and the Birkhoff theorem in all the approaches. It is worth stressing that, in extending the formalisms to f(R), f(T), and f(Q) gravities respectively, the dynamical equivalence is lost opening the discussion on the different number of **degrees of freedom** intervening in the various representations of gravitational theories.

NOTATIONS

We are going to use the following notations

- Metric signature (-,+,+,+)
- Greek indices take values 0,1,2,3, while the lowercase Latin ones 1,2,3.
- Capital Latin letters indicate tetrad indices.
- The flat metric is indicated by $\eta_{\mu\nu}= \eta^{\mu\nu}$=diag(-1,1,1,1)
- The determinant of the metric $g_{\mu\nu}$ is denoted by g.
- Round (square) brackets around a pair of indices stands for symmetrization (antisymmetrization) procedure, i.e.,

$$A_{(ij)} = A_{ij} + A_{ji} \quad (A_{[ij]} = A_{ij} - A_{ji}).$$

- *over-circles* refer to quantities built up on the Levi-Civita connection. $\longrightarrow \overset{\circ}{A}$
- *over-hats* denote quantities related to the teleparallel connection. $\longrightarrow \hat{A}$
- *over-diamonds* denote quantities involving non-metricity. $\longrightarrow \overset{\diamond}{A}$
- GR = General Relativity.
- TG = Teleparallel Gravity.
- TEGR = Teleparallel Equivalent of General Relativity.
- STEGR = Symmetric Teleparallel Equivalent of General Relativity.
- LIF = Local Inertial Frame.
- DoFs = Degrees of Freedom.

Non-Euclidean geometries and the formulations of geometry

In the 19th century, Newtonian mechanics was considered as the best theory to describe gravity.

In this period, there was a great cultural ferment around *non-Euclidean geometries* starting from fundamental works by Gauss, Lobachevsky, Bolyai, Riemann, Bianchi, Ricci-Curbastro, and several others.

The Euclidean framework, the arena for classical Physics, was overtaken by the formulation of elliptical and hyperbolic geometries, stemming out from a rigorous axiomatic reformulation of the geometry foundations.

Two main formulations emerged for geometries:

(1) ***affine geometry***, introduced by Euler in 1748 and after promoted by Möbius, Klein, and Weyl. It focuses on the study of *parallel lines*, based on the fifth Euclid postulate, and on the *affine transformations;*

(2) ***metric geometry***, introduced by Fréchet and Hausdorff, relies on a *metric function* defining the concept of distance between any two points, members of a non-empty set.

Ingredients of metric-affine geometry

A first extension of Einstein gravity starts by generalizing the affine connections which cannot be strictly Levi-Civita.

A metric-affine theory is defined by the following triplet

$$\{\mathcal{M}, g_{\mu\nu}, \Gamma^{\rho}{}_{\mu\nu}\}$$

four-dimensional
spacetime manifold

rank-two symmetric tensor
(10 independent components)

affine connection
(64 independent components)

a-priori there is
no relation
between them

CASUAL STRUCTURE

GEODESIC STRUCTURE

coincide if the
Equivalence Principle
holds

General linear affine connection

We consider

a system of coordinates $\{\overbrace{x_0,}^{\text{TIME}}\overbrace{x_1, x_2, x_3}^{\text{SPACE}}\}$

line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$

covariant derivative $\nabla_\mu A^\alpha{}_\beta := \partial_\mu A^\alpha{}_\beta - \Gamma^\rho{}_{\beta\mu} A^\alpha{}_\rho + \Gamma^\alpha{}_{\rho\mu} A^\rho{}_\beta$

The components of the general linear affine connection can be uniquely decomposed as follows

$$\Gamma^\rho{}_{\mu\nu} := \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} + K^\rho{}_{\mu\nu} + L^\rho{}_{\mu\nu}$$

Levi-Civita connection

contortion tensor

disformation tensor

$$\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} := \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \quad K^\rho{}_{\mu\nu} := \frac{1}{2}(T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T^\rho{}_{\mu\nu}) \quad L^\rho{}_{\mu\nu} := \frac{1}{2}(Q^\rho{}_{\mu\nu} - Q_\mu{}^\rho{}_\nu - Q_\nu{}^\rho{}_\mu)$$

Three main geometric objects: definitions

The three main geometric objects (related to the dynamics) are:

CURVATURE TENSOR $$R^{\mu}{}_{\nu\rho\sigma} := \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho}$$

TORSION TENSOR $$T^{\mu}{}_{\nu\rho} := 2\Gamma^{\mu}{}_{[\rho\nu]} \equiv \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho}$$

NON-METRICITY TENSOR $$Q_{\mu\nu\rho} := \nabla_\mu g_{\nu\rho} \equiv \partial_\mu g_{\nu\rho} - \Gamma^{\lambda}{}_{(\nu|\mu} g_{\rho)\lambda} \neq 0$$

These tensors show the following symmetries

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= -R^{\mu}{}_{\nu\sigma\rho}, \\ T^{\mu}{}_{\nu\rho} &= -T^{\mu}{}_{\rho\nu}, \\ Q_{\mu\nu\rho} &= Q_{\mu\rho\nu}. \end{aligned}$$

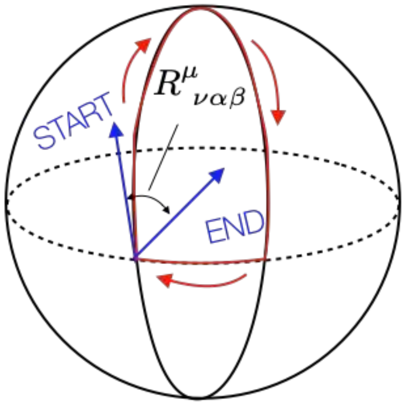
In general, the following *Bianchi identities* hold

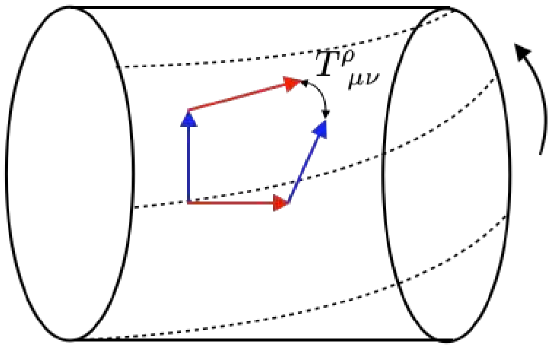
$$\begin{aligned} R^{\mu}{}_{[\nu\rho\sigma]} &= \nabla_{[\nu} T^{\mu}{}_{\rho\sigma]} + T^{\mu}{}_{\alpha[\nu} T^{\alpha}{}_{\rho\sigma]}, \\ \nabla_{[\alpha} R^{\mu}{}_{|\nu|\rho\sigma]} &= -R^{\mu}{}_{\nu\tau[\alpha} T^{\tau}{}_{\rho\sigma]}, \end{aligned}$$

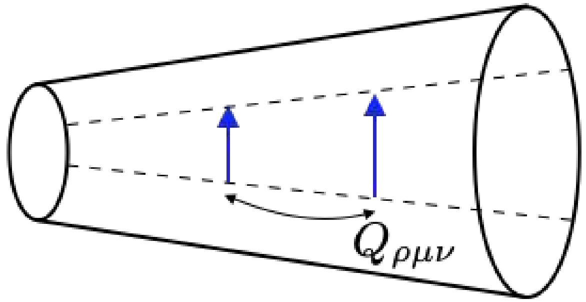
(involve only curvature and torsion tensors)

Three main geometric objects

The above geometric quantities, differently affect the parallel transport of a vector on a manifold.







CURVATURE manifests its presence when a vector is parallel transported along a closed curve on a non-flat background and come back to its starting point forming a non-null angle with its initial position

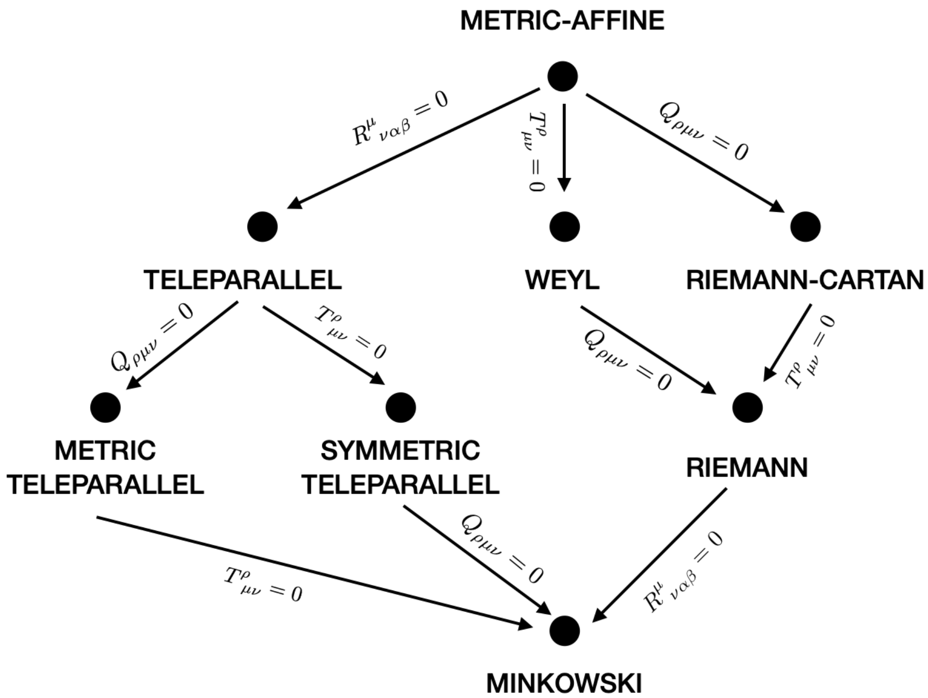
TORSION entails a rotational geometry, where the parallel transport of two vectors is antysimmetric by exchanging the transported vectors and the direction of transport. This property results in the non-closure of parallelograms

NON-METRICITY is responsible to alter the length of the vectors along the transport

In a metric-affine theory, these objects can work together and could have further meanings linked to physical quantities

Three main geometric objects

Metric-affine geometries are a broad class of theories that can be generally classified as follows:



(1) The *Riemann-Cartan geometry* is expressed in terms of metric compatible curvature and torsion tensors, which is deputed to model the quantum spin effects present in the matter.

(2) The *Weyl geometry* is constructed by vanishing the torsion, where curvature and non-metricity are the only surviving geometric objects. It is the origin of the U(1) gauge theory.

(3) *Teleparallel geometries* are curvatureless and are based on the concept of *parallelism at distance*, since the parallel transport of vectors becomes independent of the path.
They admit two special subclasses, represented by

- (3.1) *metric teleparallel theories* expressed only in terms of the torsion tensor;
- (3.2) *symmetric teleparallel theories* described only by the non-metricity tensor.

(4) The *Riemannian geometry* is the arena of GR theory, constructed only upon the curvature.

(5) The *Minkowski geometry* is obtained by setting curvature, torsion, and non-metricity to zero. This is the arena of Special Relativity.

5

LECTURE

Tetrads and Spin Connection

Mathematical tools

Considerations on the mathematical structure of Theories of Gravity

To define a theory of gravity, we need to fix the **underlying geometry**, the **transformation properties**, and the **set of observables.** GR is based on the **metric tensor** $g_{\mu\nu}$ from which we can construct the **Levi-Civita connection**, and finally the **curvature**, which encodes the gravitational dynamics. The possibility to relate the metric and the geodesic structure, which essentially coincide, rely on the validity of the **Equivalence Principle**. However, GR can be reformulated also in terms of **tetrads** and **spin connection** formalisms, giving rise to the teleparallel equivalent theories of GR.

We are going to analyse:

- THE TETRAD FORMALISM
- THE SPIN CONNECTION

Tetrad formalism

The geometric setting of any theory of gravity occurs in the **tangent bundle**, a natural construction always present in any smooth spacetime. In fact, at each point of the spacetime, it is possible to construct the tangent vector space. Any vector or covector can be expressed in terms of a general linear orthonormal frame called ***tetrads*** or ***vielbeine***.

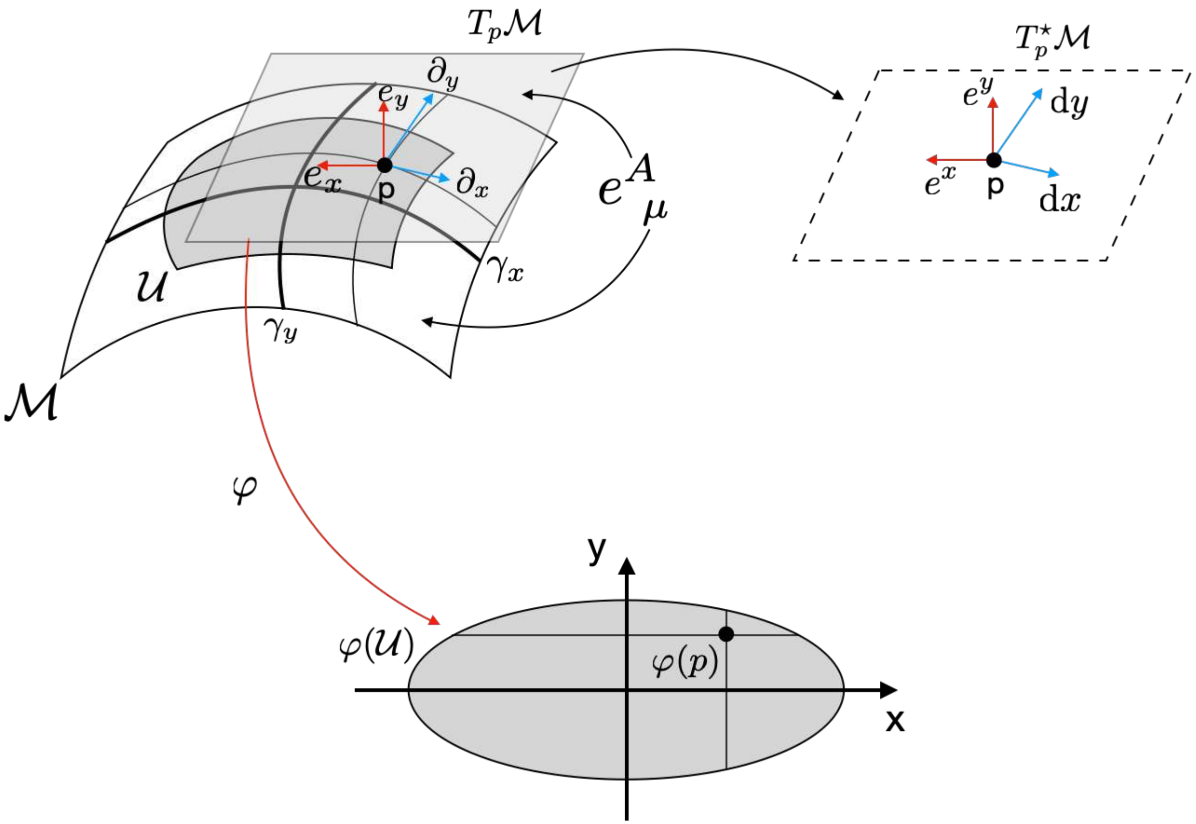
A tetrad field is a geometric construction, which permits to carry out the calculations on the tangent space.

Physically, **tetrads** represent the **standard laboratory-apparatus** of the observer for carrying out the measurements in space and time. Using a tetrad field means to adopt a *Lagrangian point of view*. A tetrad field establishes a relationship between the manifold and its tangent spaces. This geometric structure is always present, independently of any prior gravity-model assumption. The theoretical framework intervenes to characterize the occurring gravitational effects.

We will present:

1. definition and properties of tetrads
2. anhonolonomy structure of tetrads
3. the first Cartan structure equation
4. inertial class and trivial tetrads

Tetrads: definition and properties



holonomic basis in $T_p\mathcal{M}$

$$\partial_\mu := \left(\frac{\partial}{\partial x^\mu}\right)_p$$

basis in $T^*_p\,\mathcal{M}$

$$\mathrm{d}x^\mu \partial_\nu = \delta^\mu_\nu$$

tetrads in $T_p\,\mathcal{M}$

$$e_A := e_A^{\ \mu}\partial_\mu,$$
$$e^A := e^A_{\ \mu}\mathrm{d}x^\mu,$$
$$g_{\mu\nu} = \eta_{AB}e^A_{\ \mu}e^B_{\ \nu},$$
$$\eta_{AB} = g_{\mu\nu}e_A^{\ \mu}e_B^{\ \nu}.$$

Tetrads act as a *soldering agent* between the general manifold and the Minkowski spacetime.

$$e = \sqrt{-g}$$

Anholonomy of tetrad frames

A general tetrad basis $\{e^A{}_\mu\}$ satisfies the folllowing commutation relation

$$[e_A, e_B] := e_A e_B - e_B e_A = (e_A^{\ \mu}\partial_\mu)(e_B^{\ \nu}\partial_\nu) - (e_B^{\ \nu}\partial_\nu)(e_A^{\ \mu}\partial_\mu)$$
$$= \left[e_A^{\ \mu} e^C{}_\nu (\partial_\mu e_B^{\ \nu}) - e_B^{\ \nu} e^C{}_\mu (\partial_\nu e_B^{\ \mu})\right] e_C = \underbrace{e_A^{\ \mu} e_B^{\ \nu}\left[\partial_\nu e^C{}_\mu - \partial_\mu e^C{}_\nu\right]}_{f^C{}_{AB}} e_C = f^C{}_{AB} e_C,$$

structure constants
or *coefficients of anholonomy*
related to the frame $\{e^A{}_\mu\}$

They quantify the failure of the parallelogram closure generated by the vectors $\mathbf{e}_A$ and $\mathbf{e}_B$.

The coefficients of anholonomy specify how much they depart from being holonomic.

This approach reveals important properties of the underlying geometric framework on which we are working.

In GR, they can be used in the ***Bianchi classification***, which leads to 11 possible different cosmological spacetimes.

The first Cartan structure equation

Given a 1-form ω and defined dω as the exterior derivative, it can be written in components as

$$\mathrm{d}\omega = \partial_\mu \omega_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

exterior product

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu$$

tensorial product

Due to the antisymmetry of the exterior product and the Schwarz theorem, we have $\mathrm{d}^2\omega = 0$ thanks to the ***Poincaré lemma***.

The 2-form dω applied can be written as $u = u^\mu \partial_\mu,\ v = v^\nu \partial_\nu$ $\quad \mathrm{d}\omega(u,v) = u\omega(v) - v\omega(u) - \omega([u,v]_{\mathcal{L}})$

scalar

where

$$\mathrm{d}\omega(u,v) := \partial_\mu \omega_\nu (u^\mu v^\nu - u^\nu v^\mu),$$
$$u\omega(v) := u^\mu v^\nu \partial_\mu \omega_\nu + u^\mu \omega_\nu \partial_\mu v^\nu,$$
$$\omega([u,v]_{\mathcal{L}}) := \omega_\nu (u^\mu \partial_\mu v^\nu - v^\mu \partial_\mu u^\nu),$$
$$\omega = \omega_\mu \mathrm{d}x^\mu,$$
$$[u,v]_{\mathcal{L}} \equiv (\mathcal{L}_u v) := (u^\mu \partial_\mu v^\nu - v^\mu \partial_\mu u^\nu)\partial_\nu$$

Lie bracket (derivative) of the vector field **v** with respect to the vector field **u**

The first Cartan structure equation

If we consider the tetrad basis {$e^A{}_\mu$} and take **ω** =**e**^{A}, then we have the following relation

$$\left\{\mathrm{d}e^C(e_A, e_B)\right\} e_C = \left\{e_A[e^C(e_B)] - e_B[e^C(e_A)] - e^C([e_A, e_B]_{\mathcal{L}})\right\} e_C$$
$$= -e^C([e_A, e_B]^L_{\mathcal{L}} e_L) e_C = -[e_A, e_B]_{\mathcal{L}}.$$

Assigned a general metric-compatible affine connection $\Gamma^\mu{}_{\alpha\beta}$ and the associated covariant derivative ∇, we have

$$\nabla_{e_A} e_B = \underbrace{\gamma^C{}_{AB}}\, e_C$$

The ***Ricci rotation coefficients*** measure the rotation of all frame tetrads when moved in various directions, encoding thus gravitational and non-inertial effects. When we use the natural basis, they reduce to the affine connection $\Gamma^\mu{}_{\alpha\beta}$. We note that such coeffiecients arise also in a flat spacetime when, generally, non-liner coordinates are exploited, since they give rise to non-inertial effects.

In particular, in the considered tetrad basis, they assume the following expression and symmetries

$$\gamma_{\lambda\nu\mu} := e^A_\mu e^B{}_\lambda \nabla_A (e_\nu)_B = -e^A{}_\mu (e_\nu)_B \nabla_A e^B_\lambda$$
$$= -e^A_\mu e^B{}_\nu \nabla_A (e_\lambda)_B = -\gamma_{\nu\lambda\mu},$$

The first Cartan structure equation

$\gamma^C{}_{AB}$ can be seen as the action of the *connection 1-forms* $\omega^C{}_B$ on the tetrad basis $\mathbf{e}_A$ $\gamma^C{}_{AB} = \omega^C{}_B(e_A) \quad \Leftrightarrow \quad \omega^C{}_B = \gamma^C{}_{AB} e^A$

Since we know that $\nabla_\mu \partial_\nu = \Gamma^\lambda{}_{\mu\nu} \partial_\lambda$, if we consider the commutator of ∇_μ and ∂_ν we obtain

$$[\nabla_\mu, \partial_\nu] = \nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu = \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) \partial_\lambda = T^\lambda{}_{\mu\nu} \partial_\lambda$$

torsion tensor

$$T(v,u) := \nabla_v u - \nabla_u v - [v,u]_{\mathcal{L}}$$

Applying the torsion tensor to $\{e^A{}_\mu\}$ and using $\omega^C{}_B(e_A) = (\omega^C{}_D \otimes e^D)(e_A, e_B)$, we obtain

$$\begin{aligned} T(e_A, e_B) &= \nabla_{e_A} e_B - \nabla_{e_B} e_A - [e_A, e_B]_{\mathcal{L}} \\ &= \left[\omega^C{}_B(e_A) - \omega^C{}_A(e_B) + \mathrm{d}e^C(e_A, e_B)\right] e_C = \left[(\omega^C{}_D \wedge e^D + \mathrm{d}e^C)(e_A, e_B)\right] e_C . \end{aligned}$$

torsion differential 2-fom

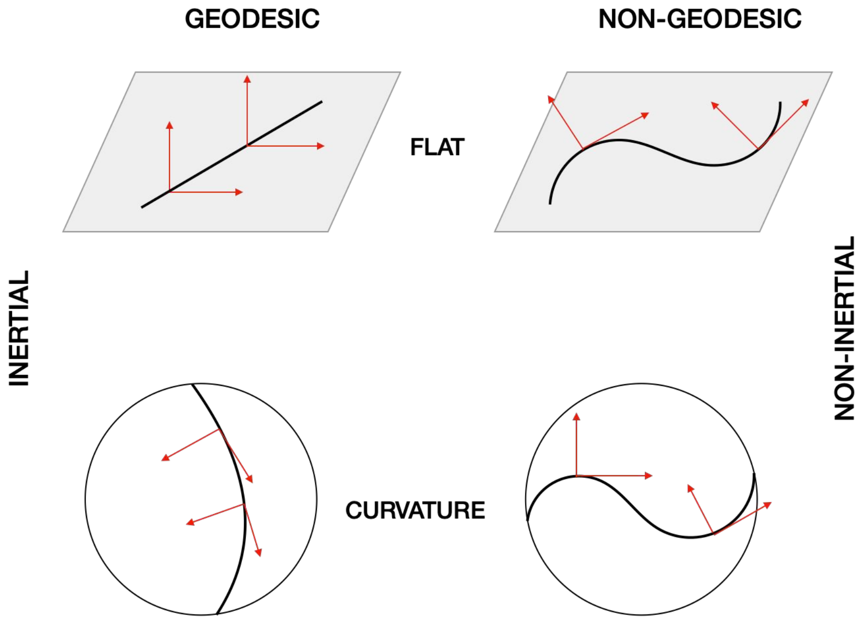
Defined $\Omega^C := \omega^C{}_D \wedge e^D + \mathrm{d}e^C$ and $T = \Omega^C \otimes e_C$, we derive the ***first Cartan structure equation***

$$\boxed{\begin{aligned} de^C &:= -\omega^C{}_A \wedge e^A = -\frac{1}{2}\left(\gamma^C{}_{AB} - \gamma^C{}_{BA}\right) e^A \wedge e^B \\ &= -\frac{1}{2} e_A{}^\mu e_B{}^\nu (\partial_\nu e^C{}_\mu - \partial_\mu e^C{}_\nu) e^A \wedge e^B = -\frac{1}{2} f^C{}_{AB} e^A \wedge e^B . \end{aligned}}$$

the anhonolonomy coefficients emerge as antisymmetric combination of the Ricci rotation coefficients. They are also related to the curls of the tetrad vector derivatives

Inertial frames and trivial tetrads

The *inertial frames* {**e**'$_A$} locally satisfying $f'^{C}{}_{AB} = 0$.
We have d**e**'$_A$=**0** (locally exact) and we have **e**'$_A$ =dx'^A (holonomic)



All coordinate bases belong to this family. It is worth noting that *this is not a local property, but it holds everywhere for all frames being part of this inertial class*

In absence of gravitation ($g_{\mu\nu}$ --> $\eta_{\mu\nu}$), the anholonomy is only caused by inertial forces present in these frames. In all coordinate systems, $\eta_{\mu\nu}$ is a function of the spacetime point, and independently of whether {**e** $_A$} is holonomic (inertial) or not we have

$$\eta_{AB} = \eta_{\mu\nu} e_A^{\mu} e_B^{\nu}$$

These are the frames appearing in Special Relativity, which are usually called ***trivial frames*** or ***trivial tetrads***. Of course, in absence of inertial forces, the class of inertial frames is, consequently, represented by vanishing structure coefficients.

Spin connection

The spin connection plays a fundamental role when we deal with tetrads, because it **encodes the inertial effects** occurring in the considered frame.

We will present:

1. fundamental properties of the Lorentz group
2. properties of the associated Lorentz algebra
3. mathematical derivation of Lorentz connections
4. tetrad postulate
5. physical meaning of Lorentz connections

Lorentz Group

Electromagnetism is framed under the standard of Special Relativity by postulating:

(1) the optical isotropy principle: all inertial frames are optically isotropic, i.e., the light propagates in these frames with velocity c in any direction;

(2) ***the principle of relativity***: the laws of physics assume the same form in all inertial reference frames.

Given two inertial frames and assuming that one is moving with respect to the other with uniform velocity **v**:=(v^1,v^2,v^3), the ***Lorentz transformation*** is a linear (affine) map relating the temporal and spatial coordinates of the two inertial observers

$$\Lambda^\mu{}_\nu : x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu(x)x^\nu$$

which leaves invariant the following quadratic form $\eta_{\mu\nu}x^\mu x^\nu = -t^2 + x^2 + y^2 + z^2$

A general Lorentz transformation is given by

$$\Lambda^\alpha{}_\beta = \mathcal{G}\cdot \begin{array}{c} \begin{array}{cc} \text{Lorentz factor} & \text{rotation matrix} \quad v=|\mathbf{v}| \end{array} \\ \begin{bmatrix} \gamma & -\gamma \mathcal{R}^i{}_j \frac{v^j}{c} \\ -\gamma \mathcal{R}^i{}_j \frac{v^j}{c} & \mathcal{R}^i{}_j \left(\delta^i_j + (\gamma - 1)\frac{v^i v^j}{v^2}\right) \end{bmatrix} \end{array}$$

$$\mathcal{G} = \{1, \mathbb{P}, \mathbb{T}, \mathbb{P}\cdot\mathbb{T}\} \quad \begin{aligned} 1 &:= \mathrm{diag}(1,1,1,1), \\ \mathbb{P} &:= \mathrm{diag}(1,-1,-1,-1), \\ \mathbb{T} &:= \mathrm{diag}(-1,1,1,1), \end{aligned}$$

A **Lorentz transformation** is defined in terms of **six parameters**:
three related to the rotation angles and the other three to the components of the spatial velocity **v**.

Lorentz Group

The set of all Lorentz transformations of Minkowski spacetime forms the *(homogeneous) Lorentz orthogonal group* O(1,3).

The Lorentz transformation entails, in matrix notation, that $\eta = \Lambda^T \eta \Lambda$, and thus $\det^2 \Lambda = 1$, namely

	$\det\Lambda = 1$	$\det\Lambda = -1$
$\Lambda^0{}_0 \geq 1$	proper orthochronous	improper orthochronous
$\Lambda^0{}_0 \leq -1$	proper non-orthochronous	improper non-orthochronous

The proper orthochronous Lorentz transformations form the ***restricted Lorentz special orthogonal group*** $SO^+(1,3)$.

The Lorentz group is a six-dimensional, non-compact, non-Abelian, and real Lie group endowed with four connected components.

The Lorentz group is involved in all known fundamental laws of Nature, describing the related symmetries of space and time. In GR, the ***local Lorentz invariance holds***, because in every small enough regions of spacetime, the gravitational effects can be neglected (Equivalence Principle), namely in the local *inertial frame (LIF)*, which recovers the Special Relativity physics.

At each point of a Riemannian spacetime, the metric $g_{\mu\nu}$ determines a tetrad up to the local Lorentz transformations in the tangent space. In other words, a tetrad vector (covector) base is not unique, because

$$\bar{e}^A{}_\mu = \Lambda^A{}_B e^B{}_\mu \quad \text{such that} \quad g_{\mu\nu} = \eta_{AB}\bar{e}^A{}_\mu \bar{e}^B{}_\nu \qquad \eta_{AB} = g_{\mu\nu}\bar{e}_A^{\ \mu}\bar{e}_B^{\ \nu}.$$

Lorentz Algebra

Another important feature of the Lorentz group is that it admits a *Lorentz algebra* $\mathcal{L}$.

If we consider an infinitesimal transformation in $SO^+(1,3)$ we have

$$\Lambda^\alpha{}_\beta = \delta^\alpha_\beta + \omega^\alpha{}_\beta + \mathcal{O}[(\omega^\alpha{}_\beta)^2]$$

Applying $\eta = \Lambda^T \eta \Lambda$ at linear order in $\omega^\alpha{}_\beta$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is an antisymmetric 4x4 matrix with six independent indices.

Therefore, we can associate six generators to the Lorentz algebra labeled by J_{AB}, with $J_{AB} = -J_{BA}$, where each of them can be expressed in the *four-vector representation* by a 4x4 matrix as follows

$$(J_{AB})^C{}_D := 2i\eta_{[B|D}\delta^C_{A]} = i(\eta_{BD}\delta^C_A - \eta_{AD}\delta^C_B)$$

Each element of the Lorentz group can be written as

$$\Lambda = e^{\frac{i}{2}\omega_{AB}J^{AB}}$$

Mathematical derivation of Lorentz connections

Some geometric objects with an established behaviour may loose the covariant character under point-dependent transformations, e.g., ordinary derivative of covariant objects. In order to supply for this defective behaviour, it is fundamental to introduce *connections* ω_μ fulfilling the following properties:
(i) they behave like vectors in the spacetime indices;
(ii) they act as non-tensor in the algebraic indices to compensate this effect and to reestablish the correct tensorial trend.

The *linear connections* fulfilling these requirements belong to the subgroup $SO^+(1,3)$ of GL(4,R), and they are dubbed *Lorentz connections*. It is worth noticing that *all Lorentz connections exhibit the presence of torsion*.

A Lorentz connection, also known as *spin connection*, ω_μ is a 1-form acting in the Lorentz algebra, namely

$$\omega_\mu : J_{AB} \in \mathfrak{L} \longrightarrow \omega_\mu := \frac{1}{2}\underbrace{\omega^{AB}{}_\mu}_{\text{spin connection components}} J_{AB}$$

$$\omega^{AB}{}_\mu = -\omega^{BA}{}_\mu$$

This permits to introduce the *Fock-Ivanenko covariant derivative* $\underbrace{\mathcal{D}_\mu}_{\text{acts only on tangent (algebraic) space indices}} := \partial_\mu - \omega_\mu = \partial_\mu - \frac{i}{2}\omega^{AB}{}_\mu \overbrace{J_{AB}}^{\text{generator in the appropriate representation of the Lorentz group}}$

Mathematical derivation of Lorentz connections

If we apply the *Fock-Ivanenko covariant derivative* to the field **e** ^{C} we obtain

$$\mathcal{D}_\mu e^C = \partial_\mu e^C - \frac{i}{2}\omega^{AB}{}_\mu \left[i(\eta_{BD}\delta^C_A - \eta_{AD}\delta^C_B)\right] e^D = \partial_\mu e^C + \frac{1}{2}\left[\omega^A{}_{D\mu}\delta^C_A + \omega^B{}_{D\mu}\delta^C_B\right] e^D = \partial_\mu e^C + \omega^C{}_{D\mu} e^D.$$

Splitting **e** ^{A} by $e_A := e_A{}^\mu \partial_\mu, \quad e^A := e^A{}_\mu \mathrm{d}x^\mu$, we obtain the following expressions

$$\mathcal{D}_\mu(e^C{}_\lambda dx^\lambda) = \mathcal{D}_\mu(e^C{}_\lambda)\mathrm{d}x^\lambda + e^C{}_\lambda \mathcal{D}_\mu(\mathrm{d}x^\lambda) = \mathcal{D}_\mu(e^C{}_\lambda)\mathrm{d}x^\lambda + e^C{}_\lambda(\delta^\lambda_\mu + e_E{}^\lambda e^D{}_\mu \omega^E{}_{D\rho}\mathrm{d}x^\rho) = \mathcal{D}_\mu(e^C{}_\lambda)\mathrm{d}x^\lambda + e^C{}_\mu,$$

$$\mathcal{D}_\mu(e^C{}_\lambda dx^\lambda) = \partial_\mu(e^C{}_\lambda dx^\lambda) + \omega^C{}_{D\mu} e^D{}_\lambda \mathrm{d}x^\lambda = \partial_\mu(e^C{}_\lambda) dx^\lambda + e^C{}_\mu + \omega^C{}_{D\mu} e^D{}_\lambda \mathrm{d}x^\lambda.$$

Equating the above equations we obtain $\mathcal{D}_\mu(e^C{}_\lambda) = \partial_\mu(e^C{}_\lambda) + \omega^C{}_{D\mu} e^D{}_\lambda$

Tetrad postulate

In tetrad bases {**e** $_A$}, the covariant derivative $\tilde{\nabla}$ of an algebraic (1,1) tensor **X** $^A{}_B$ can be written in terms of the spin connection as

$$\tilde{\nabla}_\mu X^A_{\ B} := \partial_\mu + \omega^A{}_{C\mu} X^C_{\ B} - \omega^C{}_{B\mu} X^A_{\ C}$$

Instead, the covariant derivative of a vector **V**, considered in the coordinate bases { ∂_μ}, is

$$\nabla V = (\nabla_\mu V^\nu)\mathrm{d}x^\mu \otimes \partial_\nu = (\partial_\mu V + \Gamma^\nu{}_{\mu\lambda} V^\lambda)\mathrm{d}x^\mu \otimes \partial_\nu$$

If we consider now the same vector **V** written in a mixed basis, tetrad and coordinate basis, gives

$$\begin{aligned}
\tilde{\nabla} V &= (\tilde{\nabla}_\mu V^A)\mathrm{d}x^\mu \otimes e_A = (\partial_\mu V^A + \omega^A{}_{B\mu} V^B)\mathrm{d}x^\mu \otimes e_A \\
&= [\partial_\mu (e^A_{\ \lambda} V^\lambda) + \omega^A{}_{B\mu} e^B_{\ \lambda} V^\lambda]\mathrm{d}x^\mu \otimes (e_A^{\ \nu} \partial_\nu) \\
&= [\partial_\mu V^\nu + (e_A^{\ \nu} \partial_\mu e^A_{\ \lambda} + \omega^A{}_{B\mu} e_A^{\ \nu} e^B_{\ \lambda}) V^\lambda]\mathrm{d}x^\mu \otimes \partial_\nu \\
&= [\partial_\mu V^\nu + (e_A^{\ \nu} \mathcal{D}_\mu e^A_{\ \lambda}) V^\lambda]\mathrm{d}x^\mu \otimes \partial_\nu .
\end{aligned}$$

This is a crucial point, because the above operations are in principle distinct. However, it is reasonable to assume $\nabla = \tilde{\nabla}$, because the same covariant derivative of a vector cannot change in terms of which type of basis one chooses. This is the so-called *tetrad postulate*, which is valid for any affine connection, defined on a smooth manifold M, and no metric is involved.

Tetrad postulate

Therefore, we have $\Gamma^{\lambda}{}_{\mu\nu} \equiv e_A{}^{\lambda} \mathcal{D}_\mu e^A{}_\nu$ This identity entails several significant implications on the spin connections:

(i) since it is not a tensor, it acquires a non-homogeneous term under $\mathcal{D}_\mu$ owed to the affine connection;
(ii) a spin connection is naturally induced by the affine connection;
(iii) it can be also regarded as the gauge field generated by local Lorentz transformations;
(iv) inverting the above identity with respect to the spin connection, we obtain

$$\omega^A{}_{B\mu} = e^A{}_\lambda e_B{}^\nu \Gamma^\lambda{}_{\mu\nu} + e^A{}_\sigma \partial_\mu e_B{}^\sigma \equiv e^A{}_\nu \nabla_\mu e_B{}^\nu$$

(v) according to the above equation, the connection 1-form $\omega^A{}_B$ can be written as $\omega^{AB} = \omega^{AB}{}_\mu \mathrm{d}x^\mu$ and *the Ricci rotation coefficients are the spacetime indices of the spin connection components*;
(vi) the covariant derivative of the tetrad, expressed in terms of the affine and spin connections, vanishes identically, namely

$$\nabla_\mu e^A_\nu = \partial_\mu e^A{}_\nu - \Gamma^\lambda{}_{\mu\nu} e^A{}_\lambda + \omega^A{}_{B\mu} e^B{}_\nu = 0;$$

(vii) we note that ∇_μ is the covariant derivative linked to the connection $\Gamma^\mu{}_{\alpha\beta}$ when acts on external indices and can be defined for tensorial fields, whereas $\mathcal{D}_\mu$ acts on internal indices and can be defined for all tensorial and spinorial fields;
(viii) from the metric compatibility condition, we obtain a sort of consistency check given by

$$\begin{aligned} 0 = \nabla_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - \Gamma^\sigma{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu} g_{\mu\sigma} = \partial_\lambda (e^A{}_\mu e^B{}_\nu \eta_{AB}) - e_A{}^\sigma g_{\sigma\nu} \mathcal{D}_\lambda e^A{}_\mu - e_A{}^\sigma g_{\mu\sigma} \mathcal{D}_\lambda e^A{}_\nu \\ &= -e^A{}_\nu e^D{}_\mu (\omega_{AD\lambda} - \omega_{DA\lambda}), \end{aligned}$$

which implies $\omega_{AB\mu} = -_{BA\mu}$, i.e., $\omega_{AB\mu}$ is Lorentzian. If the metric is not valid, the corresponding spin connection cannot assume values in the Lorentz algebra, because it is not a Lorentz connection.
Therefore, we have this equivalence: ***metric compatibility holds if and only if we choose a Lorentz connection*.**

Physical considerations on the Lorentz connection

The tetrads transform under local (point-dependent) Lorentz transformations $\Lambda^A{}_B(x)$. Now let us apply the same transformations to the spin connections. Let us first consider the inertial frames $\{\mathbf{e}'^A{}_\mu\}$\$, which, in general coordinates $\{x'^\mu\}$, can be written in the holonomic form $\mathbf{e}'^A{}_\mu = \partial_\mu x'^A$, where $x'^A = x'^A(x^\mu)$ is a point-dependent vector. Under a local transformation $x^A = \Lambda^A{}_B(x)\, x'^B$\$, we have $\mathbf{e}^A{}_\mu = \Lambda^A{}_B(x)\, \mathbf{e}'^B{}_\mu$ by transforming the vectors x^A and x'^A in the coordinate base $\{\partial_\mu\}$.

Let us evaluate $\partial_\mu x'^A$, which gives ($\partial'_A = \partial / \partial x'^A$)

$$\partial_\mu x'^A = \partial_\mu(\Lambda^A{}_B(x) x^B) = (\partial_\mu x^B)\Lambda^A{}_B(x) + x^B(\partial_\mu \Lambda^A{}_B(x)),$$
$$\partial_\mu x'^A = e'^C{}_\mu \partial'_C x'^A = e'^A{}_\mu = e^C{}_\mu \Lambda^A{}_C(x).$$

Therefore, gathering together the above results, we have $e^A{}_\mu = \partial_\mu x^A + \overset{\bullet}{\omega}{}^A{}_{B\mu} x^B \equiv \mathcal{D}_\mu x^A$, where $\overset{\bullet}{\omega}{}^A{}_{B\mu} := \Lambda^A{}_C(x)\partial_\mu \Lambda^C{}_B(x)$

purely inertial spin connection, because it physically manifests the inertial effects occurring in the Lorentz rotated frame $e^A{}_\mu$

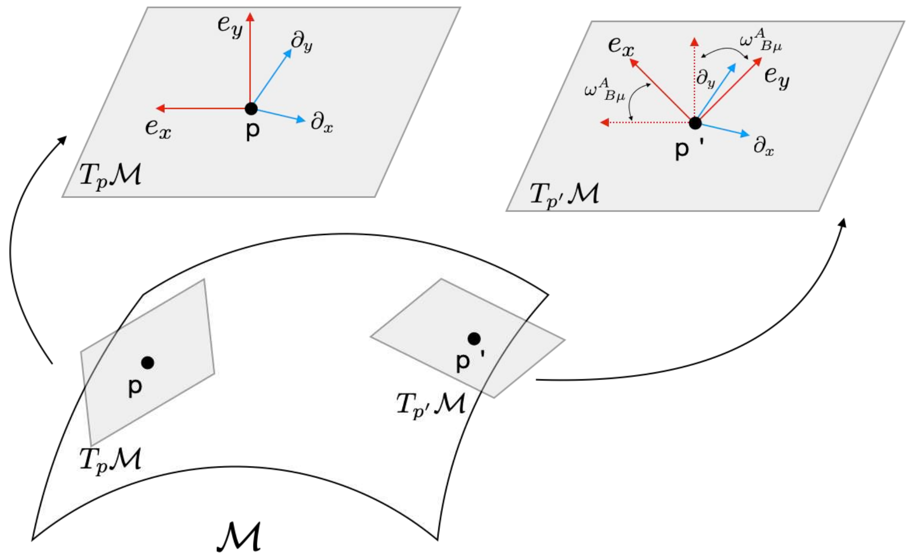
The Lorentz connections physically represent the inertial effects present in a given frame. In the inertial frames these effects are absent since the Lorentz connections vanish

Physical meaning of Lorentz connections

To better understand these results, let us consider the transformation of the spin connection under local Lorentz transformations

$$\omega^{A}{}_{B\mu} = \underbrace{\Lambda^{A}{}_{C}(x)\omega'^{C}{}_{D\mu}\Lambda^{D}{}_{C}}_{\text{non inertial}} + \underbrace{\Lambda^{A}{}_{C}\partial_\mu\Lambda^{C}{}_{B}(x)}_{\text{inertial}}$$

When we pass from a frame to another one, there are two distinct contributions: (1) *non-inertial* *effects* connected with the new frame; (2) *inertial contributions* due to the rotation of the new frame with respect to the previous one. Therefore, starting from inertial frames, it is possible to obtain a class of non-inertial frames via local Lorentz transformations. It is important to note that all these infinite frames are related through global (point-independent) Lorentz transformations



The coefficients of anholonomy can be written as

$$f^{C}{}_{AB} = \overset{\bullet}{\omega}{}^{C}{}_{BA} - \overset{\bullet}{\omega}{}^{C}{}_{AB}$$

From this relation, the spin reads as

$$\overset{\bullet}{\omega}{}^{A}{}_{BC} = \frac{1}{2}(f_B{}^{A}{}_{C} + f_C{}^{A}{}_{B} - f^{A}{}_{BC})$$

We can prove

$$R^{A}{}_{B\mu\nu} = \partial_\nu \overset{\bullet}{\omega}{}^{A}{}_{B\mu} - \partial_\mu \overset{\bullet}{\omega}{}^{A}{}_{B\nu} + \overset{\bullet}{\omega}{}^{A}{}_{E\nu}\overset{\bullet}{\omega}{}^{E}{}_{B\mu} - \overset{\bullet}{\omega}{}^{A}{}_{E\mu}\overset{\bullet}{\omega}{}^{E}{}_{B\nu} \equiv 0,$$

$$T^{A}{}_{\nu\mu} = \partial_\nu e^{A}{}_{\mu} - \partial_\mu e^{A}{}_{\nu} + \overset{\bullet}{\omega}{}^{A}{}_{E\nu} e^{E}{}_{\mu} - \overset{\bullet}{\omega}{}^{A}{}_{E\mu} e^{E}{}_{\nu}.$$

For trivial tetrads, the torsion tensor can be nullified.

6 LECTURE

Geometric Trinity: The Lagrangian approach

The Geometric Trinity of Gravity

Among the possible metric-affine gravity theories, Riemannian and teleparallel models are particularly interesting. GR is an example of Riemannian geometry, whereas the so-called *metric teleparallel equivalent of GR* (TEGR) and *symmetric teleparallel equivalent of GR* (STEGR) are examples of teleparallel geometries.
These three theories constitute the so-called ***Geometric Trinity of Gravity*.**

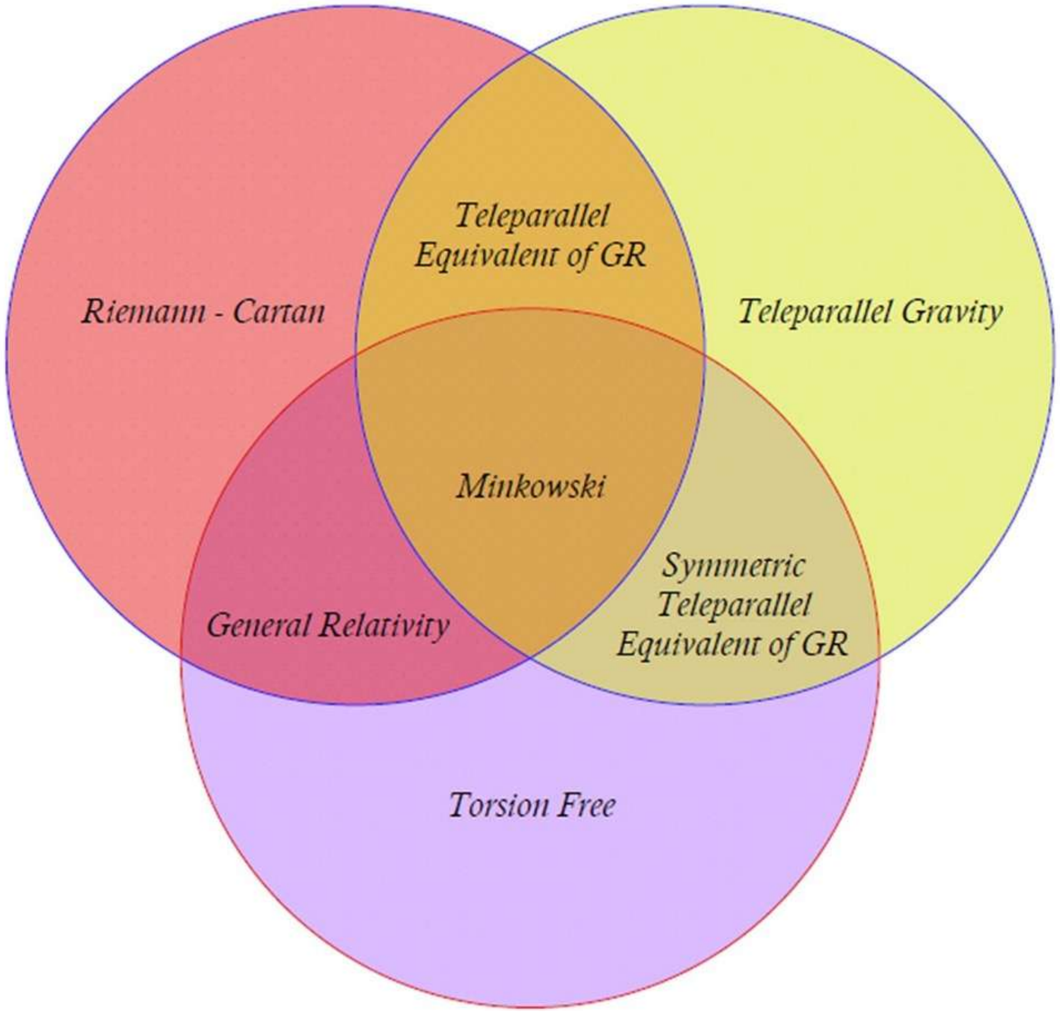
In **GR,** the geometric curvature models the gravitational force, whereas geodesics are the free-falling test particle's trajectories.

In **TEGR,** torsion replaces curvature for dynamics and it is able to provide the same descriptions of the gravitational interaction under a different perspective. The gravitational interaction emerges through the torsion tensor and acts as a (gauge) force.

STEGR shares several similar properties with TEGR. In this theory, curvature and torsion are both zero, and gravitational dynamics is attributed to the non-metricity tensor.

In the teleparallel framework, the concept of geodesics is replaced by force equations.
This is analogue to what happens in electrodynamics where the Lorentz force is present.

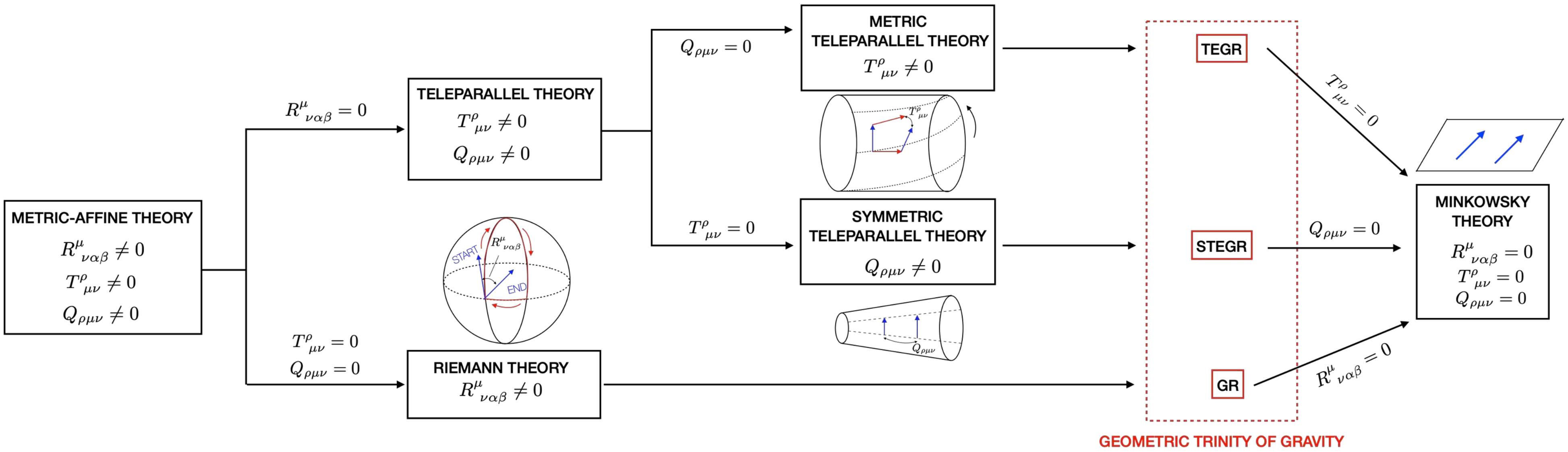
The Geometric Trinity of Gravity



Mathematical tools of the Geometric Trinity

GR is described in terms of the metric $g_{\mu\nu}$.
TEGR is based on tetrads $e^A{}_\mu$ (describing gravity) and spin connection $\omega^A{}_{B\mu}$ (flat connection outlining inertial effects).
STEGR embodies the Palatini idea where metric $g_{\mu\nu}$ and affine connection $\Gamma^\mu{}_{\alpha\beta}$ are two separated dynamical structures.

This is a scheme to understand how the Geometric Trinity of Gravity is framed within the metric-affine arena



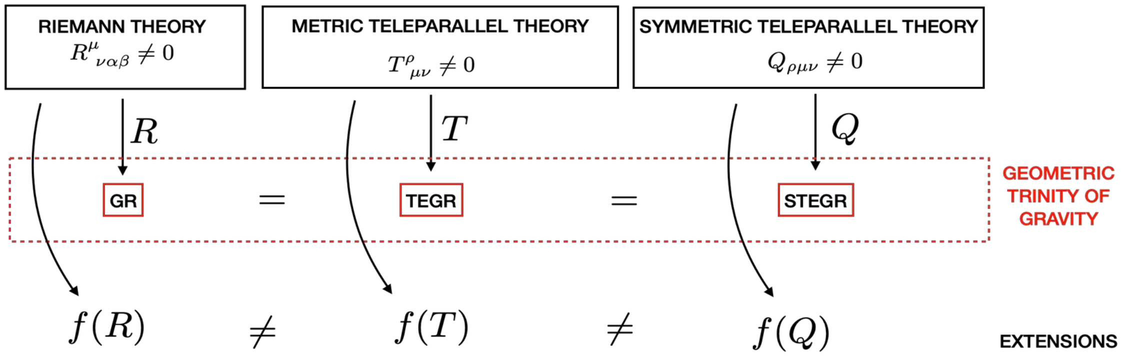
General remarks on the Geometric Trinity

Like other fundamental interactions in Nature, **gravitation** can be reformulated as **a gauge theory** through **TEGR** and **STEGR.**

The most peculiar property of gravitation seems to be its ***universal character*** (all objects regardless of their internal structure, feel this force), which is encoded in the ***Equivalence Principle***. In the teleparallel formulations, the **Equivalence Principle is recovered,** even if it does not lie at their foundation. This fact is extremely relevant because, if the Equivalence Principle were shown to be violated at some fundamental level, the final theory of gravitation could be non-metric.

We can define alternative ways of representing the gravitational field, accounting for the same DoFs, related to specific geometric invariants R, T, and Q. In this sense, GR, TEGR, and STEGR give rise to the the Geometric Trinity of Gravity.

GR can be extended to f(R), f(T) and f(Q) are the extensions of TEGR and STEGR, respectively. The equivalence among the three representations is not valid anymore among the extensions, because they give rise to dynamics with different DoFs. f(R) is of fourth order, whereas f(T) and f(Q) are of second-order. In f(T) and f(Q) we cannot choose a gauge to simplify the calculations, and we have field equations for the spin connection in f(T) and affine connection in f(Q).



Metric formulation of gravity: The case of General Relativity

The GR is the first geometric formulation of gravity in curved spacetimes.

We present:

1. principles of GR
2. implications related to the geodesic equations
3. fundamental geometric objects in GR
4. Lagrangian and field equations of GR
5. discussion on the tetrad formalism in GR

Principles of GR

The Einstein theory is essentially based on the following pillar ideas, which can be stated as follows :
(1) ***Relativity Principle***: there is no preferred inertial frames, i.e. all frames are good for Physics;
(2) ***General Covariance Principle***: the basic laws of Physics can be formulated in tensor form in any smooth four-dimensional manifold. This means that field equations must be " covariant" in form, i.e. they must be invariant under the action of spacetime diffeomorphisms;
(3) ***Equivalence Principle***: in any smooth four-dimensional manifold, it is possible to consider a small spacetime region, where spatial and temporal gravitational changes are negligible. Therefore, there always exists a LIF where gravitational effects can be nullified. In other words, inertial effects are locally indistinguishable from gravitational effects (which means the equivalence between the inertial and the gravitational masses).
(4) ***Causality Principle***: each point of spacetime has to admit a universally valid notion of past, present and future.

The first 2 principles represent a ***democracy principle*** *for all observers*, (all observers have the same right to describe the reality).

The third principle permits to locally recover the Physics of Special Relativity. Geometrically, it translates in determining the tangent plane in every point of a smooth manifold. Furthermore, gravity is the only interaction that cannot be switched off in *absolute*. It can be only locally nullified in the LIFs, physically coinciding with local free-falling frames. LIF is defined by

$$g_{\mu\nu}(\varphi(p)) = \eta_{\mu\nu}, \qquad \partial_\lambda g_{\mu\nu}(\varphi(p)) = 0.$$

This holds if we assume that inertial and gravitational mass coincides (*weak equivalence principle*).

The fourth principle is needed to ensure the uniqueness of the time notion despite of spacetime deformations and singularities.

Geodesic Equations

From the universality of free fall postulate in LIF via the coordinates $\{\xi^{\mu}\}$, a test particle will draw a straight line

$$\frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}s^2} = 0 \qquad \mathrm{d}s^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$$ line element

Since in such a frame it is not possible to experience the existence of gravitational effects, we perform a change of coordinates $\xi^{\alpha} = \xi^{\alpha}(x^{\mu})$, with $\{x^{\mu}\}$ the new coordinates. Applying this transformation, we obtain

$$\frac{\mathrm{d}^2x^\lambda}{\mathrm{d}s^2} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0$$

GEODESIC EQUATION
in a generic coordinate system $\{x^{\mu}\}$

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

$$\overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} := \frac{\partial x^\lambda}{\partial \xi^\sigma}\frac{\partial^2\xi^\sigma}{\partial x^\mu \partial x^\nu}$$

affine connection responsabile of the geodesic spacetime structure, which arises from the gravitational force acting on the test particle and being responsible of the departure from the straight trend. It explicitly shows that it is not a tensor. Physically they are the apparent forces acting on the body due to the curved geometric background induced by gravity.

In a metric compatible and torsion-free spacetime, we have that the unique affine symmetric connection is the *Levi-Civita one*.

The condition $\overset{\circ}{\nabla}_\lambda g_{\mu\nu} = 0$ gives $\overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} \equiv \left\{ {\lambda \atop \mu\nu} \right\}$

Fundamental geometric objects in GR

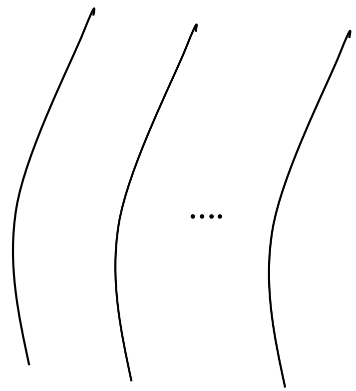
To quantify it as a field we have to introduce the *Riemann curvature tensor* arising from $[\mathring{\nabla}_\mu, \mathring{\nabla}_\nu] v^\alpha = \mathring{R}^\alpha{}_{\beta\mu\nu} v^\beta$

This equation is telling us that the Schwarz theorem, applied to covariant derivatives, does not hold, otherwise we have a flat spacetime. The GR gravitational field is fully encoded in this tensor.

The Riemann tensor has the following symmetries $\mathring{R}_{\mu\nu\alpha\beta} = -\mathring{R}_{\mu\nu\beta\alpha}, \quad \mathring{R}_{\mu\nu\alpha\beta} = -\mathring{R}_{\nu\mu\alpha\beta}, \quad \mathring{R}_{\mu\nu\alpha\beta} = \mathring{R}_{\alpha\beta\mu\nu}$

Due to the above symmetries, we can define the symmetric Ricci tesor $\mathring{R}_{\alpha\beta} = \mathring{R}^\mu{}_{\alpha\mu\beta}$ and the scalar curvature $\mathring{R} = \mathring{R}^\alpha{}_\alpha$

We consider a one-parameter family of geodesics $\gamma_s(t)$, where t is the affine parameter along the geodesic, and s labels the curves. We assume that the collection of these curves defines a smooth two-dimensional surface $x^\mu(t,s)$ embedded in M.



$\gamma_1(t)$ $\gamma_2(t)$ $\gamma_N(t)$

A natural vector basis adapted to the coordinate system is $T^\mu = \dfrac{\partial x^\mu}{\partial t}, \qquad S^\mu = \dfrac{\partial x^\mu}{\partial s}$

We define the relative velocity V^μ and acceleration A^μ along the geodesics as follows

$$V^\mu = T^\nu \mathring{\nabla}_\nu T^\mu, \quad A^\mu = T^\nu \mathring{\nabla}_\nu V^\mu$$

We then obtain the *geodesic deviation equation* $A^\mu = \mathring{R}^\mu{}_{\lambda\alpha\beta} T^\lambda T^\alpha S^\beta$

where A^μ between two close geodesics is proportional to the Riemann curvature tensor, which characterizes the behaviour of a one-parameter family of neighbouring geodesics.

Lagrangian and field equations of GR

The GR dynamics is derived from the *Hilbert-Einstein action* $S_{\rm GR} := \frac{c^4}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\,(\underbrace{\mathcal{L}_{\rm GR}}_{\overset{\circ}{R}(g)} + \underbrace{\mathcal{L}_{\rm m}}_{\text{matter}})$

The total DoFs are the 10 independent components of the metric tensor, from which we must subtract the 4-parameter diffeomorphisms underlying the invariance (*gauge symmetries' freedom*) and other 4 by a suitable choice of the coordinates (*gauge fixing*). Therefore, the gravitational dynamical DoFs becomes 2, corresponding to the *graviton*, massless spin-2 particle.

Applying the principle of least action $S_{\rm GR}$, we derive the GR field equations in presence of matter

$$\boxed{\underbrace{\overset{\circ}{G}_{\mu\nu}}_{\text{Einstein tensor}} := \overset{\circ}{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overset{\circ}{R} = \frac{8\pi G}{c^4}\underbrace{\mathfrak{T}_{\mu\nu}}}$$

$$\mathfrak{T}^{\mu\nu} = -\frac{1}{2\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}} \qquad \overset{\circ}{\nabla}_\mu\mathfrak{T}^{\mu\nu} = 0$$

(second-order) energy-momentum tensor, which is symmetric, satisfies the conservation equations , and physically represents the source of gravitational field.

Particular consideration has to be devoted to matter fields and gravity, because some subtleties can arise, like ambiguity in the matter coupling; treatment of bosonic and fermionic fields. For example in GR, fermions require tetrads and spin connection.

Discussion on the tetrad formalism in GR

GR conceives the gravitational interaction as a change in the geometry of spacetime itself. We pass from the Minkowski $\eta_{\mu\nu}$ to the Riemannian metric $g_{\mu\nu}$, and from partial ∂ to covariant derivatives ∇. In order to study how gravitation couples with others fields, we have to introduce the tetrads to deal with spinors in curved spacetimes. Tetrads encode the Equivalence Principle, since they are locally defined, as gravitation is locally equivalent to an accelerated frame.

Einstein's vierbein theory becomes thus a gauge field theory for gravity.

Given a general (anhonolomic) tetrad $\{e^A{}_\mu\}$, we can rewrite the Riemann tensor according to the Cartan structure equations as

$$\mathrm{d}e^C + \overset{\circ}{\omega}{}^A{}_B \wedge e^B = 0, \quad \overset{\circ}{\omega}_{AB} + \overset{\circ}{\omega}_{BA} = \mathrm{d}g_{AB}, \quad \mathrm{d}\overset{\circ}{\omega}{}^A{}_B + \overset{\circ}{\omega}{}^A{}_C \wedge \overset{\circ}{\omega}{}^C{}_B = \frac{1}{2}\underbrace{\overset{\circ}{r}{}^A{}_{BCD}}_{\text{Riemann curvature tensor in the tetrad frame}} e^C \wedge e^D$$

where

$$\overset{\circ}{\omega}{}^A{}_{B\mu} := e^A{}_\nu \overset{\circ}{\nabla}_\mu e_B^\nu, \quad \overset{\circ}{f}{}^A{}_{BC} := \overset{\circ}{\gamma}{}^A{}_{BC} - \overset{\circ}{\gamma}{}^A{}_{CB}, \quad \mathrm{d}g_{AB} = \partial_C\, g_{AB} e^C, \quad \overset{\circ}{\gamma}{}^A{}_{BC} = \frac{1}{2}(\overset{\circ}{f}{}^A{}_{BC} - g_{CL}g^{AM}\overset{\circ}{f}{}^L{}_{BM} - g_{BL}g^{AM}\overset{\circ}{f}{}^L{}_{CM}) + \overset{\circ}{\Gamma}{}^A{}_{BC}$$

We note that we can uniquely associate the Lorentz connection to the Levi-Civita connection.

In addition, if we consider the natural basis, then we have $\Rightarrow \overset{\circ}{\omega}{}^A{}_{BC} = 0 \Rightarrow \overset{\circ}{\gamma}{}^A{}_{BC} \equiv \overset{\circ}{\Gamma}{}^A{}_{BC} \Rightarrow$ Therefore, we have

$$\overset{\circ}{r}{}^A{}_{BCD} = \partial_D \overset{\circ}{\gamma}{}^A{}_{BC} - \partial_C \overset{\circ}{\gamma}{}^A{}_{BD} + \overset{\circ}{\gamma}{}^A{}_{CM}\overset{\circ}{\gamma}{}^M{}_{DB} - \overset{\circ}{\gamma}{}^A{}_{DM}\overset{\circ}{\gamma}{}^M{}_{CB} - \overset{\circ}{\gamma}{}^A{}_{MB}\overset{\circ}{\gamma}{}^M{}_{CD}$$

In the natural basis, we re-obtain the standard definition of the Riemann curvature tensor.

Gauge formulation of gravity: The case of Teleparallel Gravity

A gauge formulation of gravity is possible in the Teleparallel Gravity Theory.

We present:

1. Teleparallel Gravity as translation gauge theory
2. geodesics and autoparallel curves
3. Metric Teleparallel Gravity and TEGR
4. Symmetric Teleparallel Gravity and STEGR

Teleparallel Gravity as translation gauge theory

In a modern vision of physics, it is very important to settle theories in a gauge framework. We have seen that also GR can be converted in a gauge theory. Let us now sketch how GR can be formulated as a *gauge theory of translations*.

This picture of GR can be achieved by both invoking the Nöther theorem and recalling that the source of the gravitational field is given by the energy and momentum. Since the gravitational Lagrangian is invariant under spacetime translations, the energy-momentum current is covariantly conserved.

We will see that a metric teleparallel theory is more suitable to express gravity in this context, because it entails more benefits, and the introduction of tetrads reveals to be more natural.

This approach was first proposed by Lasenby, Doran, and Gull in 1998. Its geometric setting is the tangent bundle, where the gauge transformations take place. We introduce $\{x^\mu\}$$ and $\{x^A\}$$ as the coordinates on M and T_pM, respectively. Now, let us consider

$$x^A \longrightarrow \bar{x}^A = x^A + \varepsilon^A(x^\mu)$$
infinitesimal local translation

$\Rightarrow$ O(1,3)
translation Lie group

$\Rightarrow$ $P_A := \partial_A$
generators

$\Rightarrow$ *ABELIAN TRANLSATION ALGEBRA*
$$[P_A, P_B] \equiv [\partial_A, \partial_B] = 0$$

$$\delta\bar{x}^A = \varepsilon(x^\mu)^B \partial_B x^A = \varepsilon(x^\mu)^A$$
infinitesimal transformation, written in terms of the generators

Teleparallel Gravity as translation gauge theory

$$\Psi = \Psi(\bar{x}^A(x^\mu)) \xrightarrow{x^A \longrightarrow \bar{x}^A = x^A + \varepsilon^A(x^\mu)} \delta_\varepsilon \Psi = \epsilon^A(x^\mu)\partial_A \Psi$$

general source field

If ε^A=*const* be a global translation, then $\partial_\mu\psi$ transforms covariantly

$$\longrightarrow \partial_\varepsilon(\partial_\mu \Psi) = \varepsilon^A \partial_A(\partial_\mu \Psi)$$

For a local translational transformation $\varepsilon^A(x^\mu)$, $\partial_\mu\psi$ does not transform covariantly

$$\longrightarrow \partial_\varepsilon(\partial_\mu \Psi) = \underbrace{\varepsilon^A(x^\mu)\partial_A(\partial_\mu \Psi)}_{\text{correct}} + \underbrace{(\partial_\mu \varepsilon^A(x^\mu))\partial_A \Psi}_{\text{spurious}}$$

To save the gauge covariance of the gravity theory, we set forth the *translational gauge potential 1-form* B_μ, assuming values in the Lie algebra of the translation group.

For trivial tetrads we have, we introduce

$$e'_\mu \Psi \equiv \partial_\mu \Psi = \partial_\mu + B^A{}_\mu \partial_A \Psi \xrightarrow{\text{transforms}} \delta_\varepsilon B^A{}_\mu = -\partial_\mu \varepsilon^A(x^\mu) \longrightarrow \partial_\varepsilon(e'_\mu \Psi) = \underbrace{\varepsilon^A(x^\mu)\partial_A(\partial_\mu \Psi)}_{\text{correct}}$$

gauge covariant derivative

For a general non-trivial tetrad field, we introduce

$$e_\mu \Psi = \partial_\mu + \overset{\bullet}{\omega}{}^A{}_{B\mu} x^B \partial_A \Psi + B^A{}_\mu \partial_A \Psi \xrightarrow{\text{transforms}} \delta_\varepsilon B^A{}_\mu = -\overset{\bullet}{\mathcal{D}}_\mu \varepsilon^A(x^\mu) \longrightarrow \partial_\varepsilon(e_\mu \Psi) = \underbrace{\varepsilon^A(x^\mu)\partial_A(\partial_\mu \Psi)}_{\text{correct}}$$

In general, we have $\boxed{e^A{}_\mu = \partial_\mu x^A + \overset{\bullet}{\omega}{}^A{}_{B\mu} x^B + B^A{}_\mu = \overset{\bullet}{\mathcal{D}}_\mu x^A + B^A{}_\mu}$

Teleparallel Gravity as translation gauge theory

In the context of teleparallel gravity, we have applied

$$e'^A{}_\mu \longrightarrow e^A{}_\mu \xrightarrow{\text{naturally emerges}} \eta_{\mu\nu} \longrightarrow g_{\mu\nu}$$

translation coupling prescription inTG *gravitational coupling prescription in GR*

The local Lorentz invariance is a fundamental symmetry respected by all physical laws in Nature, therefore, we must impose that our new theory be *locally Lorentz invariant*. Such a requirement entails the additional *Lorentz gravitational coupling prescription*, direct consequence of the strong Equivalence Principle.

This prescription is based on the General Covariance Principle, which can be seen as an active version of the strong Equivalence Principle. Given an equation valid in presence of gravitation, the corresponding special relativistic equation is locally recovered (at a point or along a trajectory). In formulae we have

$$\underbrace{\partial_\mu \Psi}_{\text{general field}} \to \mathcal{D}'_\mu \Psi = \partial_\mu \Psi + \frac{1}{2} e'^A{}_\mu \left(f'_B{}^C{}_A + f'_A{}^C{}_B - f'^C{}_{BA} \right) \underbrace{S^B{}_C}_{\text{generators of the Lorentz group in the same representation to which } \psi \text{ belongs}} \Psi$$

In presence of gravitation, we obtain $\partial_\mu \Psi \to \mathcal{D}_\mu \Psi = \partial_\mu \Psi + \frac{1}{2} e^A{}_\mu \left(f_B{}^C{}_A + f_A{}^C{}_B - f^C{}_{BA} \right) S^B{}_C \Psi$

full (Lorentz plus translational) gravitational coupling prescription in teleparallel gravity

We have therefore the following scheme

$$\boxed{\underbrace{\begin{Bmatrix} e'^A{}_\mu \longrightarrow e^A{}_\mu \\ \partial_\mu \longrightarrow \mathcal{D}_\mu \end{Bmatrix}}_{\text{grav. coupling prescription in TG}} \Leftrightarrow \underbrace{\eta_{\mu\nu} \longrightarrow g_{\mu\nu}}_{\text{grav. coupling prescription in GR}}}$$

Geodesics and autoparallel curves

Let us consider the equation of motion of a free test particle first described in the inertial frames e' $^{A}{}_{\mu}$

anholonomic four–velocity of the test particle

$$\frac{\mathrm{d}u'^A}{\mathrm{d}\sigma} = 0 \xrightarrow{\text{local Lorentz transformation}} \frac{\mathrm{d}u'^A}{\mathrm{d}\sigma} = \underbrace{\Lambda^A{}_B(x)\frac{\mathrm{d}u^B}{\mathrm{d}\sigma}}_{\text{correct}} + \underbrace{\frac{\mathrm{d}\Lambda^A{}_B(x)}{\mathrm{d}\sigma}u^B}_{\text{spurious}}$$

$$\mathrm{d}\sigma^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

we consider the anhonolomic frame e$^{A}{}_{\mu}$, associated to e' $^{A}{}_{\mu}$ through local Lorentz transformation

$$\frac{\mathrm{d}u'^A}{\mathrm{d}\sigma} = 0 \longrightarrow \frac{\mathrm{d}u^B}{\mathrm{d}\sigma} + \overset{\bullet}{\omega}{}^A{}_{B\mu}u^B u^\mu = 0$$

This restores the covarinace

Geodesics and autoparallel curves

The notion of geodesic equation must be revised in the parallel framework. Let us consider a chart (U, φ) on the manifold M and let $\gamma^\mu(\tau)$ be the parametric equation of a curve γ contained in U, where τ is the affine parameter along γ.

$$\dot{\gamma}(\tau) := \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau}\partial_\mu \xrightarrow{Y^\mu(\tau) \text{ is defined to be } \textit{parallel transported along } \gamma} \frac{\mathrm{d}Y^\mu}{\mathrm{d}\tau} := \nabla_\gamma Y^\mu \equiv \frac{\mathrm{d}Y^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\alpha\beta} Y^\alpha \frac{\mathrm{d}\gamma^\beta}{\mathrm{d}\tau} = 0$$

tangent vector to γ in the natural basis $\{\partial_\mu\}$ along γ

$Y^\mu(\gamma(\tau))$ depends on the curve γ

A curve $\gamma(\tau)$ is said to be *autoparallel* if its tangent vector satisfies

$$\nabla_\gamma \dot{\gamma} \equiv \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

it remains parallel to itself along $\gamma(\tau)$

In GR autoparallels and geodesic equations coincide, whereas in TG they are two distinct structures. The autoparallels are related to the affine connection, whereas the geodesic to the metric (measuring the minimal lengths between two or more points.

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \overset{\circ}{\Gamma}{}^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -K^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau},$$

Lorentz force-like interaction

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \overset{\circ}{\Gamma}{}^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -L^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}.$$

kinetic energy-like interaction

They are sensitive to parameter changes. If $\gamma(\tau)$ autoparallel then $\mu(\tau) = \gamma(\lambda(\tau))$ gives

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2 \frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau}$$

linear transformations preserve the autoparallelism

Metric Teleparallel Gravity and TEGR

Metric teleparallel gravity is obtained by assuming the metric compatibility. It is geometrically described only by the torsion tensor. Tetrads and spin connection play a fundamental role in describing gravity. Indeed, GR can be recast as a translational gauge theory, where the related gravitational field strength arises from the commutation relation of the covariant derivatives

$$[e_\mu, e_\nu] = \underbrace{\hat{T}{}^A{}_{\nu\mu}}_{\text{torsion field strength}} \partial_A$$

$$\hat{T}{}^A{}_{\mu\nu} = \partial_\nu B^A{}_\mu - \partial_\mu B^A{}_\nu + \overset{\bullet}{\omega}{}^A{}_{B\nu} B^B{}_\mu - \overset{\bullet}{\omega}{}^A{}_{B\mu} B^B{}_\nu$$

adding $\overset{\bullet}{\mathcal{D}}_\mu(\overset{\bullet}{\mathcal{D}}_\nu x^A) - \overset{\bullet}{\mathcal{D}}_\nu(\overset{\bullet}{\mathcal{D}}_\mu x^A) \equiv 0$

$$\hat{T}{}^A{}_{\mu\nu} = \partial_\nu e^A{}_\mu - \partial_\mu e^A{}_\nu + \overset{\bullet}{\omega}{}^A{}_{B\nu} e^B{}_\mu - \overset{\bullet}{\omega}{}^A{}_{B\mu} e^B{}_\nu$$

$$\boxed{\hat{T}{}^\lambda{}_{\mu\nu} = e_A{}^\lambda \hat{T}{}^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}}$$

The spin connection is linked to the inertial effects present in the tetrad frame, it is covariant under both diffeomorphisms and local Lorentz transformations, assuring the same properties also for the torsion tensor.

It is important to *associate at each tetrad the related spin connection*, therefore we have always to provide the couple $\{e^A{}_\mu, \overset{\bullet}{\omega}{}^A{}_{B\mu}\}$

$\{e^A{}_\mu, 0\}$ ⟶ *Weitzenböck gauge* ⟶ *Weitzenböck connection*

proper frames

$$\overset{\wedge}{\Gamma}{}^\lambda{}_{\nu\mu} = e_A{}^\lambda \partial_\mu e^A{}_\nu$$

distant parallelism condition from where TG takes its name

Metric Teleparallel Gravity and TEGR

A natural question spontaneously arises: *given a tetrad frame, how do we operatively associate the related spin connection?*

The simplest solution is tho choose proper frames, but, a priori, we do not know which are the related tetrads. Therefore, we have to find a strategy to answer this question. Determining them from the field equations is, in general, not a simple task.

$$e^A_{(\mathrm{r})\mu} \xrightarrow{\text{gravity is switched off}} e^A_{(\mathrm{r})\mu} := \lim_{G\to 0} e^A{}_\mu$$

reference *tetrad*

We are basically exploiting the Equivalence Principle or the inverse translational coupling prescription

This has the effect to consider a trivial tetrad, where the anhonomaly coefficients are zero and therefore the torsion tensor vanishes.

$$\hat{T}^A{}_{BC}(e^A{}_\mu, \overset{\bullet}{\omega}{}^A{}_{B\mu}) = \overset{\bullet}{\omega}{}^A{}_{BC} - \overset{\bullet}{\omega}{}^A{}_{BC} - f^A{}_{BC}(e_{(\mathrm{r})}) = 0,$$

Since they differ only by the gravitational content, they represent the gravitational effects inside the tetrad frame.

$$\overset{\bullet}{\omega}{}^A{}_{BC} = \frac{1}{2} e^C_{(\mathrm{r})\mu} \left[f_B{}^A{}_C(e_{(\mathrm{r})}) + f_C{}^A{}_B(e_{(\mathrm{r})}) - f^A{}_{BC}(e_{(\mathrm{r})}) \right]$$

This approach can be schematized as follows

$$\begin{tikzcd}
\overbrace{e^A{}_\mu}^{\text{general tetrad}} \arrow[r, "\text{gravity off}"] & \overbrace{e^A_{(\mathrm{r})\mu}}^{\text{reference tetrad}} \arrow[d, "\text{reduce to}"] \\
 & \underbrace{e'^A{}_\mu}_{\text{trivial tetrad}} \arrow[ul, "\overset{\bullet}{\omega}{}^A{}_{B\mu}"]
\end{tikzcd}$$

Metric Teleparallel Gravity and TEGR

The coefficients of anhonolonomy, in presence of torsion, read as

$$\hat{K}^{C}{}_{BA} = \frac{1}{2}\left(\hat{T}_{B}{}^{C}{}_{A} + \hat{T}_{A}{}^{C}{}_{B} - \hat{T}^{C}{}_{BA}\right)$$

contortion tensor

$$\overset{\bullet}{\omega}{}^{C}{}_{AB} - \overset{\bullet}{\omega}{}^{C}{}_{BA} = f^{C}{}_{AB} + T^{C}{}_{AB} \longrightarrow \frac{1}{2}\left(f_{B}{}^{C}{}_{A} + f_{A}{}^{C}{}_{B} - f^{C}{}_{BA}\right) = \overset{\bullet}{\omega}{}^{C}{}_{BA} - \hat{K}^{C}{}_{BA}$$

Using the fundamental identity of the theory of Lorentz connections, we obtain

Its derivation is also not trivial at all. It can be provide an intuitive proof, although a more rigorous demonstration can be found in Sec. II.6 of *Kobayashi (1996)*.

$$\boxed{\overset{\bullet}{\omega}{}^{C}{}_{B\mu} - \hat{K}^{C}{}_{B\mu} = \overset{\circ}{\omega}{}^{C}{}_{B\mu}}$$

It joins together GR and TG in a single compact expression. We remark that this «combined» coupling prescription has been obtained from the General Covariance Principle, and it is thus consistent with the strong Equivalence Principle.

Inertial effects in TG

Gravitaty in TG

Gravitational and inertial effects in GR

This is a new and elegant perspective to see the strong Equivalence Principle in TG.

in a local frame where the GR spin connection vanishes

$$\boxed{\overset{\bullet}{\omega}{}^{C}{}_{B\mu} = \hat{K}^{C}{}_{B\mu}}$$

inertial effects compensate gravitation, resembling the free-falling cabin' situation.

Metric Teleparallel Gravity and TEGR

Another fundamental ingredient of TG theory is represented by the *superpotential* $\hat{S}_A{}^{\mu\nu} := \hat{K}^{\mu\nu}{}_A - e_A{}^{\nu}\hat{T}^{\mu} + e_A{}^{\mu}\hat{T}^{\nu}$,

$\hat{T}^{\alpha\mu}{}_{\alpha} := \hat{T}^{\mu}$

torsion vector

torsion scalar

$$\hat{T} := \frac{1}{2}\hat{S}_A{}^{\mu\nu}\hat{T}^A{}_{\mu\nu} = \underbrace{\frac{1}{4}\hat{T}^{\rho}{}_{\mu\nu}\hat{T}_{\rho}{}^{\mu\nu}} + \underbrace{\frac{1}{2}\hat{T}^{\rho}{}_{\mu\nu}\hat{T}^{\nu\mu}{}_{\rho} - \hat{T}_{\mu}\hat{T}^{\mu}}$$

resembles that of the usual Lagrangian of internal gauge theories

stem out from the tetrad soldered character allowing thus to set at the same level internal and external indices

$$\hat{R} = \overset{\circ}{R} + \hat{T} + \frac{2}{e}\partial_{\mu}\left(e\hat{T}^{\mu}\right) = 0 \longrightarrow \boxed{\overset{\circ}{R} = -\hat{T} - \underbrace{\frac{2}{e}\partial_{\mu}\left(e\hat{T}^{\mu}\right)}_{\text{boundary term}}}$$

TG LAGRANGIAN

$$S_{\mathrm{TEGR}} = -\frac{c^4}{16\pi G}\int \mathrm{d}^4x\, e\underbrace{\mathcal{L}_{\mathrm{TEGR}}}_{-\hat{T}} + \int \mathrm{d}^4x e\,\mathcal{L}_m \xrightarrow{\text{dynamically equivalent of GR}} \boxed{S_{\mathrm{TEGR}} = S_{\mathrm{GR}}}$$

TEGR FIELD EQUATIONS

$$\hat{G}_{\mu\nu} := \frac{1}{e}e^A{}_{\mu}g_{\nu\rho}\partial_{\sigma}(e\hat{S}_A{}^{\rho\sigma}) - \hat{S}_B{}^{\sigma}{}_{\nu}\hat{T}^B{}_{\sigma\mu} + \frac{1}{2}\hat{T}g_{\mu\nu} - e^A{}_{\mu}\omega^B{}_{A\sigma}\hat{S}_{B\nu}{}^{\sigma} = \frac{8\pi G}{c^4}\mathfrak{T}_{\mu\nu}$$

An important issue is related to the matter couplings, because the presence of torsion introduces some difficulties when dealing with fermions and bosons.

Metric Teleparallel Gravity and TEGR

It is possible to obtain new theories by considering the following general definition of torsion scalar

c_1, c_2, c_3 are some free real constants

$$\hat{T}_{\text{gen}} := -\frac{c_1}{4}\hat{T}_{\alpha\mu\nu}\hat{T}^{\alpha\mu\nu} - \frac{c_2}{2}\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu} + c_3\hat{T}_{\alpha}\hat{T}^{\alpha}$$

three-parameter Hayashi-Shirafuji theory

The general torsion is invariant under both general coordinates and local Lorentz transformations, independently of the numerical values of the coefficients, because it relies only on the properties of the torsion tensor. On the contrary, the equivalence with GR, and then TEGR, is achieved only for $c_1=c_2=c_3=1$, which is naturally obtained within the TG gauge paradigm, without resorting to hypotheses related to GR. This crucial aspect makes TG a self-consistent theory.

$$\partial_\mu \mathfrak{T}^{\mu\nu} = 0 \longrightarrow Q^\mu := \int e\mathrm{d}^3x \mathfrak{T}^{0\mu}$$

conservation

spacetime charges are conserved

The antisymmetric part of the energy-momentum tensor is vanishing

$$\mathfrak{T}_{[\mu\nu]} = e^{A}{}_{[\mu}g_{\nu]\rho}\hat{T}{}_{A}{}^{\rho} = 0$$

Another way to see this identity is through the invariance of the action under local Lorentz transformations. In TEGR, the covariance eliminates six of the sixteen equations, which means that we are able to determine the tetrads up to a local Lorentz transformation, which is equivalent to determine the metric tensor.

Metric Teleparallel Gravity and TEGR

dynamically equivalent to the Hilbert-Einsten action

Let us consider the following TG Lagrangians $\mathcal{L}_{\mathrm{TEGR}}(e^A{}_\mu, 0), \quad \mathcal{L}_{\mathrm{TEGR}}(e^A{}_\mu, \overset{\bullet}{\omega}{}^A{}_{B\mu})$

$$\mathcal{L}_{\mathrm{TEGR}}(e^A{}_\mu, \overset{\bullet}{\omega}{}^A{}_{B\mu}) + \partial_\mu\left[\frac{ec^4}{8\pi G}\hat{T}{}^\mu(e^A{}_\mu, \overset{\bullet}{\omega}{}^A{}_{B\mu})\right] = \mathcal{L}_{\mathrm{TEGR}}(e^A{}_\mu, 0) + \partial_\mu\left[\frac{ec^4}{8\pi G}\hat{T}{}^\mu(e^A{}_\mu, 0)\right] \longrightarrow \hat{T}{}^\mu(e^A{}_\mu, \overset{\bullet}{\omega}{}^A{}_{B\mu}) = \hat{T}{}^\mu(e^A{}_\mu, 0) - \overset{\bullet}{\omega}{}^\mu$$

$$\mathcal{L}_{\mathrm{TEGR}}(e^A{}_\mu, \overset{\bullet}{\omega}{}^A{}_{B\mu}) = \mathcal{L}_{\mathrm{TEGR}}(e^A{}_\mu, 0) + \partial_\mu\left[\frac{ec^4}{8\pi G}\overset{\bullet}{\omega}{}^\mu\right]$$

This proves that the spin connection enters in the Lagrangian as a total derivative, justifying also the possibility to reduce the calculations in TEGR by adopting the Weitzenböck gauge. If we vary the Lagrangian with respect to the spin connection, we obtain an identically vanishing equation. Therefore, the spin connection does not contribute to the TG field equations, representing non-dynamical DoFs. This fact shows also that TEGR can be considered as a *pure tetrad teleparallel gravity*, which assumes that for whatever tetrad one chooses, the spin connection is zero, treating these two objects as independent structures.

The spin connection is not relevant, if we are interested in searching for the solutions of TEGR field equations. However, formally, its presence fulfills a paramount role, because: (i) it guarantees the covariance of the action under local Lorentz transformations and diffeomorphisms; (ii) it is endowed with a regularizing power, because it removes the divergent inertial effects from the Lagrangian (*renormalized action*); (iii) it permits to obtain a regular field theory and naturally produces, in its action, a Gibbons-Hawking-York term, which permits to be coherently related to the formulation of a quantum gravity theory.

Analysing the DoFs of TEGR, we start from the vierbeine with 16 components. We have to subtract 6 DoFs related to the inertial effects due to the spin connection and other 8 non-dynamical DoFs due to diffeomorphisms (the same as in GR). The result is 2 DoFs as in the case of GR.

Symmetric Teleparallel Gravity and STEGR

Symmetric teleparallel gravity (STG) is a formulation of gravitational interaction described only in terms of non-metricity. This theory can be either formulated in terms of metric tensor or tetrads. The symmetric affine connection assumes a fundamental dynamical role and represents an independent structure (*Palatini formalism*).

The presence of non-metricity entails particular geometric effects, which gives rise to counterintuitive implications:

- raising up or lowering down indices of vectors or tensors under the covariant derivative $g_{\nu\lambda}\overset{\diamond}{\nabla}_\mu v^\lambda = \overset{\diamond}{\nabla}_\mu v_\nu - v^\lambda \overset{\diamond}{Q}_{\mu\nu\lambda};$
- non-metricity does not preserve the length of vectors $v = v^\mu \partial_\mu, \qquad w = w^\mu \partial_\mu$ parallel along a curve γ

$$T = \underbrace{T^\mu}_{\dot{\gamma}^\mu}\partial_\mu \quad \left.\begin{aligned} T^\lambda \overset{\diamond}{\nabla}_\lambda v^\mu &= 0 \\ T^\lambda \overset{\diamond}{\nabla}_\lambda w^\mu &= 0 \end{aligned}\right\} \longrightarrow T^\lambda \overset{\diamond}{\nabla}_\lambda v \cdot w = T^\lambda v^\mu w^\nu \overset{\diamond}{Q}_{\lambda\mu\nu} \longrightarrow T^\lambda \overset{\diamond}{\nabla}_\lambda \left(\frac{v\cdot w}{|v||w|}\right) \neq 0$$

- angles, between two vectors, do not in general conserve
- STG is, in general, not a *conformal theory*, but it can be made
- impossibility to define a proper time along a curve as in GR

- four-velocity is not anymore orthogonal to the four-acceleration

acceleration $a^\mu := u^\lambda \overset{\diamond}{\nabla}_\lambda u^\mu$

anomalous acceleration $\tilde{a}_\mu := u^\lambda \overset{\diamond}{\nabla}_\lambda u_\mu = a_\mu + \overset{\diamond}{Q}_{\lambda\nu\mu} u^\lambda u^\nu$

Symmetric Teleparallel Gravity and STEGR

$$a^\mu := u^\lambda \mathring{\nabla}_\lambda u^\mu \longrightarrow u_\mu a^\mu = u_\mu u^\lambda \mathring{\nabla}_\lambda u^\mu = u^\lambda \mathring{\nabla}_\lambda (u_\mu u^\mu) - u^\mu u^\lambda \mathring{\nabla}_\lambda u_\mu = \mathring{Q}_{\lambda\mu\nu} u^\lambda u^\mu u^\nu + 2u_\mu a^\mu - \tilde{a}_\mu u^\mu$$

$$\downarrow$$

$$a^\mu u_\mu = \tilde{a}_\mu u^\mu - \mathring{Q}_{\lambda\mu\nu} u^\lambda u^\mu u^\nu$$

$$\tilde{a}_\mu := u^\lambda \mathring{\nabla}_\lambda u_\mu = a_\mu + \mathring{Q}_{\lambda\nu\mu} u^\lambda u^\nu \longrightarrow (\tilde{a}_\mu - a_\mu)\, u^\mu = \mathring{Q}_{\lambda\mu\nu} u^\lambda u^\mu u^\nu$$

The non-metricity tensor expresses how much the anomalous acceleration deviates from the standard acceleration, and it is also responsible to depart the acceleration from the spatial hypersurface orthogonal to the four-velocity;

• the acceleration of autoparallels in STG becomes $a^\mu = 0, \qquad \tilde{a}_\mu = \mathring{Q}_{\lambda\nu\mu} u^\lambda u^\nu;$

• to recover the length conservation and the autoparallel definition, we have to impose $\mathring{Q}_{(\lambda\mu\nu)} = 0, \qquad \mathring{Q}_{(\lambda\mu)\nu} = 0$

These two conditions are too strict constraints. These issues can be solved by resorting to the *Weyl conformal transformations*.

Symmetric Teleparallel Gravity and STEGR

Let us consider now the STG action $$S_{\mathrm{STEGR}} := \int \mathrm{d}^4x\, \sqrt{-g}\left[\frac{c^4}{16\pi G}\underbrace{\mathcal{L}_{\mathrm{STEGR}}}_{\overset{\diamond}{Q}} + \mathcal{L}_m\right]$$

non-metricity scalar

$$\overset{\diamond}{Q} := g^{\mu\nu}\left(\overset{\diamond}{L}{}^{\alpha}{}_{\beta\mu}\overset{\diamond}{L}{}^{\beta}{}_{\nu\alpha} - \overset{\diamond}{L}{}^{\alpha}{}_{\beta\alpha}\overset{\diamond}{L}{}^{\beta}{}_{\mu\nu}\right) = \frac{1}{4}\left(\overset{\diamond}{Q}_{\alpha}\overset{\diamond}{Q}{}^{\alpha} - \overset{\diamond}{Q}_{\alpha\beta\gamma}\overset{\diamond}{Q}{}^{\alpha\beta\gamma}\right) + \frac{1}{2}\left(\overset{\diamond}{Q}_{\alpha\beta\gamma}\overset{\diamond}{Q}{}^{\beta\alpha\gamma} - \overset{\diamond}{Q}_{\alpha}\overset{\diamond}{\bar{Q}}{}^{\alpha}\right)$$

$$\overset{\diamond}{Q}_{\alpha} := \overset{\diamond}{Q}_{\alpha\lambda}{}^{\lambda}$$

$$\overset{\diamond}{\bar{Q}}_{\alpha} := \overset{\diamond}{Q}{}^{\lambda}{}_{\lambda\alpha}$$

STEGR theory

$$\boxed{\overset{\diamond}{Q} = \overset{\circ}{R} + \overset{\circ}{\nabla}_{\mu}(\overset{\diamond}{Q}{}^{\mu} - \overset{\diamond}{\bar{Q}}{}^{\mu})} \xrightarrow{\overset{\circ}{\nabla}_{\mu}(\overset{\diamond}{Q}{}^{\mu} - \overset{\diamond}{\bar{Q}}{}^{\mu}) \equiv \frac{1}{\sqrt{-g}}\partial_{\mu}\left[\sqrt{-g}(\overset{\diamond}{Q}{}^{\mu} - \overset{\diamond}{\bar{Q}}{}^{\mu})\right]}$$

dynamically equivalent to GR
up to a boundary term

general

$$\overset{\diamond}{Q}_{\mathrm{gen}} := c_1\overset{\diamond}{Q}_{\alpha\beta\gamma}\overset{\diamond}{Q}{}^{\alpha\beta\gamma} + c_2\overset{\diamond}{Q}_{\alpha\beta\gamma}\overset{\diamond}{Q}{}^{\beta\alpha\gamma} + c_3\overset{\diamond}{Q}_{\alpha}\overset{\diamond}{Q}{}^{\alpha} + c_4\overset{\diamond}{\bar{Q}}_{\alpha}\overset{\diamond}{\bar{Q}}{}^{\alpha} + c_5\overset{\diamond}{Q}_{\alpha}\overset{\diamond}{\bar{Q}}{}^{\alpha}.$$

c_1, c_2, c_3, c_4, c_5 are real free constant parameters

five-parameter family of quadratic theories or the so-called *New GR*

Symmetric Teleparallel Gravity and STEGR

We can introduce a *superpotential* or the *non-metricity conjugate* as

$$\overset{\diamond}{P}{}^{\alpha}{}_{\mu\nu} := \frac{1}{2\sqrt{-g}}\frac{\partial(\sqrt{-g}\overset{\diamond}{Q})}{\partial \overset{\diamond}{Q}{}_{\alpha}{}^{\mu\nu}} = \frac{1}{4}\overset{\diamond}{Q}{}^{\alpha}{}_{\mu\nu} - \frac{1}{4}\overset{\diamond}{Q}{}_{(\mu}{}^{\alpha}{}_{\nu)} - \frac{1}{4}g_{\mu\nu}\overset{\diamond}{Q}{}^{\alpha\beta}{}_{\beta} + \frac{1}{4}\left[\overset{\diamond}{Q}{}_{\beta}{}^{\beta\alpha}g_{\mu\nu} + \frac{1}{2}\delta^{\alpha}_{(\mu}\overset{\diamond}{Q}{}_{\nu)}{}^{\beta}{}_{\beta}\right]$$

We can introduce also the further quantity

$$\frac{1}{\sqrt{-g}}\overset{\diamond}{q}_{\mu\nu} := \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{-g}\overset{\diamond}{Q})}{\partial g^{\mu\nu}} + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = \frac{1}{4}\left(2\overset{\diamond}{Q}_{\alpha\beta\mu}\overset{\diamond}{Q}{}^{\alpha\beta}{}_{\nu} - \overset{\diamond}{Q}_{\mu\alpha\beta}\overset{\diamond}{Q}{}_{\nu}{}^{\alpha\beta}\right) - \frac{1}{4}\left(2\overset{\diamond}{Q}{}_{\alpha}{}^{\beta}{}_{\beta}\overset{\diamond}{Q}{}^{\alpha}{}_{\mu\nu} - \overset{\diamond}{Q}{}_{\mu}{}^{\beta}{}_{\beta}\overset{\diamond}{Q}{}_{\nu}{}^{\beta}{}_{\beta}\right) - \frac{1}{2}\left(\overset{\diamond}{Q}_{\alpha\beta\mu}\overset{\diamond}{Q}{}^{\beta\alpha}{}_{\nu} - \overset{\diamond}{Q}{}_{\beta}{}^{\beta}{}_{\alpha}\overset{\diamond}{Q}{}^{\alpha}{}_{\mu\nu}\right)$$

- Varying the STEGR action with respect to the metric tensor

$$\overset{\diamond}{G}_{\mu\nu} := \frac{2}{\sqrt{-g}}\nabla_{\alpha}\left(\sqrt{-g}\overset{\diamond}{P}{}^{\alpha}{}_{\mu\nu}\right) - \frac{1}{\sqrt{-g}}\overset{\diamond}{q}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = \frac{8\pi G}{c^4}\mathfrak{T}_{\mu\nu}$$

METRIC FIELD EQUATIONS

- Varying the STEGR action with respect to the affine connection

$$\nabla_{\mu}\nabla_{\nu}\left(\sqrt{-g}\overset{\diamond}{P}{}_{\alpha}{}^{\mu\nu}\right) = 0$$

AFFINE FIELD EQUATIONS

Symmetric Teleparallel Gravity and STEGR

It is possible to recast the STEGR connection via the tetrads and the curvatureless hypothesis in the following form

$$\Gamma^{\alpha}{}_{\mu\nu} := (e^{-1})^{\alpha}{}_{\beta}\partial_{\mu}e^{\beta}{}_{\nu} \xrightarrow{\text{torsionless}} T^{\alpha}{}_{\mu\nu} := (e^{-1})^{\alpha}{}_{\beta}\partial_{[\nu}e^{\beta}{}_{\mu]} = 0 \longrightarrow \partial_{\mu}e^{\beta}{}_{\nu} = \partial_{\nu}e^{\beta}{}_{\mu} \overset{\text{holonomic}}{\Leftrightarrow} e^{\alpha}{}_{\beta} \equiv e'^{\alpha}_{\beta} := \partial_{\beta}\xi^{\alpha}$$

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \xi^{\lambda}}\partial_{\mu}\partial_{\nu}\xi^{\lambda}$$

affine (gauge) roto-translational transformation of coordinates

$$\xi^{\alpha} := M^{\alpha}_{\beta}x^{\beta} + \xi^{\alpha}_{0}$$

coincident gauge

$$\boxed{\Gamma^{\alpha}{}_{\mu\nu} = 0}$$

Physically, this means that the origin of the tangent space is coincident with the spacetime origin. This gauge is defined up to a linear affine transformation.

The affine connection is an integrable translation, and the coincident gauge embodies the strong Equivalence Principle of GR.

It is worth noticing that the STEGR affine connection is purely inertial and it does not contain any information about gravitation. Another important implication of the coincident gauge is the explicit breaking of diffeomorphism invariance due to the particular choice of coordinates, which does not occur in other frames. The use or not of the coincident gauge affects only the boundary, which has no influence on the ensuing dynamics and therefore neither on the evolution of the metric tensor.

Symmetric Teleparallel Gravity and STEGR

This particular gauge form permits to considerably simplify the calculations. In addition, the affine field equations are trivially satisfied. In TG the local Lorentz transformations are gauged through the spin connection and the calculations are simplified via the Weitzenböck choice, whereas, in STG, the diffeomorphism of coordinates become the new gauge and the calculations are easily carried out through the coincident gauge.

$$
\begin{tikzcd}
 & \text{loc. Lorentz trans.} \arrow[r] & \text{Weitzenböck} \\
\text{gauge} \arrow[ur, "\text{TEGR}"] \arrow[dr, "\text{STEGR}"'] & & \\
 & \text{diff. of coord.} \arrow[r] & \text{coincident}
\end{tikzcd}
$$

There are also other two beneficial effects considering, which are

$$\overset{\diamond}{\nabla}_\mu = \partial_\mu, \qquad \overset{\diamond}{L}{}^{\lambda}{}_{\mu\nu} = -\overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\nu}$$

It is worth noticing that, in generic STG theories, it is not possible to require that the coincident gauge holds a priori. More specifically, it is not possible, in general, to use a coordinate system which simultaneously simplifies metric and connection. When this can be achieved, it holds for a restricted set of geometries or it reduces the class of solutions.

In STEGR we have that the total DoFs are enconded in the metric tensor, having 10 components from which we have to subtract 8 diffeomorphisms as in GR, having therefore again 2 DoFs as in GR. Here we have that the 4 diffeomorphisms of coordinates become the gauge diffeomorphism symmetries. While in TEGR metric and connection are related, in STEGR the connection becomes essentially a pure gauge and all the dynamics is enclosed in the metric, which is trivially connected.

Discussion on Trinity Gravity at Lagrangian level

GR, TEGR, and STEGR are nothing else but particular cases of wide classes of geometries. According to their formulation, they are three non-communicating theories, because they start from distinct hypotheses and different dynamical-geometric objects. GR is usually conceived as the geometric formulation of gravity, whereas TEGR and STEGR as the gauge approaches to gravity, albeit also GR can be formulated in a gauge way via the use of tetrads and spin connection.

Covariance and (strong) Equivalence Principles are at the foundations of GR. The former postulate has a more general character, which can be easily recognized also in TEGR and STEGR. The latter provides mathematical and physical coherence for TEGR and STEGR theories, emerging naturally and guaranteeing thus the equivalence among these theories.

$$\begin{tikzcd}
 & \text{foundation} \\
\text{strong EP} \arrow[ur, "\text{GR}"] \arrow[r, "\text{TEGR}"] \arrow[dr, "\text{STEGR}"'] & \overset{\bullet}{\omega}^{C}{}_{B\mu} - \hat{K}^{C}{}_{B\mu} = \overset{\circ}{\omega}^{C}{}_{B\mu} \\
 & \text{coincident gauge}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overbrace{\mathscr{L}_{\mathrm{GR}}}^{\overset{\circ}{R}} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\underbrace{\mathscr{L}_{\mathrm{TEGR}}}_{-\hat{T}-\frac{2}{e}\partial_\mu\left(e\hat{T}^{\mu}\right)} \arrow[rr, leftrightarrow] & & \underbrace{\mathscr{L}_{\mathrm{STEGR}}}_{\overset{\diamond}{Q}-\overset{\circ}{\nabla}_\mu(\overset{\diamond}{Q}^{\mu}-\overset{\diamond}{Q}^{\mu})}
\end{tikzcd}$$

The equivalence does not hold for extensions like f(R), f(T), and f(Q) for different DoFs. However, equivalence can be restored also in extensions considering appropriate boundary terms, like f(R), f(T,B), and f(Q,B).

7 LECTURE

Geometric Trinity: The field equations

Preamble

In the previous lecture, equivalent representations of gravity have been compared at the level of actions and Lagrangians. Here we want to develop the same comparison at the level of field equations.

Let us start from the Bianchi identities, having the pivotal role to link the field equations with the conservation laws of the gravity tensor invariants and with the energy-momentum tensor.
We start from the second Bianchi identity, whose more explicit expression is

$$\nabla_\lambda R^\alpha{}_{\beta\mu\nu} + \nabla_\mu R^\alpha{}_{\beta\nu\lambda} + \nabla_\nu R^\alpha{}_{\beta\lambda\mu} = T^\rho{}_{\mu\lambda} R^\alpha{}_{\beta\nu\rho} + T^\rho{}_{\nu\lambda} R^\alpha{}_{\beta\mu\rho} + T^\rho{}_{\nu\mu} R^\alpha{}_{\beta\lambda\rho}$$

We prove the equivalence in terms of the field equations among
1. GR
2. TEGR
3. STEGR

GR field equations

second Bianchi identity

By covariance Principle, we can exploit the LIF's coordinates

$$\overset{\circ}{\nabla}_\lambda \overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu} + \overset{\circ}{\nabla}_\mu \overset{\circ}{R}{}^{\alpha}{}_{\beta\nu\lambda} + \overset{\circ}{\nabla}_\nu \overset{\circ}{R}{}^{\alpha}{}_{\beta\lambda\mu} = 0 \longrightarrow \partial_\lambda \overset{\circ}{R}{}^{\lambda}{}_{\beta\mu\nu} + \partial_\mu \overset{\circ}{R}{}^{\lambda}{}_{\beta\nu\lambda} + \partial_\nu \overset{\circ}{R}{}^{\lambda}{}_{\beta\lambda\mu} = 0$$

$$\overset{\circ}{R}_{\mu\nu\alpha\beta} = -\overset{\circ}{R}_{\mu\nu\beta\alpha}$$

contracting β and μ

$$\partial_\mu \overset{\circ}{R}{}^{\mu}{}_{\nu} - \frac{1}{2}\partial_\nu \overset{\circ}{R} = 0 \longleftarrow -\partial_\lambda \overset{\circ}{R}{}^{\lambda}{}_{\nu} - \partial_\beta \overset{\circ}{R}{}^{\beta}{}_{\nu} + \partial_\nu \overset{\circ}{R} = 0 \longleftarrow \partial_\lambda \overset{\circ}{R}{}^{\lambda}{}_{\beta\mu\nu} - \partial_\mu \overset{\circ}{R}{}^{\lambda}{}_{\beta\lambda\nu} + \partial_\nu \overset{\circ}{R}{}^{\lambda}{}_{\beta\lambda\mu} = 0$$

Einstein field equations in vacuum

$$\partial_\mu(\overset{\circ}{R}{}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\hat{R}) = 0 \;\Rightarrow\; \overset{\circ}{\nabla}_\mu(\overset{\circ}{R}{}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\hat{R}) = 0$$

$$\overset{\circ}{\nabla}_\mu \overset{\circ}{G}{}^{\mu\nu} = 0, \quad \Leftrightarrow \quad \overset{\circ}{\nabla}_\mu \mathfrak{T}^{\mu\nu} = 0$$

TEGR field equations

Since in TEGR curvature and non-metricity vanish, we can further simplify the calculations via the the Weitzenböck gauge

$$\overset{\wedge}{\nabla}_\lambda R^\alpha{}_{\beta\mu\nu} + \overset{\wedge}{\nabla}_\nu R^\alpha{}_{\beta\lambda\mu} + \overset{\wedge}{\nabla}_\mu R^\alpha{}_{\beta\nu\lambda} = 0$$

$$R^\alpha{}_{\beta\mu\nu} \equiv \overset{\circ}{R}{}^\alpha{}_{\beta\mu\nu} + \overset{\wedge}{\mathcal{K}}{}^\alpha{}_{\beta\mu\nu} = 0$$

$$\overset{\wedge}{\mathcal{K}}{}^\alpha{}_{\beta\mu\nu} := \overset{\circ}{\nabla}_\mu \hat{K}{}^\alpha{}_{\beta\nu} - \overset{\circ}{\nabla}_\nu \overset{\circ}{K}{}^\alpha{}_{\beta\mu} + \hat{K}{}^\alpha{}_{\sigma\mu} \hat{K}{}^\sigma{}_{\beta\nu} - \hat{K}{}^\alpha{}_{\sigma\nu} \hat{K}{}^\sigma{}_{\beta\mu}$$

$$\overset{\wedge}{\mathcal{K}}{}^\alpha{}_{\beta\mu\nu} = -\overset{\wedge}{\mathcal{K}}{}_\beta{}^\alpha{}_{\mu\nu}, \qquad \overset{\wedge}{\mathcal{K}}{}^\alpha{}_{\beta\mu\nu} = -\overset{\wedge}{\mathcal{K}}{}^\alpha{}_{\beta\nu\mu}$$

contracting α and λ

$$\overset{\wedge}{\nabla}_\lambda \hat{R}{}^\lambda{}_{\beta\mu\nu} + \overset{\wedge}{\nabla}_\mu \hat{R}{}^\lambda{}_{\beta\nu\lambda} + \overset{\wedge}{\nabla}_\nu \hat{R}{}^\lambda{}_{\beta\lambda\mu} + \overset{\wedge}{\nabla}_\lambda \overset{\wedge}{\mathcal{K}}{}^\lambda{}_{\beta\mu\nu} + \overset{\wedge}{\nabla}_\mu \overset{\wedge}{\mathcal{K}}{}^\lambda_{\beta\nu\lambda} + \overset{\wedge}{\nabla}_\nu \overset{\wedge}{\mathcal{K}}{}^\lambda_{\beta\lambda\mu} = 0$$

metric compatibility of TEGR

$$\overset{\wedge}{\mathcal{K}}_{\mu\nu} := \overset{\wedge}{\mathcal{K}}{}^\lambda{}_{\mu\lambda\nu}$$

$$\overset{\wedge}{\nabla}_\mu (\hat{R}{}^\mu{}_\nu + \overset{\wedge}{\mathcal{K}}{}^\mu{}_\nu) - \frac{1}{2} \overset{\wedge}{\nabla}_\nu (\hat{R} + \overset{\wedge}{\mathcal{K}}) = 0 \longrightarrow$$

$$\boxed{\begin{aligned} &\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} = -\overset{\wedge}{\mathcal{K}}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \overset{\wedge}{\mathcal{K}}, \\ &\overset{\wedge}{\mathcal{K}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\wedge}{\mathcal{K}} = 0, \end{aligned}}$$

TEGR field equations

We can prove that TEGR field equations are equivalent to GR ones.

$$\hat{\mathcal{K}}_{\mu\nu} = \hat{\nabla}_\alpha \hat{K}^{\alpha}{}_{\mu\nu} - \hat{\nabla}_\nu \hat{K}^{\alpha}{}_{\mu\alpha} + \hat{K}^{\sigma}{}_{\mu\nu}\hat{K}^{\alpha}{}_{\sigma\alpha} - \hat{K}^{\sigma}{}_{\mu\alpha}\hat{K}^{\alpha}{}_{\sigma\nu} = \hat{\nabla}_\alpha \hat{K}^{\alpha}{}_{\mu\nu} + \hat{\nabla}_\nu \hat{T}_\mu - \hat{K}_{\sigma\mu\nu}\hat{T}^{\sigma} - \hat{K}^{\sigma}{}_{\mu\alpha}\hat{K}^{\alpha}{}_{\sigma\nu} = \hat{\nabla}_\alpha \hat{S}_\nu{}^{\alpha}{}_{\mu} + \hat{\nabla}_\alpha \hat{T}^{\alpha} g_{\mu\nu} - \hat{K}^{\alpha}{}_{\sigma\nu}\hat{S}_\alpha{}^{\sigma}{}_{\mu}$$

$$\hat{K}^{\alpha}{}_{\mu\alpha} = -\hat{T}_\mu,$$
$$\hat{K}^{\alpha}{}_{\alpha\mu} = 0,$$
$$\hat{K}^{\mu}{}_{\nu\lambda} = \hat{S}_\lambda{}^{\mu\nu} + \delta^{\nu}_{\lambda}\hat{T}^{\mu} - \delta^{\mu}_{\lambda}\hat{T}^{\nu}.$$

$$\hat{\mathcal{K}} = 2\hat{\nabla}_\lambda \hat{T}^{\lambda} + \hat{T} = \frac{2}{e}\partial_\lambda(e\hat{T}^{\lambda}) + \hat{T}$$

$$\hat{\mathcal{K}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{\mathcal{K}} = 0 \longrightarrow \boxed{\hat{\nabla}_\alpha \hat{S}_{\nu\mu}{}^{\alpha} + \hat{K}^{\alpha}{}_{\sigma\nu}\hat{S}_\alpha{}^{\sigma}{}_{\mu} + \frac{1}{2}g_{\mu\nu}\hat{T} = 0}$$

$$-\hat{K}^{\alpha}{}_{\nu\sigma}\hat{S}_{\mu\alpha}{}^{\sigma} - \hat{K}_{\nu\rho\sigma}\hat{S}_\mu{}^{\rho\sigma} = 0,$$
$$\hat{K}^{\sigma}{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^{\alpha} + \hat{T}_\sigma \hat{S}_{\mu\nu}{}^{\sigma} = 0.$$

$$\hat{\nabla}_\sigma \hat{S}_{\mu\nu}{}^{\sigma} - \hat{K}^{\alpha}{}_{\mu\sigma}\hat{S}_{\alpha\nu}{}^{\sigma} - \hat{K}^{\alpha}{}_{\nu\sigma}\hat{S}_{\mu\alpha}{}^{\sigma} + \hat{K}^{\sigma}{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^{\alpha} - \hat{K}_{\nu\rho\sigma}\hat{S}_\mu{}^{\rho\sigma} + \hat{T}_\sigma \hat{S}_{\mu\nu}{}^{\sigma} - \hat{S}_\alpha{}^{\sigma}{}_{\nu}\hat{T}^{\alpha}{}_{\sigma\mu} + \frac{1}{2}g_{\mu\nu}\hat{T} = 0$$

metric compatibility & tetrad postulate

$$e^{-1}\partial_\sigma e \equiv \partial_\sigma \ln(\sqrt{-g}) = \hat{\Gamma}^{\alpha}{}_{\alpha\sigma}$$

splitting GR and contrion tensor contributions

$$\boxed{e^{A}{}_{\mu}g_{\nu\rho}\partial_\sigma \hat{S}_A{}^{\rho\sigma} + e^{-1}e^{A}{}_{\mu}g_{\nu\rho}\partial_\sigma e - \hat{S}_B{}^{\sigma}{}_{\nu}T^{B}{}_{\sigma\mu} + \frac{1}{2}g_{\mu\nu}\hat{T} = 0} \longrightarrow \partial_\sigma \hat{S}_{\mu\nu}{}^{\sigma} - \hat{S}_{\alpha\nu}{}^{\sigma}\Gamma^{\alpha}{}_{\nu\sigma} - \hat{S}_{\alpha\mu}{}^{\sigma}\Gamma^{\alpha}{}_{\nu\sigma} - \hat{S}_\mu{}^{\rho\sigma}\Gamma^{\alpha}{}_{\rho\sigma}g_{\nu\alpha} - \hat{S}_\alpha{}^{\sigma}{}_{\nu}\hat{T}^{\alpha}{}_{\sigma\mu} + \Gamma^{\alpha}{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^{\sigma} + \frac{1}{2}\hat{T}g_{\mu\nu} = 0$$

STEGR field equations

coincident gauge

$$\partial_\lambda R^\alpha{}_{\beta\mu\nu} + \partial_\nu R^\alpha{}_{\beta\lambda\mu} + \partial_\mu R^\alpha{}_{\beta\nu\lambda} = 0$$

$$R^\alpha{}_{\beta\mu\nu} \equiv \overset{\circ}{R}\alpha{}_{\beta\mu\nu} + \overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\beta\mu\nu} = 0$$

$$\overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\beta\mu\nu} = \overset{\diamond}{\nabla}_\mu \overset{\diamond}{L}{}^\alpha{}_{\beta\nu} - \overset{\diamond}{\nabla}_\nu \overset{\diamond}{L}{}^\alpha{}_{\beta\mu} + \overset{\diamond}{L}{}^\alpha{}_{\sigma\mu} \overset{\diamond}{L}{}^\sigma{}_{\beta\nu} - \overset{\diamond}{L}{}^\alpha{}_{\sigma\nu} \overset{\diamond}{L}{}^\sigma{}_{\beta\mu}$$

$$\overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\beta\mu\nu} = -\overset{\diamond}{\mathcal{L}}{}^\beta{}_{\alpha\mu\nu}, \qquad \overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\beta\mu\nu} = -\overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\beta\nu\mu}$$

contracting α and λ

$$\partial_\lambda \overset{\diamond}{R}{}^\lambda{}_{\beta\mu\nu} + \partial_\mu \overset{\diamond}{R}{}^\lambda{}_{\beta\nu\lambda} + \partial_\nu \overset{\diamond}{R}{}^\lambda{}_{\beta\lambda\mu} + \partial_\lambda \overset{\diamond}{\mathcal{L}}{}^\lambda{}_{\beta\mu\nu} + \partial_\mu \overset{\diamond}{\mathcal{L}}{}^\lambda{}_{\beta\nu\lambda} + \partial_\nu \overset{\diamond}{\mathcal{L}}{}^\lambda{}_{\beta\lambda\mu} = 0$$

same strategy of GR

$$\overset{\diamond}{\mathcal{L}} := \overset{\diamond}{\mathcal{L}}{}^\mu{}_\mu$$

$$\overset{\diamond}{\mathcal{L}}_{\mu\nu} := \overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\mu\alpha\nu}$$

resembling formally the expression of Ricci tensor and scalar curvature

$$\boxed{\begin{aligned} &\overset{\diamond}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\diamond}{R} = -\overset{\diamond}{\mathcal{L}}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \overset{\diamond}{\mathcal{L}}, \\ &\overset{\diamond}{\mathcal{L}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\diamond}{\mathcal{L}} = 0, \end{aligned}}$$

$$\overset{\diamond}{L}{}^\alpha{}_{\mu\nu} = -\overset{\diamond}{\Gamma}{}^\alpha{}_{\mu\nu}$$

it is easy to see that it is equivalent to GR

STEGR field equations

We prove that STEGR field equations are equivalent to GR ones.

$$\overset{\diamond}{\mathcal{L}}_{\mu\nu} = \overset{\diamond}{\nabla}_\alpha \overset{\diamond}{L}^\alpha{}_{\mu\nu} - \overset{\diamond}{\nabla}_\nu \overset{\diamond}{L}^\alpha{}_{\mu\alpha} + \overset{\diamond}{L}^\sigma{}_{\mu\nu} \overset{\diamond}{L}^\alpha{}_{\sigma\alpha} - \overset{\diamond}{L}^\sigma{}_{\mu\alpha} \overset{\diamond}{L}^\alpha{}_{\sigma\nu} = \overset{\diamond}{\nabla}_\alpha \overset{\diamond}{L}^\alpha{}_{\mu\nu} + \frac{1}{2}\overset{\diamond}{\nabla}_\nu \overset{\diamond}{Q}_\mu - \frac{1}{2}\overset{\diamond}{Q}_\alpha \overset{\diamond}{L}^\alpha{}_{\mu\nu} - \frac{1}{4}\left[\overset{\diamond}{Q}_\mu{}^\sigma{}_\alpha \overset{\diamond}{Q}_\nu{}^\alpha{}_\sigma + 2\overset{\diamond}{Q}^\alpha{}_{\sigma\nu}(\overset{\diamond}{Q}^\sigma{}_{\alpha\mu} - \overset{\diamond}{Q}_\alpha{}^\sigma{}_\mu)\right]$$

$$\overset{\diamond}{L}^\alpha{}_{\mu\alpha} = -\frac{1}{2}\overset{\diamond}{Q}_\mu,$$

$$\overset{\diamond}{L}^\alpha{}_{\mu\nu} = 2\overset{\diamond}{P}^\alpha{}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\bar{Q}}^\alpha) - \frac{1}{4}(\delta^\alpha_\mu \overset{\diamond}{Q}_\nu + \delta^\alpha_\nu \overset{\diamond}{Q}_\mu).$$

$$\overset{\diamond}{\mathcal{L}} = \overset{\diamond}{\nabla}_\alpha(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\bar{Q}}^\alpha) + \frac{1}{4}\overset{\diamond}{Q}_{\alpha\beta\gamma}\overset{\diamond}{Q}^{\alpha\beta\gamma} - \frac{1}{2}\overset{\diamond}{Q}_{\alpha\beta\gamma}\overset{\diamond}{Q}^{\gamma\beta\alpha} - \frac{1}{4}\overset{\diamond}{Q}_\alpha \overset{\diamond}{Q}^\alpha + \frac{1}{2}\overset{\diamond}{Q}_\alpha \overset{\diamond}{\bar{Q}}^\alpha = \overset{\diamond}{\nabla}_\alpha(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\bar{Q}}^\alpha) - \overset{\diamond}{Q}$$

$$\partial_\alpha \overset{\diamond}{Q}^\alpha = \overset{\diamond}{\nabla}_\alpha \overset{\diamond}{Q}^\alpha + \overset{\diamond}{L}^\alpha{}_{\sigma\alpha}\overset{\diamond}{Q}^\sigma = \overset{\diamond}{\nabla}_\alpha \overset{\diamond}{Q}^\alpha - \frac{1}{2}\overset{\diamond}{Q}_\alpha \overset{\diamond}{Q}^\alpha$$

$$2\partial_\alpha P^\alpha{}_{\mu\nu} + \frac{1}{2}\overset{\diamond}{Q}_{\alpha\mu\nu}(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha) + \frac{1}{2}g_{\mu\nu}\partial_\alpha(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha) + \frac{1}{2}\overset{\diamond}{L}^\sigma{}_{\mu\nu}\overset{\diamond}{Q}_\sigma + \frac{1}{4}\overset{\diamond}{Q}_\mu{}^\alpha{}_\sigma \overset{\diamond}{Q}_\nu{}^\sigma{}_\alpha + \frac{1}{2}\overset{\diamond}{Q}^\alpha{}_{\sigma\mu}(\overset{\diamond}{Q}^\sigma{}_{\nu\alpha} - \overset{\diamond}{Q}_\alpha{}^\sigma{}_\nu) - \frac{1}{2}g_{\mu\nu}\overset{\diamond}{\nabla}_\alpha(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha) + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = 0$$

$$\partial_\alpha(\sqrt{-g}\overset{\diamond}{P}^\alpha{}_{\mu\nu}) = \partial_\alpha(\sqrt{-g})\overset{\diamond}{P}^\alpha{}_{\mu\nu} + \partial_\alpha(\overset{\diamond}{P}^\alpha{}_{\mu\nu})\sqrt{-g} \qquad \frac{\partial_\alpha\sqrt{-g}}{\sqrt{-g}} = \overset{\diamond}{\Gamma}^\sigma{}_{\alpha\sigma} = -\overset{\diamond}{L}^\sigma{}_{\alpha\sigma} = \frac{1}{2}\overset{\diamond}{Q}_\alpha$$

$$\boxed{\frac{2}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\overset{\diamond}{P}^\alpha{}_{\mu\nu}) - \frac{1}{\sqrt{-g}}\overset{\diamond}{q}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = 0}$$

The equivalence holds only for the theories stemming out from R, T, and Q (having second order equations). This is not true for extensions (non-linear functions of these invariants). GR, and its equivalent representations, are very peculiar cases among the theories of gravity, as we will see below..

8 LECTURE

Geometric Trinity: The solutions

Schwarzschild solution & Birkhoff theorem

The equivalence of GR, TEGR, and STEGR has to be proven also at the solution level. The same field equations have been obtained, and then the same exact solutions, under the same symmetries and boundary conditions, have to be achieved.

Recently, it has been proposed a *3+1 splitting formalism in the Geometric Trinity of Gravity* with the following advantages:
(1) simplicity in carrying out numerical analyses;
(2) solving some theoretical issues existing in the various formulations of GR at the fundamental level;
(3) broadening this methodology also in extended and alternative gravity frameworks.

Here, we focus the attention on the *Schwarazschild spacetime*. Soon after the publication of GR theory, Schwarzschild determined the solution, describing the spacetime metric outside a spherically symmetric mass-energy distribution. This result is in perfect agreement with the weak field approximation.

Jebsen, in 1921, and Birkhoff, in 1923, independently proved that the Schwarzschild solution holds outside a spherically symmetric mass distribution, even if this varies over time. This is now known as the *(Jebsen) Birkhoff theorem*, stating:

Any spherically symmetric solution of the GR field equations in vacuum has to be static and asymptotically flat. In addition, the Schwarzschild solution is the unique solution satisfying these hypotheses.

Schwarzschild solution & Birkhoff theorem

The ***Birhoff theorem*** entails several significant implications:
(1) uniqueness of the Schwarzschild solution in GR by imposing the spherical symmetry as starting hypothesis;
(2) no emission of gravitational waves;
(3) It can be compared with the Gauss theorem implications in electromagnetism and in classical Newtonian gravity.

We consider a generic spherically symmetric metric, whose line element, written in spherical coordinates {t,r, θ, φ}, in the equatorial plane $\theta = \pi/2$, and in geometric units (G=c=1), reads as

$$\mathrm{d}s^2 = -e^{\nu(t,r)}\mathrm{d}t^2 + e^{\lambda(t,r)}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2$$

unknown functions to be determined

weak field limit on the metric time component $-e^{\nu(t,r)} \approx -1 + \underbrace{\frac{2M}{r}}_{\text{Newtonian gravitational potential}}$ compact object mass

We want to solve the field equations in vacuum in GR, TEGR, and STEGR, recovering also the Birkhoff theorem.

The Schwarzschild metric $\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2$

Given a function f(t,r), we use the following notations

$$\dot{f}(t,r) := \frac{\mathrm{d}f(t,r)}{\mathrm{d}t}, \qquad f'(t,r) := \frac{\mathrm{d}f(t,r)}{\mathrm{d}r}$$

Spherically symmetric solutions in GR

The vacuum field equations in GR can be recast also as $\mathring{G}_{\mu\nu} \equiv \mathring{R}_{\mu\nu} = 0 \longrightarrow \mathring{R} = 0$

$$\mathring{G}_{tr} \equiv \frac{\dot{\lambda}(t,r)}{r} = 0, \quad \Rightarrow \quad \lambda = \lambda(r)$$

$$\mathring{G}_{rr} \equiv -e^{\lambda(r)} + r\nu'(t,r) + 1 = 0, \longrightarrow \text{metric static} \longrightarrow \boxed{\text{Birkhoff theorem}}$$

$$\mathring{G}_{tt} \equiv e^{-\lambda(r)}\left(r\lambda'(r) - 1\right) + 1 = 0. \longrightarrow \left[e^{-\lambda(r)} r\right]' = 1 \quad \Rightarrow \quad e^{-\lambda(r)} = 1 - \frac{C_1}{r}$$

integration constant

Multiplying thefirst by e$^{\lambda(r)}$ and summing it to the second

integration constant

$$\lambda'(r) + \nu'(r) = 0, \quad \Rightarrow \quad \lambda(r) + \nu(r) = C_2 \xrightarrow[\text{asymptotic flatness}]{} C_2 = 0$$

weak field limit

$$\boxed{-e^{\nu(r)} = 1 - \frac{2M}{r}, \qquad e^{\lambda(r)} = \frac{1}{1 - \frac{2M}{r}}}$$

Spherically symmetric solutions in TEGR

For solving the TEGR field equations, we adopt the tetrad formalism.

We know that each tetrad field must be associated to the related spin connection. However, in TEGR, we can drastically simplify the calculations resorting to the Weitzenböck gauge. Therefore, we can choose the diagonal tetrad

$$e^A{}_\mu = \begin{pmatrix} \sqrt{-e^{\nu(r)}} & 0 & 0 & 0 \\ 0 & \sqrt{e^{\lambda(r)}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}$$

non-zero torsion tensor components

$$\hat{T}{}^t{}_{tr} = -\frac{1}{2}\nu'(r) = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1},$$

$$\hat{T}{}^\varphi{}_{r\varphi} = \frac{1}{r}.$$

redshifted radial gravitational force

classical centrifugal force occurring in the tetrad frame

non-zero superpotential components

$$\hat{S}{}_{\hat{t}}{}^{tr} = \frac{2e^{-\lambda(r)}\sqrt{e^{-\nu(r)}}}{r} = \frac{2}{r}\sqrt{1-\frac{2M}{r}},$$

$$\hat{S}{}_{\hat{\varphi}}{}^{r\varphi} = -\frac{e^{-\lambda(r)}\left(r\nu'(r)+2\right)}{2r^2} = \frac{M-r}{r^3}.$$

non-zero contorsion tensor components

$$\hat{K}{}_{ttr} = \frac{1}{2}e^{\nu(r)}\nu'(r) = \frac{M}{r^2},$$

$$\hat{K}{}_{\varphi r\varphi} = r,$$

torsion scalar

$$\hat{T} = -\frac{2e^{-\lambda(r)}\left(r\nu'(r)+1\right)}{r^2} = -\frac{2}{r^2}$$

dynamically active part of the scalar curvature, whereas the remaining part is included in the *dynamically passive boundary term*.

Combining these elements, it is easy to obtain the Schwarzschild solution and the Birkhoff theorem also in TEGR.

Spherically symmetric solutions in STEGR

Regarding the STEGR field equations, we adopt the coincident gauge to ease the calculations.

In this case, it is immediate to get the same results of GR. However let us calculate the fundamental terms occurring in the equations for extracting the physical information.

non-metricity tensor

$$\overset{\diamond}{Q}_{r\mu\nu} = \begin{pmatrix} -e^{\nu(r)}\nu'(r) & 0 & 0 \\ 0 & e^{\lambda(r)}\lambda'(r) & 0 \\ 0 & 0 & 2r \end{pmatrix} = \begin{pmatrix} -\frac{2M}{r^2} & 0 & 0 \\ 0 & -\frac{2M}{r^2\left(1-\frac{2M}{r}\right)^2} & 0 \\ 0 & 0 & 2r \end{pmatrix},$$

the derivative of gravitational potential represents the gravitational force acting on the observer and producing the disformations

conjugate potential

$$\overset{\diamond}{P}{}^{t}{}_{tr} = \frac{r\lambda'(r) - r\nu'(r) + 4}{8r} = \frac{1}{8}\left(\lambda'(r) + \nu'(r)\right),$$

$$\overset{\diamond}{P}{}^{r}{}_{rr} = \frac{e^{\nu(r)-\lambda(r)}}{r} = \frac{1}{r}\left(1 - \frac{2M}{r}\right)^2,$$

$$\overset{\diamond}{P}{}^{r}{}_{\varphi\varphi} = -\frac{1}{4}re^{-\lambda(r)}\left(r\nu'(r) + 2\right) = \frac{M-r}{2},$$

$$\overset{\diamond}{P}{}^{\varphi}{}_{r\varphi} = \frac{1}{8}\left(\lambda'(r) + \nu'(r)\right) = 0,$$

$$\frac{\overset{\diamond}{q}_{\mu\nu}}{\sqrt{-g}} = \begin{pmatrix} \frac{2e^{\nu(r)-\lambda(r)}\nu'(r)}{r} & 0 & 0 \\ 0 & \frac{2r\nu'(r)+2}{r^2} & 0 \\ 0 & 0 & -\frac{r\nu'(r)+2}{e^{\lambda(r)}} \end{pmatrix} = \begin{pmatrix} \frac{4M}{r^3}\left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & \frac{2}{r^2\left(1-\frac{2M}{r}\right)} & 0 \\ 0 & 0 & \frac{2M}{r} - 2 \end{pmatrix}.$$

We recover the Schwarzschild solution and the validity of the Birkhoff theorem also in STEGR

→

It is worth stressing that, also at this level, we cannot expect the same solutions for f(R), f(T), and f(Q) extensions.

9 LECTURE

Extensions of Geometric Trinity

The Extended Geometric Trinity of Gravity

It is possible to consider generalizations of the Geometric Trinity
replacing the scalars R, T, and Q with arbitrary smooth functions in the
respective actions; this led to three dynamical equivalent exten

- Extended Metric Gravity, $f(R)$;
- Extended Teleparallel Gravity, $f(T, B)$;
- Extended Symmetric Teleparallel Gravity, $f(Q, B)$.

The equivalence is based on the choice of the smooth function f.

The Extended Geometric Trinity of Gravity

In fact, the equivalence of the Ricci scalar with the torsion and the non-metricity scalar up to the respective boundary terms leds to:

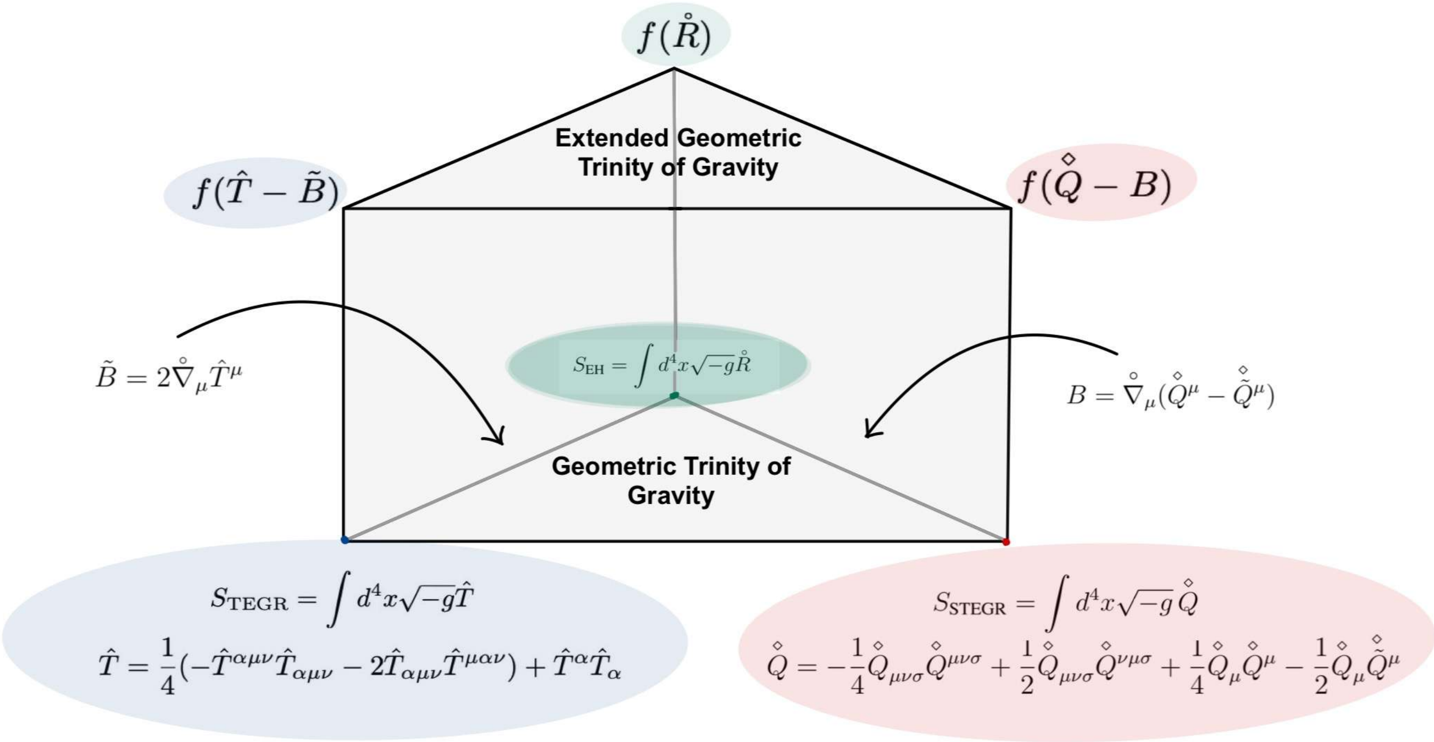
$$R = \mathring{R} \longrightarrow f(\mathring{R})$$

$$R = 0 \to \mathring{R} = T - B \longrightarrow f(\mathring{R}) = f(T, B) = f(T - B)$$

$$R = 0 \to \mathring{R} = Q + B \longrightarrow f(\mathring{R}) = f(Q, B) = f(Q + B)$$

These conditions define the equivalence between $f(\mathring{R})$ Lagrangian and field equations with the $f(T, B)$, and $f(Q, B)$ theories.
In this perspective, $f(\mathring{R})$, $f(T, B)$, and $f(Q, B)$ form the so-called **Extended Geometric Trinity of Gravity**.

The Extended Geometric Trinity of Gravity



The Extended Geometric Trinity of Gravity

The alternative approach to discuss extensions of gravity theories is constr
stress-energy tensors containing the further geometric DoFs. In this picture
geometry appear as additional **matter fields**.
It is based on:

- The second Bianchi Identity: $\nabla_{-} \mathrm{R}^{1}{}_{./0} + \nabla_{/} \mathrm{R}^{1}{}_{.-0} + \nabla_{0} \mathrm{R}^{1}{}_{./-} = (T^{4}_{23} R^{7}_{546} + T^{4}_{63} R^{7}_{524} + T^{4}_{62} R^{7}_{543});$

- the covariant conservation of the stress-energy tensor: $\nabla_{2} T^{26} = 0.$

Thus, starting from the second Bianchi Identity for a general linear affine co
possible to infer the field equations in TEGR and STEGR framework, which a
Einstein field equations.

The Extended Geometric Trinity of Gravity

Curvature, torsion, and non-metricity are simply alternative ways of represen
gravitational field, accounting for the same DoFs.

The conservation laws can also be applied to $f(Q)$ EGTG. In fact, the field equations can be rewritten
w.r.t. the Einstein tensor:

$$G_{\mu\nu} = \chi\left(T_{\mu\nu} + T^{eff}_{\mu\nu}\right),$$

separating the usual matter energy-momentum tensor and the effective ener
which is given by the geometrical contribution:

$$T^{eff}_{\mu\nu} = \frac{1}{f'(\mathring{R})}\left(\nabla_\mu \nabla_\nu f'(\mathring{R}) - f'(\mathring{R})\, g_{\mu\nu} + g_{\mu\nu}\frac{f(\mathring{R}) - \mathring{R} f'(\mathring{R})}{2}\right).$$

The Einstein tensor as usual is $G_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R}$.

The Extended Geometric Trinity of Gravity

Since for the second Bianchi Identity $\nabla_{!} G^{!"} = 0$, as well as the energy-momentum tensor is divergence free, we get that:

$$\nabla_2 T^{26}_{9::} = 0$$

Hence, the continuity equation holds also for the $T^{26}_{9::}$. As a consequence, generalized conservation will be also valid for the field equations of $f(T - B)$ and $f(Q + B)$, according to their dynamical equivalence to $f(R)$.

10 LECTURE

A summary and open issues

Summary

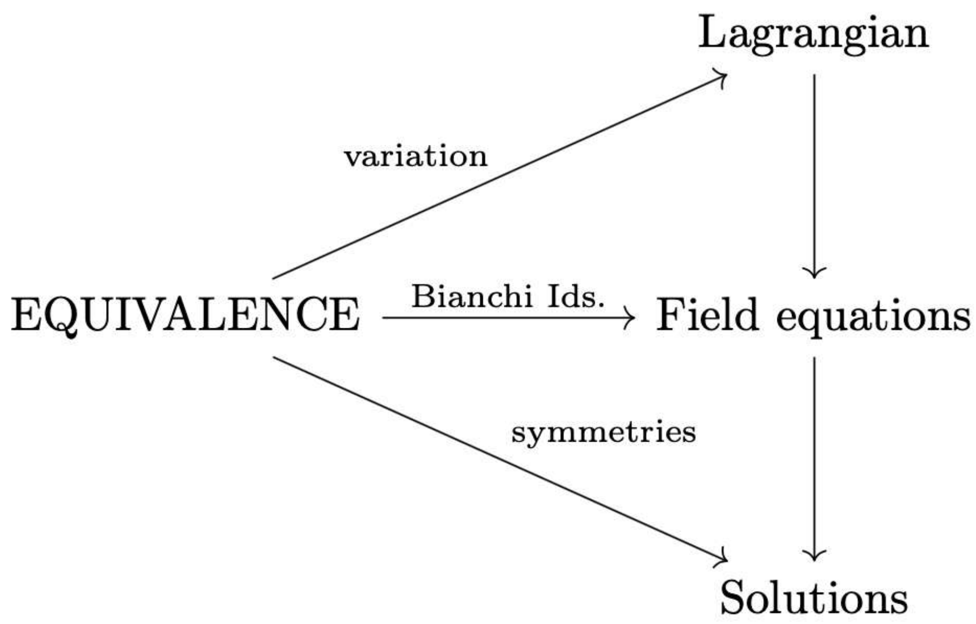
We have gathered together basic concepts of Geometric Trinity of Gravity and derived the related dynamics pointing out analogies and differences of metric, affine, and non-metric approaches. We tried to give a self-consistent picture of the three representations of gravitational field. The main statement is that equivalence is strictly achieved for GR, TEGR, and STEGR and not for any extension of these theories.

Firstly, we introduced the geometric arena of metric-affine gravity, where metric tensor and affine connection are two separate and independent structures. After, we provided the fundamental geometric objects, that are tetrads and spin connection. The former represents the observer laboratory, which solders the tangent space to the spacetime manifold. This procedure gives rise to anholonomic frames. They become holonomic when we are dealing with inertial frames, where a particular role is fulfilled by trivial tetrads of Special Relativity. The latter are intimately related with general tetrads, because they represent the inertial effects and they are generated by local Lorentz transformations. They form the Lorentz group, which, in turn, can be proved to give rise to a Lorentz algebra. This is a crucial aspect for defining the Fock-Ivanenko covariant derivative, useful to characterize the spin connection in terms of tetrads and to introduce the tetrad postulate. The physical interpretation is also provided.

Summary

These mathematical tools allow to describe the Geometric Trinity of Gravity. Specifically, a metric formulation (encoded in the Riemannian geometry) and a gauge approach (encoded in Teleparallel Gravity) are possible.

GR, TEGR, and STEGR are dynamically equivalent from the variation of their Lagrangians up to a boundary term. Furthermore, starting from the second Bianchi identity, it is possible to infer the field equations, which are identical in the three representations. Finally, we analysed spherically symmetric solutions in the three theories deriving the Schwarzschild spacetime and the Birkhoff theorem. The approaches can be summarized as follows:



Metric-Affine Theories of Gravity

Metric tensor and **affine connection** are two independent geometrical structures **.**

$$\Gamma^{-}{}_{+,} = \mathrm{I}\!\!\#_{+,} + K^{-}{}_{+,} + L^{-}{}_{+,}$$

- $\mathrm{I}\!\!\#_{+,} = \frac{\cdot}{/} g^{0-} \left(g_{+0,,} + g_{0,,+} - g_{+,,0}\right)$ **Levi-Civita connection**;
- $K^{-}{}_{+,} = \frac{\cdot}{/} g^{0-} \left(T_{+0,} + T_{,0+} - T_{0+,}\right)$ **Contortion tensor**;
- $L^{-}{}_{+,} = \frac{\cdot}{/} g^{0-} \left(Q_{0+,} - Q_{+0,} - Q_{,0+}\right)$ **Disformation tensor**.

Metric-Affine Theories of Gravity

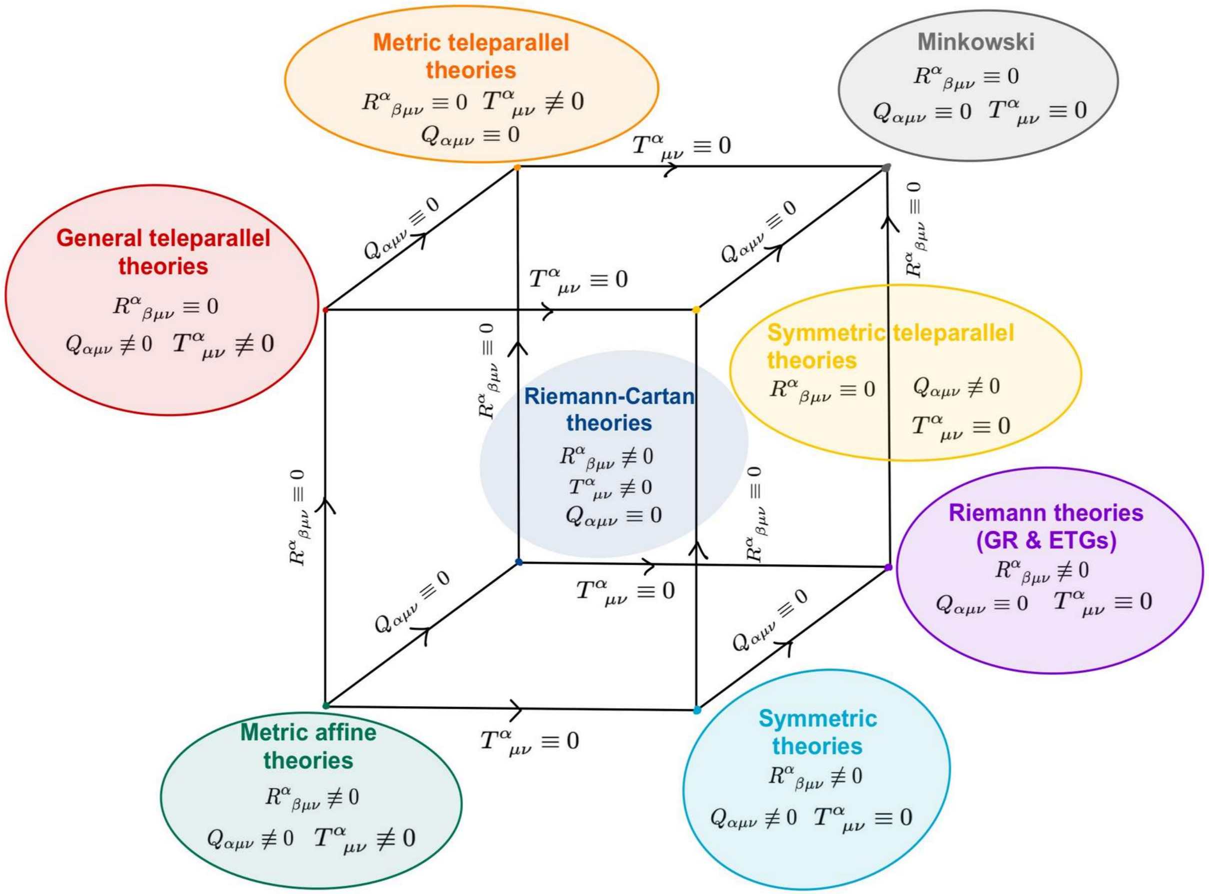
Through the affine connection, it is possible to define:

- the **curvature tensor** $R^0{}_{1+,} = \partial_+ \Gamma^0_{1,} - \partial_, \Gamma^0_{1+} + \Gamma^0_{2+} \Gamma^2_{1,} - \Gamma^0_{2,} \Gamma^2_{1+}$;

- the **torsion tensor** $T^0_{+,} = \Gamma^0_{,+} - \Gamma^0_{+,}$;

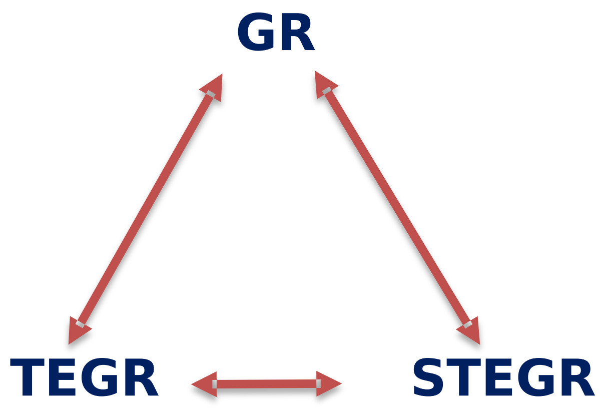
- the **nonmetricity tensor** $Q_{0+,} = \nabla_0 g_{+,} = \partial_0 g_{+,} - \Gamma^2_{34} g_{25} - \Gamma^2_{,4} g_{+2}$.

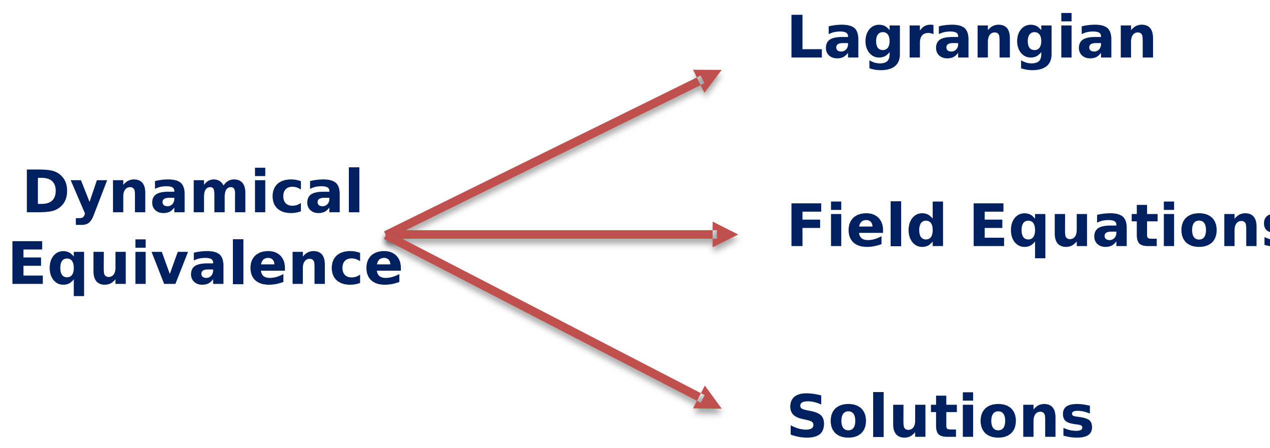
Metric-Affine Theories of Gravity



The Geometric Trinity of Gravity

General Relativity (GR) can be formulated also through torsion or non-n
Equivalent of GR (TEGR) and Symmetric TEGR (STEGR), respectively. Th
Geometric Trinity of Gravity.





The Geometric Trinity of Gravity

The equivalence of the three representations is first shown via the Lagr

$$R = \mathring{R} \rightarrow \mathcal{L}_{67} = \mathring{R}$$

$$R = \mathring{R} - T + \overset{0}{B} = 0 \rightarrow \mathcal{L}_{8967} = T - \overset{0}{B}$$

$$R = \mathring{R} - Q - B = 0 \rightarrow \mathcal{L}_{:8967} = Q + B$$

TEGR and STEGR Lagrangian are equivalent to the GR Hilbert-Einstein a
terms B.

Perspectives

- Investigation of metric field equations and connection field equations for quadratic non-metric gravity;
- Constrained Hypermomentum under projective invariance;
- Investigation on how a projective transformation could eventually allow to connect MAGs and Teleparallel and Symmetric Teleparallel Theories;
- The role of boundary terms B to establish the correct relation among gravity theories;
- Autoparallel as equations of motion for MAGs.

Open problems

Although mathematical results are equivalent, the physical interpretation can be different depending on the considered variables and observables. This fact opens several questions. Some of them can be listed as follows:

• Are there other equivalent formulations of gravity, outside of the Geometric Trinity within the metric-affine arena or, more in general, identifying other fundamental variables?

• From an observational point of view, what does it mean that these three theories are dynamically equivalent? This issue translates in extracting observables from each gravity theory.

• How can we construct observational apparatuses to test different theories dynamically equivalent to GR? Is it possible to discriminate different observables for equivalent descriptions of gravity from observations?

• STG theories are the less analysed among the three approaches. A general tetrad formulation is necessary in view of physical implications. In particular, the interpretation of gravity as a gauge theory could be particularly relevant to uniform gravity under the same standard of other fundamental theories.

• The Equivalence Principle is a fundamental aspect of GR. It can be recovered in TEGR and STEGR. If it were violated at some level (e.g. at quantum level), would it be possible to state that TEGR and STEGR are more fundamental theories than GR because they do not require it as a basic principle?

References

§ *New class of ghost-and tachyon-free metric affine gravities*
R. Percacci, R. Sezgin, *Physical Review D*, 101 (2020) 8, 084040.

§ *Projective transformations in metric-affine and Weylian geometries*
D. Sauro, R. Martini, O. Zanusso, *IJGMMP*, *20* (2023) 13, 2350237.

§ *Scale transformations in metric-affine geometry*
D. Iosifidis and T. Koivisto, 5 (2019) 3: 82.

§ *Extended Geometric Trinity of Gravity*
S. Capozziello, S. Cesare, C. Ferrara, *Eur.Phys.J.C* 85 (2025) 9, 932.

§ *The role of the boundary term in f(Q, B) symmetric teleparallel gravity*
S. Capozziello, V. De Falco, C. Ferrara, *Eur.Phys.J.C 83 (2023) 10, 915*

§ *Comparing equivalent gravities: common features and differences*
S. Capozziello, V. De Falco, C. Ferrara, *Eur.Phys.J.C 82 (2022) 10, 865*

§ *Equivalent Gravities and Equivalence Principle: Foundations and Experimental Implications*
Ch. Mancini, G. M. Tino, S. Capozziello, *Found. Phys. 55 (2025) 5, 69.*