Teleparallel Regularization of Gravitational Action

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Introduction and Motivation

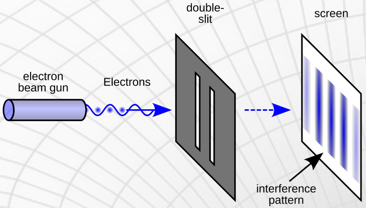
- Global aspects of field theories in physics?
- A classical field theory is defined by the local field equations

$$E(\psi) = 0$$

  usually obtained from the variational principle $\delta S = 0$
- These solutions have global properties, which can be studied:
  - Conserved quantities and Noether theorems
  - Asymptotical properties and boundary effects
  - Gravity: casual structure and restriction on solutions
  - ...
  - Path integrals in quantum theory

Global Aspects and Quantum Theory

- Classical mechanics: principle of the least action $\delta\mathcal{S} = 0$
- Quantum mechanics: particle "follows" multiple paths, each with certain amplitude



Feynman (path integral) formulation: all path with amplitude $\propto e^{\frac{i}{\hbar}\mathcal{S}}$

- Classical limit: $h \to 0$ leaves the classical path, i.e. $\delta\mathcal{S} = 0$
- Semiclassical approximation: expansion in terms of $\hbar$ around critical points of $\mathcal{S}$ (classical trajectories)

Global Aspects of Field Theories

- For a field theory we integrate over all possible field configurations

$$Z = \int_{\mathcal{M}} d[\psi] e^{i\mathcal{S}}$$

- Euclidean path integrals: we usually consider Wick rotation $t \to -i\tau$ that turns oscillating path integrals into "nicely" convergent ones

$$Z = \int_{\mathcal{M}} d[\psi] e^{-\mathcal{S}^E}$$

- Classical finite Euclidean action solutions (Instantons) dominate the path integral
- **Non-perturbative effects in QFT that are "essentially classical"**
- Well-studied in Yang-Mills theories: non-trivial structure of vacuum, topology....
- Not so straightforward in gravity...

Gravitational Instantons

- Total gravitational action on the Euclidean manifold $\mathcal{E}$ All quantities with $\circ$ are related to Riemannian geometry

$$\mathcal{S}_{\text{total}} = \frac{1}{2\kappa}\int_{\mathcal{E}}\sqrt{-g}\,\overset{\circ}{R} + \frac{1}{\kappa}\oint_{\partial\mathcal{E}}\sqrt{-\gamma}K + \mathcal{S}_{\text{counter}}$$

- Several possible counterterms:
  - **Background Subtraction** Gibbons& Hawking 1977

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\sqrt{-\gamma}K^0,$$

  - **Holographic Renormalization:**
    - AdS case Balasubramanian Kraus 1999

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\sqrt{-\gamma}\left(\frac{2}{l} - \frac{l}{2}\mathcal{R}\right),$$

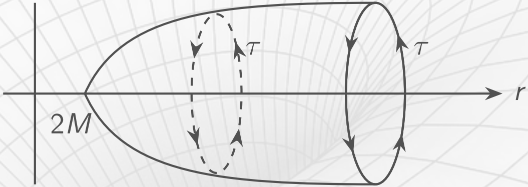
    - Asymptotically flat case Mann, Marolf 2006

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\sqrt{-\gamma}\hat{K}$$

    where $\hat{K}$ is a solution to $\mathcal{R}_{ik} = \hat{K}_{ik}\hat{K} - \hat{K}_i^m\hat{K}_{mk}$
  - **Kounterterms, topological regularization...** Olea 2007

Example: Gibbons-Hawking Instanton 1977

- Wick rotated $t \longrightarrow -i\tau$ Schwarzschild solution
- To avoid conic singularities $\Longrightarrow$ Periodic time $\beta = 8\pi M$ $\Longrightarrow$ Euclidean manifold $\mathcal{E}$



- Traditional viewpoint: Euclidean BH is regular at the horizon and only boundary is $r \to \infty$

$$\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}} = \frac{1}{2\kappa}\int_{\mathcal{E}} \sqrt{-g}\,\mathring{R} + \frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{-\gamma}\,(K - K_0) = \frac{\beta M}{2}$$

- Leads to the correct area entropy law $A/4$
- Considerably different from what we do in Yang-Mills theory

Yang-Mills and General Relativity

Yang-Mills Theory

- Action

$$\mathcal{S}_{\mathrm{YM}} = \int_{\mathcal{M}} \mathrm{Tr}\, F \wedge \star F$$

- Field Equations

$$DF = 0$$
$$D \star F = 0$$

General Relativity

- Action

$$\mathcal{S}_{\mathrm{GR}} = -\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R}$$

- Field Equations

$$\begin{aligned} \mathring{R}^{\rho}{}_{[\sigma\mu\nu]} &= 0 \\ \mathring{R}_{\mu\nu} &= 0 \end{aligned}$$

Do not look alike!

Gravity à la Yang-Mills: Teleparallel Gravity

- Alternative formulation of general relativity formulated in **tetrad formalism**

$$\{g_{\mu\nu}, \Gamma^{\rho}{}_{\mu\nu}\} \Longrightarrow \{h^a, \omega^a{}_b\}$$

- Riemannian geometry[1]

$$\begin{aligned} 0 &= dh^a + \overset{\circ}{\omega}{}^a{}_b \wedge h^b, && \text{(zero torsion)} \\ \overset{\circ}{R}{}^a{}_b &= d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b && \text{(non-zero curvature)} \end{aligned}$$

- **Teleparallel geometry**

$$\begin{aligned} 0 &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b && \text{(zero curvature)} \\ T^a &= dh^a + \omega^a{}_b \wedge h^b && \text{(non-zero torsion)} \end{aligned}$$

- Leads to the pure-gauge teleparallel (Weitzenböck) connection

$$\omega^a{}_b = \Lambda^a{}_c d(\Lambda^{-1})^c{}_b, \qquad \Lambda^a{}_b \in SO(1,3) \text{ or } SO(4)$$

[1] Notation: All quantities with $\circ$ are related to Riemannian geometry

Teleparallel Equivalent of General Relativity

- Relation between Riemannian $\overset{\circ}{\omega}{}^a{}_b$ and teleparallel $\omega^a{}_b$ connection

$$\boxed{\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b + K^a{}_b} \qquad T^a = K^a{}_b \wedge h^b$$

- Allows to rewrite any Riemannian expression in terms of teleparallel ones

$$-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(h T^{\nu\mu}{}_\nu \right)$$

where T is torsion scalar

$$T = \frac{1}{4} T^a{}_{\mu\nu} T_a{}^{\mu\nu} + \frac{1}{2} T^a{}_{\mu\nu} T^{\nu\mu}{}_a - T^{\rho\mu}{}_\rho T^\nu{}_{\mu\nu}$$

- **Teleparallel Equivalent of General Relativity (TEGR)**

$$\boxed{\mathcal{S}_{\mathrm{TG}} = \frac{1}{2\kappa} \int_{\mathcal{M}} h\, T}$$

- Is equivalent to GR (dynamically): same solutions ...

Regularization of Teleparallel Action

- What is not equivalent? Global aspects
- Teleparallel action $\mathcal{S}_{\mathsf{TG}} \propto \text{Torsion}^2$, no 2nd derivatives, no GHY and counterterms?
- Reformulating gravity, preserving difficulty: Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations and enters the action as Krssak 2016

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b) = \mathcal{S}_{\mathsf{TG}}(h^a, 0) + \int_{\mathcal{M}} \partial_\mu (h \omega^{\rho\mu}{}_\rho)$$

- Finite actions are obtained when specific $\tilde{\omega}^a{}_{b\mu}$ is matched to the tetrad $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Finding $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$ is equivalent to finding the counterterm: **teleparallel regularization**

Example: Teleparallel Schwarzschild Instanton

- Schwarzschild static tetrad

$$h^a{}_\mu \equiv \mathrm{diag}\big(f, f^{-1}, r, r\sin\theta\big) \qquad f^2 = 1 - \frac{2M}{r}$$

- Require that torsion vanishes at infinity $r \to \infty$ Obukhov&Pereira 2002, Krssak&Pereira 2015

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta.$$

- Alternative: canonical frame (Kerr-Schild) Beltran Jimenez, Heisenberg, Koivisto 2019
- Naive **bulk** Euclidean action Krssak&Pereira 2015

$$\mathcal{S}^E_{TG}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = \frac{1}{2\kappa}\int_{\mathcal{E}} hT = \beta M = 2\,\mathcal{S}^E_{GR}$$

- However, in vacuum we can use the identity

$$\boxed{h\,T = -2\,\partial_\mu(hT^\mu)} \qquad T^\mu = T^{\nu\mu}{}_\nu (\text{vectorial torsion})$$

- Stokes' theorem: calculate the **quasi-local action**

$$\boxed{\tilde{\mathcal{S}}^E_{\mathrm{TG}}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu = \frac{\beta M}{2} = \mathcal{S}^E_{\mathrm{GR}}}$$

Flashbacks: Electrostatics

- Recall calculation of a total charge using Gauss's Law in

$$Q = \int_V \nabla.\mathbf{E}\, d^3x = \oint_{\partial V} \mathbf{E}.\mathbf{A}$$

- For a point charge

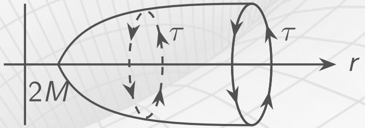
$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$$

- Total charge is $Q = q$, while $\nabla.\mathbf{E} = 0$
- We can
  - Add the Dirac delta function as a source
  - Consider the volume integral as a charge where $\mathbf{E}$ is regular, while surface integral being a total charge bounded by ∂V (including singularity)

Role of Singularities in Euclidean Action

$$S^E_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{\mathcal{S}}^E_{\mathsf{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu$$

- are equal iff T^μ is regular on $\mathcal{E}$ and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$ corresponds to only the black hole exterior



- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: bulk and quasilocal actions do not agree
- Black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- Physics insights: We can quantify how interior and singularity contributes to path integraland we detect holes in our manifold
- Mathematical insights: role of topology, cohomology, ...

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider transformations

$$\bar{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- with asymptotic behaviour

$$\alpha \sim \frac{\alpha_0}{r^p}, \qquad r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean quasilocal action $\beta\frac{M}{2} \longrightarrow \beta\frac{M-\alpha_0^2}{2}$
- See Faddeev 1982 analysis of symmetries of the gravitational action or Regge and Teitelboim 1974 Hamiltonian analysis

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^a{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes, i.e. give fall-off conditions for torsion
- Two options, fall-off for:
  - **vectorial torsion**

$$T^\mu = \mathcal{O}\left(\frac{1}{r^2}\right),$$

    - $p = 1/2$ boosts are then large "gauge" transformations
  - **torsion**

$$T^\rho{}_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

    - $p = 1/2$ boosts are excluded and not a symmetry

Conclusions

- Solutions of field equations have local and global properties
- Instantons (classical solutions with a finite Euclidean action) play an important role in quantum gravity
- Gravity: we need to evaluate $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ on solutions
- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions and the role of singularities and the under horizon geometry
- Teleparallel regularization is determined only up to some ambiguity related to remnant symmetries Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev