# 2D vector calculus and forms on 4D space-time

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- How 4D forms can be 1+3 decomposed w.r.t. $V$

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- Coordinate free approach:
- How 4D forms can be 1+3 decomposed w.r.t. $V$
- How 4D forms can be 1+1+2 decomposed w.r.t. $V, W$

## Contents

# 3D vector calculus and forms on $E^3$

On any $(M, g, o)$ we have canonical isomorphisms $\sharp_g$, $\flat_g$ and $*_g$.
Happy coincidence on $(M^3, g, o)$: We can "forget about forms":

$$
\begin{tikzcd}
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \\
\mathcal{F} \arrow[u, "id"] \arrow[r, "\mathrm{grad}"'] & \mathfrak{X} \arrow[u, "\flat"'] \arrow[r, "\mathrm{curl}"'] & \mathfrak{X} \arrow[u, "*\flat"'] \arrow[r, "\mathrm{div}"'] & \mathcal{F} \arrow[u, "*"']
\end{tikzcd}
$$

We can make do with just functions and vector fields.
Then $\mathrm{grad}$, $\mathrm{curl}$ and $\mathrm{div}$ represent effective exterior derivative.

# 3D vector calculus and forms on $E^3$ (2)

There is a convenient notation for the action of the three canonical isomorphisms:

$$
\begin{aligned}
\flat : \mathfrak{X} \to \Omega^1 &\qquad \boldsymbol{A} \mapsto \boldsymbol{A} \cdot d\boldsymbol{r} \\
* \circ \flat : \mathfrak{X} \to \Omega^2 &\qquad \boldsymbol{A} \mapsto \boldsymbol{A} \cdot d\boldsymbol{S} \\
* : \mathcal{F} \to \Omega^3 &\qquad h \mapsto h dV
\end{aligned}
$$

Directly from the diagram we can read differential identities

$$
\begin{aligned}
df &= \operatorname{grad} f \cdot d\boldsymbol{r} \\
d(\boldsymbol{A} \cdot d\boldsymbol{r}) &= (\operatorname{curl} \boldsymbol{A}) \cdot d\boldsymbol{S} \\
d(\boldsymbol{A} \cdot d\boldsymbol{S}) &= (\operatorname{div} \boldsymbol{A}) dV
\end{aligned}
$$

# 3D vector calculus and forms on $E^3$ (3)

Integrating and using Stokes theorem (for forms) we get the following (well-known) integral identities

| | |
|---|---|
| gradient theorem | $\int_c \mathrm{grad}\, f \cdot d\boldsymbol{r} := f(B) - f(A)$ |
| Stokes theorem | $\int_S (\mathrm{curl}\, \boldsymbol{A}) \cdot d\boldsymbol{S} := \oint_{\partial S} \boldsymbol{A} \cdot d\boldsymbol{r}$ |
| Gauss's theorem | $\int_V (\mathrm{div}\, \boldsymbol{A}) dV := \oint_{\partial V} \boldsymbol{A} \cdot d\boldsymbol{S}$ |

# 2D vector calculus and forms on $E^2$

Again we have canonical isomorphisms $\sharp_g$, $\flat_g$ and $*_g$.
Also on $(M^2, g, o)$ we can "forget about forms",
we can still make do with just functions and vector fields.
But now a new feature arises: $* : \Omega^1 \to \Omega^1$
$\Rightarrow$ we have two commutative diagrams (see Fecko 2021):

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] \arrow[d, "id"'] & \Omega^1 \arrow[r, "d"] \arrow[d, "\sharp"] & \Omega^2 \arrow[d, "*"] \\
\mathcal{F} \arrow[r, "\text{grad}"'] & \mathfrak{X} \arrow[r, "\text{curl}_3"'] & \mathcal{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^0 \arrow[r, "d"] \arrow[d, "id"'] & \Omega^1 \arrow[r, "d"] \arrow[d, "\sharp *"] & \Omega^2 \arrow[d, "*"] \\
\mathcal{F} \arrow[r, "\text{Ham}"'] & \mathfrak{X} \arrow[r, "-\text{div}"'] & \mathcal{F}
\end{tikzcd}$$

So there are as many as four effective exterior derivative operators.

# 2D vector calculus and forms on $E^2$ (2)

Now we can read off four differential identities

$$df = \operatorname{grad} f \cdot d\boldsymbol{r} \qquad d(\boldsymbol{A} \cdot d\boldsymbol{r}) = (\operatorname{curl}_3 \boldsymbol{A})dS$$

$$df = - * (\operatorname{Ham} f \cdot d\boldsymbol{r}) \qquad d * (\boldsymbol{A} \cdot d\boldsymbol{r}) = (\operatorname{div} \boldsymbol{A})dS$$

and obtain corresponding four integral identities.

$$\int_C \operatorname{grad} f \cdot d\boldsymbol{r} = f(B) - f(A) \qquad \int_S (\operatorname{curl}_3 \boldsymbol{A})dS = \oint_{\partial S} \boldsymbol{A} \cdot d\boldsymbol{r}$$

$$\int_C *(\operatorname{Ham} f \cdot d\boldsymbol{r}) = f(A) - f(B) \qquad \int_S (\operatorname{div} \boldsymbol{A})dS = \oint_{\partial S} *(\boldsymbol{A} \cdot d\boldsymbol{r})$$

# 2D vector calculus and forms on $E^2$ (3)

A useful observation about $* : \Omega^1 \to \Omega^1$ :

$$*(\boldsymbol{A} \cdot d\boldsymbol{r}) =: \boldsymbol{A}_{\pi/2} \cdot d\boldsymbol{r}$$

where $\boldsymbol{A}_{\pi/2}$ is the vector field rotated by $\pi/2$ (in positive sense):

$$\boldsymbol{A} \mapsto \boldsymbol{A}_{\pi/2} \qquad \leftrightarrow \qquad (A_x, A_y) \mapsto (-A_y, A_x)$$

So in $E^2$, contrary to $E^3$, there is a canonical operation on vector fields

$$(\ )_{\pi/2} : \mathfrak{X} \to \mathfrak{X} \qquad \boldsymbol{A} \mapsto \boldsymbol{A}_{\pi/2}$$

(In $E^3$ no canonical axis - or a 2-plane - is available.)

# 2D vector calculus and forms on $E^2$ (4)

This observation also provides an alternative way
how to "understand" the two new (and strange looking)
differential operations present in 2D vector calculus:

$$\mathrm{Ham}\, f = (\mathrm{grad}\, f)_{\pi/2} \qquad \mathrm{curl}\,_3\boldsymbol{A} = -\mathrm{div}\, \boldsymbol{A}_{\pi/2}$$

So $\mathrm{Ham}\, f$ can be produced in two steps:
- compute "just" the good old gradient of $f$
- rotate the resulting vector field, in each point, by $\pi/2$.

Similarly $\mathrm{curl}\,_3\boldsymbol{A}$:
- first rotate $\boldsymbol{A}$, in each point, by $\pi/2$, then
- compute "just" the good old divergence of the resulting field.

## 3D vector calculus and forms on Minkowski space-time $E^{1,3}$

Consider now Minkowski space-time $E^{1,3}$
with Cartesian coordinates $x^\mu = (t, x, y, z)$.
So 1-form coordinate basis is $dx^\mu = (dt, dx, dy, dz)$.

Then, in coordinate expression of any $p$-form $\alpha$ on $E^{1,3}$,
any term contains $dt$ exactly once or not at all
(since $dt \wedge dt = 0$ :-).
Then, we can write

$$\boxed{\alpha = dt \wedge \hat{s} + \hat{r}}$$

where $\hat{s}$ and $\hat{r}$ only contain $dx, dy, dz$ (but not $dt$).
Such forms are called spatial forms.

# Spatial forms and 3D vector calculus

Since $dx, dy, dz$ may be also treated as a basis of 1-forms on $E^3$, each spatial form may be written in the notation from $E^3$.

As an example, a two-form $\alpha$ on Minkowski space-time reads

$$\alpha = dt \wedge \boldsymbol{a} \cdot d\boldsymbol{r} + \boldsymbol{b} \cdot d\boldsymbol{S}$$

Notice that $\boldsymbol{a} \cdot d\boldsymbol{r}$ and $\boldsymbol{b} \cdot d\boldsymbol{S}$ are not literally forms on $E^3$, since they are in general depend on time $t$.

So, when working with forms on Minkowski space-time $E^{1,3}$, we can instead switch to work with time-dependent objects from good-old 3D vector calculus.

# Operations $d$, $*$ and $\delta$ on forms on $E^{1,3}$

Now a computation shows that

$$\begin{aligned} d(dt\wedge\hat{s}+\hat{r}) &= dt\wedge(\mathcal{L}_{\partial_t}\hat{r}-\hat{d}\hat{s})+\hat{d}\hat{r} \\ *(dt\wedge\hat{s}+\hat{r}) &= dt\wedge\hat{*}\hat{r}+\hat{*}\hat{\eta}\hat{s} \\ \delta(dt\wedge\hat{s}+\hat{r}) &= dt\wedge\hat{\delta}\hat{s}+(-\mathcal{L}_{\partial_t}\hat{s}-\hat{\delta}\hat{r}) \end{aligned}$$

where

- $\delta = *^{-1}d * \hat{\eta}$ is the codifferential
- hatted operations ($\hat{d}, \hat{*}, \hat{\delta}$) only act on hatted forms

That means, we get operations from 3D vector calculus.

# $d$, $*$ and $\delta$ and 3D vector calculus

The only operation which is not from 3D vector calculus
is the Lie derivative $\mathcal{L}_{\partial_t}$ along the time axis (it comes from 4D $d$).
This operation is completely foreign
to the proper 3D vector calculus!

So, all operations on forms on Minkowski space boil down to

- standard operations of 3D vector calculus
- plus $\mathcal{L}_{\partial_t}$ playing role of time derivative.

## Example: Electromagnetism on Minkowski space-time $E^{1,3}$

The best known example:

$$\begin{aligned} F &= dt \wedge \boldsymbol{E} \cdot d\boldsymbol{r} - \boldsymbol{B} \cdot d\boldsymbol{S} \\ j &= \rho dt - \boldsymbol{j} \cdot d\boldsymbol{r} \end{aligned}$$

The general formulas lead to

$$\begin{aligned} d(dt \wedge \boldsymbol{E} \cdot d\boldsymbol{r} - \boldsymbol{B} \cdot d\boldsymbol{S}) &= dt \wedge (-\partial_t \boldsymbol{B} - \operatorname{curl} \boldsymbol{E}) \cdot d\boldsymbol{S} - (\operatorname{div} \boldsymbol{B}) dV \\ \delta(dt \wedge \boldsymbol{E} \cdot d\boldsymbol{r} - \boldsymbol{B} \cdot d\boldsymbol{S}) &= (-\operatorname{div} \boldsymbol{E}) dt + (-\partial_t \boldsymbol{E} + \operatorname{curl} \boldsymbol{B}) \cdot d\boldsymbol{r} \end{aligned}$$

Then, the two Maxwell equations

$$\delta F = -j \qquad dF = 0$$

translate to four equations from electromagnetism course

$$\begin{aligned} \operatorname{div} \boldsymbol{E} &= \rho & \qquad -\partial_t \boldsymbol{E} + \operatorname{curl} \boldsymbol{B} &= \boldsymbol{j} \\ \operatorname{div} \boldsymbol{B} &= 0 & \qquad -\partial_t \boldsymbol{B} - \operatorname{curl} \boldsymbol{E} &= 0 \end{aligned}$$

# 2D vector calculus and forms on Minkowski space-time $E^{1,3}$

Consider again Minkowski space-time $E^{1,3}$
with Cartesian coordinates $x^\mu = (t, x, y, z)$.
We already know that

$$\boxed{\alpha = dt \wedge \hat{s} + \hat{r}}$$

where $\hat{s}$ and $\hat{r}$ only contain $dx, dy, dz$ (but not $dt$).

Now we can iterate the procedure with (say) $dz$ present in spatial forms:

$$\hat{s} = dz \wedge \breve{A} + \breve{B} \qquad \hat{r} = dz \wedge \check{C} + \check{D}$$

Here $\breve{A}$, $\breve{B}$, $\check{C}$, $\check{D}$ only contain $dx, dy$ (but neither $dz$ nor $dt$).
We can call them 2D-spatial forms.

# 2D-spatial forms and 2D vector calculus

So, for a general form on Minkowski space
we have a finer decomposition (with four 2D-spatial terms)

$$\boxed{\alpha = dt \wedge dz \wedge \breve{A} + dt \wedge \breve{B} + dz \wedge \breve{C} + \breve{D}}$$

Each 2D-spatial form may be written in the notation from $E^2$.

So, when working with forms on Minkowski space-time $E^{1,3}$,
we can also decide to switch to work
with $z$-and-time-dependent objects from 2D vector calculus.

# 2D-spatial forms and 2D vector calculus

As an example, in electromagnetism, it turns out that

$$F = dt \wedge dz \wedge E_z + dt \wedge \mathbf{E} \cdot d\mathbf{r} + dz \wedge \breve{*}(\mathbf{B} \cdot d\mathbf{r}) - B_z dS$$

where

$$\begin{aligned} \mathbf{E} \cdot d\mathbf{r} &= E_x dx + E_y dy \\ \mathbf{B} \cdot d\mathbf{r} &= B_x dx + B_y dy \\ \breve{*}(\mathbf{B} \cdot d\mathbf{r}) &= B_x dy - B_y dx \\ B_z dS \equiv \breve{*} B_z &= B_z dx \wedge dy \end{aligned}$$

# 2D-vector calculus version of Maxwell equations

The Maxwell equations

$$\delta F = -j \qquad dF = 0$$

translate to their not so well known 2D-version

$$\begin{aligned} -\boldsymbol{\nabla}_{\pi/2} B_z + \partial_z \boldsymbol{B}_{\pi/2} - \partial_t \boldsymbol{E} &= \boldsymbol{j} \\ -\operatorname{div} \boldsymbol{B}_{\pi/2} - \partial_t E_z &= j_3 \\ \operatorname{div} \boldsymbol{E} + \partial_z E_z &= \rho \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla}_{\pi/2} E_z - \partial_z \boldsymbol{E}_{\pi/2} - \partial_t \boldsymbol{B} &= 0 \\ \operatorname{div} \boldsymbol{E}_{\pi/2} - \partial_t B_z &= 0 \\ \operatorname{div} \boldsymbol{B} + \partial_z B_z &= 0 \end{aligned}$$

## Dimensional reduction of electrodynamics w.r.t. $\partial_t$

Return to description of electrodynamics in the language of 3D-spatial objects, i.e. in terms of 3D vector calculus.

Consider the situation when all quantities happen to be invariant w.r.t. time direction

$$\boxed{\mathcal{L}_{\partial_t}(\dots) = 0}$$

This means that the time direction becomes in a sense superfluous, the quantities become "true" objects from 3D-vector calculus rather than time-dependent objects from 3D-vector calculus, as was the case before.

The procedure is known as dimensional reduction (from 4D to 3D) or (sometimes) as Hadamard method of descent.

# Dimensional reduction of electrodynamics w.r.t. $\partial_t$ (2)

Then in the "full" set of Maxwell equations

$$\operatorname{div} \boldsymbol{E} = \rho \qquad -\partial_t \boldsymbol{E} + \operatorname{curl} \boldsymbol{B} = \boldsymbol{j}$$
$$\operatorname{div} \boldsymbol{B} = 0 \qquad -\partial_t \boldsymbol{B} - \operatorname{curl} \boldsymbol{E} = 0$$

the highlighted terms vanish and the following equations survive

$$\operatorname{div} \boldsymbol{E} = \rho \qquad \operatorname{curl} \boldsymbol{B} = \boldsymbol{j}$$
$$\operatorname{curl} \boldsymbol{E} = 0 \qquad \operatorname{div} \boldsymbol{B} = 0$$

Notice that variables $\boldsymbol{E}$ and $\boldsymbol{B}$ no longer interact.
We get two separate sectors: $(\boldsymbol{E}, \rho)$ and $(\boldsymbol{B}, \boldsymbol{j})$.

What we get physically is (historically) a step back when separate disciplines - electrostatics and magnetostatics - existed.

## Dimensional reduction of electrodynamics w.r.t. $\partial_z$

Now, consider the situation when all quantities
happen to be invariant w.r.t. $z$ direction

$$\boxed{\mathcal{L}_{\partial_z}(\dots) = 0}$$

This means that $z$ direction becomes in a sense superfluous,
the quantities become time-dependent objects
from 2D-vector calculus, namely living in the $(x, y)$-plane.

# Dimensional reduction of electrodynamics w.r.t. $\partial_z$ (2)

Then in the full set of (2D-version!) of Maxwell equations

$$\begin{aligned} -\boldsymbol{\nabla}_{\pi/2} B_z + \partial_z \boldsymbol{B}_{\pi/2} - \partial_t \boldsymbol{E} &= \boldsymbol{j} \\ -\operatorname{div} \boldsymbol{B}_{\pi/2} - \partial_t E_z &= j_3 \\ \operatorname{div} \boldsymbol{E} + \partial_z E_z &= \rho \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla}_{\pi/2} E_z - \partial_z \boldsymbol{E}_{\pi/2} - \partial_t \boldsymbol{B} &= 0 \\ \operatorname{div} \boldsymbol{E}_{\pi/2} - \partial_t B_z &= 0 \\ \operatorname{div} \boldsymbol{B} + \partial_z B_z &= 0 \end{aligned}$$

the highlighted terms vanish.

## Dimensional reduction of electrodynamics w.r.t. $\partial_z$ (3)

The following equations survive:

$$\begin{aligned} \operatorname{div} \boldsymbol{E} &= \rho \\ -\boldsymbol{\nabla}_{\pi/2} B_z - \partial_t \boldsymbol{E} &= \boldsymbol{j} \\ \operatorname{div} \boldsymbol{E}_{\pi/2} - \partial_t B_z &= 0 \end{aligned}$$

$$\begin{aligned} -\operatorname{div} \boldsymbol{B}_{\pi/2} - \partial_t E_z &= j_3 \\ \boldsymbol{\nabla}_{\pi/2} E_z - \partial_t \boldsymbol{B} &= 0 \\ \operatorname{div} \boldsymbol{B} &= 0 \end{aligned}$$

Again degrees of freedom separate to two distinct sectors: $(E_x, E_y, B_z \leftrightarrow \rho, j_x, j_y)$ and $(B_x, B_y, E_z \leftrightarrow j_3)$.

# Dimensional reduction of electrodynamics w.r.t. $\partial_z$ (4)

The first half

$$\begin{aligned} \operatorname{div} \boldsymbol{E} &= \rho \\ -\boldsymbol{\nabla}_{\pi/2} B_z - \partial_t \boldsymbol{E} &= \boldsymbol{j} \\ \operatorname{div} \boldsymbol{E}_{\pi/2} - \partial_t B_z &= 0 \end{aligned}$$

turns out to describe standard 1+2 electrodynamics
(see Lapidus 1982, Moreno,Rivas 1984, Boito et.al 2020, ...)

# Dimensional reduction of electrodynamics w.r.t. $\partial_z$ (5)

The second half

$$\begin{aligned} -\operatorname{div} \boldsymbol{B}_{\pi/2} - \partial_t E_z &= j_3 \\ \boldsymbol{\nabla}_{\pi/2} E_z - \partial_t \boldsymbol{B} &= 0 \\ \operatorname{div} \boldsymbol{B} &= 0 \end{aligned}$$

describes another possible 1+2 electrodynamics.
(See Maggi, Ercolessi, Facchi, Marmo, Pascazio, Pepe 2022).

# Decomposition w.r.t. observer field

Decomposition of forms on Minkowski space-time used above

$$\alpha = dt \wedge \hat{s} + \hat{r}$$

is just the simplest version of a much more general decomposition

$$\alpha = \tilde{V} \wedge \hat{s} + \hat{r}$$

w.r.t. a general observer field $V$ on a Lorentzian manifold $(M, g)$. Here

$$\tilde{V} := g(V, \,\cdot\,) \qquad g(V, V) \equiv ||V||^2 = 1$$

The first formula comes from Minkowski space-time $(\mathbb{R}^4, \eta)$ with

$$V = \partial_t \qquad \tilde{V} = \eta(\partial_t, \,\cdot\,) = dt$$

# Decomposition w.r.t. observer field (2)

The decomposition is based on elementary identity

$$i_V j_V + j_V i_V = g(V,V)\hat{1} = \hat{1} \qquad j_V \alpha := \tilde{V} \wedge \alpha$$

So

$$\alpha = j_V i_V \alpha + i_V j_V \alpha \equiv \tilde{V} \wedge \hat{s} + \hat{r}$$

The field $V$ encodes important kinematical characteristics of the reference frame. In general (see e.g. Fecko 1997)

$$d\tilde{V} = \tilde{V} \wedge \hat{a} + \hat{y}$$

where $\hat{a}$ is acceleration 1-form and $\hat{y}$ is vorticity 2-form ($\hat{y} \neq 0$ means time synchronization is not possible).

# Operations $d$, $*$ and $\delta$ on forms on $E^{1,3}$

Recall that for $V = \partial_t$ we had

$$
\begin{aligned}
d(dt \wedge \hat{s} + \hat{r}) &= dt \wedge (\mathcal{L}_{\partial_t} \hat{r} - \hat{d}\hat{s}) + \hat{d}\hat{r} \\
\delta(dt \wedge \hat{s} + \hat{r}) &= dt \wedge \hat{\delta}\hat{s} + (-\mathcal{L}_{\partial_t} \hat{s} - \hat{\delta}\hat{r})
\end{aligned}
$$

Now (for general $V$) life becomes more complex and we get

$$
\begin{aligned}
d(\tilde{V} \wedge \hat{s} + \hat{r}) &= \tilde{V} \wedge (\mathcal{L}_V \hat{r} - \hat{d}\hat{s} + \hat{a} \wedge \hat{s}) + \hat{d}\hat{r} + \hat{y} \wedge \hat{s} \\
\delta(\tilde{V} \wedge \hat{s} + \hat{r}) &= \tilde{V} \wedge (\hat{\delta}\hat{s} + \hat{*}(\hat{y} \wedge \hat{*}\hat{r})) + (-\hat{*}\mathcal{L}_V \hat{*}\hat{s} - \hat{\delta}\hat{r} + \hat{*}(\hat{a} \wedge \hat{*}\hat{\eta}\hat{r}))
\end{aligned}
$$

# Example: Rotating reference frame in Minkowski space

If $(t, r, \varphi, z)$ is cylindrical coordinate system in Minkowski space-time, then

$$\tau \mapsto \gamma(\tau) = (t_0 + \tau, r_0, \varphi_0 + \omega\tau, z_0)$$

is a world-line describing uniform rotation around $z$-axis.
All such world-lines happen to be integral curves of the vector field

$$\begin{aligned} v &= \partial_t + \omega\partial_\varphi && \text{(in cylindrical coordinates)} \\ &= \partial_t + \omega(x\partial_y - y\partial_x) && \text{(in Cartesian coordinates)} \end{aligned}$$

Its norm is not unity,

$$||v||^2 = g(v, v) = 1 - (\omega r)^2$$

# Example: Rotating reference frame in Minkowski space (2)

So, the vector field $V$ which we look for is just the normalized vector

$$V := \frac{v}{||v||} = \gamma(\partial_t + \omega\partial_\varphi) \qquad \gamma(r) := \frac{1}{\sqrt{1-(\omega r)^2}}$$

Thus normalization just adds overall, position dependent, relativistic factor $\gamma$.
Here we get

$$\hat{a} = \gamma^2(\omega^2 r)dr \qquad \hat{y} = -2\gamma^3\omega r dr \wedge (d\varphi - \omega dt)$$

So, in rotating reference frame we feel radial acceleration (clear :-) and we cannot perform time synchronization (known).

# Example: Rotating reference frame in Minkowski space (3)

For electromagnetic field 2-form $F$ we have

$$F = j_V i_V F + i_V j_V F \equiv \tilde{V} \wedge \hat{E} - \hat{B}$$

Here $\hat{E}$ and $\hat{B}$ describe electric and magnetic fields, respectively, as detected in rotating reference frame.

One can, as the "proof of the concept" (or just for fun :-), compute in this way the induced voltage for a wire rim uniformly rotating in a static magnetic field from the perspective of the observer rotating with the rim (see Fecko 2013).

## Maxwell equations w.r.t. general $V$

The 4D Maxwell equations

$$\delta F = -j \qquad dF = 0$$

translate, for general decomposition w.r.t. $V$, to 3D system

$$\begin{aligned} \hat{d}\hat{*}\hat{E} + \hat{y} \wedge \hat{*}\hat{B} &= \rho\hat{\omega} \\ \hat{d}\hat{*}\hat{B} - \mathcal{L}_V \hat{*}\hat{E} - \hat{a} \wedge \hat{*}\hat{B} &= \hat{*}\hat{j} \end{aligned}$$

$$\begin{aligned} \hat{d}\hat{E} + \mathcal{L}_V \hat{B} - \hat{a} \wedge \hat{E} &= 0 \\ \hat{d}\hat{B} - \hat{a} \wedge \hat{E} &= 0 \end{aligned}$$

Notice the “non-inertial” terms (containing $\hat{a}$ and $\hat{y}$), which are usually not present (for $\tilde{V} = dt$ we have $d\tilde{V} = 0$).

# Decomposition w.r.t. two fields $V$ and $W$

More detailed decomposition of forms on Minkowski space-time used above

$$\alpha = dt \wedge dz \wedge \breve{A} + dt \wedge \breve{B} + dz \wedge \breve{C} + \breve{D}$$

is again just the simplest version of a general decomposition

$$\alpha = \tilde{V} \wedge \tilde{W} \wedge \breve{A} + \tilde{V} \wedge \breve{B} + \tilde{W} \wedge \breve{C} + \breve{D}$$

w.r.t. a pair of fields $V, W$ on a Lorentzian manifold $(M, g)$. Namely we get the first formula for

$$V = \partial_t \quad W = -\partial_z \qquad \tilde{V} = \eta(\partial_t, \,\cdot\,) = dt \quad \tilde{W} = \eta(-\partial_z, \,\cdot\,) = dz$$

Computations (in particular exterior derivative) become fairly complex :-(

# For Further Reading (1)

M. Fecko.
*Differential geometry and Lie groups for physicists.*
Cambridge University Press 2006 (2011)

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- M. Fecko.
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  *On 3+1 decompositions w.r.t an observer field via diff. forms.*
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# The End :-)

Thank You for Your attention!