A Variational Framework for Linearised Theories

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Introduction and Motivations

The usual approach to perturbation theory, and, in particular, to linearisation presents some mathematical issues. In particular, in the case of gravitational waves (GW) in standard General Relativity (GR), one usually imposes the weak-field approximation (see, e.g., [1]):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

However, the bundle $\mathrm{Lor}(M)$ of Lorentzian metrics on a smooth manifold M is neither a vector bundle nor an affine bundle, which means that there is no globally well-defined addition operation (see [2, 3]). Moreover, it is trivially known that solutions at first order are not exact solutions of Einstein equations.

Starting from these considerations, a new approach for perturbation theory has been proposed, which is completely variational.

The framework

Let us consider a classical field theory with a Lagrangian $\boldsymbol{L}$ on a bundle $[B \to M]$:

$$\boldsymbol{L} = L(x^\mu, y^i, y^i_\mu, \dots).$$

One can compute its first variation $\delta \boldsymbol{L}$:

$$\delta \boldsymbol{L} = p_i \delta y^i + p^\mu_i d_\mu(\delta y^i) + p^{\mu\nu}_i d_{\mu\nu}(\delta y^i) + \dots$$

and see it as an auxiliary Lagrangian, called *Jacobi Lagrangian*:

$$\boldsymbol{L}_J = p_i X^i + p^\mu_i d_\mu X^i + p^{\mu\nu}_i d_{\mu\nu} X^i + \dots.$$

Clearly, it is no longer a Lagrangian on B, but it is a Lagrangian on the composite bundle $[V(B) \to B \to M]$. Finally, one can compute field equations of $\boldsymbol{L}_J$, which include field equations of $\boldsymbol{L}$ and the so-called *linearised equations*. For a general, second-order Lagrangian $\boldsymbol{L}$, linearised equations are

$$\begin{aligned}
&\left(\partial_i p_k - d_\mu(\partial^\mu_i p_k) + d_{\mu\nu}(\partial^{\mu\nu}_i p_k)\right) X^k + \\
&+ \left(\partial_i p^\mu_k - \partial^\mu_i p_k - d_\alpha(\partial^\alpha_i p^\mu_k) + d_{\alpha\beta}(\partial^{\alpha\beta}_i p^\mu_k)\right) d_\mu X^k + \\
&+ \left(\partial_i p^{\mu\nu}_k - \partial^\nu_i p^\mu_k - d_\alpha(\partial^\alpha_i p^{\mu\nu}_k) + \partial^{\mu\nu}_i p_k + d_{\alpha\beta}(\partial^{\alpha\beta}_i p^{\mu\nu}_k)\right) d_{\mu\nu} X^k + \\
&+ \left(-\partial^\alpha_i p^{\mu\nu}_k + \partial^{\mu\nu}_i p^\alpha_k\right) d_{\mu\nu\alpha} X^k + \\
&+ \ \partial^{\mu\nu}_i p^{\alpha\beta}_k d_{\alpha\beta\mu\nu} X^k = 0.
\end{aligned}$$

These equations can be recast in a more elegant and clear form as

$$\begin{aligned}
&X^k \partial_k(p_i - d_\alpha p^\alpha_i + d_{\alpha\beta} p^{\alpha\beta}_k) + d_\mu X^k \partial^\mu_k(p_i - d_\alpha p^\alpha_i + d_{\alpha\beta} p^{\alpha\beta}_i) + \\
&+ d_{\mu\nu} X^k \partial^{\mu\nu}_k(p_i - d_\alpha p^\alpha_i + d_{\alpha\beta} p^{\alpha\beta}_i) + d_{\mu\nu\rho} X^k \partial^{\mu\nu\rho}_k(p_i - d_\alpha p^\alpha_i + d_{\alpha\beta} p^{\alpha\beta}_i) + \\
&+ d_{\mu\nu\rho\sigma} X^k \partial^{\mu\nu\rho\sigma}_k(p_i - d_\alpha p^\alpha_i + d_{\alpha\beta} p^{\alpha\beta}_i) = 0,
\end{aligned}$$

which in turn can be written more compactly as $j^4 X(\mathbb{E}_k(L)) = 0$. In particular, a solution of linearised equations, which is called a *Jacobi field*, is a vertical field X whose fourth-order jet prolongation $j^4 X$ is tangent to the submanifold of $J^4 B$ representing field equations of $\boldsymbol{L}$. In other words, one could say that Jacobi fields drag solutions of field equations onto other solutions (see [2, 3, 4]).

Why this is useful

This framework presents numerous advantages: it can be applied to any classical field theory (generality); it does not involve any approximation in the computations (exactness); and, most importantly, it allows to study linearisation around all solutions of the theory at the same time (globality).

Moreover, solutions of linearised equations in our framework are vector fields, which produce a flow: this is not any more tangent to the space of solutions of field equations, but an object dragging points of that space (i.e., solutions of field equations) onto other points without leaving the space itself. Besides, one can still recover the usual approach by taking the variational linearised equations and simply fixing a solution of field equations as a background (see [2, 3, 4]).

We can summarise the process with the following diagram:

$$\begin{tikzcd}
\text{field theory} \arrow[r, "\substack{\text{variational}\\ \text{linearisation}}"] \arrow[d, "\substack{\text{compute}\\ \text{a solution } \bar{y}}"] & \text{linearised theory} \arrow[r, "\substack{\text{solve}\\ \text{linearised equations}}"] \arrow[d, dashed, "\substack{\text{compute it}\\ \text{around a solution } \bar{y}}"] & \text{linearised solution} \arrow[d, dashed, "\substack{\text{compute it}\\ \text{around a solution } \bar{y}}"] \\
\text{solution} \arrow[r, "\substack{\text{usual linearisation}\\ (\bar{y} + \varepsilon X)}"'] & \text{linearised equations} \arrow[r, "\substack{\text{solve}\\ \text{linearised equations}}"'] & \text{first-order solution}
\end{tikzcd}$$

Application to standard GR

As a first example in field theories, we chose standard GR, both because it is the simplest and most fundamental gravitational theory, and because our original motivations came from the study of GW in standard GR.

Let us consider the Hilbert-Einstein Lagrangian (with cosmological constant, for the sake of generality):

$$\boldsymbol{L} = \sqrt{g}\,(R - 2\Lambda)\, d\sigma.$$

Field equations are the well-known Einstein equations (with cosmological constant)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu}.$$

The Jacobi Lagrangian is

$$L_J = \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\right) X^{\mu\nu} + \sqrt{g}\left(-\nabla_\varepsilon \nabla_\mu X^{\varepsilon\mu} + g_{\mu\nu} \Box X^{\mu\nu}\right).$$

Linearised Einstein Equations (LEE) are

$$\begin{aligned}
&-\frac{1}{2} g_{\mu\nu} R_{\alpha\beta} X^{\alpha\beta} + \frac{1}{4} g_{\mu\nu} R X - \frac{1}{2} g_{\mu\nu} \Lambda X + \\
&- \nabla_\lambda \nabla_{(\nu} X^\lambda{}_{\mu)} + \frac{1}{2} \nabla_{\mu\nu} X + \frac{1}{2} \Box X_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \Box X + \\
&- \frac{1}{2} R_{\mu\nu} X + \frac{1}{2} X_{\mu\nu} R - \Lambda X_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \nabla_\alpha \nabla_\beta X^{\alpha\beta} = 0
\end{aligned}$$

(see [2, 3]).

These equations are rather complicated, but they can be significantly simplified through natural steps. First, they are paired with Einstein equations; thus we can restrict them on shell. Second, one can apply the same ideas used to study GW around a Minkowski background: one can rewrite the equations for

$$\bar{X}^{\mu\nu} = X^{\mu\nu} - \frac{1}{2} X g^{\mu\nu}$$

and impose the *De Donder gauge*:

$$\nabla_\mu X^{\mu\nu} = \frac{1}{2} g^{\mu\nu} \nabla_\mu X \quad \Leftrightarrow \quad \nabla_\mu \bar{X}^{\mu\nu} = 0.$$

Then, linearised Einstein equations reduce to

$$\frac{1}{2} \Box \bar{X}_{\mu\nu} - R_{\mu\alpha\lambda\nu} \bar{X}^{\lambda\alpha} + \frac{2\Lambda}{m-2}\left(\bar{X}_{\mu\nu} - g_{\varepsilon(\mu} g_{\nu)\alpha} \bar{X}^{\alpha\varepsilon}\right) = 0.$$

In particular, if we are considering the theory without cosmological constant ($\Lambda = 0$), the equations further simplify to a form that generalises the equations of GW around a general background:

$$\frac{1}{2} \Box \bar{X}_{\mu\nu} - R_{\mu\alpha\lambda\nu} \bar{X}^{\lambda\alpha} = 0$$

(see, e.g., [6]).

Further results and perspectives

- In [3], a few solutions of LEE were found mainly using a generalisation of Utiyama theorems. Despite being interesting, such solutions were non-physical, since they were either globally well-defined and non-trivial vector fields reducing to the null field on-shell, or a gauge transformation.
- Regarding solutions, the purpose is that of finding physically relevant solutions, or at least studying their existence, through canonical analysis.
- It has been proven that the framework is applicable to Lagrangians of any finite order k.
- A field theory is said to be *linear* if the bundle upon which it is defined is a vector or an affine bundle, and the space of its solutions is a vector space. If a theory is linear, its linearised equations are expected to be two copies of field equations. This has been proven for Klein-Gordon, Maxwell, quadratic Lagrangians.
- Study of conservation laws and symmetries.
- Application to extended theories of gravity and cosmological perturbation theory.

References

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