Geometric generalization of the Vainberg-Tonti Lagrangian

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28th International Summer School on Global Analysis
and Its Applications - 18. August 2026

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Jet bundles over fibered manifolds

- Consider a fibered manifold (Y, π, X) with surjective submersion

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- Prolongation of a local section $\gamma : U \to Y$ on $U \subset X$:

$$J^r\gamma : (x^\mu) \mapsto \left(x^\mu, y^A(x), \partial_{\mu_1}y^A(x), \ldots, \partial_{\mu_1}\cdots\partial_{\mu_r}y^A(x)\right)\,. \tag{3}$$

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$\rightsquigarrow$ Coordinates on the jet bundle J^rY:

$$(x^\mu, y^A, y^A{}_{\mu_1}, \ldots, y^A{}_{\mu_1 \cdots \mu_r})\,. \tag{4}$$

Differential forms on jet bundles

- Important classes of differential k-forms:
  - Contact one-forms:
$$\omega^A{}_{\nu_1\cdots\nu_k} = \mathrm{d}y^A{}_{\nu_1\cdots\nu_k} - y^A{}_{\nu_1\cdots\nu_k\mu}\mathrm{d}x^\mu \,. \tag{5}$$
  - Horizontal forms:
$$\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_k} \,. \tag{6}$$
  - Lagrangians:
$$\lambda = \mathcal{L}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m = \mathcal{L}\mathrm{d}^m x \,. \tag{7}$$
  - Source forms:
$$\varepsilon = \varepsilon_A \omega^A \wedge \mathrm{d}^m x \,. \tag{8}$$

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- Projection operators (linear, distributive over $\wedge$):
  - Horizontal projection operator $h : \Omega(J^r Y) \to \Omega(J^{r+1} Y)$:
  $$h\,\mathrm{d}x^\mu = \mathrm{d}x^\mu\,, \quad h\,\mathrm{d}y^A{}_{\nu_1\cdots\nu_k} = y^A{}_{\nu_1\cdots\nu_k\mu}\mathrm{d}x^\mu\,. \tag{9}$$
  - Vertical projection operator $v : \Omega(J^r Y) \to \Omega(J^{r+1} Y)$:
  $$v\,\mathrm{d}x^\mu = 0\,, \quad v\,\mathrm{d}y^A{}_{\nu_1\cdots\nu_k} = \omega^A{}_{\nu_1\cdots\nu_k}\,. \tag{10}$$

Euler operator and Euler-Lagrange equations

- Euler-Lagrange form $E(\lambda) = E_A(\lambda)\omega^A \wedge \mathrm{d}^m x$ of a Lagrangian λ:

$$E_A(\lambda) = \frac{\partial \mathcal{L}}{\partial y^A} + \sum_{k=1}^{r} (-1)^k \mathrm{d}_{\mu_1} \cdots \mathrm{d}_{\mu_k} \frac{\partial \mathcal{L}}{\partial y^A{}_{\mu_1 \cdots \mu_k}} \,. \tag{11}$$

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  - Interior Euler operator $\varrho : \Omega^{m+s}(J^r Y) \to \Omega^{m+s}(J^{2r} Y)$:

$$\varrho(\eta \wedge \mathrm{d}^m x) = \frac{1}{s}\omega^A \wedge \left(\mathbf{i}_{\partial/\partial y^A}\eta + \sum_{k=1}^{r}(-1)^k \mathrm{D}_{\mu_1} \cdots \mathrm{D}_{\mu_k} \mathbf{i}_{\partial/\partial {y^A}_{\mu_1 \cdots \mu_k}} \eta\right) \wedge \mathrm{d}^m x . \tag{13}$$

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Given a source form ε on $J^r Y$:

1. Is $\varepsilon = E(\lambda_\varepsilon)$ (locally) the Euler-Lagrange form of a Lagrangian λ_ε?
2. How can λ_ε be obtained from ε?

Local variationality and Helmholtz condition

- Some useful relations for any Lagrangian λ on $J^r Y$:
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- ε is locally variational if and only if $H(\varepsilon) = 0$ (Helmholtz condition).

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- If ε is variational, then it is the Euler-Lagrange expression of λ_ε:

$$H(\varepsilon) = 0 \quad \Rightarrow \quad E(\lambda_\varepsilon) = \varepsilon . \tag{22}$$

Coordinate-free formulation

- Change integration parameter $t = e^u$ and $\mathrm{d}t = e^u \mathrm{d}u$:

$$\mathcal{L}_\varepsilon(x^\mu, y^A, y^A{}_{\nu_1}, \ldots, y^A{}_{\nu_1\cdots\nu_k}) = \int_{-\infty}^{0} (e^u y^A)\varepsilon_A(x^\mu, e^u y^A, e^u y^A{}_{\nu_1}, \ldots, e^u y^A{}_{\nu_1\cdots\nu_k})\mathrm{d}u\,. \tag{23}$$

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$$\lambda_\varepsilon = \int_{-\infty}^{0} (J^r\varphi_u)^*(\mathbf{i}_{J^r\mathbb{C}}\varepsilon)\mathrm{d}u\,. \tag{26}$$

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$\Rightarrow$ Cannot obtain Einstein-Maxwell Lagrangian from energy-momentum tensor.

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$$\Xi = \xi^A \frac{\partial}{\partial y^A} + \mathrm{d}_\mu \xi^A \frac{\partial}{\partial y^A{}_\mu} + \mathrm{d}_\mu \mathrm{d}_\nu \xi^A \frac{\partial}{\partial y^A{}_{\mu\nu}} + \ldots \,. \tag{29}$$

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Euler-Lagrange equations and Helmholtz conditions

- Vertical derivative:

$$\mathrm{d}_V\lambda_\varepsilon = \mathrm{d}\lambda_\varepsilon = \int_{t_0}^{0}\Phi_t^*(\mathrm{d}\mathbf{i}_{\Xi}\varepsilon)\mathrm{d}t = \int_{t_0}^{0}\Phi_t^*(\mathcal{L}_{\Xi}\varepsilon - \mathbf{i}_{\Xi}\mathrm{d}\varepsilon)\mathrm{d}t\,. \tag{32}$$

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$\Rightarrow$ $E(\lambda_\varepsilon) = \varepsilon$ if $H(\varepsilon) = 0$ and $\Phi_{t_0}^*\varepsilon = 0$.

Outline

Summary

- Lagrange theory on fibered manifolds (Y, π, X):
  - Jets of local sections give rise to jet bundles $J^r Y$.
  - Lagrangian $\lambda = \mathcal{L}\mathrm{d}^m x$ defined on jet bundle.
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$\rightsquigarrow$ Geometric generalization of Vainberg-Tonti Lagrangian:
  - Replace prolongation of Liouville vector field by evolutionary vector field.
  - Reproduces Euler-Lagrange equations under convergence condition.

Outlook

- Relation with gauge symmetries:
  - Consider source form ε satisfying $\mathcal{L}_\Theta \varepsilon = 0$ for evolutionary vector field Θ.
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- Further examples surpassing limitations of Vainberg-Tonti Lagrangian?