Bianchi cosmologies with four-dimensional groups of motion in New General Relativity

Paul Martin Kull, Manuel Hohmann, Margus Saal
Institute of Physics, University of Tartu, Estonia

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1-parameter New General Relativity

A theory that attempts to eliminate ghosts from New General Relativity, 1PNGR has one defining parameter, δ.[1, 2] Like other teleparallel theories, it exhibits strong coupling, which makes perturbative analysis impossible in highly symmetric spacetimes.[3] The field equations

$$E_{\mu\nu} = G_{\mu\nu} + \delta\left(\frac{1}{2}a^\rho a_{(\rho} g_{\mu\nu)} - \frac{4}{9}\epsilon_{\nu\alpha\beta\gamma}a^\alpha t_\mu{}^{\beta\gamma} - \frac{2}{9}\epsilon_{\mu\nu\rho\sigma}a^\rho v^\sigma - \frac{1}{3}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho a^\sigma\right) = \kappa T_{\mu\nu}\,, \tag{1}$$

show that all difference from (TE)GR is concentrated in the term proportional to δ. Here, a_μ, v_μ and $t_{\rho\mu\nu}$ are the axial, vector and tensor torsions, which form the irreducible decomposition of the general torsion tensor.

Bianchi spacetime geometry

The Bianchi geometries of types II, III and IX admit 4D groups of motions and have tetrads that can be parametrised in terms of 6 free functions of time.[4] This can be further refined with a 2D group of time-dependent diffeomorphisms, which eliminates 2 of these functions. The final spacetime is anisotropic, with 2 scale factors λ and μ parallel and orthogonal to the single isotropy direction of spacetime, respectively.

Energy-momentum tensor

An anisotropic Universe may (but does not necessarily) demand anisotropic energy-momentum. An anisotropic fluid is then the most natural choice:

$$T_{\mu\nu} = g_{\mu\nu}p_\perp + (\rho + p_\perp)U_\mu U_\nu + (p_\parallel - p_\perp)W_\mu W_\nu\,, \tag{2}$$

where W_μ and U_μ are two eigenvector fields of $T_{\mu\nu}$. An equation of state is also needed for later. The simplest choice is a barotropic equation $p_i = w_i\rho\,,\ i = \parallel, \perp$, where the barotropic index w_i can be 0 (dust) or $1/3$ (radiation).

Field equations

The field equations are best behaved when presented as a first-order dynamical system. This system is 6-dimensional: 4 functions with two (λ and μ) having 2nd order derivatives. The systems are not analytically solvable for any type, but some special cases can be isolated.

Restricted case

One option is to demand that the scale factors behave identically, i.e. $\lambda = \mu$, and it is also natural to do the same with energy-momentum ($p_\parallel = p_\perp$). These together render the field equations much more easily solvable, with different results based on the Bianchi type:

- Types II and III produce solutions with $\mu =$ const., which is a Universe that does not contract or expand - undesirable.
- Type IX is much more interesting - a Friedmann-like solution:

$$\frac{\kappa\rho}{3} = \left(\frac{\mu'}{\mu}\right)^2 + \frac{1 + 2\delta/3}{\mu^2}\,, \qquad \frac{\mu''}{\mu} = -\frac{\kappa\rho}{6}(1 + 3w)\,, \tag{3}$$

  with δ functioning as a "curvature correction." Here, the symmetry group is expanded to 6 dimensions, like in FLRW.

A more general treatment requires numerical solutions.

Fixed points (or lack thereof)

None of the general systems admit fixed points. Type IX gets close: by setting $\gamma = \pm\pi/2$ and $\beta = 0$, we get $\gamma' = \beta' = 0$, but the other functions remain dynamic. By setting $\gamma = -\pi/2$ and $\beta = 0$ for type IX, we can also achieve $\lambda/\mu =$ const..

Numerics: initial parameter values

For numerics, initial values are needed. Some natural choices:

- Shift the time coordinate so that $t = 0$ corresponds to the present.
- For the scale factors, $\mu(0) = \lambda(0) = 1$.
- For their derivatives, $\mu'(0) = \lambda'(0) = 1$ to account for the current expansion of the Universe.
- Solutions can be filtered with $\rho \geq 0$ to avoid the existence of exotic matter.

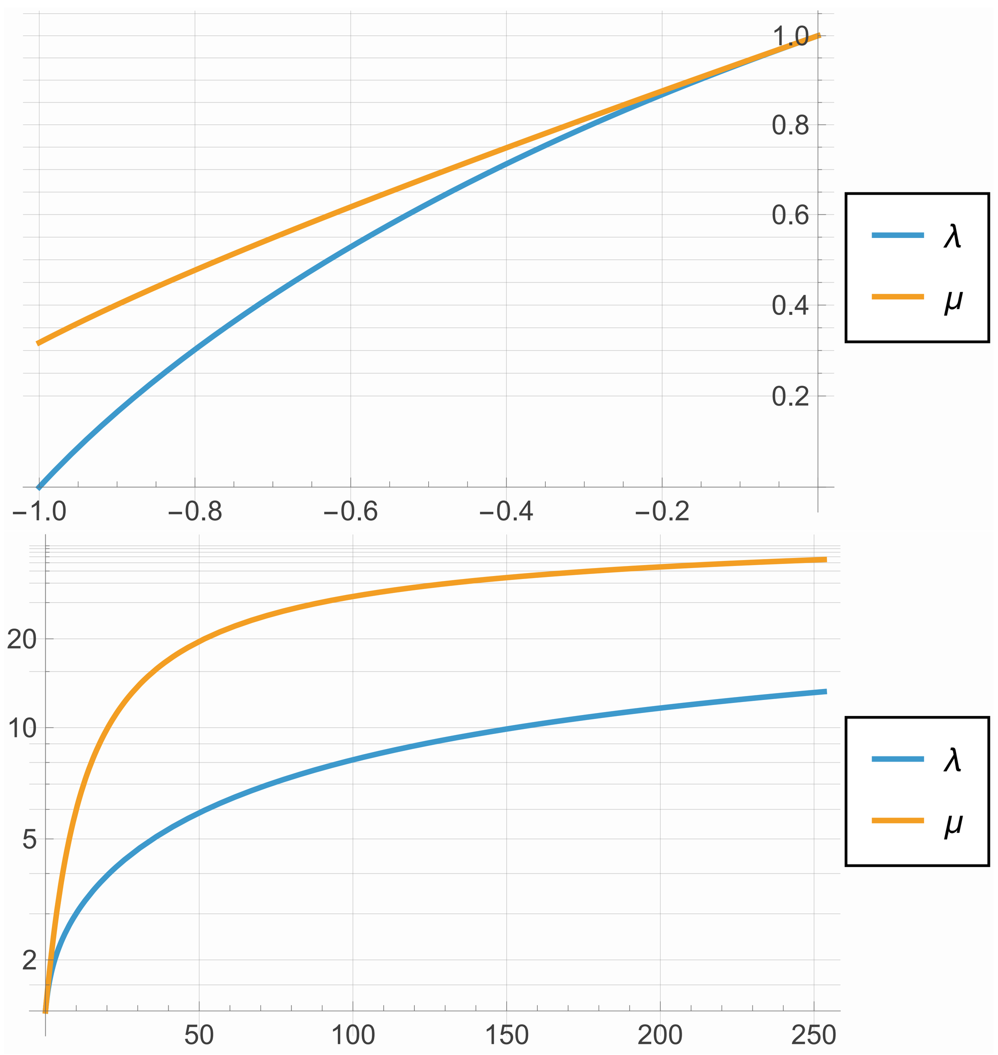
Two tetrad functions γ and β remain that must be treated type-by-type. Some values make singular field equations:

- $\sin\gamma = 0$ for type II,
- $\beta = 0$ for type III,

but everything else is fair game.

General numerical solutions

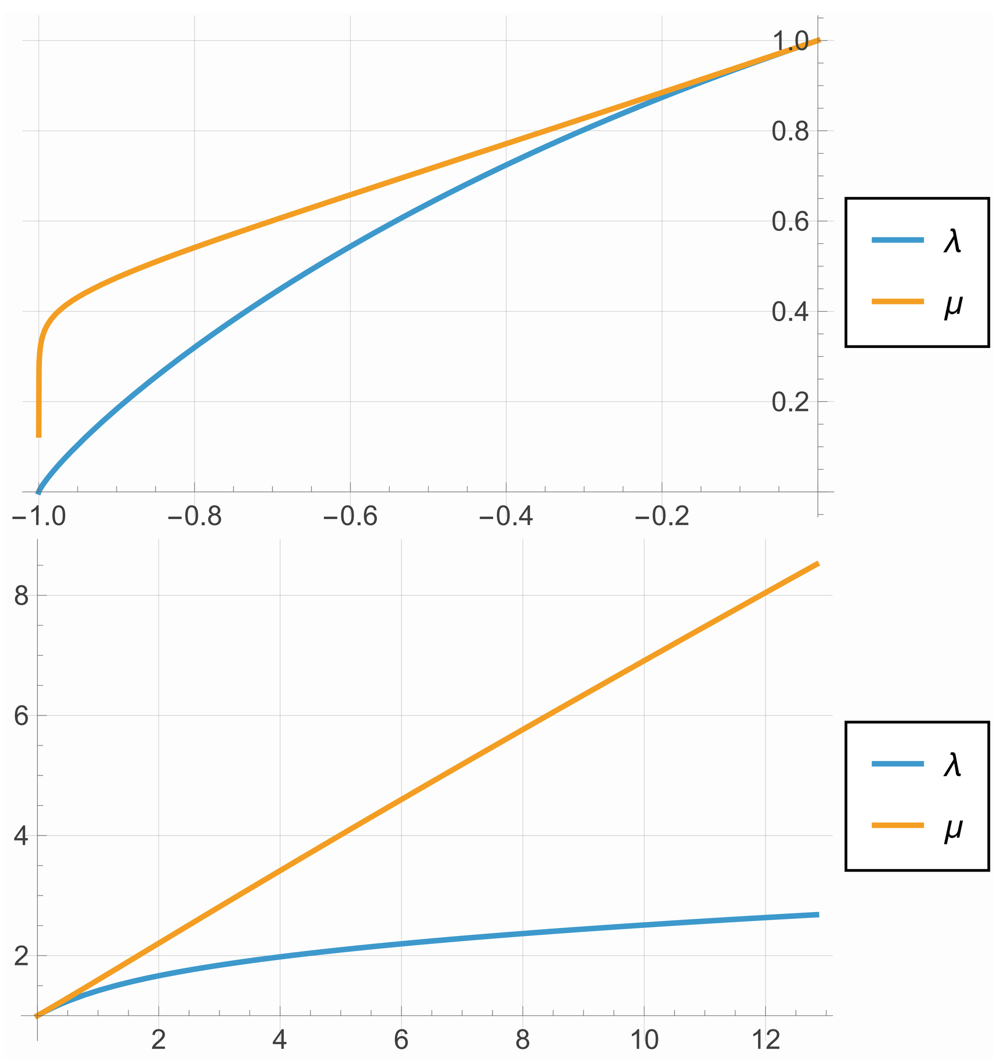
We begin with types II and III, which have fairly similar behaviours. Type II:



The scale factors of type II from $t = -1$ (the beginning of the Universe) to $t = 0$ (present day) and from the present day to the end of the numerical simulation ($t \approx 253$). β and γ both begin at ∞, but γ approaches 0 while β returns to ∞.
Initial parameters:

$$(\delta, w_\parallel, w_\perp, \gamma(0), \beta(0), \lambda'(0), \mu'(0)) = (-0.01, 0, 0, \pi/2, 0, 1, 1)$$

Type III:

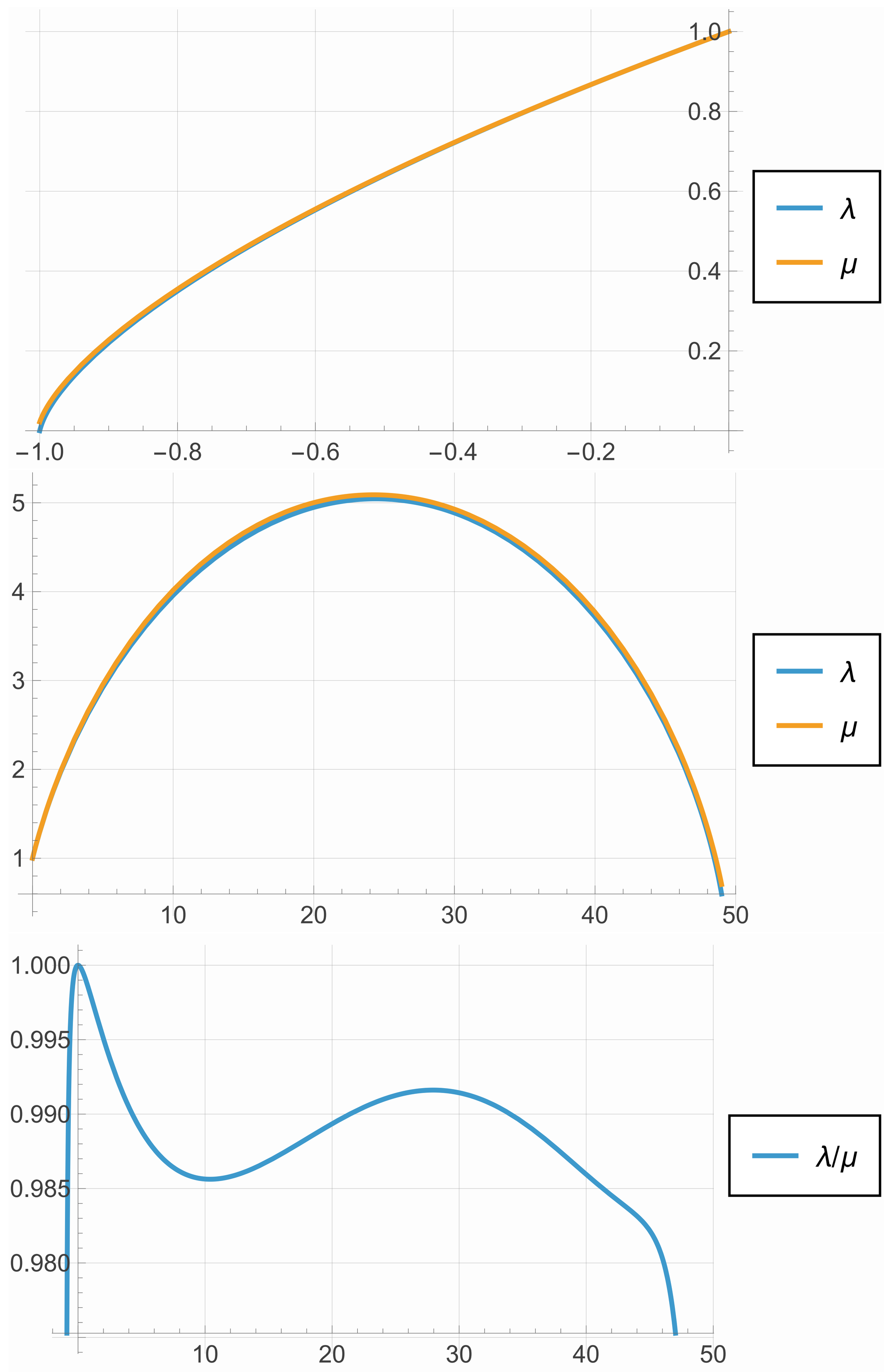


The scale factors of type III from $t = -1$ to $t = 0$ and from $t = 0$ to the end of the numerical simulation ($t \approx 12.8$). β begins at 0 and γ at $-\infty$, the former finishes at ∞ and the latter at a finite value.
Initial parameters:

$$(n, \delta, w_\parallel, w_\perp, \gamma(0), \beta(0), \lambda'(0), \mu'(0)) = (1/2, -0.01, 0, 0, \pi/2, 0, 1, 1)$$

We see that the type II and III universes continue to expand to the end of the numerical simulation, and that the scale factors evolve very differently. The numerical simulation (and therefore the universe) is bounded by singularities in β and γ.

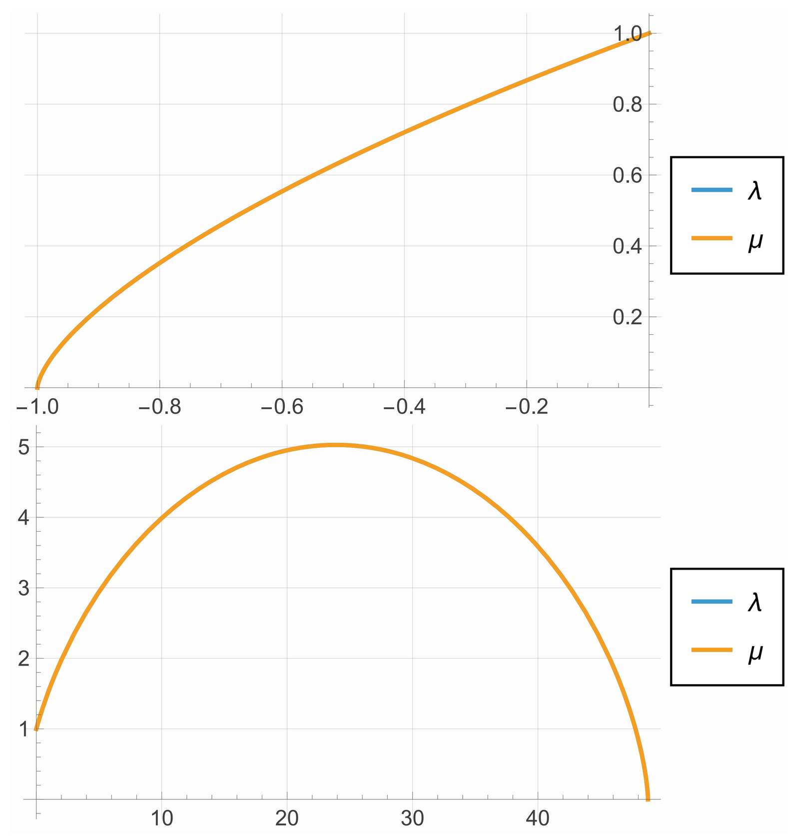
Type IX is somewhat richer in behaviour. By choosing the initial values appropriately, one can have a universe with differing scale factors:



The scale factors of unrestricted type IX from $t = -1$ to $t = 0$, from $t = 0$ to the end of the numerical simulation ($t \approx 49$), and the scale factor ratio λ/μ for the entire timeline.
Initial parameters:

$$(\delta, w_\parallel, w_\perp, \gamma(0), \beta(0), \lambda'(0), \mu'(0)) = (-0.01, 0, 0, \pi/2, 0.01, 1, 1)$$

or a universe with identical scale factors:



The scale factors of restricted type IX from $t = -1$ to $t = 0$ and from $t = 0$ to the end of the numerical simulation ($t \approx 49$).
Initial parameters:

$$(\delta, w_\parallel, w_\perp, \gamma(0), \beta(0), \lambda'(0), \mu'(0)) = (-0.01, 0, 0, -\pi/2, 0, 1, 1)$$

Here, both universes start and end with a singularity. In both cases γ begins at a finite value and ends at $-\infty$, but β begins at ∞ and ends at $-\infty$. This indicates that the type IX universe is also bounded by singularities in β.

Take home message

Bianchi geometries have great potential in modelling anisotropic and also (with Bianchi type IX) isotropic universes.

References

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[3] A. Golovnev and T. Koivisto, "Cosmological perturbations in modified teleparallel gravity models", Journal of Cosmology and Astroparticle Physics **2018**, arXiv:1808.05565 [gr-qc], 012–012 (2018).

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