INTERACTION OF FLUIDS DESCRIBED THROUGH RELATIVE MOTION AND APPLICATION TO BIANCHI TYPE-I SPACETIMES

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Perfect Fluid Action

perfect fluid action with thermodynamics described through Lagrange multipliers [1]

$$S_{\mathrm{m}} = \int d^4x \Big[-\sqrt{-g}\rho(n,s) + J^\mu \left(\varphi_{,\mu} + s\vartheta_{,\mu} + \beta_A \mathcal{X}^A{}_{,\mu}\right) \Big]$$

ρ - energy density
n - particle number density
- defined through body metric $N^{AB} = g^{\mu\nu}\mathcal{X}^A{}_{,\mu}\mathcal{X}^B{}_{,\nu}$ as $n \propto \sqrt{\det N}$

$\mathcal{X}^A$ - Lagrange (material) coordinates of volume elements, $A = 1, 2, 3$
s - entropy per particle
$J^\mu = \sqrt{-g}j^\mu$, where j^μ is particle flux density $j^\mu = nu^\mu$, and u^μ is 4-velocity field of volume elements

φ, ϑ, β_A - Lagrange multipliers:

variation with respect to $\varphi \Longrightarrow$ continuity equation $j^\mu{}_{;\mu} = 0$
φ is a Lagrange multiplier for conservation of the number of particles.
chemical free energy: $f = u^\mu\varphi_{,\mu}$

variation with respect to $\vartheta \Longrightarrow$ entropy exchange constraint $(sj^\mu)_{;\mu} = 0$
ϑ is a Lagrange multiplier for conservation of the entropy exchange.
temperature: $T = u^\mu\vartheta_{,\mu}$, chemical potential: $\mu = sT + f$

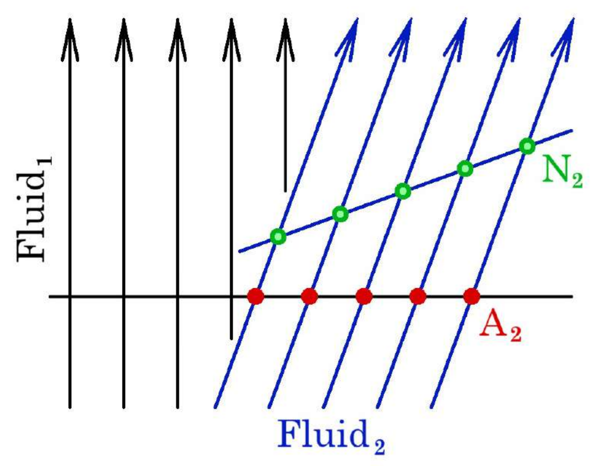
variation with respect to $\beta_A \Longrightarrow u^\mu \mathcal{X}^A{}_{,\mu} = 0$
β^A are Lagrange multipliers for correspondence between 4-velocities of the volume elements and Lagrange coordinates. β^A are constant along worldlines of volume elements of the fluid.

variation with respect to $g^{\mu\nu} \Longrightarrow$ stress-energy tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$

Interaction Constructed Through Cross-Projections

alternative definition of the body metric: $N^{AB} = (g^{\mu\nu} + u^\mu u^\nu)\,\mathcal{X}^A{}_{,\mu}\mathcal{X}^B{}_{,\nu}$ (projection)

two fluids: body metrics N_1^{AB} and N_2^{AB} defined by projections to hypersurfaces perpendicular to u_1^μ and u_2^μ, and interaction described through A_1^{AB} and A_2^{AB} defined by cross-projections



$$N_1^{AB} = (g^{\mu\nu} + u_1^\mu u_1^\nu)\,\mathcal{X}^A_{1,\mu}\mathcal{X}^B_{1,\nu}$$
$$N_2^{AB} = (g^{\mu\nu} + u_2^\mu u_2^\nu)\,\mathcal{X}^A_{2,\mu}\mathcal{X}^B_{2,\nu}$$
$$A_1^{AB} = (g^{\mu\nu} + u_2^\mu u_2^\nu)\,\mathcal{X}^A_{1,\mu}\mathcal{X}^B_{1,\nu}$$
$$A_2^{AB} = (g^{\mu\nu} + u_1^\mu u_1^\nu)\,\mathcal{X}^A_{2,\mu}\mathcal{X}^B_{2,\nu}$$

(non-thermodynamic part of) action describing two interacting fluids

$$S_{\mathrm{m}} = \int \sqrt{-g}d^4x \Big[-\rho_1(n_1, a_1) - \rho_2(n_2, a_2) \Big]$$
$$n_1 = n_0\sqrt{\det N_1} \quad n_2 = n_0\sqrt{\det N_2} \quad a_1 = n_0\sqrt{\det A_1} \quad a_2 = n_0\sqrt{\det A_2}$$

The interaction is described through quantities a_1 and a_2 that measure the local particle density of one fluid through the metric tensor induced on hypersurfaces perpendicular to the 4-velocity field of the other fluid – the particle density of one fluid measured in reference frames of volume elements of the other fluid – as opposed to the standardly defined particle density of a fluid n measured in reference frames of its own volume elements.

Stress-Energy Tensor

variation of the action with respect to the spacetime metric yielding

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} = T_{1\mu\nu} + T_{2\mu\nu}$$
$$T_{1\mu\nu} = \left(n_1\frac{\partial\rho_1}{\partial n_1} + \frac{3}{2 + (u_1\cdot u_2)^2}a_1\frac{\partial\rho_1}{\partial a_1}\right)(u_{1\mu}u_{1\nu} + g_{\mu\nu}) - \rho_1 g_{\mu\nu}$$
$$= (\rho_1 + p_1)(u_{1\mu}u_{1\nu} + g_{\mu\nu}) - \rho_1 g_{\mu\nu}$$
$$T_{2\mu\nu} = \left(n_2\frac{\partial\rho_2}{\partial n_2} + \frac{3}{2 + (u_1\cdot u_2)^2}a_2\frac{\partial\rho_2}{\partial a_2}\right)(u_{2\mu}u_{2\nu} + g_{\mu\nu}) - \rho_2 g_{\mu\nu}$$
$$= (\rho_2 + p_2)(u_{2\mu}u_{2\nu} + g_{\mu\nu}) - \rho_2 g_{\mu\nu}$$

scalar product of two 4-velocities expressible through relative speed of two fluids

$$u_1\cdot u_2 = -\frac{1}{\sqrt{1 - v^2}} = -\gamma(v)$$

pressure-to-energy density ratio of each fluid

$$w_1 = \frac{n_1}{\rho_1}\frac{\partial\rho_1}{\partial n_1} + \frac{1 - v^2}{1 - \frac{2}{3}v^2}\frac{a_1}{\rho_1}\frac{\partial\rho_1}{\partial a_1} - 1, \qquad w_2 = \frac{n_2}{\rho_2}\frac{\partial\rho_2}{\partial n_2} + \frac{1 - v^2}{1 - \frac{2}{3}v^2}\frac{a_2}{\rho_2}\frac{\partial\rho_2}{\partial a_2} - 1$$

Application to Bianchi Type-I Spacetimes

spacetime metric with residual isotropy in y-z-plane

$$ds^2 = e^{2\alpha}\Big[-d\eta^2 + \left(e^{-4\sigma} - \gamma^2\right)dx^2 + e^{2\sigma}\left(dy^2 + dz^2\right) \Big]$$

action with fluid$_1$ driving the expansion and fluid$_2$ causing small anisotropy

$$S_{\mathrm{m}} = \int \sqrt{-g}d^4x \left\{ -\frac{\rho_0}{1+\epsilon}\left[\left(\frac{n_1}{n_0}\right)^{w_1+1} + \epsilon\left(\frac{n_2}{n_0}\right)^{\omega_2+1}\left(\frac{a_2}{n_0}\right)^{\lambda_2} \right] \right\}$$

fluid$_2$ moving with speed v in x-direction and slowing down with friction

$$\frac{d\chi}{d\tau} = -\,(\text{a positive constant})\; n_1\,\chi, \qquad \chi = \frac{v}{\sqrt{1 - v^2}}$$

speed v decreasing from initial v_0 to a limit value v_{lim}

$$\chi = \frac{v_0}{\sqrt{1 - v_0^2}}\exp\left\{ \ln\left(\frac{v_0}{v_{\mathrm{lim}}}\frac{\sqrt{1 - v_{\mathrm{lim}}^2}}{\sqrt{1 - v_0^2}}\right)\left[\left|\frac{\eta}{\eta_0}\right|^{\frac{3(w_1-1)}{3w_1+1}} - 1\right] \right\}$$

anisotropy parameter describing evolution of small anisotropy

$$\delta = \frac{\dot\sigma}{\dot\alpha} = \epsilon\,(\text{a constant})\;\widehat{\delta}, \qquad \widehat{\delta} = \frac{3w_1 + 1}{2}\widehat{\eta}\frac{d\widehat{\sigma}}{d\widehat{\eta}}$$

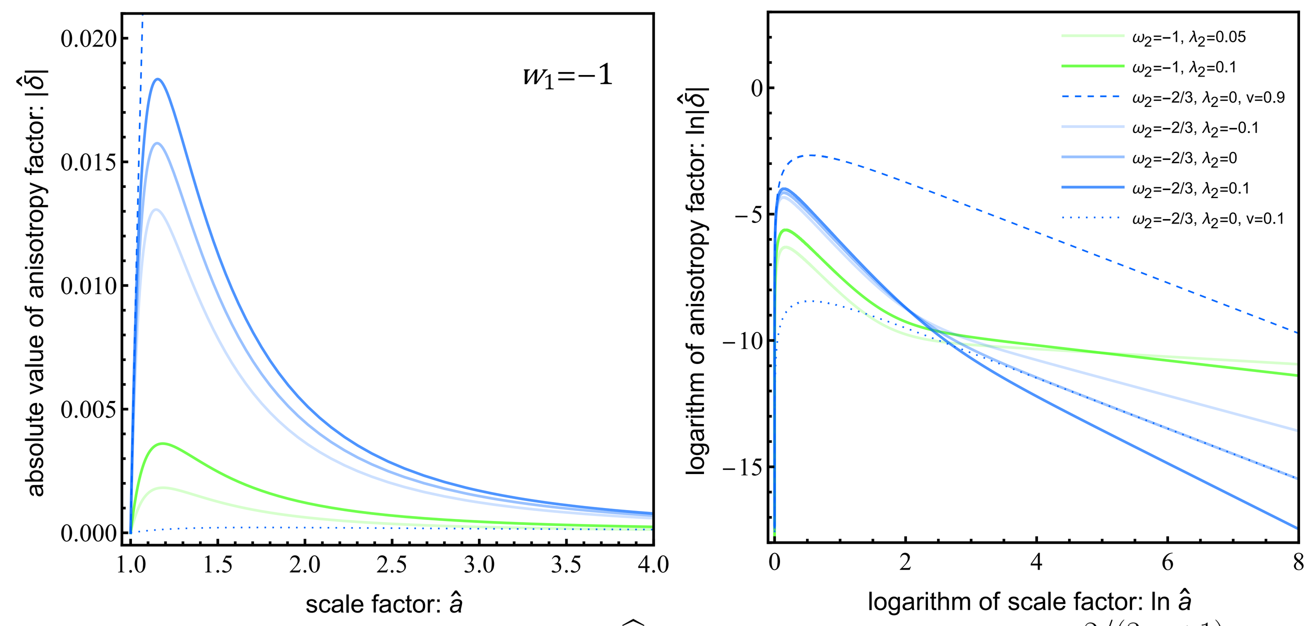


Figure 1: Dependence of the anisotropy parameter $\widehat{\delta}$ on the scale factor $\widehat{a} = |\widehat{\eta}|^{2/(3w_1+1)}$ for $w_1 = -1$ and various values of ω_2 and λ_2. On the right panel, the natural logarithms of these quantities capture the late-time asymptotic behavior. Dashed lines correspond to constant relative speed $v = 0.9$, dotted lines are for $v = 0.1$, and full lines represent cases with the relative speed decreasing from 0.9 to 0.1 in the late-time limit.

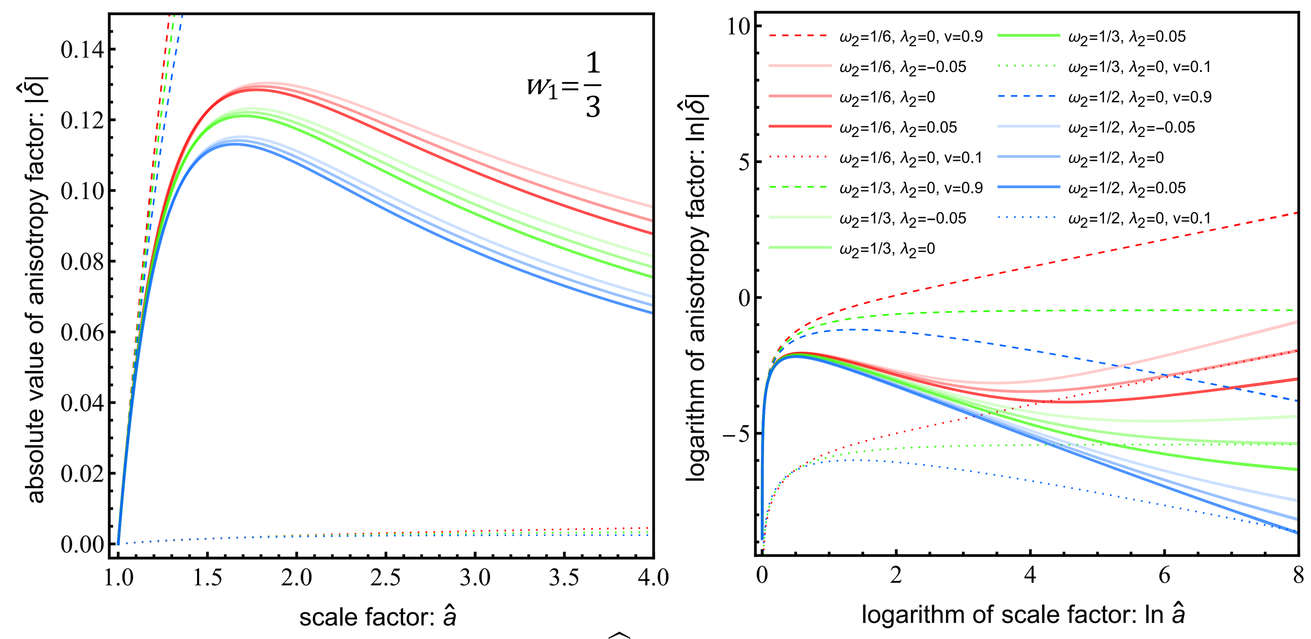


Figure 2: Dependence of the anisotropy parameter $\widehat{\delta}$ on the scale factor $\widehat{a}$ for $w_1 = 1/3$ and various values of ω_2 and λ_2, with the same conventions as in figure 1.

Novelty of the Approach in a Broader Context

Interactions of a fluid with other matter fields or with gravity are usually described in terms of Lagrange multipliers [2], algebraic couplings [3], or through scalar products of particle fluxes [4, 5] or 4-velocities [6, 7] of different fluids.

We modified the body metric N^{AB}, finding pressure-to-energy density ratio being dependent on the scalar product of 4-velocities of the two fluids $u_1\cdot u_2$.

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