Conservation laws in natural and gauge natural field theories

Lecture 1: Field theory and Noether theorem

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Contents

Why one should consider Noether theorem?

- It is an algorithmic way of extracting information from variational systems
- We have a complete and geometric framework covering
  - mechanics (first integrals) and field theories (continuity equations)
  - superpotentials in natural and gauge natural theories
  - a number of generalizations
- Insights to physical meaning and solution space

Many quantities relevant for physics are defined in terms of Noether theory

In mechanics we know how to use first integrals

In field theories the relation between conservation laws and solutions is looser

(Almost) Everything we know in physics is stated in terms of symmetries.

The variational setting

$$\begin{array}{cc} j^k_x y = (x^\mu, y^i, y^i_\mu, \dots) & J^k B \\ & \Big\downarrow \pi^k_0 \\ (x^\mu, y^i) & B \\ & \Big\downarrow \pi \\ x^\mu & M \end{array}$$

The *Lagrangian* $L = \mathcal{L}(j^k_x y) d^m x$ is a horizontal form on $J^k B$.
(or a bundle map $L : J^k B \to A_m(M)$, or a class of forms, ...)

Global sections $\sigma : M \to B \in \Gamma(\pi)$ can be lift to $j^k\sigma : M \to J^k B$ then $(j^k\sigma)^* L$ is an m-form on M.

For any compact region $D \subset M$, the *action* is

$$A_D[\sigma] = \int_D (j^k\sigma)^* L \tag{1}$$

A *deformation* (on D) is a vertical vector field X on B (supported on $\pi^{-1}(D)$), its flow being $\Phi_s : B \to B$.
Let the *deformed configuration* be $\sigma_s = \Phi_s \circ \sigma$, the *deformed action* be $A_D[\sigma_s]$.

Principle of critical action: a configuration σ is a *critical section*
iff, for all regions D and for all defomations X on D, the action is *stationary*, i.e.

$$\delta_X A_D[\sigma] = \frac{d}{ds}\left(A_D[\sigma_s]\right)\Big|_{s=0} = 0 \tag{2}$$

A section σ is critical iff, locally, it is a solution of the Euler-Lagrange equations

$$\mathbb{E}_i \circ j^{2k}\sigma = \left(\frac{\partial \mathcal{L}}{\partial y^i} - d_\mu \frac{\partial \mathcal{L}}{\partial y^i_\mu} + \ldots + (-1)^k d_{\mu_1 \ldots \mu_k} \frac{\partial \mathcal{L}}{\partial y^i_{\mu_1 \ldots \mu_k}} \right) \circ j^{2k}\sigma = 0 \tag{3}$$

Global first variation formula

The local computation can be captured in global morphisms:

$$
\begin{tikzcd}
J^kB \arrow[r, "\delta L"] \arrow[d, "\pi^k_0"] & V^*(J^kB)\otimes A_m(M) \arrow[d] \\
B \arrow[r, equal] \arrow[d, "\pi"] & B \arrow[d] \\
M \arrow[r, equal] & M
\end{tikzcd}
$$

$$\delta L(j^k_x y) = \left(p_i \bar{d}y^i + p_i^\mu \bar{d}y^i_\mu + \ldots + p_i^{\mu_1\ldots\mu_k}\bar{d}y^i_{\mu_1\ldots\mu_k}\right)\otimes d^m x$$

$$\mathbb{E}(j^{2k}_x y) = \mathbb{E}_i \bar{d}y^i \otimes d^m x$$

$$\mathbb{F}(j^{2k-1}_x y) = \left(f_i^\mu \bar{d}y^i + f_i^{\mu\alpha}\bar{d}y^i_\alpha + \ldots\right)\otimes (d^m x)_\mu$$

Naive momenta: $p_i = \frac{\partial \mathcal{L}}{\partial y^i}$, $p_i^\mu = \frac{\partial \mathcal{L}}{\partial y^i_\mu}$, $\ldots$, $p_i^{\mu_1\ldots\mu_k} = \frac{\partial \mathcal{L}}{\partial y^i_{\mu_1\ldots\mu_k}}$

Formal momenta: $f_i^\mu = p_i^\mu - d_\alpha p_i^{\mu\alpha} + \ldots,\ ,\ \ldots,\ f_i^{\mu\alpha_1\ldots\alpha_{k-1}} = p_i^{\mu\alpha_1\ldots\alpha_{k-1}}$

$$
\begin{tikzcd}
J^{2k}B \arrow[r, "\mathbb{E}"] \arrow[d, "\pi^{2k}_0"] & V^*(B)\otimes A_m(M) \arrow[d] \\
B \arrow[r, equal] \arrow[d, "\pi"] & B \arrow[d] \\
M \arrow[r, equal] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
J^{2k-1}B \arrow[r, "\mathbb{F}"] \arrow[d, "\pi^{2k-1}_0"] & V^*(J^{k-1}B)\otimes A_{m-1}(M) \arrow[d] \\
B \arrow[r, equal] \arrow[d, "\pi"] & B \arrow[d] \\
M \arrow[r, equal] & M
\end{tikzcd}
$$

We have the *global first variation formula* on $J^{2k}B$

$$\langle \delta L | j^k X\rangle = \langle \mathbb{E} | X\rangle + d\langle \mathbb{F} | j^{k-1}X\rangle \tag{4}$$

No functional spaces in variational calculus

Exercises

Let $L = \mathcal{L}(j^1_x y) d^m x$ be a first order Lagrangian. $L = \mathcal{L}(x^\mu, y^i, y^i_\mu) d^m x$

On the configuration bundle $[B \to M]$ we consider fibered coordinates (x^μ, y^i)

$$\langle \delta L | j^1 X \rangle = \left(p_i X^i + p^\mu_i d_\mu X^i \right) d^m x = \left(p_i - d_\mu p^\mu_i \right) X^i d^m x + d \left(p^\mu_i X^i (d^m x)_\mu \right) \tag{5}$$

$$\mathbb{E} = \left(p_i - d_\mu p^\mu_i \right) \bar{d} y^i \otimes d^m x \qquad \mathbb{F} = p^\mu_i \bar{d} y^i \otimes (d^m x)_\mu \tag{6}$$

Consider fibered transformations

$$x'^\mu = x'^\mu(x) \qquad y'^i = Y^i(x, y) \qquad \left(J^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}(x) \quad Y^i_\nu = \frac{\partial y'^i}{\partial x^\nu}(x, y) \quad Y^i_k = \frac{\partial y'^i}{\partial y^k}(x, y) \right) \tag{7}$$

$$J \mathcal{L}'(x'^\mu, y'^i, y'^i_\mu) = \mathcal{L}(x^\mu, y^i, y^i_\mu) \tag{8}$$

$$J p'_k Y^k_i + J p'^\mu_k d'_\mu Y^k_i = p_i \qquad J \bar{J}^\mu_\alpha p'^\alpha_k Y^k_i = p^\mu_i \tag{9}$$

$$\mathbb{E}'_i \bar{d} y'^i \otimes d^m x' = \left(p'_i - d'_\mu p'^\mu_i \right) \bar{d} y'^i \otimes d^m x' = \mathbb{E}_i \bar{d} y^i \otimes d^m x \qquad \mathbb{F}' = p'^\mu_i \bar{d} y'^i \otimes (d^m x')_\mu = \mathbb{F} \tag{10}$$

$$\mathbb{E} = \left(p_i - d_\mu p^\mu_i \right) \omega^i \wedge d^m x \qquad \mathbb{F} = p^\mu_i \omega^i \wedge (d^m x)_\mu \tag{11}$$

Everything is in transformation laws.

Exercises

For a second order Lagrangian, one can find the computation in [1]

Exercise: check that there

$$f_i^\mu = p_i^\mu - d_\alpha p_i^{\mu\alpha} \qquad f_i^{\mu\nu} = p_i^{\mu\nu} \tag{12}$$

and that they transform as

$$f_i^\mu = J\bar{J}^\mu_\alpha \left(f'^{\alpha}_k Y^k_i + f'^{\alpha\beta}_k d'_\beta Y^k_i \right) \qquad\qquad f_i^{\mu\nu} = J\bar{J}^\mu_\alpha \bar{J}^\nu_\beta f'^{\alpha\beta}_k Y^k_i \tag{13}$$

Exercise: show that

$$\mathbb{E} := \mathbb{E}_i \, \omega^i \wedge d^m x \qquad\qquad \mathbb{F}(L) := f_i^\mu \omega^i \wedge (d^m x)_\mu + f_i^{\mu\nu} \omega^i_\nu \wedge (d^m x)_\mu \tag{14}$$

are still global.

Exercise: show that the maps

$$\mathbb{E} = \mathbb{E}_i \, \bar{d}y^i \otimes d^m x \qquad\qquad \mathbb{F} = f_i^\mu \bar{d}y^i \otimes (d^m x)_\mu + f_i^{\mu\nu} \bar{d}y^i_\nu \otimes (d^m x)_\mu \tag{15}$$

are still global.

Exercise: Find $\mathbb{E}$ and $\mathbb{F}$ for the Klein-Gordon Lagrangian $\mathcal{L} = -\frac{\sqrt{|g|}}{2} \left(g^{\mu\nu} d_\mu \varphi^\dagger d_\nu \varphi + m^2 \varphi^\dagger \varphi \right)$.

Exercise: Find $\mathbb{E}$ and $\mathbb{F}$ for the Maxwell Lagrangian $\mathcal{L} = -\frac{\sqrt{|g|}}{4} F_{\alpha\beta} F_{\mu\nu} g^{\alpha\mu} g^{\nu\beta}$ with $F_{\alpha\beta} = d_\alpha A_\beta - d_\beta A_\alpha$.

Lagrangian Symmetries

A fibered transformation $\Phi : B \to B$ is a *Lagrangian symmetry* iff $(J^1\Phi)^* L = L$, i.e.

$$J\mathcal{L}(j^k_x y') = \mathcal{L}(j^k_x y) \tag{16}$$

A projectable vector field $\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x,y)\partial_i$ is an *infinitesimal symmetry* iff the *covariance identity* is satisfied

$$p_i \pounds_\Xi y^i + p_i^\mu d_\mu \pounds_\Xi y^i = d_\mu(\xi^\mu \mathcal{L}) \tag{17}$$

where we set

$$\pounds_\Xi \sigma = T\sigma(\xi) - \Xi \circ \sigma = \left(\xi^\mu y^i_\mu - \xi^i\right)\partial_i \in V(B) \qquad \left(\pounds_\Xi y = \left(\xi^\mu y^i_\mu - \xi^i\right)\partial_i : J^1B \to V(B)\right) \tag{18}$$

for the *Lie derivative of a section.* (*formal Lie derivative*)

$$\begin{tikzcd}
J^1B \arrow[r, "\pounds_\Xi y"] \arrow[d, "\pi^1_0"] & V(B) \arrow[d] \\
B \arrow[r, equal] \arrow[d, "\pi"] & B \arrow[d] \\
M \arrow[r, equal] \arrow[uu, bend left, "j^1\sigma"] & M \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

$$\pounds_\Xi y \circ j^1\sigma = \pounds_\Xi \sigma$$

$$[\pounds_{\Xi_1}, \pounds_{\Xi_2}] y = \pounds_{[\Xi_1, \Xi_2]} y$$

Noether theorem

We can set recast the covariance identity globally as

$$\langle \delta L | j^k \pounds_\Xi y \rangle = d(\xi \rfloor L) \tag{19}$$

and use the first variation formula

$$\begin{gathered} \langle \delta L | j^k \pounds_\Xi y \rangle = \langle \mathbb{E} | \pounds_\Xi y \rangle + d \langle \mathbb{F} | j^{k-1} \pounds_\Xi y \rangle = d(\xi \rfloor L) \\ d \left(\langle \mathbb{F} | j^{k-1} \pounds_\Xi y \rangle - \xi \rfloor L \right) = - \langle \mathbb{E} | \pounds_\Xi y \rangle \end{gathered} \tag{20}$$

Let us define the *Noether current* and the *work current*

$$\mathcal{E} = \langle \mathbb{F} | j^{k-1} \pounds_\Xi y \rangle - \xi \rfloor L : J^{2k-1} B \to A_{m-1}(M) \qquad \mathcal{W} = - \langle \mathbb{E} | \pounds_\Xi y \rangle : J^{2k} B \to A_m(M) \tag{21}$$

which obey the *conservation law*

$$d\mathcal{E} = \mathcal{W} \tag{22}$$

– if evaluated along a solution, $d(\mathcal{E} \circ j^{2k-1}\sigma) = 0$, *continuity equations*
– in mechanics, $m = 1$, $\mathcal{E}$ is a *first integral*

Symmetries group

For a Lagrangian system defined on $[B \to M]$, we can define a group of *symmetries* as well as a Lie sub-algebra of *infinitesimal symmetries*

$$\mathrm{Sym} \subset \mathrm{Aut}(B) \qquad \mathfrak{sym} \subset \mathfrak{X}_\pi(B) \tag{23}$$

Lie group of symmetries: (First Noether theorem)

$\mathfrak{sym}$ is a finite dimensional Lie algebra Sym is a Lie group

$\Xi_A = \xi^\mu_A \partial_\mu + \xi^i_A \partial_i$ is a base of $\mathfrak{sym}$ $\qquad [\Xi_A, \Xi_B] = c^C{}_{AB}\Xi_C \qquad \Xi = \zeta^A \Xi_A \subset \mathfrak{sym}, \quad \zeta^A \in \mathbb{R}$

$$\pounds_\Xi y = \zeta^A \left(\xi^\mu_A y^i_\mu - \xi^i_A\right) \partial_i = \zeta^A \pounds_{\Xi_A} y \qquad \begin{cases} \mathcal{E} = \left(\langle \mathbb{F} | j^{k-1} \pounds_{\Xi_A} y\rangle - \xi_A \lrcorner L\right) \zeta^A = \mathcal{E}_A \zeta^A \\ \mathcal{W} = -\langle \mathbb{E} | \pounds_{\Xi_A} y\rangle \zeta^A = \mathcal{W}_A \zeta^A \end{cases} \tag{24}$$

Localizable symmetries: (Second Noether theorem)

$\mathfrak{sym}$ are parameterized by local functions $\zeta^A(x)$ and their derivatives up to some finite order r

$$\begin{aligned} \pounds_\Xi y =& L_A \zeta^A + L^\mu_A d_\mu \zeta^A + \ldots + L^{\mu_1 \ldots \mu_r}_A d_{\mu_1 \ldots \mu_r} \zeta^A = \\ =& \hat{L}_A \zeta^A + \hat{L}^\mu_A \nabla_\mu \zeta^A + \ldots + \hat{L}^{\mu_1 \ldots \mu_r}_A \nabla_{\mu_1 \ldots \mu_r} \zeta^A \end{aligned} \tag{25}$$

$$\begin{aligned} \pounds_\xi g &= \left(\xi^\alpha d_\alpha g_{\mu\nu} + d_\mu \xi^\alpha g_{\alpha\nu} + d_\nu \xi^\alpha g_{\alpha\mu}\right) \frac{\partial}{\partial g_{\mu\nu}} = \\ &= \left(\xi^\alpha \nabla_\alpha g_{\mu\nu} + \nabla_\rho \xi^\alpha \left(\delta^\rho_\mu g_{\alpha\nu} + \delta^\rho_\nu g_{\alpha\mu}\right)\right) \frac{\partial}{\partial g_{\mu\nu}} \end{aligned} \tag{26}$$

Examples of First Noether theorem

Ex 1: consider the Kepler Lagrangian in space ($r^2 = x^2 + y^2 + z^2$)

$$L = \left(\tfrac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2) + \tfrac{\alpha}{r}\right) dt \tag{27}$$

$X_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ is a symmetry $j^1X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} + u^2\frac{\partial}{\partial u^3} - u^3\frac{\partial}{\partial u^2}$

$$j^1X_1(\mathcal{L}) = \tfrac{yz}{r^2} - \tfrac{zy}{r^2} + m\left(u^2u^3 - u^3u^2\right) = 0 \qquad J_1 = z\dot y - y\dot z \quad \text{is conserved} \tag{28}$$

Analogously, for $X_2 = x\partial_z - z\partial_x$ and $X_3 = y\partial_x - x\partial_y$. $\langle X_i\rangle = \mathfrak{so}(3)$

$$[X_i, X_j] = \epsilon^k{}_{ij}X_k \qquad [X_1, X_2] = y\partial_x - x\partial_z = X_3 \tag{29}$$

Ex 2: consider the Klein-Gordon Lagrangian

$$L = -\tfrac{\sqrt{g}}{2}\left(\varphi^\dagger_\mu g^{\mu\nu}\varphi_\nu + m^2\varphi^\dagger\varphi\right) d^mx \tag{30}$$

$\Xi = -i\varphi^\dagger\frac{\partial}{\partial\varphi^\dagger} + i\varphi\frac{\partial}{\partial\varphi}$ is a symmetry $j^1\Xi = -i\varphi^\dagger\frac{\partial}{\partial\varphi^\dagger} + i\varphi\frac{\partial}{\partial\varphi} - i\varphi^\dagger_\mu\frac{\partial}{\partial\varphi^\dagger_\mu} + i\varphi_\mu\frac{\partial}{\partial\varphi_\mu}$

$$j^1\Xi(\mathcal{L}) = i\tfrac{\sqrt{g}}{2}m^2\left(\varphi^\dagger\varphi - \varphi^\dagger\varphi\right) - i\tfrac{\sqrt{g}}{2}\left(\varphi^\dagger_\mu g^{\mu\nu}\varphi_\nu - \varphi^\dagger_\mu g^{\mu\nu}\varphi_\nu\right) = 0 \tag{31}$$

$$\mathcal{Q} = i\tfrac{\sqrt{g}}{2}g^{\mu\nu}\left(\varphi^\dagger\varphi_\nu - \varphi^\dagger_\nu\varphi\right)(d^mx)_\mu \qquad \text{is conserved along solutions} \tag{32}$$

Second Noether theorem

With localizable symmetries $(\pounds_\Xi y^i = L^i_A\zeta^A + L^{i\mu}_A d_\mu\zeta^A + \ldots + L^{i\mu_1\ldots\mu_r}_A d_{\mu_1\ldots\mu_r}\zeta^A)$

$$\begin{aligned}\mathcal{E} &= \left(f^\mu_i \pounds_\Xi y^i + f^{\mu\alpha}_i d_\alpha \pounds_\Xi y^i + \ldots + f^{\mu\alpha_1\ldots\alpha_{k-1}}_i d_{\alpha_1\ldots\alpha_{k-1}}\pounds_\Xi y^i - \xi^\mu \mathcal{L}\right)(d^m x)_\mu = \\ &= \left(\mathcal{E}^\mu_A\zeta^A + \mathcal{E}^{\mu\alpha}_A d_\alpha\zeta^A + \ldots + \mathcal{E}^{\mu\alpha_1\ldots\alpha_{k+r-1}}_A d_{\alpha_1\ldots\alpha_{k+r-1}}\zeta^A\right)(d^m x)_\mu = \\ &= \left(\hat{\mathcal{E}}^{\ \mu}_A\zeta^A + \hat{\mathcal{E}}^{\ \mu\alpha}_A \nabla_\alpha\zeta^A + \ldots + \hat{\mathcal{E}}^{\ \mu\alpha_1\ldots\alpha_{k+r-1}}_A \nabla_{\alpha_1\ldots\alpha_{k+r-1}}\zeta^A\right)(d^m x)_\mu\end{aligned} \tag{33}$$

$$\begin{aligned}\mathcal{W} &= -\left(p_i - d_\mu p^\mu_i + \ldots + (-1)^k d_{\mu_1\ldots\mu_k} p^{\mu_1\ldots\mu_k}_i\right)\pounds_\Xi y^i (d^m x) = \\ &= \left(\mathcal{W}_A\zeta^A + \mathcal{W}^\alpha_A d_\alpha\zeta^A + \ldots + \mathcal{W}^{\alpha_1\ldots\alpha_r}_A d_{\alpha_1\ldots\alpha_r}\zeta^A\right)(d^m x) = \\ &= \left(\hat{\mathcal{W}}_A\zeta^A + \hat{\mathcal{W}}^\alpha_A \nabla_\alpha\zeta^A + \ldots + \hat{\mathcal{W}}^{\alpha_1\ldots\alpha_r}_A \nabla_{\alpha_1\ldots\alpha_r}\zeta^A\right)(d^m x)\end{aligned} \tag{34}$$

Conservation laws for a first order current

$$\begin{aligned}&\left(\nabla_\mu \hat{\mathcal{E}}^{\ \mu}_A + \tfrac{1}{2}\hat{\mathcal{E}}^{\ \mu\alpha}_B R^B{}_{A\mu\alpha}\right)\zeta^A + \left(\nabla_\alpha \hat{\mathcal{E}}^{\ \alpha\mu}_A + \hat{\mathcal{E}}^{\ \mu}_A\right)\nabla_\mu\zeta^A + \hat{\mathcal{E}}^{\ \mu\alpha}_A \nabla_{\mu\alpha}\zeta^A = \\ &= \hat{\mathcal{W}}_A\zeta^A + \hat{\mathcal{W}}^\alpha_A \nabla_\alpha\zeta^A + \hat{\mathcal{W}}^{\alpha\beta}_A \nabla_{\alpha\beta}\zeta^A\end{aligned} \tag{35}$$

where we set $[\nabla_\mu, \nabla_\alpha]\zeta^A = R^A{}_{B\mu\alpha}\zeta^B$.

$$\begin{cases}\nabla_\mu \hat{\mathcal{E}}^{\ \mu}_A + \frac{1}{2}\hat{\mathcal{E}}^{\ \mu\alpha}_B R^B{}_{A\mu\alpha} = \hat{\mathcal{W}}_A \\ \nabla_\alpha \hat{\mathcal{E}}^{\ \alpha\mu}_A + \hat{\mathcal{E}}^{\ \mu}_A = \hat{\mathcal{W}}^\alpha_A \\ \hat{\mathcal{E}}^{\ (\mu\alpha)}_A = \hat{\mathcal{W}}^{\alpha\beta}_A\end{cases} \tag{36}$$

Bianchi identities

For a second order work current $\mathcal{W} = \left(\hat{\mathcal{W}}_A\zeta^A + \hat{\mathcal{W}}_A^\alpha \nabla_\alpha \zeta^A + \hat{\mathcal{W}}_A^{\alpha\beta}\nabla_{\alpha\beta}\zeta^A\right)(d^m x)$ we have

$$\begin{aligned}
&\left(\hat{\mathcal{W}}_A\zeta^A + \hat{\mathcal{W}}_A^\alpha \nabla_\alpha \zeta^A + \hat{\mathcal{W}}_A^{\alpha\beta}\nabla_{\alpha\beta}\zeta^A\right) = \\
=&\left(\hat{\mathcal{W}}_A - \nabla_\alpha \hat{\mathcal{W}}_A^\alpha + \nabla_{\alpha\beta}\hat{\mathcal{W}}_A^{\alpha\beta}\right)\zeta^A + \nabla_\alpha\left(\left(\hat{\mathcal{W}}_A^\alpha - \nabla_\beta \hat{\mathcal{W}}_A^{\alpha\beta}\right)\zeta^A + \hat{\mathcal{W}}_A^{\alpha\beta}\nabla_\beta\zeta^A\right) = \\
=&\mathcal{B} + d\tilde{\mathcal{E}}
\end{aligned} \tag{37}$$

where we defined the *Bianchi current* $\mathcal{B} = B\, d^m x \equiv 0$ which vanishes (off-shell)

$$\begin{aligned}
B =& \left(\hat{\mathcal{W}}_A - \nabla_\alpha \hat{\mathcal{W}}_A^\alpha + \nabla_{\alpha\beta}\hat{\mathcal{W}}_A^{\alpha\beta}\right)\zeta^A = \\
&= \nabla_\mu \hat{\mathcal{E}}^{\,\mu}_{A} + \tfrac{1}{2}\hat{\mathcal{E}}^{\,\mu\alpha}_{B} R^B{}_{A\mu\alpha} - \nabla_\mu(\nabla_\alpha \hat{\mathcal{E}}^{\,\alpha\mu}_{A} + \hat{\mathcal{E}}^{\,\mu}_{A}) + \nabla_{\alpha\beta}\hat{\mathcal{E}}^{\,(\mu\alpha)}_{A} = \\
&= \nabla_\mu \hat{\mathcal{E}}^{\,\mu}_{A} + \tfrac{1}{2}\hat{\mathcal{E}}^{\,\mu\alpha}_{B} R^B{}_{A\mu\alpha} - \nabla_{\mu\alpha}\hat{\mathcal{E}}^{\,\alpha\mu}_{A} - \tfrac{1}{2}[\nabla_\mu, \nabla_\alpha]\hat{\mathcal{E}}^{\,\alpha\mu}_{A} - \nabla_\mu \hat{\mathcal{E}}^{\,\mu}_{A} + \nabla_{\alpha\beta}\hat{\mathcal{E}}^{\,(\mu\alpha)}_{A} = 0
\end{aligned} \tag{38}$$

and the *reduced current* $\tilde{\mathcal{E}} = \left(\left(\hat{\mathcal{W}}_A^\alpha - \nabla_\beta \hat{\mathcal{W}}_A^{\alpha\beta}\right)\zeta^A + \hat{\mathcal{W}}_A^{\alpha\beta}\nabla_\beta\zeta^A\right)(d^m x)_\alpha$ which vanishes on-shell.

$$\tilde{\mathcal{E}} = \left(\left(\hat{\mathcal{E}}^{\,\alpha}_{A} + \nabla_\beta \hat{\mathcal{E}}^{\,[\beta\alpha]}_{A}\right)\zeta^A + \hat{\mathcal{E}}^{\,(\alpha\beta)}_{A}\nabla_\beta\zeta^A\right)(d^m x)_\alpha \tag{39}$$

The work current is exact $\mathcal{W} = d\tilde{\mathcal{E}}$

The current $d(\mathcal{E} - \tilde{\mathcal{E}}) = 0$ is conserved off-shell and it coincides on-shell with the Noether current.

Exercises

Ex 1: check that $[\nabla_\mu, \nabla_\nu]\hat{\mathcal{E}}^{\mu\nu}_{\ \alpha} = \hat{\mathcal{E}}^{\mu\nu}_{\ \epsilon} R^\epsilon{}_{\alpha\mu\nu}$ where $\hat{\mathcal{E}}^{\mu\nu}_{\ \alpha}$ is a tensor density of weight 1.

Exercise: Consider a Lagrangian on $[\mathbb{R} \times \mathbb{A}^m \to \mathbb{R}]$ with coordinates (s, x^μ), $L = \mathcal{L}(x^\mu, u^\mu)ds$ homogeneous of degree 1, i.e. such that

$$\frac{\partial \mathcal{L}}{\partial u^\mu} u^\mu = \dot{p}_\mu u^\mu = \mathcal{L} \tag{40}$$

e.g. $L = \sqrt{g_{\mu\nu}(x)u^\mu u^\nu}ds$ or $L = A_\mu(x)u^\mu ds$.

Consider the transformation $\qquad x^\mu(s') = x^\mu(s(s'))$

$$s' = s'(s) \qquad\qquad x'^\mu = x^\mu \qquad\qquad u'^\mu = \frac{ds}{ds'}u^\mu \tag{41}$$

The infinitesimal generator is

$$j^1\Xi = \xi\frac{\partial}{\partial s} - \frac{d\xi}{ds}u^\mu\frac{\partial}{\partial u^\mu} \qquad\qquad \pounds_\Xi x^\mu = \xi u^\mu \tag{42}$$

This is a symmetry $\qquad \left(\frac{\partial \mathcal{L}}{\partial s} = 0\right)$

$$\begin{aligned} p_\mu \xi u^\mu + \dot{p}_\mu \frac{d}{ds}(\xi u^\mu) &= \xi p_\mu u^\mu + \frac{d\xi}{ds}\dot{p}_\mu u^\mu + \xi \dot{p}_\mu \dot{u}^\mu = \\ &= \frac{d\xi}{ds}\mathcal{L} + \xi\frac{\partial \mathcal{L}}{\partial s} + \xi p_\mu u^\mu + \xi \dot{p}_\mu \dot{u}^\mu = \frac{d}{ds}(\xi\mathcal{L}) \end{aligned} \tag{43}$$

The corresponding Noether current is

$$\mathcal{H} = \dot{p}_\mu u^\mu - \mathcal{L} = 0 \tag{44}$$

It is constant along solutions (:o)) $\qquad$ It is constant along *any* curve (:o( )

Solutions

Ex 1:

$$[\nabla_\mu, \nabla_\nu]\hat{\mathcal{E}}\,_\alpha^{\mu\nu} = R^\nu{}_{\epsilon\mu\nu}\hat{\mathcal{E}}\,_\alpha^{\mu\epsilon} + R^\mu{}_{\epsilon\mu\nu}\hat{\mathcal{E}}\,_\alpha^{\epsilon\nu} - R^\epsilon{}_{\alpha\mu\nu}\hat{\mathcal{E}}\,_\epsilon^{\mu\nu} \tag{45}$$

Since $\hat{\mathcal{E}}$ is a density of weight 1 and covariant derivatives are for Levi-Civita connection we have

$$\begin{aligned}\nabla_\alpha\nabla_\beta\hat{\mathcal{E}} =& d_\alpha\nabla_\beta\hat{\mathcal{E}} - \Gamma^\epsilon_{\beta\alpha}\nabla_\epsilon\hat{\mathcal{E}} - \Gamma^\epsilon_{\epsilon\alpha}\nabla_\beta\hat{\mathcal{E}} = \\ =& d_{\alpha\beta}\hat{\mathcal{E}} - d_\alpha\Gamma_\beta\hat{\mathcal{E}} - \Gamma_\beta d_\alpha\hat{\mathcal{E}} - \Gamma^\epsilon_{\beta\alpha}\nabla_\epsilon\hat{\mathcal{E}} - \Gamma_\alpha d_\beta\hat{\mathcal{E}} + \Gamma_\alpha\Gamma_\beta\hat{\mathcal{E}} \\ [\nabla_\alpha, \nabla_\beta]\hat{\mathcal{E}} =& \left(d_\beta\Gamma^\epsilon_{\epsilon\alpha} - d_\alpha\Gamma^\epsilon_{\epsilon\beta} + \Gamma^\epsilon_{\sigma\beta}\Gamma^\sigma_{\epsilon\alpha} - \Gamma^\epsilon_{\sigma\alpha}\Gamma^\sigma_{\epsilon\beta}\right)\hat{\mathcal{E}} = R^\epsilon{}_{\epsilon\beta\alpha}\hat{\mathcal{E}} = 0\end{aligned} \tag{46}$$

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Lorenzo Fatibene,
Relativistic Theories, Gravitational Theories, and General Relativity,
`www.fatibene.org > Book`

Marcella Palese, Olga Rossi, Fabrizio Zanello,
Geometric integration by parts and Lepage equivalents
Differential Geometry and its Applications, **81**, April 2022, 101866

Summary

Variational calculus X deformation Stokes + boundary conditions

$$\langle \delta L | j^k X \rangle = \langle \mathbb{E} | X \rangle + d\langle \mathbb{F} | j^{k-1} X \rangle \qquad \Rightarrow \qquad \mathbb{E} \circ j^{2k}\sigma = 0 \text{ field equations} \tag{47}$$

Noether theorem $\pounds_\Xi y$ symmetry covariance + along solutions

$$\langle \delta L | j^k \pounds_\Xi y \rangle = \langle \mathbb{E} | \pounds_\Xi y \rangle + d\langle \mathbb{F} | j^{k-1} \pounds_\Xi y \rangle \qquad \Rightarrow \qquad d\mathcal{E} \circ j^{2k-1}\sigma = 0 \text{ conservation laws} \tag{48}$$

Conservation laws in natural and gauge natural field theories

Lecture 2: Natural theories

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Summary

Variational calculus X deformation   Stokes + boundary conditions

$$\langle \delta L | j^k X \rangle = \langle \mathbb{E} | X \rangle + d\langle \mathbb{F} | j^{k-1} X \rangle \qquad \Rightarrow \qquad \mathbb{E} \circ j^{2k}\sigma = 0 \text{ field equations} \tag{1}$$

Noether theorem $\pounds_\Xi y$ symmetry   covariance + along solutions

$$\langle \delta L | j^k \pounds_\Xi y \rangle = \langle \mathbb{E} | \pounds_\Xi y \rangle + d\langle \mathbb{F} | j^{k-1} \pounds_\Xi y \rangle \qquad \Rightarrow \qquad d\mathcal{E} \circ j^{2k-1}\sigma = 0 \text{ conservation laws} \tag{2}$$

Contents

How about field theories that are covariant for any spacetime diffeomorphism? as GR

- Fields need to support a representation of the group $\mathrm{Diff}(M)$ Natural bundles
- Any vector field $\xi \in \mathfrak{X}(M)$ is an infinitesimal Lagrangian symmetry Second Noether theorem
- If $\hat{\cdot} : \mathrm{Diff}(M) \to \mathrm{Aut}(C)$, then it splits the group sequence

$$0 \to \mathrm{Aut}_V(C) \to \mathrm{Aut}(C) \to \mathrm{Diff}(M) \to 0 \tag{3}$$

- If one has a group homomorphism $\hat{\cdot} : \mathrm{Diff}(M) \to \mathrm{Aut}(C) : \phi \mapsto \hat{\phi}$,
  then $\mathrm{Diff}(M)$ acts on sections $\sigma' = \hat{\phi} \circ \sigma \circ \varphi^{-1}$
- Infinitesimally one has $\hat{\cdot} : \mathfrak{X}(M) \to \mathfrak{X}(C) : \xi \mapsto \hat{\xi}$

$$0 \to \mathfrak{X}_V(C) \to \mathfrak{X}(C) \to \mathfrak{X}(M) \to 0 \tag{4}$$

splits the Lie algebra sequence. $[\xi, \zeta]\hat{} = [\hat{\xi}, \hat{\zeta}]$

The bundle TM has these properties (with $\hat{\phi} = T\phi$).

What is the most general bundle $[C \to M]$ with these properties? Natural bundles
What extra properties has a field theory like that? Natural theories

...

The groups

Consider maps $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ such that $\alpha(0) = 0$ and locally invertible around the origin.

$j^r\alpha \in \mathrm{GL}^r(m)$ form a Lie group $\qquad j^r\alpha \cdot j^r\beta = j^r(\alpha \circ \beta) \qquad (j^r\alpha)^{-1} = j^r\bar\alpha$

For $\mathrm{GL}^2(m)$ consider

$$\alpha \simeq \partial_b\alpha^a x^b + \frac{1}{2}\partial_{bc}\alpha^a x^b x^c + \dots \qquad \alpha^a_b = \partial_b\alpha^a \qquad \alpha^a_{bc} = \partial_{bc}\alpha^a \tag{5}$$

and expand $\alpha \circ \beta : x^b \mapsto x'^a = \alpha(\beta(x))$

$$\alpha \circ \beta \simeq \partial_p\alpha^a\partial_b\beta^p x^b + \frac{1}{2}\left(\partial_{pq}\alpha^a\partial_b\beta^p\partial_c\beta^q + \partial_p\alpha^a\partial_{bc}\beta^p\right) x^b x^c + \dots \qquad (\alpha^a_b, \alpha^a_{bc}) \cdot (\beta^d_e, \beta^d_{ef}) = (\alpha^a_p\beta^p_b, \alpha^a_{pq}\beta^p_b\beta^q_c + \alpha^a_p\beta^p_{bc}) \tag{6}$$

For the inverse $(\alpha^a_b, \alpha^a_{bc})^{-1} = (\bar\alpha^a_b, \bar\alpha^a_{bc})$

$$(\alpha^a_b, \alpha^a_{bc}) \cdot (\bar\alpha^d_e, \bar\alpha^d_{ef}) = (\alpha^a_p\bar\alpha^p_b, \alpha^a_{pq}\bar\alpha^p_b\bar\alpha^q_c + \alpha^a_p\bar\alpha^p_{bc}) = (\delta^a_b, 0) \qquad \alpha^a_p\bar\alpha^p_b = \delta^a_b \qquad \bar\alpha^a_{bc} = -\bar\alpha^a_d\alpha^d_{pq}\bar\alpha^p_b\bar\alpha^q_c \tag{7}$$

$$(\alpha^a_b, \alpha^a_{bc}) = (J^a_b, J^a_{bc})$$

The group $\mathrm{GL}^1(m)$ is the usual general linear group $\mathrm{GL}(m)$

Tensors transform as $\qquad$ read this as a group left action

$$t'^{\mu\nu}_\alpha = J^\mu_\rho J^\nu_\sigma t^{\rho\sigma}_\epsilon \bar J^\epsilon_\alpha \qquad \lambda : \mathrm{GL}(m) \times T \to T : \left(J^a_b, t^{ab}_c\right) \mapsto t'^{ab}_c = J^a_p J^b_q t^{pq}_d \bar J^d_c$$

Tensor densities (of weight w) transform as $\qquad$ read this as a group representation

$$\tau'^{\mu\nu}_\alpha = \bar J^w J^\mu_\rho J^\nu_\sigma \tau^{\rho\sigma}_\epsilon \bar J^\epsilon_\alpha \qquad \lambda : \mathrm{GL}(m) \times T^w \to T^w : \left(J^a_b, \tau^{ab}_c\right) \mapsto \tau'^{ab}_c = \bar J^w J^a_p J^b_q \tau^{pq}_d \bar J^d_c$$

Connections on M transform as $\qquad$ read this as a group representation

$$\Gamma'^\mu_{\alpha\beta} = J^\mu_\epsilon\left(\Gamma^\epsilon_{\rho\sigma}\bar J^\rho_\alpha\bar J^\sigma_\beta + \bar J^\epsilon_{\alpha\beta}\right) \qquad \lambda : \mathrm{GL}^2(m) \times C \to C : ((J^a_b, J^a_{bc}), \Gamma^a_{bc}) \mapsto \Gamma'^a_{bc} = J^a_d\left(\Gamma^d_{pq}\bar J^p_b\bar J^q_c + \bar J^d_{bc}\right)$$

Crush course on principal bundles

$[P \to M]$ is a *principal bundle* iff a vertical, transitive on fibers, and free right action of a Lie group G on P.

$\Phi \in \mathrm{Aut}(P)$ $\qquad R_g : P \to P : p \mapsto p \cdot g$

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d] & P \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d, "R_g"'] & P \arrow[d, "R_g"] \\ P \arrow[r, "\Phi"] & P \end{tikzcd}$$

Automorphisms: $(\Phi, \varphi) : P \to P \in \mathrm{Aut}(P)$, $\qquad R_g \circ \Phi = \Phi \circ R_g \qquad (\Phi, \varphi) \in \mathrm{Aut}_V(P) : \varphi = \mathrm{id}_M$

$$\mathbb{I} \to \mathrm{Aut}_V(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M) \to \mathbb{I} \tag{8}$$

Let $\sigma : M \to P$ a local section on $U \subset M$ $\qquad p = \sigma(x) \cdot g \qquad t : \pi^{-1}(U) \to U \times G : p \mapsto (x, g)$

$\sigma'(x) = \sigma(x) \cdot \overline{f(x)}$ $\qquad$ Transition functions (or automorphisms) on P are $x' = x'(x)$ and $g' = f(x) \cdot g$

A *connection* on a bundle $[C \to M]$ is a sub-bundle $[H \to C]$ of $[TC \to C]$ such that $H_b \oplus V_b = T_b C$.

It is a *principal connection* if $TR_g(H_p) = H_{p \cdot g}$ $\qquad$ Horizontal lift $\omega = dx^\mu \otimes (\partial_\mu - \omega^A_\mu(x)\rho_A) : T_x M \to H_p \subset T_p P$

For any action $\lambda : G \to \mathrm{Diff}(F)$, one can define a functor $(\cdot) \times_\lambda F$ $\qquad P \times_\lambda F = (P \times F)\backslash_\lambda$ is *associated bundle*

$\Phi_\lambda : P \times_\lambda F \to P \times_\lambda F : [p, f]_\lambda \mapsto [\Phi(p), f]_\lambda$ $\qquad \mathrm{Aut}(P) \subset \mathrm{Aut}(P \times_\lambda F)$

The structure bundles

Consider maps $\epsilon : \mathbb{R}^m \to M$ such that $\epsilon(0) = x$ and locally invertible around the origin.

$j^r_x\epsilon \in L^r(M)$ form a principal bundle $\qquad j^r_x\epsilon \cdot j^r\alpha = j^r_x(\epsilon \circ \alpha)$

Natural lift $\varphi : M \to M$ $\qquad \hat\varphi = L^s(\varphi) : L^r(M) \to L^r(M) : j^r_x\epsilon \mapsto j^r_{\varphi(x)}(\varphi \circ \epsilon)$

$(\varphi_1 \circ \varphi_2)\hat{} = \hat\varphi_1 \circ \hat\varphi_2$ $\qquad$ lifts flows on M to flows on $L^r(M)$ $\qquad$ Infinitesimally, $\qquad [\xi_1, \xi_2]\hat{} = [\hat\xi_1, \hat\xi_2]$

$$\begin{tikzcd} \mathbb{I} \arrow[r] & \mathrm{Aut}_V(L^r(M)) \arrow[r] & \mathrm{Aut}(L^r(M)) \arrow[r] & \mathrm{Diff}(M) \arrow[r] \arrow[l, bend left, "\hat{(\cdot)}"] & \mathbb{I} \end{tikzcd} \tag{9}$$

and, infinitesimally, we have

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{X}_V(L^r(M)) \arrow[r] & \mathfrak{X}(L^r(M)) \arrow[r] & \mathfrak{X}(M) \arrow[r] \arrow[l, bend left, "\hat{(\cdot)}"] & 0 \end{tikzcd} \tag{10}$$

If we consider an associated bundle $C := L^r(M) \times_\lambda F$

we have the group $\varphi \in \mathrm{Diff}(M)$ acting on the structure bundle $\hat\varphi \in \mathrm{Aut}(L^r(M))$,

on the associated bundle $(\hat\varphi)_\lambda \in \mathrm{Aut}(C)$,

and on fields $\varphi_* : \mathrm{Sec}(C) \to \mathrm{Sec}(C)$ $\qquad$ *that's all functorial, folks!*

$$\sigma' = \varphi_*\sigma = (\hat\varphi)_\lambda \circ \sigma \circ \bar\varphi \tag{11}$$

If one wants general covariance, Diff(M) needs to act on fields in the first place

Lie derivatives and covariant derivatives

A spacetime vector field $\xi \in \mathfrak{X}(M)$, can be lifted to the structure bundle $\hat{\xi} \in \mathfrak{X}(L^r(M))$ just functoriality
Then $\hat{\xi}$ on the structure bundle $L^r(M)$, induces a vector field Ξ_λ on any associated bundle $C = L^r(M) \times_\lambda F$.
The vector field Ξ_λ has a fibered flow $\Phi_\epsilon : C \to C$.
It *drags* a section $\sigma : M \to C$ as

$$\sigma_\epsilon = \Phi_\epsilon \circ \sigma \circ \bar{\varphi}_\epsilon \qquad \frac{d\sigma_\epsilon}{d\epsilon}\Big|_{\epsilon=0} = T\sigma(\xi) - \Xi_\lambda \circ \sigma = \pounds_\xi \sigma \tag{12}$$

which is called the *Lie derivative* of the section (wrt ξ).
The Lie derivative is a vertical vector field on σ, i.e. a section of $[V(C) \to C \to M]$.
It captures how much the section is deformed when dragged along ξ.
It is natural

$$[\pounds_\xi, \pounds_\zeta]\sigma \simeq \pounds_{[\xi,\zeta]}\sigma \tag{13}$$

Given a connection $\hat{\omega}$ on $L^r(M)$, that induces a connection ω on any associated bundle $C \to M$.
We define the *covariant derivative*

$$\nabla_\xi \sigma = T\sigma(\xi) - \omega(\xi) \qquad \nabla_\xi y^i = \xi^\mu \left(y^i_\mu - \omega^i_\mu(x, y) \right) \tag{14}$$

This is also a vertical vector on σ.

Geometric objects of order r

A *geometric object* of order r is a section of an associated bundle $[C \to M]$ to $L^r(M)$
Transformation laws on $L^r(M)$ depend on Jacobians up to order r $\quad (J^\alpha_\mu, J^\alpha_{\mu_1\mu_2}, \dots, J^\alpha_{\mu_1\mu_2\dots\mu_r})$
The vector field $(\hat{\xi})_\lambda$ expands on $\xi^\alpha, d_\mu\xi^\alpha, \dots, d_{\mu_1\mu_2\dots\mu_r}\xi^\alpha$. When we have a connection Γ on M we can

$$\nabla_\mu\xi^\alpha = d_\mu\xi^\alpha + \Gamma^\alpha_{\beta\mu}\xi^\beta \qquad \nabla_{\mu_1\mu_2}\xi^\alpha = \nabla_{(\mu_1}\nabla_{\mu_2)}\xi^\alpha \qquad \nabla_{\mu_1\mu_2\mu_3}\xi^\alpha \qquad \dots \tag{15}$$

and we have $(\hat{\xi})_\lambda = \xi^\mu(x)\partial_\mu + \xi^i(x,y)\partial_i$

$$\xi^i = \hat{A}^i_\alpha\xi^\alpha + \hat{A}^{i\mu}_\alpha\nabla_\mu\xi^\alpha + \hat{A}^{i\mu_1\mu_2}_\alpha\nabla_{\mu_1\mu_2}\xi^\alpha + \dots + \hat{A}^{i\mu_1\dots\mu_r}_\alpha\nabla_{\mu_1\dots\mu_r}\xi^\alpha \tag{16}$$

The Lie derivative

$$\pounds_\xi\sigma = T\sigma(\xi) - \hat{\xi}\circ\sigma = \left(\hat{L}^i_\alpha\xi^\alpha + \hat{L}^{i\mu}_\alpha\nabla_\mu\xi^\alpha + \dots + \hat{L}^{i\mu_1\dots\mu_r}_\alpha\nabla_{\mu_1\dots\mu_r}\xi^\alpha\right)\partial_i \tag{17}$$

Natural Theories

A *natural theory* is a field theory with fields which are geometric objects
Sections on a bundle $[C \to M]$ associated to some $L^r(M)$ it supports an action of $\mathrm{Diff}(M)$

- $\varphi \in \mathrm{Diff}(M)$ are Lagrangian symmetries
- $\Gamma : J^1C \to \mathrm{Con}(M)$ one can define a connection out of fields (and their derivatives)

The covariance identity holds true for *any* spacetime vector field ξ

$$\langle \delta L | j^k \pounds_\xi y \rangle = d(\xi \rfloor L) \tag{18}$$

we have a Noether current for *any* spacetime vector field ξ

$$\mathcal{E} = \langle \mathbb{F} | j^{k-1} \pounds_\xi y \rangle - \xi \rfloor L \qquad\qquad \mathcal{W} = -\langle \mathbb{E} | \pounds_\xi y \rangle \tag{19}$$

Noether currents expand up to at most order $k + r - 1$ work currents up to order r

$$\begin{aligned} \mathcal{E} &= \left(\mathcal{E}_i^{\,\epsilon} \pounds_\Xi y^i + \mathcal{E}_i^{\,\epsilon\mu} d_\mu \pounds_\Xi y^i + \ldots + \mathcal{E}_i^{\,\epsilon\mu_1\ldots\mu_{k-1}} d_{\mu_1\ldots\mu_{k-1}} \pounds_\Xi y^i \right) (d^m x)_\epsilon = \\ &= \left(\hat{T}^\epsilon_\alpha \xi^\alpha + \hat{T}^{\epsilon\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha \right) (d^m x)_\epsilon \end{aligned} \tag{20}$$

$$\begin{aligned} \mathcal{W} &= \left(\mathcal{W}_i \pounds_\Xi y^i + \mathcal{W}_i^{\,\mu} d_\mu \pounds_\Xi y^i + \ldots + \mathcal{W}_i^{\,\mu_1\ldots\mu_{k-1}} d_{\mu_1\ldots\mu_{k-1}} \pounds_\Xi y^i \right) (d^m x) = \\ &= \left(\hat{W}_\alpha \xi^\alpha + \hat{W}^\mu_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{W}^{\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha \right) (d^m x) \end{aligned} \tag{21}$$

Currents and their splittings

We consider *currents*

$$
\begin{aligned}
\mathcal{D} &= \left(D^{\epsilon\sigma}_\alpha \xi^\alpha + D^{\epsilon\sigma\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + D^{\epsilon\sigma\mu_1\ldots\mu_k}_\alpha \nabla_{\mu_1\ldots\mu_k}\xi^\alpha\right)(d^m x)_{\epsilon\sigma}\\
\mathcal{C} &= \left(C^{\epsilon}_\alpha \xi^\alpha + C^{\epsilon\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + C^{\epsilon\mu_1\ldots\mu_k}_\alpha \nabla_{\mu_1\ldots\mu_k}\xi^\alpha\right)(d^m x)_{\epsilon}\\
\mathcal{B} &= \left(B_\alpha \xi^\alpha + B^{\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + B^{\mu_1\ldots\mu_k}_\alpha \nabla_{\mu_1\ldots\mu_k}\xi^\alpha\right) d^m x
\end{aligned}
\tag{22}
$$

Any splitting is not uniquely defined since

$$
\mathcal{C} = \tilde{\mathcal{C}} + d\mathcal{D} = \left(\tilde{\mathcal{C}} + d\alpha\right) + d\left(\mathcal{D} - \alpha + d\theta\right) \tag{23}
$$

A current is Γ-*reduced* iff

$\mathcal{B}$ is always reduced

$$
C^{[\epsilon\mu]}_\alpha = 0 \qquad C^{[\epsilon\mu_1]\ldots\mu_k}_\alpha = 0 \qquad\qquad D^{\epsilon[\sigma\mu]}_\alpha = 0 \qquad D^{\epsilon[\sigma\mu_1]\ldots\mu_k}_\alpha = 0 \tag{24}
$$

$k = 1$

$\tilde{\Gamma} = \Gamma + K$

$$
\mathcal{C} = C^\epsilon_\alpha \xi^\alpha + C^{\epsilon\mu}_\alpha \nabla_\mu \xi^\alpha = \left(C^\epsilon_\alpha - C^{\epsilon\mu}_\beta K^\beta_{\alpha\mu}\right)\xi^\alpha + C^{\epsilon\mu}_\alpha \tilde{\nabla}_\mu \xi^\alpha = \tilde{C}^\epsilon_\alpha \xi^\alpha + \tilde{C}^{\epsilon\mu}_\alpha \tilde{\nabla}_\mu \xi^\alpha \tag{25}
$$

$k = 2$

$$
\tilde{\mathcal{C}}^{\mu\gamma}_\epsilon = \mathcal{C}^{\mu\gamma}_\epsilon + \mathcal{C}^{\mu\alpha\beta}_\epsilon K^\gamma_{\beta\alpha} \qquad\Rightarrow\qquad \tilde{\mathcal{C}}^{[\mu\gamma]}_\epsilon = \mathcal{C}^{[\mu\underline{\alpha\beta}}_\epsilon K^{\gamma]}_{\beta\alpha} \neq 0 \tag{26}
$$

For any reduced current $\tilde{\mathcal{C}}$

$\mathrm{div}\,\tilde{\mathcal{C}} = 0$ iff $\tilde{\mathcal{C}} = 0$

Any current $\mathcal{C}$ splits as $\tilde{\mathcal{C}} + d\mathcal{U}$ with $\tilde{\mathcal{C}}$ a reduced current

the splitting is unique up to a divergenceless current in $\mathcal{U}$

Noether theorem and superpotentials

The work current is always reduced it splits as $\mathcal{W} = \mathcal{B} + d\tilde{\mathcal{E}}$

$\mathcal{B}$ is reduced and $d\mathcal{W} = 0$, hence $d\mathcal{B} = 0$ then $\mathcal{B} = 0$ $\mathcal{W} = d\tilde{\mathcal{E}}$

The Noether current $\mathcal{E}$ can split as $\mathcal{E} = \hat{\mathcal{E}} + d\mathcal{U}$

By conservation

$$d\hat{\mathcal{E}} = d\mathcal{E} = \mathcal{W} = d\tilde{\mathcal{E}} \qquad \Rightarrow \qquad d(\hat{\mathcal{E}} - \tilde{\mathcal{E}}) = 0 \qquad \Rightarrow \qquad \hat{\mathcal{E}} \equiv \tilde{\mathcal{E}} \tag{27}$$

The Noether current uniquely splits as $\mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$ and the work current as $\mathcal{W} = d\tilde{\mathcal{E}}$.

Any natural theory has its own Bianchi identities. The current $\tilde{\mathcal{E}}$ is called the reduced current it vanishes on-shell

Then $\mathcal{E} - \tilde{\mathcal{E}}$ agrees with the Noether current on-shell, it is strongly conserved $d(\mathcal{E} - \tilde{\mathcal{E}}) = \mathcal{B} \equiv 0$

The current $\mathcal{U}$ is called the superpotential

for higher order currents it depends on the choice of the connection

For currents of second order

$$\mathcal{U} = \frac{1}{2}\left(\frac{4}{3}\mathcal{E}_{\ \epsilon}^{\mu\alpha\beta}\nabla_\beta\xi^\epsilon + \left(\mathcal{E}_{\ \epsilon}^{\mu\alpha} - \frac{2}{3}\nabla_\beta\mathcal{E}_{\ \epsilon}^{\mu\alpha\beta}\right)\xi^\epsilon\right)(d^m x)_{\mu\alpha} \tag{28}$$

Examples of natural theories

See notes

Exercises

Exercise: Define $\mathrm{Con}(M)$ as an associated bundle to $L^2(M)$.
Check how $\Phi \in \mathrm{Aut}(L^2(M))$ acts on $\mathrm{Con}(M)$

Define the natural lift $(\hat{\cdot}) : \mathrm{Diff}(M) \to \mathrm{Aut}(L^2(M)) : \varphi \mapsto \hat{\varphi}$
Check how $\varphi \in \mathrm{Diff}(M)$ acts on $\mathrm{Con}(M)$
Define a basis of right invariant vector fields on $L^2(M)$
Define a general principal connection ω on $L^2(M)$
Define the induced connection on $\mathrm{Con}(M)$
Define a *covariant derivative* of a connection on M !!!

If you need warming up first replace $L^2(M)$ with $L^1(M)$ and $\mathrm{Con}(M)$ with $T(M)$

Summary

$[L^r(M) \to M]$ principal bundles with group $\mathrm{GL}^r(m)$ Structure bundle

Natural lift: $\hat{(\cdot)} : \mathrm{Diff}(M) \to \mathrm{Aut}(L^r(M))$ $\hat{(\cdot)} : \mathfrak{X}(M) \to \mathfrak{X}_\pi(L^r(M))$

$[C := L^r(M) \times_\lambda F \to M]$ Associated bundle

Natural lift: $\hat{(\cdot)}_\lambda : \mathrm{Diff}(M) \to \mathrm{Aut}(C)$ $\hat{(\cdot)} : \mathfrak{X}(M) \to \mathfrak{X}_\pi(C)$

Lie derivative $\pounds_\xi \sigma = T\sigma(\xi) - \hat{\xi} \circ \sigma$ (about) $\pounds_\xi \sigma \in \mathfrak{X}_V(C)$

Connection on $L^1(M)$: $\Gamma = dx^\mu \otimes \left(\partial_\mu - \Gamma^\alpha_{\beta\mu} \rho^\beta_\alpha\right)$ Covariant derivative $\nabla_\xi \sigma = T\sigma(\xi) - \omega(\xi)$

Conservation laws and currents $d\mathcal{E} = \mathcal{W}$

$$\mathcal{W} = \mathcal{B} + d\tilde{\mathcal{E}} \qquad \mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$$

$$\begin{aligned} \mathcal{E} &= \left(\hat{T}^\epsilon_\alpha \xi^\alpha + \hat{T}^{\epsilon\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha\right) (d^m x)_\epsilon \\ \mathcal{W} &= \left(\hat{W}_\alpha \xi^\alpha + \hat{W}^\mu_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{W}^{\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha\right) (d^m x) \end{aligned} \tag{29}$$

Conservation laws in natural and gauge natural field theories

Lecture 3: Gauge natural theories

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Summary

$[L^r(M) \to M]$ principal bundles with group $\mathrm{GL}^r(m)$ — Structure bundle

Natural lift: $\hat{(\cdot)} : \mathrm{Diff}(M) \to \mathrm{Aut}(L^r(M))$ — $\hat{(\cdot)} : \mathfrak{X}(M) \to \mathfrak{X}_\pi(L^r(M))$

$[C := L^r(M) \times_\lambda F \to M]$ — Associated bundle

Natural lift: $\hat{(\cdot)}_\lambda : \mathrm{Diff}(M) \to \mathrm{Aut}(C)$ — $\hat{(\cdot)} : \mathfrak{X}(M) \to \mathfrak{X}_\pi(C)$

Lie derivative $\pounds_\xi \sigma = T\sigma(\xi) - \hat{\xi} \circ \sigma$ — (about) $\pounds_\xi \sigma \in \mathfrak{X}_V(C)$

Connection on $L^1(M)$: $\Gamma = dx^\mu \otimes \left(\partial_\mu - \Gamma^\alpha_{\beta\mu} \rho^\beta_\alpha\right)$ — Covariant derivative $\nabla_\xi \sigma = T\sigma(\xi) - \omega(\xi)$

Conservation laws and currents — $d\mathcal{E} = \mathcal{W}$

$\mathcal{W} = \mathcal{B} + d\tilde{\mathcal{E}} \qquad \mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$

$$\begin{aligned} \mathcal{E} &= \left(\hat{T}^\epsilon_\alpha \xi^\alpha + \hat{T}^{\epsilon\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha\right) (d^m x)_\epsilon \\ \mathcal{W} &= \left(\hat{W}_\alpha \xi^\alpha + \hat{W}^\mu_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{W}^{\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha\right) (d^m x) \end{aligned} \tag{1}$$

Example: standard GR on $[\mathrm{Lor}(M) \to M]$

How about gauge field theories that have an extra *internal* symmetry group? as Maxwell and Yang–Mills

- Fields need to support a representation of the group $\mathrm{Aut}(P)$ gauge natural bundles
- Any right invariant vector fields $\Xi \in \mathrm{Aut}(P)$ is a Lagrangian symmetry Second Noether theorem
- One has the group sequence

$$0 \to \mathrm{Aut}_V(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M) \to 0 \tag{2}$$

- One has a group homomorphism $\hat{\ } : \mathrm{Aut}(P) \to \mathrm{Aut}(C) : \Phi \mapsto \hat{\Phi}$,
  then $\mathrm{Aut}(P)$ acts on sections $\sigma' = \hat{\Phi} \circ \sigma \circ \varphi^{-1}$
- Infinitesimally one has $\hat{\ } : \mathfrak{X}_R(P) \to \mathfrak{X}_\pi(C) : \Xi \mapsto \hat{\Xi}$

$$0 \to \mathfrak{X}_V(C) \to \mathfrak{X}_\pi(C) \to \mathfrak{X}(M) \to 0 \tag{3}$$

No canonical splitting. $[\Xi_1, \Xi_2]\hat{\ } = [\hat{\Xi}_1, \hat{\Xi}_2]$

What is the most general bundle $[C \to M]$ with these properties? Gauge natural bundles
What extra properties has a field theory like that? Gauge natural theories

...

The groups

Consider maps $a:\mathbb{R}^m \to G$ such that $a(0)=g$.

$j^s a \in G^s(m)$ form a Lie group $\qquad j^s a \cdot j^s b = j^s(a\cdot b) \qquad (j^s a)^{-1} = j^s \bar a$

For $G^1(m)$ consider

$$a^a(x) \simeq a^a(0) + \partial_b a^a(0) x^b + \dots \qquad a^a = a^a(0) \qquad a^a_b = \partial_b a^a(0) \tag{4}$$

and expand $a\cdot b : x^b \mapsto c^a(x) = \pi^a(a(x), b(x))$

$$a\cdot b \simeq \pi^a(a,b) + \left(\partial^1_p \pi^a(a,b) a^p_b + \partial^2_p \pi^a(a,b) b^p_b\right) x^b + \dots \qquad (a^a, a^a_b)\cdot(b^a, b^d_e) = (\pi^a(a,b), \partial^1_p\pi^a(a,b)a^p_b + \partial^2_p \pi^a(a,b) b^p_b) \tag{5}$$

Spinor fields transform as $\psi' = S\psi$

read this as a group left action $\lambda : \mathrm{Spin}(r,s)\times V \to V : (S,\psi)\mapsto \psi' = S\psi$

frame fields transform as $e'^a_\mu = \ell^a_b(S) e^b_\alpha \bar J^\alpha_\mu$

read this as a group left action $\lambda : \mathrm{Spin}(r,s)\times \mathrm{GL}(m)\times V \to V : (S, J, e)\mapsto e' = \ell(S) e \bar J$

Gauge fields transform as $\omega'^A_\mu = \bar J^\alpha_\mu \left(\mathrm{ad}^A_B(\varphi)\omega^B_\alpha - \bar R^A_a(\varphi) d_\alpha \varphi^a\right)$

read this as a group left action $\lambda : G^1 \times \mathrm{GL}(m)\times T \to T : \left(\varphi, \varphi_a, J^a_b, \omega^A_a\right)\mapsto \omega'^A_a = \dots$

The structure bundles

We define $W^{(s,r)}P = J^s P \times_M L^r(M)$ with $s \le r$ which is a principal bundle for the group $G^s \rtimes \mathrm{GL}^r(m)$

The semi-direct product is defined using the canonical right action of $\mathrm{GL}^r(m)$ on G^s that one has when $s \le r$

$$j^s_0 a \cdot j^r \alpha = j^s_0 (a \circ \alpha) \tag{6}$$

A point in $W^{(s,r)}P = J^s P \times_M L^r(M)$ is a pair $(j^s_x \sigma, j^r_x \epsilon)$ and we have a canonical right action

$$(j^s_x \sigma, j^r_x \epsilon) \cdot (j^s_0 a, j^r_0 \alpha) = (j^s_x(\sigma \cdot (a \circ \bar{\alpha} \circ \bar{\epsilon})), j^r_x(\epsilon \circ \alpha)) \tag{7}$$

Canonical lift of $\Phi \in \mathrm{Aut}(P)$ to $W^{(s,r)}P$

$$W^{(s,r)}\Phi : W^{(s,r)}P \to W^{(s,r)}P : (j^s_x \sigma, j^r_x \epsilon) \mapsto (j^s_x(\Phi \circ \sigma \circ \bar{\varphi}), j^r_x(\varphi \circ \epsilon)) \tag{8}$$

$$\mathbb{I} \longrightarrow \mathrm{Aut}_V(P) \longrightarrow \mathrm{Aut}(P) \longrightarrow \mathrm{Diff}(M) \longrightarrow \mathbb{I} \tag{9}$$

and, infinitesimally, we have

$$0 \longrightarrow \mathfrak{X}^R_V(P) \longrightarrow \mathfrak{X}^R_\pi(P) \longrightarrow \mathfrak{X}(M) \longrightarrow 0 \tag{10}$$

If we consider an associated bundle $C := W^{(s,r)}(P) \times_\lambda F$

we have the group $\Phi \in \mathrm{Aut}(P)$ acting on the structure bundle $\hat{\Phi} \in \mathrm{Aut}(W^{(s,r)}P)$,

and on the associated bundle $(\hat{\Phi})_\lambda \in \mathrm{Aut}(C)$,

and on fields $\Phi_* : \mathrm{Sec}(C) \to \mathrm{Sec}(C)$ *that's all functorial, folks!*

$$\sigma' = \Phi_* \sigma = (\hat{\Phi})_\lambda \circ \sigma \circ \bar{\varphi} \tag{11}$$

If one wants gauge covariance, Aut(P) needs to act on fields in the first place.

Lie derivatives and covariant derivatives

A projectable right invariant vector field $\Xi \in \mathfrak{X}^R_\pi(P)$ can be lifted to $\hat{\Xi} \in \mathfrak{X}^R_\pi(W^{(s,r)}P)$ just functoriality
Then $\hat{\Xi}$, induces a vector field Ξ_λ on any associated bundle $C = W^{(s,r)}(P) \times_\lambda F$.
The vector field Ξ_λ has a fibered flow $\Phi_\epsilon : C \to C$.
It *drags* a section $\sigma : M \to C$ as

$$\sigma_\epsilon = \Phi_\epsilon \circ \sigma \circ \bar{\varphi}_\epsilon \qquad\qquad \frac{d\sigma_\epsilon}{d\epsilon}\Big|_{\epsilon=0} = T\sigma(\xi) - \Xi_\lambda \circ \sigma = \pounds_\Xi \sigma \tag{12}$$

which is the *Lie derivative* of the section (wrt a projectable right invariant field Ξ on the structure bundle P).
The Lie derivative is a vertical vector field on σ, i.e. a section of $[V(C) \to C \to M]$.
It captures how much the section is deformed when dragged along Ξ.
It is natural

$$[\pounds_{\Xi_1}, \pounds_{\Xi_2}]\sigma = \pounds_{[\Xi_1, \Xi_2]}\sigma \tag{13}$$

Given a connection $\hat{\omega}$ on $W^{(s,r)}P$, that induces a connection ω on any associated bundle $[C \to M]$.
We define the *covariant derivative*

$$\nabla_\xi \sigma = T\sigma(\xi) - \omega(\xi) \qquad\qquad \nabla_\xi y^i = \xi^\mu \left(y^i_\mu - \omega^i_\mu(x,y)\right) \tag{14}$$

This is also a vertical vector on σ.

Gauge natural objects of order (r, s)

A *gauge natural object* of order (s, r) is a section of an associated bundle $[C \to M]$ to $W^{(s,r)}(P)$
Transformation laws on $W^{(s,r)}(P)$ depend on Jacobians up to order r $\qquad (J^\alpha_\mu, J^\alpha_{\mu_1\mu_2}, \dots, J^\alpha_{\mu_1\mu_2\dots\mu_r})$
and derivatives up to order s of the group element $\qquad (g^a, g^a_\mu, \dots, g^a_{\mu_1\dots\mu_s})$
The vector field $(\hat{\xi})_\lambda$ expands on $\xi^\alpha, d_\mu\xi^\alpha, \dots, d_{\mu_1\mu_2\dots\mu_r}\xi^\alpha$. When we have a connection Γ on M we can

$$\nabla_\mu\xi^\alpha = d_\mu\xi^\alpha + \Gamma^\alpha_{\beta\mu}\xi^\beta \qquad \nabla_{\mu_1\mu_2}\xi^\alpha = \nabla_{(\mu_1}\nabla_{\mu_2)}\xi^\alpha \qquad \nabla_{\mu_1\mu_2\mu_3}\xi^\alpha \qquad \dots \tag{15}$$

and we have $(\hat{\Xi})_\lambda = \xi^\mu(x)\partial_\mu + \xi^i(x, y)\partial_i$

$$\begin{aligned}\Xi &= \xi^\mu(x)\partial_\mu + \xi^A(x)\rho_A\\ &= \xi^\mu(x)(\partial_\mu - \omega^A_\mu\rho_A) + (\xi^A + \xi^\mu\omega^A_\mu)\rho_A\end{aligned}$$

$$\begin{aligned}\xi^i =&\hat{A}^i_\alpha\xi^\alpha + \hat{A}^{i\mu}_\alpha\nabla_\mu\xi^\alpha + \hat{A}^{i\mu_1\mu_2}_\alpha\nabla_{\mu_1\mu_2}\xi^\alpha + \dots + \hat{A}^{i\mu_1\dots\mu_r}_\alpha\nabla_{\mu_1\dots\mu_r}\xi^\alpha +\\ &+\hat{A}^i_A\xi^A_{(V)} + \hat{A}^{i\mu}_A\nabla_\mu\xi^A_{(V)} + \hat{A}^{i\mu_1\mu_2}_A\nabla_{\mu_1\mu_2}\xi^A_{(V)} + \dots + \hat{A}^{i\mu_1\dots\mu_s}_A\nabla_{\mu_1\dots\mu_s}\xi^A_{(V)}\end{aligned} \tag{16}$$

The Lie derivative

$$\begin{aligned}\pounds_\xi y^i =&\hat{L}^i_\alpha\xi^\alpha + \hat{L}^{i\mu}_\alpha\nabla_\mu\xi^\alpha + \dots + \hat{L}^{i\mu_1\dots\mu_r}_\alpha\nabla_{\mu_1\dots\mu_r}\xi^\alpha +\\ &+\hat{L}^i_A\xi^A_{(V)} + \hat{L}^{i\mu}_A\nabla_\mu\xi^A_{(V)} + \dots + \hat{L}^{i\mu_1\dots\mu_s}_A\nabla_{\mu_1\dots\mu_s}\xi^A_{(V)}\end{aligned} \tag{17}$$

Gauge Natural Theories

A *gauge natural theory* is a field theory with fields which are geometric objects
Sections on a bundle $[C \to M]$ associated to some structure bundle P it supports an action of $\mathrm{Aut}(P)$

- $\Phi \in \mathrm{Aut}(P)$ are Lagrangian symmetries
- $\Gamma : J^1C \to \mathrm{Con}(M)$ one can define a connection on M out of fields (and their derivatives)
- $\omega : J^1C \to \mathrm{Con}(P)$ one can define a connection of P out of fields (and their derivatives)

The covariance identity holds true for *any* spacetime vector field ξ

$$\langle \delta L | j^k \pounds_\xi y \rangle = d(\xi \rfloor L) \tag{18}$$

we have a Noether current for *any* right invariant vector field Ξ on $[P \to M]$

$$\mathcal{E} = \langle \mathbb{F} | j^{k-1} \pounds_\Xi y \rangle - \xi \rfloor L \qquad\qquad \mathcal{W} = -\langle \mathbb{E} | \pounds_\Xi y \rangle \tag{19}$$

Noether currents expand up to at most order $k + r - 1$ work currents up to order r

$$\begin{aligned}\mathcal{E} = &\left(\hat{T}^\epsilon_\alpha \xi^\alpha + \hat{T}^{\epsilon\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha \right) (d^m x)_\epsilon + \\ &+ \left(\hat{T}^\epsilon_A \xi^A_{(V)} + \hat{T}^{\epsilon\mu}_A \nabla_\mu \xi^A_{(V)} + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+s-1}}_A \nabla_{\mu_1\ldots\mu_{k+s-1}} \xi^A_{(V)} \right) (d^m x)_\epsilon\end{aligned} \tag{20}$$

Noether theorem and superpotentials

Nothing changes for superpotentials, except one starts from double long currents.

A current is *reduced* iff $\hat{T}_{\alpha}^{[\epsilon\mu_1]\dots\mu_n} = 0$ and $\hat{T}_{A}^{[\epsilon\mu_1]\dots\mu_n} = 0$ being reduced depends on the connection one chosses

The work current is always reduced

Bianchi identities: any reduced current $\tilde{\mathcal{W}}$ is exact $\tilde{\mathcal{W}} = d\tilde{\mathcal{E}}$

One can algorithmically split any current $\mathcal{W} = \mathcal{B} + d\tilde{\mathcal{E}}$, uniquely if $\mathcal{B}$ is required to be reduced.

In particular, the Noether current $\mathcal{E}$ can split as $\mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$ and the work term splits as $\mathcal{W} = d\tilde{\mathcal{E}}$ (since it is reduced)

The Noether current uniquely splits as $\mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$ and the work current as $\mathcal{W} = d\tilde{\mathcal{E}}$.
Any gauge natural theory has its own Bianchi identities.
The current $\tilde{\mathcal{E}}$ is called the reduced current
it vanishes on-shell
Then $\mathcal{E} - \tilde{\mathcal{E}}$ agrees with the Noether current on-shell, it is strongly conserved $d(\mathcal{E} - \tilde{\mathcal{E}}) = \mathcal{B} \equiv 0$
The current $\mathcal{U}$ is called the superpotential it depends on the choice of the connection

Summary

Structure bundle: $[P \to M]$ principal bundles with group G canonical lift $W^{(s,r)}P = J^sP \rtimes L^r(M)$

Canonical lift: $\hat{(\cdot)} : \mathrm{Aut}(P) \to \mathrm{Aut}(W^{(s,r)}P)$ $\hat{(\cdot)} : \mathfrak{X}_R(P) \to \mathfrak{X}_R(W^{(s,r)}P)$

$[C := W^{(s,r)}P \times_\lambda F \to M]$ Associated bundle

Natural lift: $\hat{(\cdot)}_\lambda : \mathrm{Aut}(P) \to \mathrm{Aut}(C)$ $\hat{(\cdot)} : \mathfrak{X}_R(P) \to \mathfrak{X}_\pi(C)$

Lie derivative $\pounds_\Xi \sigma = T\sigma(\xi) - \hat{\hat{\Xi}} \circ \sigma$ (about) $\pounds_\Xi \sigma \in \mathfrak{X}_V(C)$

Connection on P: $\omega = dx^\mu \otimes \left(\partial_\mu - \omega^A_\mu \rho_A\right)$ $\Xi = \xi^\mu(\partial_\mu - \omega^A \rho_A) \oplus \Xi_V$

Conservation laws and currents $d\mathcal{E} = \mathcal{W}$

$\mathcal{W} = \mathcal{B} + d\tilde{\mathcal{E}} \qquad \mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$

$$\begin{aligned}
\mathcal{E} = \big(&\hat{T}^\epsilon_\alpha \xi^\alpha + \hat{T}^{\epsilon\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha + \\
&+ \hat{T}^\epsilon_A \xi^A_V + \hat{T}^{\epsilon\mu}_A \nabla_\mu \xi^A_V + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+s-1}}_A \nabla_{\mu_1\ldots\mu_{k+s-1}} \xi^A_V\big)(d^m x)_\epsilon \\
\mathcal{W} = \big(&\hat{W}_\alpha \xi^\alpha + \hat{W}^\mu_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{W}^{\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha + \\
&+ \hat{W}_A \xi^A_V + \hat{W}^\mu_A \nabla_\mu \xi^A_V + \ldots + \hat{W}^{\mu_1\ldots\mu_{k+s-1}}_A \nabla_{\mu_1\ldots\mu_{k+s-1}} \xi^A_V\big)(d^m x)
\end{aligned} \tag{21}$$

Example: standard Yang-Mills on $[\mathrm{Lor}(M) \times_M \mathrm{Con}(P) \to M]$

Conservation laws in natural and gauge natural field theories

Lecture 4: Conserved quantities, cohomology, special relativity

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Summary

Structure bundle: $[P \to M]$ principal bundles with group G canonical lift $W^{(s,r)}P = J^sP \rtimes L^r(M)$

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Natural lift: $\hat{(\cdot)}_\lambda : \mathrm{Aut}(P) \to \mathrm{Aut}(C)$ $\hat{(\cdot)} : \mathfrak{X}_R(P) \to \mathfrak{X}_\pi(C)$

Lie derivative $\pounds_\Xi \sigma = T\sigma(\xi) - \hat{\Xi} \circ \sigma$ (about) $\pounds_\Xi \sigma \in \mathfrak{X}_V(C)$

Connection on P: $\omega = dx^\mu \otimes \left(\partial_\mu - \omega^A_\mu \rho_A\right)$ $\Xi = \xi^\mu(\partial_\mu - \omega^A \rho_A) \oplus \Xi_V$

Conservation laws and currents $d\mathcal{E} = \mathcal{W}$

$\mathcal{W} = \mathcal{B} + d\tilde{\mathcal{E}} \qquad \mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$

$$\begin{aligned}
\mathcal{E} = \big(&\hat{T}^\epsilon_\alpha \xi^\alpha + \hat{T}^{\epsilon\mu}_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha + \\
&+ \hat{T}^\epsilon_A \xi^A_V + \hat{T}^{\epsilon\mu}_A \nabla_\mu \xi^A_V + \ldots + \hat{T}^{\epsilon\mu_1\ldots\mu_{k+s-1}}_A \nabla_{\mu_1\ldots\mu_{k+s-1}} \xi^A_V\big)(d^m x)_\epsilon \\
\mathcal{W} = \big(&\hat{W}_\alpha \xi^\alpha + \hat{W}^\mu_\alpha \nabla_\mu \xi^\alpha + \ldots + \hat{W}^{\mu_1\ldots\mu_{k+r-1}}_\alpha \nabla_{\mu_1\ldots\mu_{k+r-1}} \xi^\alpha + \\
&+ \hat{W}_A \xi^A_V + \hat{W}^\mu_A \nabla_\mu \xi^A_V + \ldots + \hat{W}^{\mu_1\ldots\mu_{k+s-1}}_A \nabla_{\mu_1\ldots\mu_{k+s-1}} \xi^A_V\big)(d^m x)
\end{aligned} \tag{1}$$

Example: standard Yang-Mills on $[\mathrm{Lor}(M) \times_M \mathrm{Con}(P) \to M]$

Contents

How one can use Noether currents to compute conserved quantities?

In mechanics, Noether currents are *functions* (0-forms) constant on solutions $dF = 0$.
If the system starts on a level set $F = F_0$, it stays on the level set.

In field theories, Noether currents $\mathcal{E}$ are $(m-1)$-forms to be evaluated along a solution and closed $d\mathcal{E} = 0$
For a Lie group G of symmetries we have *first Noether theorem*

$$\Xi = \xi^A \Xi_A \in \mathfrak{g} \qquad \Rightarrow \qquad \mathcal{E} = \xi^A \mathcal{E}_A \tag{2}$$

Let $D \subset M$ be a compact m-region and compact border $\partial D \subset M$ of dimension $m-1$.

$$Q_A = \int_{\partial D} (j^1\sigma)^* \mathcal{E}_A \qquad\qquad Q = Q^A T_A \in \mathfrak{g} \tag{3}$$

is the conserved current within D.

What if one has pointlike charges?

Compare with second Noether theorem

Compare with special relativity.

Mechanics

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\alpha}{\sqrt{x^2+y^2}} \tag{4}$$

SO(2) are symmetries $\quad \Xi = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad \hat{\Xi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + \dot{x}\frac{\partial}{\partial \dot{y}} - \dot{y}\frac{\partial}{\partial \dot{x}}$

The *angular momentum* is conserved along solutions

$$J = x\dot{y} - y\dot{x} \tag{5}$$

$t' = t + \epsilon$ are symmetries $\quad \Xi = \frac{\partial}{\partial t}, \qquad \hat{\Xi} = \frac{\partial}{\partial t} \qquad$ The *energy* is conserved

$$\mathcal{H} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\alpha}{\sqrt{x^2+y^2}} \tag{6}$$

Ex: generalize to rotations in dimension 3 to the group SO(3) $\quad \Xi_i \quad \rightsquigarrow \quad J_i \qquad [\Xi_i, \Xi_j] = \epsilon_{ij}{}^k \Xi_k$

In Hamiltonian formalism $\qquad \{J_i, J_j\} = \epsilon_{ij}{}^k J_k$

First Noether theorem

$[M \times \mathbb{C} \to M]$ with coordinates (x^μ, φ) $[\mathrm{Lor}(M) \to M]$ with coordinates $(x^\mu, g_{\alpha\beta})$

$$L = -\frac{\sqrt{g}}{2} \left(\nabla_\alpha \bar{\varphi} g^{\alpha\beta} \nabla_\beta \varphi + m^2 \bar{\varphi} \varphi \right) \tag{7}$$

$$\begin{aligned} \delta L = &- \frac{\sqrt{g}}{2} T_{\mu\nu} \delta g^{\alpha\beta} - \frac{\sqrt{g}}{2} \left(\nabla_\alpha \bar{\varphi} g^{\alpha\beta} \nabla_\beta \delta\varphi + m^2 \bar{\varphi} \delta\varphi \right) - \frac{\sqrt{g}}{2} \left(\nabla_\alpha \delta\bar{\varphi} g^{\alpha\beta} \nabla_\beta \varphi + m^2 \delta\bar{\varphi} \varphi \right) \\ T_{\alpha\beta} = &\nabla_\alpha \bar{\varphi} \nabla_\beta \varphi - \frac{1}{2} \left(\nabla_\mu \bar{\varphi} g^{\mu\nu} \nabla_\nu \varphi + m^2 \bar{\varphi} \varphi \right) g_{\alpha\beta} \qquad \Box = g^{\alpha\beta} \nabla_\beta \nabla_\alpha \end{aligned} \tag{8}$$

$$\mathbb{E} = \frac{\sqrt{g}}{2} \left(\delta\bar{\varphi} \left(\Box\varphi - m^2 \varphi \right) + \left(\Box\bar{\varphi} - m^2 \bar{\varphi} \right) \delta\varphi - T_{\mu\nu} \delta g^{\mu\nu} \right) (d^m x) \tag{9}$$

$$\mathbb{F} = -\frac{\sqrt{g}}{2} \left(\nabla_\alpha \bar{\varphi} g^{\alpha\mu} \delta\varphi + \delta\bar{\varphi} g^{\mu\beta} \nabla_\beta \varphi \right) (d^m x)_\mu \tag{10}$$

$$\begin{aligned} \mathcal{E} = &- \frac{\sqrt{g}}{2} \left(\pounds_\Xi \bar{\varphi} g^{\mu\beta} \nabla_\beta \varphi + \nabla_\alpha \bar{\varphi} g^{\alpha\mu} \pounds_\Xi \varphi - \xi^\mu \left(\nabla_\alpha \bar{\varphi} g^{\alpha\beta} \nabla_\beta \varphi + m^2 \bar{\varphi} \varphi \right) \right) (d^m x)_\mu \\ \mathcal{W} = &\frac{\sqrt{g}}{2} \left(\pounds_\Xi \bar{\varphi} \boxed{\left(\Box\varphi - m^2 \varphi \right)} + \boxed{\left(\Box\bar{\varphi} - m^2 \bar{\varphi} \right)} \pounds_\Xi \varphi - T_{\mu\nu} \pounds_\xi g^{\mu\nu} \right) (d^m x) \end{aligned} \tag{11}$$

$$d\mathcal{E} = -\frac{\sqrt{g}}{2} T_{\mu\nu} \boxed{\pounds_\xi g^{\mu\nu}} (d^m x) \qquad (\xi \text{ is Killing vectors})$$

First Noether theorem 2

$$\varphi'(x') = \varphi(x) \qquad \pounds_\xi \varphi = \xi^\mu \nabla_\mu \varphi \qquad \pounds_\xi \bar\varphi = \xi^\mu \nabla_\mu \bar\varphi \qquad \pounds_\xi g_{\mu\nu} = \nabla_\mu \xi^\alpha g_{\alpha\nu} + g_{\mu\alpha}\nabla_\nu \xi^\alpha$$

$$\varphi'(x) = e^{iq\alpha}\varphi(x) \qquad \pounds_\Xi \varphi = iq\varphi\zeta \qquad \pounds_\Xi \bar\varphi = -iq\bar\varphi\zeta \qquad \pounds_\Xi g_{\mu\nu} = 0$$

$$\begin{aligned}\mathcal{E}[\xi] =& -\sqrt{g}\xi^\nu g^{\mu\epsilon}\left(\tfrac{1}{2}\left(\nabla_\nu\bar\varphi\nabla_\epsilon\varphi + \nabla_\epsilon\bar\varphi\nabla_\nu\varphi\right) - \tfrac{1}{2}\left(\nabla_\alpha\bar\varphi g^{\alpha\beta}\nabla_\beta\varphi + m^2\bar\varphi\varphi\right)g_{\epsilon\nu}\right)(d^mx)_\mu = \\ =& -\sqrt{g}g^{\mu\epsilon}T_{(\epsilon\nu)}\xi^\nu(d^mx)_\mu \qquad \begin{array}{ll} \xi = \partial_t & \text{Energy } E \\ \xi_i = \partial_i & \text{Momentum } P_i \\ R_i = \epsilon_{ij}{}^k x^j \partial_k & \text{Angular momentum } J_i \\ K_i = \delta_{ik}x^k\partial_0 + x^0\partial_i & \text{Current momentum } M_i \end{array}\end{aligned} \tag{12}$$

$$\mathcal{E}_2 = -iq\frac{\sqrt{g}}{2}g^{\mu\alpha}\left(-\bar\varphi\nabla_\alpha\varphi + \nabla_\alpha\bar\varphi\varphi\right)\zeta(d^mx)_\mu = -\zeta J \qquad \text{(Charge)}$$

Second Noether theorem

The Lagriangian $L = -\frac{\sqrt{g}}{2}\left(\nabla_\alpha\bar\varphi g^{\alpha\beta}\nabla_\beta\varphi + m^2\bar\varphi\varphi\right)$ is generally covariant! hence $\forall\xi \in \mathfrak{X}(M)$

$$\mathcal{E}[\xi] = -\sqrt{g}\xi^\nu(x)g^{\mu\epsilon}\left(\nabla_{(\nu}\bar\varphi\nabla_{\epsilon)}\varphi - \frac{1}{2}\left(\nabla_\alpha\bar\varphi g^{\alpha\beta}\nabla_\beta\varphi + m^2\bar\varphi\varphi\right)g_{\epsilon\nu}\right)(d^mx)_\mu \tag{13}$$

is conserved!

$[P \to M]$ a $U(1)$ structure bundle Second Noether theorem
On $\mathrm{Lor}(M) \times_M \mathrm{Con}(P) \times_M (M \times \mathbb{C})$ with coordinates $(x^\mu, g_{\alpha\beta}, A_\mu, \varphi)$ $\nabla_\alpha\varphi = d_\alpha\varphi - iqA_\alpha\varphi$
$\varphi' = e^{iq\alpha(x)}\varphi$ are now symmetries.

$$\mathcal{E}_2 = -iq\frac{\sqrt{g}}{2}g^{\mu\alpha}\left(-\bar\varphi\nabla_\alpha\varphi + \nabla_\alpha\bar\varphi\varphi\right)\zeta(x)(d^mx)_\mu = -\zeta(x)J \tag{14}$$

In GR any spacetime vector field is a symmetry, no need to restrict to Killing vectors

Killing Killing vectors

Consider the Lagrangian $\boldsymbol{L} = \boldsymbol{L}_H(j^2g) + \boldsymbol{L}_M(g, j^1A) + \boldsymbol{L}_{KG}(g, A, j^1\varphi)$
When we compute field equations we have

$$\begin{aligned}\delta \boldsymbol{L}_H =& \tfrac{\sqrt{g}}{2} G_{\alpha\beta}\delta g^{\alpha\beta} + d\,()\\ \delta \boldsymbol{L}_M =& -\tfrac{\sqrt{g}}{2} T^{em}_{\alpha\beta}\delta g^{\alpha\beta} - \nabla_\alpha(\sqrt{g}F^{\alpha\mu})\delta A_\mu + d\,()\\ \delta \boldsymbol{L}_{KG} =& -\tfrac{\sqrt{g}}{2} T^{KG}_{\alpha\beta}\delta g^{\alpha\beta} + \tfrac{\sqrt{g}}{2} J^\mu \delta A_\mu + \sqrt{g}(\Box\bar\varphi - m^2\bar\varphi)\delta\varphi + \sqrt{g}\delta\bar\varphi(\Box\varphi - m^2\varphi) + d\,()\end{aligned} \tag{15}$$

Field equations are then

$$G_{\alpha\beta} = T^{em}_{\alpha\beta} + T^{KG}_{\alpha\beta} \qquad \nabla_\alpha(\sqrt{g}F^{\alpha\mu}) = \tfrac{\sqrt{g}}{2} J^\mu \qquad \Box\varphi - m^2\varphi = 0 \tag{16}$$

We fix no field *all fields are dynamical*

In SR we consider $\boldsymbol{L}_M + \boldsymbol{L}_{KG}$ (or $\boldsymbol{L}_{KG}$) but we do not want to impose $T^{em}_{\alpha\beta} + T^{KG}_{\alpha\beta} = 0$ (or $J^\mu = 0$) as field eq.
Hence we impose $\delta g^{\alpha\beta} = 0$ (and $\delta A_\mu = 0$). Then field equations are only

$$\nabla_\alpha(\sqrt{g}F^{\alpha\mu}) = J^\mu \qquad \Box\varphi - m^2\varphi = 0 \tag{17}$$

or $\Box\varphi - m^2\varphi = 0$.
Accordingly, $\mathcal{W} = -\frac{\sqrt{g}}{2}\left(\left(T^{KG}_{\alpha\beta} + T^{em}_{\alpha\beta}\right)\pounds_\xi g^{\alpha\beta} + J^\mu \pounds_\Xi A_\mu\right)$. For conservation of Noether current one needs

$$\pounds_\xi g^{\alpha\beta} = 0 \qquad \pounds_\Xi A_\mu = 0 \tag{18}$$

GR interaction SR evolution on a background

Homological quantities

Forms as $\mathcal{E}$ are born to be integrated over $(m-1)$-regions

A *k-region* is a compact $\Omega_k \subset M$ of dimension k with a boundary $\partial\Omega_k = B_{k-1} \subset M$ which is a *closed* submanifold of dimension $k-1$.
We have

- k-regions $\Omega_k \subset M$ (submanifolds with boundary)
- k-submanifold $Z_k \subset M$ with no bounday (closed submanifolds)
- k-boundaries $B_k \subset M$ (closed submanifolds, boundaries of Ω_{k-1})

A disc without the origin is not a 2-region in $\mathbb{R}^2 - \{0\}$, a closed ring around the origin is. No improper integrals.

Let $\Omega_4 \subset M$ be a 4-region in spacetime. Its bundary $\partial\Omega_4 = B_3$ is a closed 3 submanifold.
Since the Noether current $\mathcal{E}$ is conserved, if B_3 is a boundary

$$\int_{B_3} \mathcal{E} = \int_{\partial\Omega_4} \mathcal{E} = \int_{\Omega_4} d\mathcal{E} = 0 \tag{19}$$

If $\mathcal{E}$ admits superpotential and Z_3 is closed

$$\int_{Z_3} \mathcal{E} = \int_{Z_3} d\mathcal{U} = \int_{\partial Z_3 = \varnothing} \mathcal{U} = 0 \tag{20}$$

On M, $\partial\Omega_4 = B_3 = \Omega_3' - \Omega_3 + \Sigma_3$ $\qquad\mathcal{E} = \rho\,(d^m x)_0 + j^i\,(d^m x)_i$

$$\begin{gathered}\int_{B_3} \mathcal{E} = \int_{\Omega_3'} \mathcal{E} - \int_{\Omega_3} \mathcal{E} + \int_{\Sigma_3} \mathcal{E} = 0 \qquad \Rightarrow \qquad Q' = Q - \int_{\Sigma_3} \mathcal{E} \\ d\mathcal{E} = (\dot\rho + \mathrm{div} j)\,(d^m x) = 0 \qquad\qquad \dot\rho + \mathrm{div}\vec{j} = 0 \qquad (\textit{continuity equation} \text{ for } Q = \textstyle\int_{\Omega_3} \mathcal{E})\end{gathered} \tag{21}$$

Homological quantities

We assumed fields to be smooth on M, we need to cut out the points where fields diverge. *(e.g. point–like particles)*

In $\mathbb{R}^2$ consider a circle $c : (x^2 + y^2 = r^2)$. That is a closed submanifold (with no boundary).
It is also the *boundary* ∂D of the disc $D \subset \mathbb{R}^2$.

Repeat the argument with $\mathbb{R}^2 - \{0\}$. The circle c is again a closed submanifold of $\mathbb{R}^2 - \{0\}$.
The interior is a punched disc $D_0 = D - \{0\}$. However the bondary of D_0 is now $\partial D_0 = c \cup \{0\}$ (or $\partial D_0 = c \cup (x^2 + y^2 = \epsilon^2)$).
Moreover, we here switch to *oriented boundaries* which are used in Stokes theorem.

If there are singularities of fields, even when they are cut out of M, integrals can be improper integrals.

Integrals work much better when we require forms to be smooth in (an openset cointaining) a compact integration region.

Defining $Q = \int_{\Omega_3} \mathcal{E}$ is not good idea. When $\mathcal{E} = d\mathcal{U}$ and Z_2 is a closed region ($\partial Z_2 = \varnothing$)

$$\boxed{Q := \int_{Z_2} \mathcal{U}} \qquad \Rightarrow \quad \int_{\Omega_3} \mathcal{E} = \int_{\Omega_3} d\mathcal{U} = \int_{Z_2' - Z_2} \mathcal{U} = \int_{Z_2'} \mathcal{U} - \int_{Z_2} \mathcal{U} = Q' - Q \tag{22}$$

Q is well defined even with pointwise particles provided they are not on Z_2

Similar to residual integrals no complex setting

Uniqueness results

Cohomology: $\mathcal{U}' = \mathcal{U} + d\alpha$ $\mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U} = d\mathcal{U}$

$$Q' = \int_{Z_2} \mathcal{U}' = \int_{Z_2} \mathcal{U} + d\alpha = \int_{Z_2} \mathcal{U} + \int_{\partial Z_2 = \varnothing} \alpha = Q \tag{23}$$

Conserved quantities are invariant.

If $\boldsymbol{L}' = \boldsymbol{L} + d_\epsilon \alpha^\epsilon$ $\mathcal{U}' = \mathcal{U} - \xi^{[\alpha}\alpha^{\beta]}(d^m x)_{\alpha\beta}$

Standard GR Komar superpotential $\mathcal{U} = \sqrt{g}\nabla^\alpha \xi^\beta (d^m x)_{\alpha\beta}$
Schwarzschild $E = \frac{1}{2}m$
Reisner-Norsdrom $E = \frac{1}{2}m$, $Q = q$
Kerr $E = \frac{1}{2}m$ and $J = ma$
...
Schwarzschild-(A)dS $E = \infty$!

Can we choose a Lgrangian representative to correct these misbehaviours?

Augmented variational principles

Consider X the infinitesimal generator of a curve tangent to the space of solutions.
The variation of the conserved current is expected to be (for many different motivations)

$$\delta_X \mathcal{Q} = \delta_X \mathcal{U} - \xi \lrcorner j^{k-1} X \lrcorner \mathbb{F} \tag{24}$$

For a theory in $[C \to M]$ we can double to configuration bundle $[C \times_M C \to M]$ with coordinates $(x, y^i, \bar{y}^i)$ and extend to it the Lagrangian

$$\boldsymbol{L}_{Aug}[y, \bar{y}] = \boldsymbol{L}[y] + d_\mu \alpha^\mu [y, \bar{y}] - \boldsymbol{L}[\bar{y}] \tag{25}$$

This is also another natural theory.
The superpotential

$$\mathcal{U}_{Aug} = \mathcal{U} - \xi \lrcorner \alpha - \bar{\mathcal{U}} \tag{26}$$

and we can require that along an integral curve of X based at $\bar{y}$ one has

$$\frac{d\alpha}{ds}(y(s), \bar{y})\Big|_{s=0} = j^{k-1} X \lrcorner \mathbb{F} \tag{27}$$

For standard GR, one has (using the affine structure on connections)

$$X \lrcorner \mathbb{F} = \sqrt{g} g^{\mu\nu} \delta \Gamma^\alpha_{\mu\nu} (d^m x)_\alpha \qquad \Rightarrow \qquad \alpha = \sqrt{g} g^{\mu\nu} (u^\alpha_{\mu\nu} - \bar{u}^\alpha_{\mu\nu}) (d^m x)_\alpha \tag{28}$$

Relative conserved quantities

The augmented variational principle for standard GR is then

$$\boldsymbol{L}_{Aug} = \left(\sqrt{g}R + \nabla_\alpha \left(\sqrt{\bar g}\bar g^{\mu\nu}(u^\alpha_{\mu\nu} - \bar u^\alpha_{\mu\nu})\right) - \sqrt{\bar g}\bar R\right) d^m x \tag{29}$$

Since $K = \Gamma - \bar\Gamma$ is a tensor, we have

$$w^\alpha_{\mu\nu} = u^\alpha_{\mu\nu} - \bar u^\alpha_{\mu\nu} = K^\alpha_{\mu\nu} - \delta^\alpha_{(\mu} K^\epsilon_{\nu)\epsilon} \tag{30}$$

When we integrate $\mathcal{U}_{Aug}$ on Z_2 and g induces on Z_2 the same metric of $\bar g$
this is interpreted as the *relative conserved quantities* i.e. needed to go from $\bar g$ to g.

Schwarzschild $E = m - m_0$
Reisner-Norsdrom $E = m - m_0,\ Q = q - q_0$
Kerr $E = m - m_0$ and $J = ma - m_0 a_0$
...
Schwarzschild-(A)dS $E = m - m0$

It works also on finite regions provided one finds another solution $\bar g$ which induces the same metric on Z_2

Going from a Lagrangian to its augmented one also restores covariance is it is ruined.

Entropy

Let us consider Kerr BH.
We want to define its entropy S (which is not conserved).
At some stage we would like to prove first law of termidynamics which, in this simple case, is

$$\delta m = \frac{1}{T}\delta S + \Omega \delta J \tag{31}$$

where T is the temperature of the Hawking radiation, Ω is the angular velocity.

More terms in more general cases.

Here m and J are Noether conserved currents.
The first law of termodynamics can be racast as

$$\delta S = T\left(\delta m - \Omega \delta J\right) \tag{32}$$

If we fix $\xi = \partial_t - \Omega \partial_\phi$, the corresponding Noether charge is

$$m - \Omega J \tag{33}$$

and one can use the first law of theormodynamics to compute S
once some definition of T is obtained from some other mean. $S = \frac{1}{4}A_H$

Notice that there is no reason to integrate Noether current on the horizon
since the Noether charge in homologically invariant

$$Q' - Q = \int_{Z_2' - Z_2} \mathcal{U} = \int_{\partial B_3} \mathcal{U} = \int_{B_3} d\mathcal{U} = \int_{B_3} \mathcal{E} \tag{34}$$

Summary

In the second Noether theorem, symmetries contains one parameter families

First Noether theorem applies as well.

One does not need Killing vectors in a relativistic (or gauge) theory.

Integral of the superpotential on a closed surface is a homological conserved quantity

One can fix the ambiguity of the Lagrangian to improve the description of conserved quantities
and eventually recovered a good framework for entropy of gravitational systems.

Watch out that the representative picked up for the Lagrangian is a different representative of the one picked in variational sequences.

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