Navigation problems in Finsler Geometry

A navigation problem requires to find curves of shortest travel time (geodesics) on a Riemannian manifold, under the influence of a wind or current (a vector field).

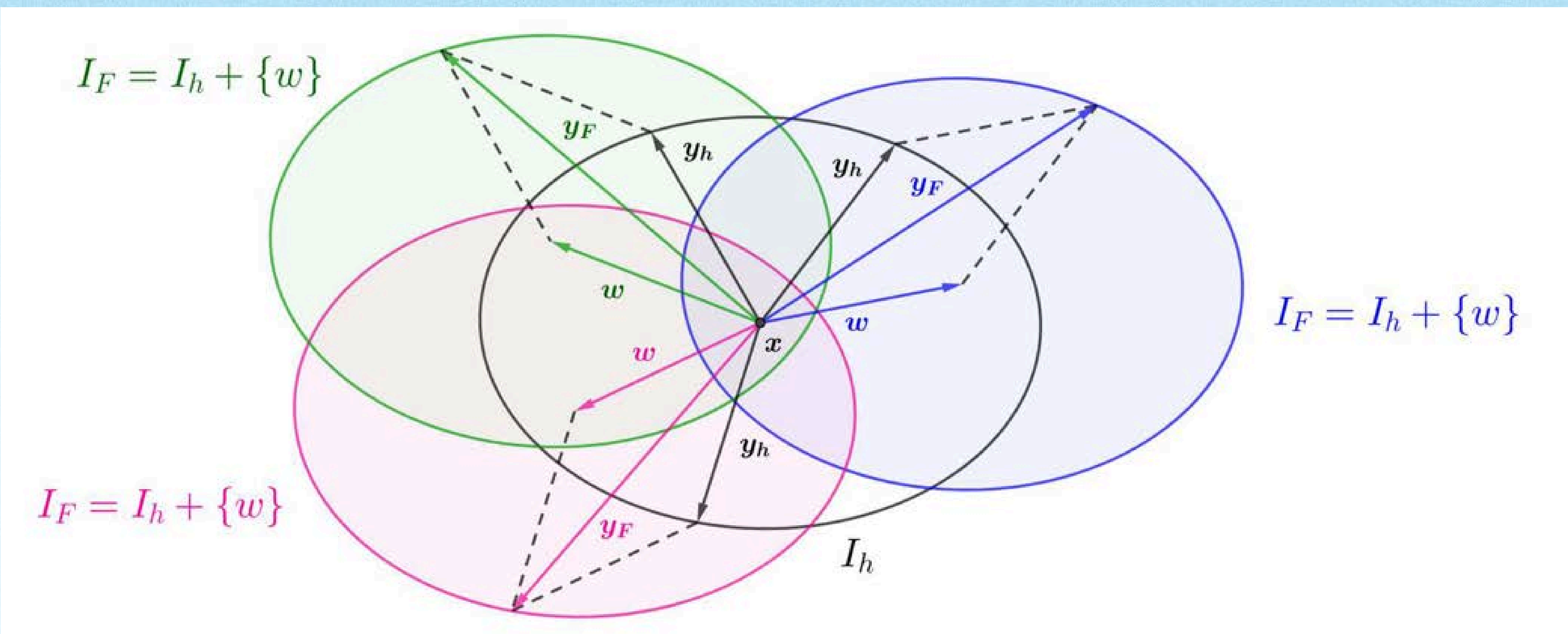
Zermelo navigation problem:

Zermelo's navigation problem aims to find the time-optimal path between two points on a Riemannian manifold (M, h) under the influence of a wind (or current), represented by a vector field w on M, with $\|w\|_h = \sqrt{h(w,w)} < 1$, known as the weak wind condition. We introduce the indicatrix of the Riemannian metric h and the indicatrix of a Finsler function F as follows:

$$\begin{aligned} I_h(x) &= \{y \in T_xM \mid h_x(y,y) = 1\} = \left\{y \in T_xM \mid \| \, y\|_{h(x)} = 1\right\}, \\ I_F(x) &= \{y \in T_xM \mid F(x,y) = 1\}. \end{aligned}$$

The following relation shows that the Finsler indicatrix can be seen as a perturbation of the Riemannian indicatrix caused by the wind w (see the figure below):

$$I_F = I_h + \{w\}.$$



Our goal is to find a Finsler function F such that: $\forall y_h \in I_h \Leftrightarrow \|y_F - w\|_h = 1, \forall y_F \in I_F$.

This yields: $\frac{\|y\|^2}{F^2} - 2h\left(\frac{y}{\|y\|}, w\right)\frac{\|y\|}{F} + \|w\|^2 - 1 = 0, \ \forall y \in T_xM.$

By introducing the notation $\lambda = 1 - \|w\|^2$ and keeping in mind that $h\left(\frac{y}{\|y\|}, w\right) = pr_y w$, the function F has the following form:

$$F = \frac{\|y\|}{pr_y w + \sqrt{\lambda + pr_y^2 w}} = \frac{\|y\| \sqrt{\lambda + pr_y^2 w} - \|y\| \, pr_y w}{\lambda}. \qquad (F.Z)$$

Let h_{ij} be a Riemannian metric, $w = w^i \frac{\partial}{\partial x^i}$, the metric F is uniquely determined by:

$$h_{ij}\left(\frac{y^i}{F} - w^i\right)\left(\frac{y^j}{F} - w^j\right) = 1, \qquad (E.F.Z_1)$$

henceforth, we shall refer to this equation as the *First Fundamental Zermelo Equation*. It is also worth observing that this equation is equivalent to $\left\|\frac{y}{F} - w\right\|_h = 1$, where a straightforward computation yields:

$$\frac{\|y\|^2}{F^2} - \frac{2}{F}h(y,w) + \|w\|^2 = 1, \qquad (E.F.Z_2)$$

The geometry of the original Riemannian metric and that of the resulting Finslerian metric can differ significantly due to the presence of an external drift.

Taking the covariant derivatives with respect to x^k and y^k:

$$h_{ij}\left(-\frac{y^i}{F^2}\frac{\delta F}{\delta x^k} - w^i_{|k}\right)\left(\frac{y^j}{F} - w^j\right) = 0, \ h_{ij}\left(\delta^i_k - \frac{y^i}{F}\frac{\partial F}{\partial y^k}\right)\left(\frac{y^j}{F} - w^j\right) = 0.$$

Theorem 1: If we take $w^i_{|k} = -\frac{\sigma}{2}\delta^i_k$, $\sigma = \sigma(x)$, we obtain:

$d_{h_0}F = \frac{\sigma}{2}F d_{\mathcal{T}}F$, which will be referred to as the almost-Funk metric.

Theorem 2: The following conditions are equivalent:

$$\begin{aligned} d_{h_0}F &= \frac{\sigma}{2}F d_{\mathcal{T}}F, && (1) \\ S_0(F) &= \frac{\sigma}{2}F^2 \text{ and } \delta_{S_0}F = 0. && (2) \end{aligned}$$

Remark: The equation $\delta_{S_0}F = 0$ means that F and h are projectively equivalent (they have the same geodesics, up to an orientation-preserving reparametrization). In this case, the projective factor is $P = \frac{\sigma}{4}F$.

We will refer to:

$$w^i_{|k} = -\frac{\sigma}{2}\delta^i_k,$$

as the *Supplementary Hypothesis* and denote it by $(I.S)$. We ask under what conditions $(I.S)$ admits solutions, for which we obtain the integrability conditions for $(I.S)$:

$$R^i_{skj}w^s = \frac{1}{2}\sigma_{|j}\delta^i_k - \frac{1}{2}\sigma_{|k}\delta^i_j.$$

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