A Real Clifford Algebra Approach to Plebanski Gravity

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1. Motivation: Plebanski formulation of GR

$S(\omega^+, \Sigma^+, \Psi) = \int_M \Sigma^+ \wedge R(\omega^+) + \Psi(\Sigma^+ \wedge \Sigma^+)$

- Internal Hodge operator $*$ on $\mathfrak{so}(1,3)$ squares to -1.
- Splitting the algebra requires complexification and projectors $\Pi^\pm = \frac{1}{2}(1 \mp i*)$
- Curvature term $F_\mathfrak{h} \wedge F_\mathfrak{h}$ from MacDowell-Mansouri action splits naturally in this decomposition
- Add Lagrange multiplier $\Psi(\Sigma^+ \wedge \Sigma^+)$ as simplicity constraint for Σ ($\Sigma = e \wedge e$)
- Artificial reality conditions needed to ensure a real metric

2. Our Approach: Clifford algebras

- (V, η) vector space, bilinear form with signature $(p, q) \in \{(4,1), (2,3)\}$, $(e_i)_{i=0,1,2,3,4}$ ONB
- $Y \in V$, $Y^2 \neq 0 \rightsquigarrow V = V_\perp \oplus V_Y$ "symmetry breaking vector"
- $\rightsquigarrow$ Clifford algebras $\mathrm{Cl}(p-1, q) \hookrightarrow \mathrm{Cl}(p, q)$ generated by V_Y and V, respectively
- Volume elements $\Omega := e_0 e_1 e_2 e_3 e_4$, $\Omega_Y := \Omega Y$. $\Omega_Y^2 = \Omega^2 = -1$ (correspond to $*$ and i, respectively)
- $\mathfrak{g} := \mathfrak{spin}(p, q) \hookrightarrow \mathrm{Cl}(p, q)$, $\mathfrak{h} := \mathfrak{spin}(p-1, q) \hookrightarrow \mathrm{Cl}(p-1, q)$. Consider forms as Clifford-valued
- $\Omega : \mathfrak{h} \to Y\mathfrak{h}$, $*M = \Omega_Y M$ on $\mathfrak{h}$

Unified setting for MacDowell-Mansouri, Stelle-West and Plebanski actions on the spin bundle!

3. The Stelle-West action

We rewrite the Stelle-West action in Clifford Terms: $S = \int_M \mathrm{Tr}(\Omega_Y F \wedge F + \lambda(Y^2 - 1))$

Variation with respect to λ:

$$Y^2 = \mathbb{1}$$

The choice of Y (gauge fixing) decides the decomposition of the Clifford and spin algebra.

4. The Real Plebanski Split

We decompose $\mathrm{Cl}(p, q)$ into the eigenspaces of left and right multiplication by Y. $v \in \mathrm{Cl}(p, q)$ which can be written as a "block matrix"

$$\begin{pmatrix} {}^+v^+ & {}^-v^+ \\ {}^+v^- & {}^-v^- \end{pmatrix}, \quad {}^i v^j = P^i v P^j,$$

where $P^\pm = \frac{1}{2}(1 \pm Y)$ are the eigenspace projectors.
For the generators $e_a \in V_\perp$ and Lorentz bivectors $M \in \mathfrak{h}$:

- $Y e_a = -e_a Y \implies$ Translations vanish in diagonal blocks.
- $YM = MY \implies$ Lorentz algebra survives in diagonal blocks.

The Clifford-Hodge Isomorphism: The above block matrix split restricted to $\mathfrak{h} + Y\mathfrak{h} \cong \mathfrak{h} \otimes \mathbb{C}$ corresponds to the classical Hodge split. The eigenvalue equations are analogous.

$$*M = iM \iff \Omega_Y M = \Omega M$$

$$\rightsquigarrow F = \begin{pmatrix} R^+ + \Sigma^+ & T^+ \\ T^- & R^- + \Sigma^- \end{pmatrix}$$

corresponds to the classical Plebanski split, but does not require complexification. We still consider $\mathrm{Cl}(p, q)$ and the subspaces as real vector spaces.

5. Conclusion & Outlook

- To complete the Plebanski action in Clifford terms, we need to express the constraint field Ψ as a bilinear form on $\mathrm{Cl}(p, q)$ (ongoing).
- Physical application: the reformulation of GR on the spin bundle, where the fields are Clifford-valued forms, yields a canonical framework to couple gravitational fields to spinors without requiring auxiliary representation mappings.