Weil contact elements and contact correspondences

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Weil bundles — definition

Let M be a smooth manifold and let $C^\infty(M)$ denote the $\mathbb{R}$-algebra of smooth functions $f\colon M \to \mathbb{R}$. For each point $p \in M$, the evaluation map

$$\mathrm{ev}_p\colon C^\infty(M) \to \mathbb{R}, \qquad \mathrm{ev}_p(f) = f(p),$$

is clearly an $\mathbb{R}$-algebra homomorphism. Indeed, it preserves addition, multiplication, and the unit.
The map $\Phi\colon M \to \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), \mathbb{R})$, $p \mapsto \mathrm{ev}_p$, is injective: if $p \neq q$, choose a smooth bump function f with $f(p) \neq f(q)$; then $\mathrm{ev}_p(f) \neq \mathrm{ev}_q(f)$.
To see that Φ is surjective, let $H\colon C^\infty(M) \to \mathbb{R}$ be any $\mathbb{R}$-algebra homomorphism. Since H is unital, its image is $\mathbb{R}$ and hence $\ker H$ is a maximal ideal of $C^\infty(M)$. A standard fact about smooth functions on a manifold says that every maximal ideal of $C^\infty(M)$ is of the form

$$\mathfrak{m}_p = \{f \in C^\infty(M) : f(p) = 0\}$$

for a unique point $p \in M$. Thus $\ker H = \mathfrak{m}_p$. For any $f \in C^\infty(M)$, the function $f - f(p)$ vanishes at p, so it belongs to $\ker H$. Consequently,

$$0 = H(f - f(p)) = H(f) - f(p),$$

because H maps constants to themselves. Hence $H(f) = f(p) = \mathrm{ev}_p(f)$, so $H = \mathrm{ev}_p$. This proves that the evaluation map gives a bijection

$$M \cong \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), \mathbb{R}).$$

This point of view has a useful generalization. For any commutative unital $\mathbb{R}$-algebra A, define the set of A-points of M by

$$M^A := \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), A).$$

For $A = \mathbb{R}$, the above bijection gives $M(\mathbb{R}) \cong M$. For the algebra of dual numbers $\mathbb{D} = \mathbb{R}[\varepsilon]/\langle \varepsilon^2 \rangle$, one obtains

$$M^{\mathbb{D}} \cong TM,$$

the tangent bundle of M. More generally, for any Weil algebra A, M^A is the Weil bundle associated to A, often denoted $T^A M$.

Categorical viewpoint

The preservation of the product:
If we denote by **WA** the category of Weil algebras and Weil algebra homomorphisms, then the problem of classification of all product preserving functors was solved in works of Kainz and Michor, Luciano and Eck in 1980's and reads as follows.
Product preserving bundle functors from the category **Mf** *of manifolds into the category* **FM** *of fibered manifolds are in bijection with objects of* **WA** *and natural transformations between two such functors are in bijection with the morphisms of* **WA***.*
Known examples: T^r_k (velocities), $T^r_k T^s_l$ (iterations). Our research by Weil approach covers even more general functors.

Contact correspondences

A correspondence between M_1 and M_2 is a set $C \subseteq M_1 \times M_2$. The projections of C are $p_i : (x_1, x_2) \mapsto x_i$ for $i = 1, 2$, where $(x_1, x_2) \in C$. A correspondence C between manifolds is a Plücker correspondence if $p_i C = M_i$ for $i = 1, 2$ and for all $x \in M_i$ the fiber $p_i^{-1} x$ is a (not necessarily connected) differentiable $(m-1)$-dimensional manifold embedded in $M_1 \times M_2$.
By constant correspondences we mean Plücker correspondences preserving contact elements. Contact correspondences were studied by Heinrich Guggenheimer. However, he focused primarily on the cases of the functors T^r_1. A description for Weil functors appears to be a new, interesting, and potentially applicable topic. It may also be beneficial for field theory.

Weil Bundles — An Informal Explanation

Figure 1: André Weil in Paris in 1953

Differential geometry studies objects that generalize curves and surfaces to arbitrary dimensions: it uses the notion of a manifold for them. When we look at a manifold locally, only in a neighborhood of some point, we find that it actually looks like a piece of $\mathbb{R}^n$, although it may be curved in some way, even though we can straighten that piece. We can imagine this using a piece of a circle. Isn't it actually just a bent line segment that we can straighten effortlessly? However, when we look at a manifold globally, as a whole, there can generally be no question of straightening it.
The need to have tangent vectors in geometry is, of course, closely connected with mechanics. For motion along some path on a manifold, tangent vectors are nothing other than velocity vectors.
We want to attach tangent vectors at each point of a manifold, that is, in fact, a vector space called the tangent space. We can then consider the union of these vector spaces, more precisely their disjoint union, which is called the tangent bundle. An elegant way of doing this was found by the French mathematician André Weil (1906–1998), who described it in 1953 in the now famous paper *Théorie des points proches sur les variétés différentiables*. Here Weil speaks of a manifold obtained by enriching the original manifold with so-called *points proches*, neighboring, close (infinitely close) points.
This is a procedure that belongs to the foundations of differential geometry. At each point of a manifold, we "hang" another manifold, which we shall call the typical fiber. This gives rise to a new manifold.
The tangent bundle in its algebraic construction is based on the algebra of dual numbers $\mathbb{D}$. Generalizing this to an arbitrary Weil algebra A, we obtain the Weil bundle, which plays a fundamental role in modern differential geometry.

Weil contact elements

Let a Weil algebra A has the width $\mathrm{w}(A) = k < m = \dim M$ and the order $\mathrm{ord}(A) = r$. Every A-velocity V determines underlying $\mathbb{D}^1_k$-velocity $\underline{V}$. We say V is *regular*, if $\underline{V}$ is regular, i.e. having maximal rank k (in its local coordinates). Let us denote $\mathrm{reg}\, T^A M$ the open subbbundle of $T^A M$ of regular velocities on M. The *contact element of type A* or briefly the *Weil contact element* on M determined by $X \in \mathrm{reg}\, T^A M$ is the equivalence class

$$\mathrm{Aut}\, A_M(X) = \{\phi(X); \phi \in \mathrm{Aut}\, A\}.$$

We denote by $K^A M$ the set of all contact elements of type A on M. Then

$$K^A M = \mathrm{reg}\, T^A M / \mathrm{Aut}\, A$$

has a differentiable manifold structure and

$$\mathrm{reg}\, T^A M \to K^A M$$

is a principal fiber bundle with the structure group $\mathrm{Aut}\, A$. Moreover, $K^A M$ is a generalization of higher order contact elements bundle

$$K^r_k M = \mathrm{reg}\, T^r_k M / G^r_k$$

introduced ever since Claude Ehresmann.