Conservation laws in field theories

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We write here a long exercise to show how to compute what we did in the lectures. The material is mainly taken from

> L. Fatibene,
> *Relativistic theories, gravitational theories, and general relativity.*
> Unpublished, 2024; http://www.fatibene.org/book.html

and

> Lecture notes

The purpose is to show you how one approaches a field theory to deal with its conservation laws.

1. Hilbert Lagrangian (purely metric formalism)

This is a theory for a Lorentian metric with field equations which are Einstein equations.

First, we want to define a configuration bundle $[\mathrm{Lor}(M) \to M]$ the sections of which are in 1-to-1 correspondence with Lorentzian metrics on M.

This is standard kinematics.
Let L be such a set of bilinear symmetric non-degenerate forms on the vector space $\mathbb{R}^m$ with signature $(m-1,1)$. Given a basis e_a in $\mathbb{R}^m$, a point in $g \in L$ is described by a matrix $g = g_{ab}\, e^a \otimes e^b$.
Lorentian metrics g, given coordinates x^μ defined on $U \subset M$ which determine a natural basis ∂_μ in T_xM, are local maps

$$g : U \to L : x \mapsto g_{\mu\nu}(x) \tag{1.1}$$

Transformation laws under changes of coordinates are

$$g'_{\mu\nu} = \bar J^\alpha_\mu g_{\alpha\beta} \bar J^\beta_\nu \tag{1.2}$$

which can be used to infer a left action

$$\lambda : \mathrm{GL}(M) \times L \to L : (J^c_a, g_{ab}) \mapsto g'_{cd} = \bar J^a_c g_{ab} \bar J^b_d \tag{1.3}$$

Check this is a left action.
The we can consider the associated bundle $\mathrm{Lor}(M) = L(M) \times_\lambda L$. If we cover the frame bundle $L(M)$ with coordinates ϵ^μ_a for the frame $\epsilon_a = \epsilon^\mu_a \partial_\mu$, the associated bundle is covered by coordinates $(x^\mu, g_{\mu\nu})$ where

$$g_{\mu\nu} = \epsilon^a_\mu g_{ab} \epsilon^b_\nu \tag{1.4}$$

For any change f coordinates $x'^\mu = x'^\mu(x)$ in M that induces a coordinate change (in $L(M)$ given by $\epsilon'^\mu_a = J^\mu_\alpha \epsilon^\alpha_a$ and) in $\mathrm{Lor}(M)$ given by

$$x'^\mu = x'^\mu(x) \qquad\qquad g'_{\mu\nu} = \bar J^\alpha_\mu g_{\alpha\beta} \bar J^\beta_\nu \tag{1.5}$$

Accordingly, a global Lorentian metric corresponds to a global section $g : M \to \mathrm{Lor}(M)$, simply by glueing together local expressions, just because we cooked things so that the transition maps on the bundle agree with transformation laws of the metrics.

A diffeomorphism $\varphi : M \to M : x^\mu \mapsto x'^\mu(x)$ can be naturally lifted to $L(M)$

$$\hat{\varphi} : L(M) \to L(M) : (x^\mu, \epsilon^\mu_a) \mapsto (x'^\mu(x), \epsilon'^\mu_a) \qquad \epsilon'^\mu_a = J^\mu_\alpha \epsilon^\alpha_a \tag{1.6}$$

and then transfered to $\mathrm{Lor}(M)$ as

$$\mathrm{Lor}(\varphi) : \mathrm{Lor}(M) \to \mathrm{Lor}(M) : (x^\mu, g_{\alpha\beta}) \mapsto (x'^\mu(x), g'_{\mu\nu} = \bar{J}^\alpha_\mu g_{\alpha\beta} \bar{J}^\beta_\nu) \tag{1.7}$$

If we drag a section $g : M \to \mathrm{Lor}(M)$ by such a lift we get transformation laws of metrics

$$g' : M \to \mathrm{Lor}(M) : x^\mu \mapsto (x'^\mu, g'_{\mu\nu}(x') = \bar{J}^\alpha_\mu(x') g_{\alpha\beta}(x(x')) \bar{J}^\beta_\nu(x')) \tag{1.8}$$

Dynamics

We can fix a second order Lagrangian

$$\boldsymbol{L}_H = \tfrac{1}{2\kappa}\sqrt{g} R \, d^m x \tag{1.9}$$

where we set

$$\{g\}^\alpha_{\beta\mu} = \tfrac{1}{2} g^{\alpha\epsilon}\left(-d_\epsilon g_{\beta\mu} + d_\mu g_{\epsilon\beta} + d_\beta g_{\mu\epsilon}\right) \qquad R^\alpha{}_{\beta\mu\nu} = d_\mu \{g\}^\alpha_{\beta\nu} - d_\nu \{g\}^\alpha_{\beta\mu} + \{g\}^\alpha_{\epsilon\mu}\{g\}^\epsilon_{\beta\nu} - \{g\}^\alpha_{\epsilon\nu}\{g\}^\epsilon_{\beta\mu} \tag{1.10}$$

and $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$.

The variation of the Lagrangian wrt a deformation $X = \delta g^{\alpha\beta} \frac{\partial}{\partial g^{\alpha\beta}}$ gives us

$$\begin{aligned}
\left\langle \delta \boldsymbol{L}_H | j^2 X \right\rangle &= \tfrac{\sqrt{g}}{2\kappa}\left(\left(R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta}\right) \delta g^{\alpha\beta} + \left(g^{\mu\nu} g_{\alpha\beta} - \delta^\mu_{(\alpha}\delta^\nu_{\beta)}\right) \nabla_{\mu\nu} \delta g^{\alpha\beta} \right) (d^m x) \\
\left\langle \mathbb{E} | X \right\rangle &= \tfrac{\sqrt{g}}{2\kappa}\left(R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta}\right) \delta g^{\alpha\beta} (d^m x) \\
\left\langle \mathbb{F} | j^1 X \right\rangle &= \tfrac{\sqrt{g}}{2\kappa}\left(g^{\mu\nu} g_{\alpha\beta} - \delta^\mu_{(\alpha}\delta^\nu_{\beta)}\right) \nabla_\nu \delta g^{\alpha\beta} (d^m x)_\mu
\end{aligned} \tag{1.11}$$

Field equations are then *Einstein equations*

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} = 0 \tag{1.12}$$

which describe the evolution of gtavitational field $g_{\alpha\beta}(x)$ in vacuum.

Lie derivatives

In view of transformation laws wrt spacetimes diffeomorohisms we have the Lie derivatives

$$\pounds_\xi g_{\alpha\beta} = \boxed{\xi^\mu \nabla_\mu g_{\alpha\beta}} + \nabla_\alpha \xi^\epsilon g_{\epsilon\beta} + g_{\alpha\epsilon} \nabla_\beta \xi^\epsilon \qquad \pounds_\xi g^{\alpha\beta} = -\nabla_\epsilon \xi^\alpha g^{\epsilon\beta} - g^{\alpha\epsilon} \nabla_\epsilon \xi^\beta \tag{1.13}$$

using Levi-Civita connection.

Noether currents

By Noether theorem we can define the currents

$$
\begin{aligned}
\mathcal{W} =& - \tfrac{\sqrt{g}}{2\kappa} \left(R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta}\right) \pounds_\xi g^{\alpha\beta} (d^m x) = \\
=& \tfrac{\sqrt{g}}{\kappa} \left(R^\epsilon{}_\beta - \tfrac{1}{2} R \delta^\epsilon_\beta\right) \nabla_\epsilon \xi^\beta (d^m x) = \sqrt{g}\, \hat{W}^\mu_\alpha \nabla_\mu \xi^\alpha (d^m x) \\
\mathcal{E} =& \tfrac{\sqrt{g}}{2\kappa} \left(\left(g^{\mu\nu} g_{\alpha\beta} - \delta^\mu_{(\alpha} \delta^\nu_{\beta)} \right) \nabla_\nu \pounds_\xi g^{\alpha\beta} - \xi^\mu R \right) (d^m x)_\mu = \\
=& \tfrac{\sqrt{g}}{2\kappa} \left(\left(\delta^\mu_\alpha g^{\rho\sigma} - g^{\mu(\rho} \delta^{\sigma)}_\alpha \right) \nabla_{\rho\sigma} \xi^\alpha + \left(\tfrac{3}{2} R^\mu{}_\alpha - R \delta^\mu_\alpha \right) \xi^\alpha \right) (d^m x)_\mu = \\
=& \sqrt{g} \left(\hat{T}^{\mu\rho\sigma}_\alpha \nabla_{\rho\sigma} \xi^\alpha + \hat{T}^\mu_\alpha \xi^\alpha \right) (d^m x)_\mu
\end{aligned}
\tag{1.14}
$$

We can integrate by parts the work current

$$
\mathcal{W} = d \left(\tfrac{\sqrt{g}}{\kappa} \left(R^\mu{}_\alpha - \tfrac{1}{2} R \delta^\mu_\alpha \right) \xi^\alpha (d^m x)_\mu \right) - \boxed{\tfrac{\sqrt{g}}{\kappa} \nabla_\mu \left(R^\mu{}_\alpha - \tfrac{1}{2} R \delta^\mu_\alpha \xi^\alpha \right) (d^m x)}
\tag{1.15}
$$

where we used Bianchi identities. That defines the *reduced current*

$$
\tilde{\mathcal{E}} = \tfrac{\sqrt{g}}{\kappa} \left(R^\mu{}_\alpha - \tfrac{1}{2} R \delta^\mu_\alpha \right) \xi^\alpha (d^m x)_\mu = \sqrt{g}\, \tilde{T}^\mu_\alpha \xi^\alpha (d^m x)_\mu
\tag{1.16}
$$

which in fact venishes along solutions.

We can also integrate by parts the Noether current

$$
\mathcal{E} = \tfrac{\sqrt{g}}{2\kappa} \left(\left(\delta^\mu_\alpha g^{\rho\sigma} - g^{\mu(\rho} \delta^{\sigma)}_\alpha \right) \nabla_{\rho\sigma} \xi^\alpha + \left(\tfrac{3}{2} R^\mu{}_\alpha - R \delta^\mu_\alpha \right) \xi^\alpha \right) (d^m x)_\mu
\tag{1.17}
$$

We can integrate the first term as

$$
\begin{aligned}
& \tfrac{\sqrt{g}}{2\kappa} \left(\delta^\mu_\epsilon g^{\rho\sigma} - g^{\mu\sigma} \delta^\rho_\epsilon \right) \nabla_{\rho\sigma} \xi^\epsilon = \tfrac{\sqrt{g}}{2\kappa} \left(\delta^\mu_\epsilon g^{\rho\sigma} - g^{\mu\sigma} \delta^\rho_\epsilon \right) \left(\nabla_\rho \nabla_\sigma \xi^\epsilon - \tfrac{1}{2} R^\epsilon{}_{\alpha\rho\sigma} \xi^\alpha \right) = \\
&= \tfrac{\sqrt{g}}{2\kappa} \left(\delta^\mu_\epsilon g^{\rho\sigma} - g^{\mu\sigma} \delta^\rho_\epsilon \right) \nabla_\rho \nabla_\sigma \xi^\epsilon - \tfrac{1}{2} \tfrac{\sqrt{g}}{2\kappa} \left(-g^{\mu\sigma} \delta^\rho_\epsilon \right) R^\epsilon{}_{\alpha\rho\sigma} \xi^\alpha = \\
&= \nabla_\rho \left(\tfrac{\sqrt{g}}{2\kappa} \left(\delta^\mu_\epsilon g^{\rho\sigma} - g^{\mu\sigma} \delta^\rho_\epsilon \right) \nabla_\sigma \xi^\epsilon \right) + \tfrac{1}{2} \tfrac{\sqrt{g}}{2\kappa} R^\mu{}_\epsilon \xi^\epsilon = \nabla_\rho \left(2 \tfrac{\sqrt{g}}{2\kappa} \nabla^{[\rho} \xi^{\mu]} \right) + \tfrac{1}{2} \tfrac{\sqrt{g}}{2\kappa} R^\mu{}_\alpha \xi^\alpha
\end{aligned}
\tag{1.18}
$$

Let us define the *Komar superpotential* $\mathcal{U} = -\tfrac{\sqrt{g}}{2k} \nabla^{[\rho} \xi^{\mu]} (d^\mu x)_{\rho\mu}$ and compute

$$
d\mathcal{U} = \nabla_\alpha \left(\tfrac{\sqrt{g}}{2k} \nabla^{[\rho} \xi^{\mu]} \right) dx^\alpha \wedge (d^\mu x)_{\mu\rho} = \nabla_\rho \left(\tfrac{\sqrt{g}}{2k} \nabla^{[\rho} \xi^{\mu]} \right) (d^m x)_\mu - \nabla_\mu \left(\tfrac{\sqrt{g}}{2k} \nabla^{[\rho} \xi^{\mu]} \right) (d^m x)_\rho = \nabla_\rho \left(2 \tfrac{\sqrt{g}}{2k} \nabla^{[\rho} \xi^{\mu]} \right) (d^m x)_\mu
\tag{1.19}
$$

Hence we have

$$
\mathcal{E} = \tfrac{\sqrt{g}}{2\kappa} \left(\left(\delta^\mu_\alpha g^{\rho\sigma} - g^{\mu(\rho} \delta^{\sigma)}_\alpha \right) \nabla_{\rho\sigma} \xi^\alpha + 2 \left(R^\mu{}_\alpha - \tfrac{1}{2} R \delta^\mu_\alpha \right) \xi^\alpha \right) (d^m x)_\mu = \tilde{\mathcal{E}} + d\mathcal{U}
\tag{1.20}
$$

Noether charges

We define the Noether charge along a closed $(m-2)$-surface B_2

$$
Q[\xi] = \frac{1}{2k} \int_{B_2} \sqrt{g}\, \nabla^{[\rho} \xi^{\mu]} (d^\mu x)_{\mu\rho}
\tag{1.21}
$$

If we fix $A(r) = 1 - \frac{\alpha}{r}$

$$g = -Adt^2 + A^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{1.22}$$

which is the Schwarzschild spacetime, and $B_2 : (t = t_0, r = r_0 > \alpha)$.

For $\xi = \partial_t$: $E = \frac{m}{2}$ $\qquad \xi = \partial_\phi$: $J = 0$

For Kerr we obtain

For $\xi = \partial_t$: $E = \frac{m}{2}$ $\qquad \xi = \partial_\phi$: $J = ma$

For Schwarzschild-AdS solution $\left(A = \frac{\Lambda r^2 + r - \alpha}{r}\right)$, Noether charges diverge.

$$\boldsymbol{L} = \frac{1}{2\kappa}\left(\sqrt{g}R\, d^m x + \nabla_\alpha\left(\sqrt{g}g^{\rho\sigma}\left(\{g\}^\alpha_{\rho\sigma} - \{\bar{g}\}^\alpha_{\rho\sigma}\right)\right) - \sqrt{\bar{g}}\bar{R}\, d^m x\right) \tag{1.23}$$

It is covariant and we have good reasons to say it gives the *relative Noether charge* if we find 2 solutions $(g, \bar{g})$ inducing the same metric on B_2.

For g Schwarzschild we can fix a finite sphere B_2 and Minkowski $\bar{g}$ so that g and $\bar{g}$ agree on B_2.

For $\xi = \partial_t$: $E = m$ $\qquad \xi = \partial_\phi$: $J = 0$

or two Schwarzschild solutions and B_2 a sphere of $\lim r_0 \to \infty$

For $\xi = \partial_t$: $E = m - m_0$ $\qquad \xi = \partial_\phi$: $J = 0$

For Kerr

For $\xi = \partial_t$: $E = m - m_0$ $\qquad \xi = \partial_\phi$: $J = ma - m_0 a_0$

For Schwarzschild-AdS solution

For $\xi = \partial_t$: $E = m - m_0$ $\qquad \xi = \partial_\phi$: $J = ma - m_0 a_0$

Same for BTZ, Taub-Bolt, ...

2. Maxwell Lagrangian

We introduce a structure bundle $[P \to M]$ to describe the $U(1)$-gauge. A gauge transformation $\Phi \in \mathrm{Aut}(P)$ reads as

$$x'^\mu = x'^\mu(x) \qquad\qquad e^{i\theta'} = e^{i(\theta + \alpha(x))} \tag{2.1}$$

A $U(1)$-connection transforms as

$$A'_\mu = \bar{J}^\alpha_\mu\left(A_\alpha + d_\alpha \alpha\right) \tag{2.2}$$

which suggests a left action $\lambda : J^1G \rtimes \mathrm{GL}(m) \times (R^m)^* \to (R^m)^*$

$$A'_a = \bar{J}^c_a\left(A_c - e^{-i\alpha}d_c e^{i\alpha}\right) = \bar{J}^c_a\left(A_c - i d_c\alpha\right) \qquad \Rightarrow \qquad iA'_a = \bar{J}^c_a\left(iA_c + d_c\alpha\right) \tag{2.3}$$

and we can define the configuration bundle $\mathrm{Con}(P) = W^{(1,1)}P \times_\lambda (R^m)^*$ which is parameterised by coordinates (x^μ, A_μ) where we set $A_\mu = \epsilon^a_\mu A_a$.

In this way we cooked a bundle $[\mathrm{Con}(P) \to M]$ which has section in 1-to-1 correspondence with connections on P, same transformations laws.

Dynamics

We can fix a first order Lagrangian on $[\mathrm{Lor}(M)\times_M \mathrm{Con}(P)\to M]$

$$\boldsymbol{L}_M = -\frac{\sqrt{g}}{4g} F^{\alpha\beta} F_{\alpha\beta}\, d^m x \tag{2.4}$$

The variation of the Lagrangian wrt a deformation $X = \delta g^{\alpha\beta}\frac{\partial}{\partial g^{\alpha\beta}} + \delta A_\mu \frac{\partial}{\partial A_\mu}$ gives us

$$\begin{aligned}\langle \delta \boldsymbol{L}_M | j^1 X\rangle =& \frac{\sqrt{g}}{g}\left(-\frac{1}{2}\left(F_{\mu\alpha}F^{\mu}{}_{\beta} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta}\right)\delta g^{\alpha\beta} - F^{\mu\nu} d_\mu \delta A_\nu\right)(d^m x)\\ \langle \mathbb{E}|X\rangle =& \left(-\frac{\sqrt{g}}{2} T^{(em)}_{\alpha\beta}\delta g^{\alpha\beta} + \frac{1}{g}\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\right)\delta A_\nu\right)(d^m x)\\ \langle \mathbb{F}|X\rangle =& -\frac{\sqrt{g}}{g} F^{\mu\nu}\delta A_\nu (d^m x)_\mu\end{aligned} \tag{2.5}$$

where we set $T^{(em)}_{\alpha\beta} = \frac{1}{g}\left(F_{\mu\alpha}F^{\mu}{}_{\beta} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta}\right)$ for the *energy-momentum tensor* of electromagnetic field.

In SR, we fix the metric (hence we have $\delta g^{\alpha\beta}=0$) and we consider the equations for A_μ, namely

$$\nabla_\alpha(\sqrt{g}F^{\alpha\mu}) = 0 \tag{2.6}$$

which describe the evolution of the electromagnetic field in a fixed gravitational field g. They are *Maxwell* equations on the Lorentzian manifold (M, g).

In GR, we consider the total Lagrangian $\boldsymbol{L}_H + \boldsymbol{L}_M$ and field equations are then *Einstein-Maxwell equations*

$$G_{\alpha\beta} = \kappa T^{(em)}_{\alpha\beta} \qquad\qquad \nabla_\alpha(\sqrt{g}F^{\alpha\mu}) = 0 \tag{2.7}$$

which describe the evolution of gravitational field $g_{\alpha\beta}$ in interaction with an electromagnetic field A_μ, which are both unknown. Einstein-Maxwell equations are equations on a manifold M, with no further structure fixed on it, and up to diffeomorphisms.

Actually, we fixed P. However, right away we consider *any* $[P\to M]$ to give solutions of Einstein-Maxwell equations. Accordingly, we only fixed M and the gauge group $U(1)$.

Lie derivatives

For $\Xi = \xi^\mu(x)\partial_\mu + \zeta(x)\rho$ $\qquad\qquad$ $\rho = \frac{\partial}{\partial\theta}$

$$\pounds_\Xi \varphi = \xi^\epsilon F_{\epsilon\mu} + \nabla_\mu \zeta_V \tag{2.8}$$

This is obtained directly by considering a flow in $\mathrm{Aut}(P)$ acting on A_μ as

$$A'_\mu = \bar{J}^\sigma_\mu\left(A_\sigma - \partial_\sigma \alpha\right) \tag{2.9}$$

to obtain $\Xi_\lambda = \xi^\mu \frac{\partial}{\partial x^\mu} + \zeta_\mu \frac{\partial}{\partial A_\mu}$

$$\zeta_\mu = -\, d_\mu \xi^\sigma A_\sigma - d_\mu \zeta \tag{2.10}$$

Then we have the Lie derivative

$$\begin{aligned}\pounds_\Xi A_\mu =& \xi^\nu d_\nu A_\mu + d_\mu\xi^\sigma A_\sigma + d_\mu(\zeta_V - \xi^\nu A_\nu) = \xi^\nu d_\nu A_\mu + d_\mu \xi^\sigma A_\sigma + d_\mu \zeta_V - d_\mu \xi^\nu A_\nu - \xi^\nu d_\mu A_\nu =\\ =& \xi^\nu F_{\nu\mu} + \nabla_\mu \zeta_V\end{aligned} \tag{2.11}$$

Noether currents

By Noether theorem we can define the currents

$$
\begin{aligned}
\mathcal{W} =& \left(\frac{\sqrt{g}}{2} T^{(em)}_{\alpha\beta} \pounds_\xi g^{\alpha\beta} - \frac{1}{g} \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \right) \pounds_\Xi A_\nu \right) (d^m x) = \\
=& \left(-\sqrt{g} (T^{(em)})^\alpha{}_\beta \nabla_\alpha \xi^\beta - \frac{1}{g} \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \right) F_{\epsilon\nu} \xi^\epsilon - \frac{1}{g} \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \right) \nabla_\nu \zeta_V \right) (d^m x) = \\
=& \left(\mathcal{W}_\epsilon \xi^\epsilon + \mathcal{W}^\alpha_\epsilon \nabla_\alpha \xi^\epsilon + \mathcal{W}^\alpha \nabla_\alpha \zeta_V \right) (d^m x) \\
\mathcal{E} =& - \frac{\sqrt{g}}{g} \left(F^{\mu\nu} \pounds_\Xi A_\nu - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \xi^\mu \right) (d^m x)_\mu = \\
=& \left(- \frac{\sqrt{g}}{g} \left(F^{\mu\nu} F_{\epsilon\nu} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta^\mu_\epsilon \right) \xi^\epsilon - \frac{\sqrt{g}}{g} F^{\mu\nu} \nabla_\nu \zeta_V \right) (d^m x)_\mu = \\
=& \left(-\sqrt{g} (T^{(em)})^\mu{}_\epsilon \xi^\epsilon - \frac{\sqrt{g}}{g} F^{\mu\nu} \nabla_\nu \zeta_V \right) (d^m x)_\mu = \\
=& \sqrt{g} \left(\hat{T}^\mu{}_\epsilon \xi^\epsilon + \hat{T}^{\mu\nu} \nabla_\nu \zeta_V \right) (d^m x)_\mu
\end{aligned}
\tag{2.12}
$$

Let us stress that the work current of $\boldsymbol{L}_M$ does not vanish. In SR there is no need for $T^{(em)}_{\alpha\beta}$ to be zero, since we required $\delta g^{\alpha\beta} = 0$ exactly to kill the corresponding equation. There is no reason why $\pounds_\xi g^{\alpha\beta}$ should have to be zero in general. The work current of $\boldsymbol{L}_M$ vanishes only if we restrict the symmetries to be $\pounds_\xi g^{\alpha\beta} = 0$, i.e. when ξ is a Killing vector. Accordingly, the Noether current is conserved along a solutions only when it is associated to a Killing vector, namely an infinitesimal isometry.

In GR what vanish is the total work current of $\boldsymbol{L}_H + \boldsymbol{L}_M$, not the work current of $\boldsymbol{L}_M$. We learn 2 things: first, we do not need to restrict symmetries in GR exactly because all fields are dynamical. Second, the Noether current of $\boldsymbol{L}_M$ is not conserved just because the electromagnetic field and the gravitational field are in interaction. Accordingly, only the total Noether current is conserved.

We can integrate by parts the work current

$$
\begin{aligned}
\mathcal{W} =& \left(\nabla_\alpha \left(\sqrt{g} (T^{(em)})^\alpha{}_\beta \right) \xi^\beta - \frac{1}{g} \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \right) F_{\beta\nu} \xi^\beta + \frac{1}{g} \nabla_\nu \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \right) \zeta_V \right) (d^m x) + \\
&+ d \left(\left(-\sqrt{g} (T^{(em)})^\alpha{}_\beta \xi^\beta - \frac{1}{g} \nabla_\mu \left(\sqrt{g} F^{\mu\alpha} \right) \zeta_V \right) (d^m x)_\alpha \right)
\end{aligned}
\tag{2.13}
$$

where we used Bianchi identities and conservation of energy-momentum tensor.

Let us check that

$$
\begin{aligned}
\nabla_\alpha (T^{(em)})^\alpha{}_\beta =& \frac{1}{g} \nabla_\alpha \left(F^{\mu\alpha} F_{\mu\beta} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \delta^\alpha_\beta \right) = \frac{1}{g} \left(-\nabla_\alpha F^{\alpha\mu} F_{\mu\beta} - \frac{1}{2} F^{\mu\nu} \left(\nabla_\nu F_{\beta\mu} + \nabla_\mu F_{\nu\beta} + \nabla_\beta F_{\mu\nu} \right) \right) = \\
=& - \frac{1}{g} \nabla_\alpha F^{\alpha\mu} F_{\mu\beta}
\end{aligned}
\tag{2.14}
$$

which vanishes along solutions (in view of Bianchi identities $dF = 0$ for the curvature F).
Accordingly, the Bianchi current

$$
\begin{aligned}
\mathcal{B} =& \left(\nabla_\alpha \left(\sqrt{g} (T^{(em)})^\alpha{}_\beta \right) \xi^\beta - \frac{1}{g} \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \right) F_{\beta\nu} \xi^\beta + \frac{1}{g} \nabla_\nu \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \right) \zeta_V \right) (d^m x) = \\
=& \left(-\frac{1}{g} \nabla_\epsilon \left(\sqrt{g} F^{\epsilon\mu} \right) F_{\mu\beta} \xi^\beta - \frac{1}{g} \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \right) F_{\beta\nu} \xi^\beta - \frac{\sqrt{g}}{2g} [\nabla_\nu, \nabla_\mu] F^{\mu\nu} \zeta_V \right) (d^m x) = \\
=& \frac{\sqrt{g}}{2g} \left(R_{\epsilon\nu} F^{\epsilon\nu} - R_{\epsilon\mu} F^{\mu\epsilon} \right) \zeta_V (d^m x) = 0
\end{aligned}
\tag{2.15}
$$

is zero off-shell.

That defines the *reduced current*

$$
\tilde{\mathcal{E}} = \left(-\sqrt{g} (T^{(em)})^\alpha{}_\beta \xi^\beta - \frac{1}{g} \nabla_\mu \left(\sqrt{g} F^{\mu\alpha} \right) \zeta_V \right) (d^m x)_\alpha = \sqrt{g} \left(\tilde{T}^\mu_\alpha \xi^\alpha + \tilde{T}^\mu \zeta_V \right) (d^m x)_\mu
\tag{2.16}
$$

This does not vanish along solutions, however, the total reduced current for $\boldsymbol{L}_H + \boldsymbol{L}_M$ does.

We can also integrate by parts the Noether current

$$\mathcal{E} = \left(-\sqrt{g}(T^{(em)})^\mu{}_\epsilon \xi^\epsilon + \frac{1}{g}\nabla_\nu\left(\sqrt{g}F^{\mu\nu}\right)\zeta_V\right)(d^m x)_\mu + d\left(\frac{1}{2g}\sqrt{g}F^{\mu\nu}\zeta_V(d^m x)_{\mu\nu}\right) \tag{2.17}$$

Hence we have

$$\mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U} \tag{2.18}$$

where we set $\mathcal{U} = \frac{1}{2g}\sqrt{g}F^{\mu\nu}\zeta_V(d^m x)_{\mu\nu}$ for the superpotential of the Maxwell Lagrangian.

Of course, when we define conserved quantities we need the Noether current to be conserved. Accordingly, we define Noether charges using the total superpotential.

Noether charges

We define the Noether charge along a closed $(m-2)$-surface B_2

$$Q[\Xi] = \frac{1}{2k}\int_{B_2}\sqrt{g}\left(\frac{1}{2\kappa}\nabla^{[\nu}\xi^{\mu]} + \frac{1}{2g}F^{\mu\nu}\zeta_V\right)(d^m x)_{\mu\nu} \tag{2.19}$$

By choosing $\zeta_V = 0$ we define energy, momentum, ... By setting $\xi = 0$ we define the electric charge.

If we use the augmented variational principle we can consider the Reissner-Nosdrom solution (which is a electrically charged black hole) and we get expected charges.

Again no need of restricting to Killing vectors in GR.

3. Complex Klein-Gordon Lagrangian

When we want to introduce a scalar field we base on the fact that the value of the field is indepenedent of the coordinates, it depends only on the point. That is translated to the transformation law

$$\varphi'(x') = \varphi(x) \tag{3.1}$$

In this case, it is easy to define a bundle that induces those transition maps, just take $[M \times \mathbb{K} \to M]$ if the scalar is valued in the field $\mathbb{K}$.

When we consider a complex Klein-Gordon field we have gauge transformations acting as

$$\varphi'(x) = e^{i\alpha(x)}\varphi(x) \tag{3.2}$$

which suggest to consider the left action

$$\lambda : U(1) \times \mathbb{C} \to \mathbb{C} : (e^{iq\alpha}, \varphi) \mapsto \varphi' = e^{i\alpha}\varphi \tag{3.3}$$

Complex Klein-Gordon fields are then section of the associated bundle $[K = P \times_\lambda \mathbb{C} \to M]$ so that we have the correct transformation laws for transformations in $\mathrm{Aut}(P)$.

When we fix a connection A on P that induces a connection on $[K \to M]$ which in turns define the gauge covariant derivative

$$\nabla_\alpha\varphi = d_\alpha\varphi - iqA_\mu\varphi \qquad \Rightarrow \qquad (\nabla_\alpha\varphi)' = e^{iq\alpha(x)}\bar{J}^\mu_\alpha\nabla_\mu\varphi \tag{3.4}$$

We consider now the configuration bundle

$$\mathrm{Lor}(M) \times_M \mathrm{Con}(P) \times_M (P \times_\lambda \mathbb{C}) \tag{3.5}$$

which has (real) fibered coodinates $(x^\mu, g_{\alpha\beta}, A_\mu, \varphi, \bar{\varphi})$.

Dynamics

We can fix a first order Lagrangian on $[\mathrm{Lor}(M) \times_M \mathrm{Con}(P) \to M] \times_M (P \times_\lambda \mathbb{C})$

$$\boldsymbol{L}_{KG} = -\frac{\sqrt{g}}{2} \left(\nabla_\alpha \bar{\varphi} g^{\alpha\beta} \nabla_\beta \varphi + m^2 \bar{\varphi} \varphi \right) \, d^m x \tag{3.6}$$

The variation of the Lagrangian wrt a deformation $X = \delta g^{\alpha\beta} \frac{\partial}{\partial g^{\alpha\beta}} + \delta A_\mu \frac{\partial}{\partial A_\mu} + \delta \bar{\varphi} \frac{\partial}{\partial \bar{\varphi}} + \delta \varphi \frac{\partial}{\partial \varphi}$ gives us

$$\begin{aligned}
\langle \delta \boldsymbol{L}_{KG} | j^1 X \rangle =& - \frac{\sqrt{g}}{2} \left(\nabla_\alpha \bar{\varphi} \nabla_\beta \varphi - \frac{1}{2} \left(\nabla_\alpha \bar{\varphi} g^{\alpha\beta} \nabla_\beta \varphi + m^2 \bar{\varphi} \varphi \right) g_{\alpha\beta} \right) \delta g^{\alpha\beta} (d^m x) + \\
& - \frac{\sqrt{g}}{2} iq \left(\bar{\varphi} \nabla_\alpha \varphi - \nabla_\alpha \bar{\varphi} \varphi \right) g^{\mu\alpha} \delta A_\mu (d^m x) + \\
& - \frac{\sqrt{g}}{2} \left(\nabla_\alpha \delta \bar{\varphi} g^{\alpha\beta} \nabla_\beta \varphi + m^2 \delta \bar{\varphi} \varphi + \nabla_\alpha \bar{\varphi} g^{\alpha\beta} \nabla_\beta \delta \varphi + m^2 \bar{\varphi} \delta \varphi \right) (d^m x) = \\
=& - \frac{\sqrt{g}}{2} \left(T^{KG}_{\alpha\beta} \delta g^{\alpha\beta} + J_\alpha g^{\mu\alpha} \delta A_\mu \right) (d^m x) + \\
& - \frac{\sqrt{g}}{2} \left(\nabla_\alpha \delta \bar{\varphi} g^{\alpha\beta} \nabla_\beta \varphi + m^2 \delta \bar{\varphi} \varphi + \nabla_\alpha \bar{\varphi} g^{\alpha\beta} \nabla_\beta \delta \varphi + m^2 \bar{\varphi} \delta \varphi \right) (d^m x) \\
\langle \mathbb{E} | X \rangle =& - \frac{\sqrt{g}}{2} \left(T^{(KG)}_{\alpha\beta} \delta g^{\alpha\beta} + J_\alpha g^{\alpha\nu} \delta A_\nu - \delta \bar{\varphi} (\Box \varphi - m^2 \varphi) - (\Box \bar{\varphi} - m^2 \bar{\varphi}) \delta \varphi \right) (d^m x) \\
\langle \mathbb{F} | X \rangle =& - \frac{\sqrt{g}}{2} (\delta \bar{\varphi} \nabla_\alpha \varphi + \nabla_\alpha \bar{\varphi} \delta \varphi) g^{\mu\alpha} (d^m x)_\mu
\end{aligned} \tag{3.7}$$

where we set $T^{(KG)}_{\alpha\beta} = \nabla_\alpha \bar{\varphi} \nabla_\beta \varphi - \frac{1}{2} \left(\nabla_\alpha \bar{\varphi} g^{\alpha\beta} \nabla_\beta \varphi + m^2 \bar{\varphi} \varphi \right) g_{\alpha\beta}$ for the *energy-momentum tensor* of the KG field and $J_\alpha = iq \left(\bar{\varphi} \nabla_\alpha \varphi - \nabla_\alpha \bar{\varphi} \varphi \right)$ for the KG current.

Now we have 3 options:

Opt 1: we fix $\delta g_{\alpha\beta} = 0$ and $\delta A_\mu = 0$ and we write field equation for KG field for the Lagrangian $\boldsymbol{L}_{KG}$

$$\Box \varphi = m^2 \varphi \tag{3.8}$$

which describes the evolution of a KG field on fixed electromagnetic and gravitational field.

Opt 2: we fix $\delta g_{\alpha\beta} = 0$ and we write field equations for Maxwell-KG for the Lagrangian $\boldsymbol{L}_M + \boldsymbol{L}_{KG}$

$$\nabla_\mu (\sqrt{g} F^\mu{}_\alpha) = g J_\alpha \qquad \qquad \Box \varphi = m^2 \varphi \tag{3.9}$$

which describe the evolution of a KG field and the interacting electromagnetic field on a fixed gravitational field.

Opt 3: we let all fields to be dynamical and we write field equations for Einstein-Maxwell-KG for the Lagrangian $\boldsymbol{L}_H + \boldsymbol{L}_M + \boldsymbol{L}_{KG}$

$$G_{\alpha\beta} = \kappa (T^{(em)}_{\alpha\beta} + T^{(KG)}_{\alpha\beta}) \qquad \qquad \nabla_\mu (\sqrt{g} F^\mu{}_\alpha) = g J_\alpha \qquad \qquad \Box \varphi = m^2 \varphi \tag{3.10}$$

which describe the evolution of interacting KG, electromagnetic, and gravitational field.

Lie derivatives

For $\Xi = \xi^\mu (x) \partial_\mu + \zeta(x) \rho$ $\qquad \rho = \frac{\partial}{\partial \theta}$

$$\pounds_\Xi \varphi = \xi^\epsilon \nabla_\epsilon \varphi - iq \varphi \zeta_V \tag{3.11}$$

This is obtained directly by considering a flow in $\mathrm{Aut}(P)$ acting on φ as

$$\varphi' = e^{iq\alpha} \varphi \tag{3.12}$$

to obtain $\Xi_\lambda = \xi^\mu \frac{\partial}{\partial x^\mu} + \hat\zeta \frac{\partial}{\partial \varphi}$

$$\hat\zeta = iq\zeta\varphi \tag{3.13}$$

Then we have the Lie derivative

$$\pounds_\Xi \varphi = \xi^\mu d_\mu \varphi - iq\zeta\varphi = \xi^\mu d_\mu\varphi - iq\zeta_V\varphi - iq\xi^\mu A_\mu \varphi = \xi^\mu \nabla_\mu \varphi - iq\varphi\zeta_V \tag{3.14}$$

Noether currents

By Noether theorem we can define the currents

$$\begin{aligned}
\mathcal{W} =& \tfrac{\sqrt{g}}{2}\left(T^{(KG)}_{\alpha\beta}\pounds_\xi g^{\alpha\beta} + J^\nu_{\cdot}\pounds_\Xi A_\nu - \pounds_\Xi\bar\varphi(\Box\varphi - m^2\varphi) - (\Box\bar\varphi - m^2\bar\varphi)\pounds_\Xi\varphi\right)(d^m x) = \\
=& \tfrac{\sqrt{g}}{2}\left(-2(T^{(KG)})^\alpha{}_\beta \nabla_\alpha\xi^\beta + \xi^\mu\left(J^\nu_{\cdot}F_{\mu\nu} - \nabla_\mu\bar\varphi(\Box\varphi - m^2\varphi) - (\Box\bar\varphi - m^2\bar\varphi)\nabla_\mu\varphi\right)\right)(d^m x) + \\
&+ \tfrac{\sqrt{g}}{2}\left(J^\nu_{\cdot}\nabla_\nu\zeta_V - iq\left(\bar\varphi(\Box\varphi - m^2\varphi) - (\Box\bar\varphi - m^2\bar\varphi)\varphi\right)\zeta_V\right)(d^m x) = \\
=& \left(\mathcal{W}_\epsilon\xi^\epsilon + \mathcal{W}^\alpha_\epsilon\nabla_\alpha\xi^\epsilon + \mathcal{W}\zeta_V + \mathcal{W}^\alpha\nabla_\alpha\zeta_V\right)(d^m x) \\
\mathcal{E} =& -\tfrac{\sqrt{g}}{2}\left(\left(\pounds_\Xi\bar\varphi\nabla_\alpha\varphi + \nabla_\alpha\bar\varphi\pounds_\Xi\varphi\right)g^{\alpha\mu} - \left(\nabla_\alpha\bar\varphi g^{\alpha\beta}\nabla_\beta\varphi + m^2\bar\varphi\varphi\right)\delta^\mu_\sigma\xi^\sigma\right)(d^m x)_\mu = \\
=& -\tfrac{\sqrt{g}}{2}\left(2\left(\nabla_{(\sigma}\bar\varphi\nabla_{\alpha)}\varphi - \tfrac12\left(\nabla_\alpha\bar\varphi g^{\alpha\beta}\nabla_\beta\varphi + m^2\bar\varphi\varphi\right)\delta^\mu_\sigma\right)\xi^\sigma + iq\left(\bar\varphi\nabla_\alpha\varphi - \nabla_\alpha\bar\varphi\varphi\right)g^{\alpha\mu}\zeta_V\right)(d^m x)_\mu = \\
=& -\tfrac{\sqrt{g}}{2}\left(2T^{(KG)}_{(\alpha\sigma)}g^{\sigma\mu}\xi^\sigma + J_\alpha g^{\alpha\mu}\zeta_V\right)(d^m x)_\mu = \sqrt{g}\left(\hat T^\mu{}_\epsilon\xi^\epsilon + \hat T^\mu\zeta_V\right)(d^m x)_\mu
\end{aligned} \tag{3.15}$$

Let us stress that the work current of $\boldsymbol{L}_{KG}$ does not vanish. In Opt 1 there is no need for $T^{(KG)}_{\alpha\beta}$ or J_α to be zero, since we required $\delta g^{\alpha\beta} = 0$ and $\delta A_\mu = 0$ exactly to kill the corresponding equations. There is no reason why $\pounds_\xi g^{\alpha\beta}$ or $\pounds_\Xi A_\mu$ should have to be zero in general. The work current of $\boldsymbol{L}_{KG}$ vanishes only if we restrict the symmetries to be $\pounds_\xi g^{\alpha\beta} = 0$, i.e. when ξ is a Killing vector, and $\pounds_\Xi A_\mu = 0$, i.e. the transformation preserves the electromagnetic field. Accordingly, the Noether current is conserved along a solutions only when it is associated to a Killing vector, namely an infinitesimal isometry, and gauge trasformations which preserve the electromagnetic field.

In Opt 2 there is no need for $T^{(KG)}_{\alpha\beta}$ to be zero, since we required $\delta g^{\alpha\beta} = 0$ exactly to kill the corresponding equations. There is no reason why $\pounds_\xi g^{\alpha\beta}$ should have to be zero in general. The work current of $\boldsymbol{L}_M + \boldsymbol{L}_{KG}$ vanishes only if we restrict the symmetries to be $\pounds_\xi g^{\alpha\beta} = 0$, i.e. when ξ is a Killing vector. Accordingly, the Noether current is conserved along a solutions only when it is associated to a Killing vector, namely an infinitesimal isometry, and gauge trasformations which preserve the electromagnetic field.

In Opt 3 (i.e. in GR) what vanish is the total work current of $\boldsymbol{L}_H + \boldsymbol{L}_M + \boldsymbol{L}_{KG}$, not the work current of $\boldsymbol{L}_M + \boldsymbol{L}_{KG}$ or $\boldsymbol{L}_{KG}$. We learn 2 things: first, we do not need to restrict symmetries in GR exactly because all fields are dynamical. Second, the Noether currents of $\boldsymbol{L}_M + \boldsymbol{L}_{KG}$ or $\boldsymbol{L}_{KG}$ are not conserved just because the electromagnetic, the KG, and the gravitational field are in interaction. Accordingly, only the total Noether current is conserved.

We can integrate by parts the work current

$$\begin{aligned}
\mathcal{W} =& \tfrac{\sqrt{g}}{2}\left(\left(2\nabla_\alpha(T^{(KG)})^\alpha{}_\mu + J^\nu_{\cdot}F_{\mu\nu} - \nabla_\mu\bar\varphi(\Box\varphi - m^2\varphi) - (\Box\bar\varphi - m^2\bar\varphi)\nabla_\mu\varphi\right)\xi^\mu\right)(d^m x) + \\
&+ \tfrac{\sqrt{g}}{2}\left(\nabla_\nu J^\nu_{\cdot} - iq\left(\bar\varphi(\Box\varphi - m^2\varphi) - (\Box\bar\varphi - m^2\bar\varphi)\varphi\right)\right)\zeta_V(d^m x) + \\
&+ d\left(\left(-2\sqrt{g}(T^{(KG)})^\alpha{}_\beta\xi^\beta + \tfrac{\sqrt{g}}{2}J^\alpha_{\cdot}\zeta_V\right)(d^m x)_\alpha\right)
\end{aligned} \tag{3.16}$$

where we used Bianchi identities and conservation of energy-momentum tensor.

Compute $\nabla_\alpha (T^{(KG)})^\alpha{}_\beta$ and $\nabla_\alpha J^\alpha$. Then show that the Bianchi current is zero off-shell.

That defines the *reduced current*

$$\tilde{\mathcal{E}} = \left(-2\sqrt{g}(T^{(KG)})^\alpha{}_\beta \xi^\beta + \frac{\sqrt{g}}{2} J^\alpha_{\cdot} \zeta_V \right) (d^m x)_\alpha = \sqrt{g} \left(\tilde{T}^\mu_\alpha \xi^\alpha + \tilde{T}^\mu \zeta_V \right) (d^m x)_\mu \tag{3.17}$$

This does not vanish along solutions, however, the total reduced current for $\boldsymbol{L}_H + \boldsymbol{L}_M + \boldsymbol{L}_{KG}$ does.

This time in $\mathcal{E}$ there is nothing to be integrated by parts and we have $\mathcal{E} = \tilde{\mathcal{E}}$, i.e. no superpotential for $\boldsymbol{L}_{KG}$.

Of course, when we define conserved quantities we need the Noether current to be conserved. Accordingly, we define Noether charges using the total superpotential.